Superkurtosis

Degiannakis, Stavros and Filis, George and Siourounis, Grigorios and Trapani, Lorenzo

Panteion University of Social and Political Sciences, Greece, Bournemouth University, UK, Brown University, US, University of Nottingham, UK

16 October 2019

Superkurtosis

Stavros Degiannakis* **George Filis**** **Grigorios Siourounis**[†] **Lorenzo Trapani**[‡]

**Panteion University of Social and Political Science, Bank of Greece*
e-mail: stavros.degiannakis@gmail.com

***Bournemouth University*
e-mail: gfilis@bournemouth.ac.uk

[†]*Brown University*
e-mail: grigorios_siourounis@brown.edu

[‡]*University of Nottingham*
e-mail: lorenzo.trapani@nottingham.ac.uk

Abstract: Risk metrics users assume that the moments of asset returns exist, irrespectively of the trading frequency, hence the observed values of these moments are used to capture the potential losses from asset trading (e.g. with Value-at-Risk (VaR) or Expected Shortfall (ES) calculations). Despite the fact that the behavior of traditional risk metrics is well-examined for high frequency data (e.g. at daily intervals), very little is known on how these metrics behave under Ultra-High Frequency Trading (UHFT). We fill this void by firstly examining the existence of the daily and intraday returns moments, and subsequently by assessing the impact of their (non)existence in a risk management framework. We find that the third and fourth moments of the distribution of asset returns do not exist. We next use both real and simulated data to show that, when daily trading is implemented, VaR or ES deliver estimates in line with what the theory predicts. We show, however, that when UHFT is considered, assuming finite higher order moments, potential losses are much bigger than what the theory predicts, and they increase exponentially as the trading frequency increases. We argue that two possible explanations affect potential loses; first, the exponential increase in the sample data points at UHFT; second, the fact that the data, which are sampled from a heavy-tailed distribution, tend to have higher sample moments than the theory suggests - we call this phenomenon *superkurtosis*. Our findings entail that traditional risk metrics are unable to properly judge capital adequacy. Hence, the use of risk management techniques such as VaR or ES, by market participants who engage with UHFT, impose serious threats to the stability of financial markets, given that capital ratios may be severely underestimated.

Keywords: ultra-high frequency trading, risk management, finite moments, superkurtosis
JEL Classification: C12, C54, F30, G10, G15, G17.

1. Introduction

Highly sophisticated algorithms and fast computer technology have originated a new class of trading known as Ultra-High Frequency Trading (UHFT). UHFT has numerous advantages: it offers a great deal of liquidity in the market; it facilitates the instantaneous transmission of information into prices, pushing markets to be more efficient; and it creates a market place for small (retail) as well as large investors (institutions).
However, UHFT also presents unique challenges,[1] having been criticised as liable to cause large market crashes[2] which may be amplified by the influx of algorithmic trading and the order clustering caused by unintended trading strategy coordination (Beddington et al., 2012). Hence, regulators[3], economists (e.g. Kirilenko and Lo, 2013) and law scholars (e.g. Yadav, 2015) have proposed measures to curb UHFT. As a consequence, market participants are required to measure and report several market risk metrics, and to take them into account when calculating their regulated capital requirements.[4,5] For instance, ECB is required to impose capital requirements (via the Capital Requirements Regulation) on institutions who engage in UHFT. Such requirements are based on risk metrics or asset volatility[6]. However, standards set by regulators are based on risk metrics that are calculated - at most - at daily frequency. Given that UHFT takes place at higher frequencies, this leaves the market risk generated by UHFT largely as a dark pool.
Very little is known about market risk associated with UHFT, and similarly little analysis has been conducted on how traditional risk metrics such as Value at Risk (VaR, hereafter) or Expected Shortfall (ES) behave at such frequencies. Nevertheless, the investigation of this issue is of immense importance, which stems from the fact that risk metrics users assume that high order moments of asset returns in ultra-high frequency are finite, hence the empirical estimation of these moments can be used to compute the VaR and ES. In turn, such risk metrics computations can adequately capture capital adequacy. We note that the validity of such important assumption has not been formally tested thus far. In this study, we fill this gap, firstly by investigating the existence of moments of intraday currency returns, and subsequently by assessing the impact of moments (non)existence in a risk management framework.[7] Specifically, we test for the existence of the first four moments, using ultra-high frequency data from the currency markets. We find that the distribution of the returns of the assets traded under UHFT does not have finite moments of order higher than 2, implying that only the mean and the variance exist at intraday frequencies. Despite this, high-frequency traders use the empirical estimation of these moments to compute their VaR or ES. Our findings, however, imply that the VaR or ES is infinite when calculated with data frequencies higher than daily, or, to put it equivalently, that the potential capital loss implied by VaR or ES is unlimited in the presence

[1]See for example the Final Project Report from The Government Office for Science, London - 2012.

[2]See for example Bloomberg article in April 21st, 2015 by Silla Brush, Tom Schoenberg and Suzi Rin: How a Mystery Trader With an Algorithm May Have Caused the Flash Crash and Kirilenko et al. (2017) for a suggested solution.

[3]See for example the Press Release, European Parliament, MEPs Vote Laws to Regulate Financial Markets and Curb High Frequency Trading (Apr. 15, 2014).

[4]For example, on January 16th, 2016 the Basel Committee on Banking Supervision published a document that revised standards for minimum capital requirements for Market Risk.

[5]Consistent with the policy rationale underpinning the Committee has three consultative papers on the Fundamental review of the trading book. (i) Fundamental review of the trading book, May 2012, (ii) A revised market risk framework, October 2013 and (iii) Fundamental review of the trading book: Outstanding issues, December 2014.

[6]For details see https://www.bankingsupervision.europa.eu/press/publications/newsletter/2019/html/ssm.nl190213-5.en.html

[7]The focus on the foreign exchange market is justified by the fact that in September 2018 the committee report of the Bank of International Settlements sets the tone of the magnitude of the so-called fast-paced electronic markets (FPMs) for heavily levered markets, like the foreign exchange. Trading has not only become increasingly electronic and automated, but the use of ultra-high frequency algorithmic trading and machine learning protocols are increasing like never before. In the foreign exchange market more than 65% of the more than 5trillion dollars daily volume is traded from algorithms. More than 80% of Central Banks monitor FPMs operations and 60% is for market stability reasons alone.

of UHFT.
We find that this is due to two sources: first, the sheer sample size, which affects potential losses. Secondly, we find clear numerical evidence that, at higher frequencies, data sampled from a distribution which has heavy tails tend to have higher sample moments than the theory would suggest; this, in turn, affects the potential losses associated with VaR or ES. More specifically, to confirm this striking empirical finding, a series of simulations are generated aiming to disentangle possible sources of the extreme (infinite) potential losses. The simulated data shows that, on one hand, the estimated kurtosis grows as the sampling frequency increases (as theory would predict), yet, on the other hand, the growth pattern exhibits a much faster rate than the theory would predict. Thus, kurtosis diverges much faster from its theoretical path as the sampling frequency increases, contributing significantly to the exponential increase of the potential losses at UHFT. Hence, the simulated data lend strong support to our empirical conclusions.
We call this phenomenon *superkurtosis*; it implies that traditional risk measures are not a good metric for the true market risk (see also Bradley and Taqqu, 2003), and should therefore not be employed to gauge capital adequacy under UHFT.
The remainder of the paper is organised as follows. Section 2 describes the methods used for the test of moments, as well as, for the assessment of the impact of these moments on risk management. Section 3 presents our empirical results from the real data, as well as, the simulated data. Finally, Section 4 concludes the study and provides ideas for future research.

2. Methodology

Our analysis is based on two steps. We start by verifying whether higher order moments exist (Section 2.1); we then turn to assessing the impact of potential non-existence of high order moments on VaR (Section 2.2).

2.1. Testing for the existence of the asset returns' moments.

We test for the existence of up to the fourth moment of a random variable X, given a sample $\{x_t\}_{t=1}^{T}$, using the test proposed by Trapani (2016) for

$$\begin{cases} H_0 : E\left|X\right|^k = \infty \\ H_A : E\left|X\right|^k < \infty \end{cases}, \tag{2.1}$$

with $k = 2$, 3 and 4. In (2.1), the null hypothesis is the *non-existence* of the k-th absolute moment. Following the guidelines in Trapani (2016), for each k, test statistics are based on

$$\mu_k = c_k \times \frac{T^{-1}\sum_{t=1}^{T}\left|x_t\right|^k}{\left(T^{-1}\sum_{t=1}^{T}\left|x_t\right|^p\right)^{k/p}} \tag{2.2}$$

where $p = \min\{k-1, 2\}$ and

$$c_k = \begin{cases} \frac{4}{\pi} & \text{when } k = 2 \\ 1 & \text{when } k = 3 \\ \frac{1}{3} & \text{when } k = 4 \end{cases}. \tag{2.3}$$

Some comments on (2.2) and (2.3) are in order. The first statistic to be employed is μ_2, which has been designed to test for $H_0 : E\left|X\right|^2 = \infty$ - i.e., the non-existence of the variance. When $k = 2$, the sample second moment (at the numerator) is made scale-invariant by dividing by the square of the mean absolue value of x_t; other rescalings would be possible (chiefly, the median, which has

the advantage of being well-defined), but the simulations in Trapani (2016) show that the mean absolute value yields better power and size. For $k = 2, 3$, rescaling is done using the sample variance, as is more natural. Turning to the multiplicative constants c_k, these follow the guidelines in Trapani (2016), where each sample moment is rescaled by the corresponding sample absolute moment of a standard normal distribution.
Based on (2.2)-(2.3), we construct the statistic

$$\psi_k = \exp(\mu_k) - 1. \tag{2.4}$$

Allowing for weak dependence in the sample, Trapani (2016) showed that

$$P\left\{\omega : \lim_{T\to\infty} \psi_k = \infty\right\} = 1, \text{ under } H_0 : E|X|^k = \infty, \tag{2.5}$$
$$P\left\{\omega : \lim_{T\to\infty} \psi_k = 0\right\} = 1, \text{ under } H_A : E|X|^k < \infty. \tag{2.6}$$

Under H_0, ψ_k diverges to positive infinity instead of having a limiting distribution. Thus, we randomise it to produce a test statistic which has a well-defined limiting law, using the following algorithm.

Step 1 Randomly generate an $i.i.d.$ $N(0,1)$ sample of size $R = \lfloor N^{1/2} \rfloor$, say $\left\{\xi_j^{(k)}\right\}_{j=1}^R$, independently across k, and define $\left\{\psi_k^{1/2} \times \xi_j^{(k)}\right\}_{j=1}^R$.

Step 2 For $u = \left\{-\sqrt{2}, \sqrt{2}\right\}$, generate $\zeta_{j,n}^{(k)}(u) = I\left(\psi_k^{1/2} \times \xi_j^{(k)} \leq u\right)$, $1 \leq j \leq r$.

Step 3 For each u, define

$$\vartheta_{n,R}^{(k)}(u) = \frac{2}{\sqrt{R}} \sum_{j=1}^{R} \left[\zeta_{j,n}^{(k)}(u) - \frac{1}{2}\right], \tag{2.7}$$

and finally the test statistic

$$\Theta_{n,R}^{(k)} = \frac{1}{2}\left[\left(\vartheta_{n,R}^{(k)}\left(-\sqrt{2}\right)\right)^2 + \left(\vartheta_{n,R}^{(k)}\left(\sqrt{2}\right)\right)^2\right]. \tag{2.8}$$

Following Horváth and Trapani (2016), it holds that

$$\Theta_{n,R}^{(k)} \xrightarrow{d^*} \chi_1^2, \text{ under } H_0, \tag{2.9}$$
$$R^{-1}\Theta_{n,R}^{(k)} \xrightarrow{P^*} 1, \text{ under } H_A, \tag{2.10}$$

as $T \to \infty$ for almost all realisations of $\{x_t\}_{t=1}^T$. In (2.9) and (2.10), "$\xrightarrow{d^*}$" and "$\xrightarrow{P^*}$" denote convergence in distribution and in probability, respectively, with respect to P^*, defined as the probability conditional on the sample $\{x_t\}_{t=1}^T$.

2.2. Assessing the impact of (non)existence of moments in risk management.

We consider a representative trader with unlimited capital, who wants to calculate her VaR measure at each point in time. Without loss of generality, each trading day t is divided in τ equidistant intraday subintervals. The observed prices at day t are denoted as P_{t_j}, for $j = 1, 2, ..., \tau$, with sample frequency defined as $m = \tau^{-1}$. We define daily log-returns as $y_t = logP_{t_\tau} - logP_{(t-1)_\tau}$, and intraday log-returns as $y_{t_j} = logP_{t_j} - logP_{t_{j-1}}$.

The VaR for a long trading position at $(1-p)$ level of confidence, at sampling (trading) frequency m, is denoted as $VaR^{(m)}_{(1-p)}$, defined such that $P(y_{t_j} \leq VaR^{(m)}_{(1-p)})$. The $VaR^{(m)}_{(1-p)}$ is computed non-parametrically as the p-quantile of the intraday log-returns at sampling frequency m: $VaR^{(m)}_{(1-p)} = f_p\left(\{y_{t_j}\}^{t=1,\ldots,T}_{j=1,\ldots,\tau}\right)$[8]. We measure the potential losses conditional to a VaR violation (i.e. the losses that occur when the returns are lower than the VaR measure) by constructing an evaluation function, $l^{(m)}_{t_j}$ that measures the absolute distance between actual returns, y_{t_j}, and the VaR measure, i.e. the potential loss ($l^{(m)}_{t_j}$):

$$l^{(m)}_{t_j} = \begin{cases} |y_{t_j} - VaR^{(m)}_{(1-p)}| & \text{if } y_{t_j} < VaR^{(m)}_{(1-p)} \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

We note that the expected shortfall, recently proposed as an alternative risk measure, is the expectation of the potential loss ($\mathbb{E}(l^{(m)}_{t_j})$), given that the VaR violation is present.

The total potential losses over the sample period are computed as $L^{(m)} = \sum_{t=1}^{T}\sum_{j=1}^{\tau} l^{(m)}_{t_j}$. To allow comparison across the different sampling frequencies we also compute the daily adjusted losses per VaR violation as $\bar{L}^{(m)} = 1361(Nm)^{-1}L^{(m)}$, for $N = \sum_{t=1}^{T}\sum_{j=1}^{\tau} I^{(m)}_{t_j}$, where:

$$I^{(m)}_{t_j} = \begin{cases} 1 & \text{if } y_{t_j} < VaR^{(m)}_{(1-p)} \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

We multiply the number of violations by the daily adjustment, $1361/m$, where 1361 reflects the 1-minute observations per day that the market is open.

3. Empirical findings and simulations

In Section 3.1 we carry out an empirical exercise where we *(i)* check whether our data have heavy tails or not (by testing for the existence of up to the fourth moment), and *(ii)* evaluate the potential losses when using a strategy that fails to acknowledge that higher order moments do not exist. We find that, when considering ultra-high frequency data, higher order moments such as the kurtosis and the third absolute moment do not exist; and that potential losses tend, in such cases, to be much higher than anticipated. Subsequently, in Section 3.2, we analyse the causes underpinning such stylised fact. In particular, we assess the impact of the sample size on potential losses, showing that as this increases, so do the potential losses. Further, we assess the impact of having heavy tails on the potential losses; strikingly, we find that sample moments, when population moments do not exist, tend to be much higher than the theory would suggest, which we show is particularly true for the sample kurtosis.

3.1. Data analysis

3.1.1. Data description

We use 1-minute data of the front-month futures contracts for the EUR/USD, GBP/USD and CAD/USD exchange rates, obtained from TickData. The period of the study spans from August 1, 2003 to August 5, 2015. We focus on the G10 exchange rate market as it is considered continuously

[8] The VaR of intraday log-returns y_{t_j} for a short trading position at $(1-p)$ level of confidence at sampling frequency m is $P\left(y_{t_j} \geq VaR^{(m)}_{(1-p)}\right) = (1-p)$.

trading, so that we do not have to take into consideration significant data alterations. The choice of the specific currency pairs is justified by the fact that *(i)* they are among the most liquid, and *(ii)* they represent the most heavily traded exchange rate for financial transactions. Our sample consists of 3028 trading days, which contain more than 16 million 1-minute data.
Table 1 presents the descriptive statistics of the currency pairs' returns for sampling (trading) frequencies from 1-minute to 1-day. We note that volatility is falling linearly as the sampling frequency is increasing. By contrast, the third moment (skewness) does not see to change materially from symmetry at the different frequencies. More importantly, the forth moment (kurtosis) shows the well documented leptokurtosis, which increases exponentially as the sampling frequency increases. Of course, the magnitude of kurtosis at the higher frequencies differs among the different crosses, with the higher being observed in the EUR/USD.

[Insert Table 1 around here]

3.1.2. Testing for the existence of higher order moments

We start our analysis with results on the existence of moments in Table 2.[9]

[Insert Table 2 around here]

The results show that only the second moment exists across all currencies and sampling frequencies. By contrast, for all intraday sampling frequencies across the three currency pairs, the null hypotheses of non-existence of the third and fourth moments cannot be rejected at conventional significance levels. Moving to lower sampling frequencies, i.e. daily, we find evidence that higher order moments exist.

3.1.3. The impact of the non-existence of moments in UHFT

The results presented in Table 2 seem to suggest that risk management tools, such as the VaR or ES, that assume the existence of moments, might be inappropriate to assess the true underline risk of loss in the ultra-high frequencies. To ascertain this, we consider a scenario where traders assume that moments do exist at the intraday frequencies. Under this scenario, we calculate the potential losses ($L^{(m)}$ and $\bar{L}^{(m)}$), conditional to VaR violation, as shown in Section 2.2 (see Figure 1).

[Insert Figure 1 around here]

In all cases considered, potential losses are decidedly higher for the higher sampling frequencies, while they decrease gradually as the sampling frequency decreases. This holds for both the total potential losses ($L^{(m)}$) and the daily adjusted potential losses ($\bar{L}^{(m)}$). For instance, in the 1-minute sampling frequency of the EUR/USD we observe that a trader would have lost 48 times more capital than anticipated by the VaR, whereas the daily adjusted losses, for the same frequency, are about 25% more. Similar figures are reported for the GBP/USD and CAD/USD, as well.
Despite the fact that we have implied that the non-existence of moments, and in particular, this of kurtosis, could be the source of the exponential potential losses as the sampling frequency increase, we must note that the actual source of these losses is unclear. To shed light in this respect we perform an experiment with simulated data, isolating different effects to pin down the most important source of the association between potential losses and frequency of trading.

[9] We have also considered lower frequencies than the daily (e.g. weekly, biweekly and so on) and the results show that all moments continue to exist. Also, we only report results for the EUR/USD exchange rate, as numbers for the other currencies are almost exactly the same. All these unreported results are available from the authors upon request.

3.2. Evidence from simulated data

3.2.1. Assessing the impact of the sample size on potential losses

We analyse - via a simulation exercise - the impact of the sample size on potential losses, considering a Gaussian data generating process where we simulate the prices P_t, for $t = 1, ..., 10^6$ as

$$P_t = P_{t-1} + z_t, \tag{3.1}$$

with $z_t \overset{i.i.d.}{\sim} N(0, 1/h^2)$ and initial price P_1=\$1000. The $h^2 \ni \mathbb{R}^+$ enables us to control for the magnitude of variance. The total sample is split in 1000 trading days with 1000 intraday prices. Without any loss of information, the sample $\{P_t\}_{t=1}^{10^6}$ mimics the P_{t_j} for $j = 1, ..., 1000$ and $t = 1, ..., 1000$, at sampling (or trading) frequency $m = 10^{-3}$. For example, at 1-minute sampling frequency, or $m = 10^{-3}$, we have $\tau = 1000$ intraday prices, for $T = 1000$ trading days, at 2-minutes sampling frequency, we have $\tau = 500$ intraday prices, for $T = 1000$ trading days, and so on.
Figure 2 illustrates the $VaR^{(m)}_{(95\%)}$, the kurtosis of log-returns, $y_t = log(P_t/P_{t-1})$, and the potential losses $L^{(m)}$, $\bar{L}^{(m)}$ for the simulated log-returns. We have set $h = 10$; we note however that, in unreported experiments, very similar results were found when using different values of h. The x-axis presents the trading frequency in minutes; from 1 up to 120 minutes. Without any loss of generality, we assume a long trading position, hence the 95% VaR measure is computed as the 5% quantile point of the empirical distribution of y_t[10].

[Insert Figure 2 around here]

The simulated results provide us with an important finding; the potential losses are dependent on the trading frequency, where higher trading frequency leads to higher potential losses, despite the fact that the kurtosis (upper right graph) is around the value of 3 across any trading frequency $m = 1, ..., 120$. So, such finding could simply suggest that the increased potential losses are not related with the effect of *superkurtosis*, but rather they are the artifact output of the effect of number of observations which increases exponentially as the sampling frequency increases.
In order to shed further light on the impact of the sample size, we investigate the analytical form of the total potential losses.
The potential loss $L^{(m)}$ is given by

$$L^{(m)} = \sum_{t=1}^{T}\sum_{j=1}^{\tau}\big(|y_t - VaR^{(m)}_{(1-p)}| \times I_{\{y_t < VaR^{(m)}_{(1-p)}\}}\big), \tag{3.2}$$

for $I_{\{.\}}$ representing the indicator variable and we compute the expected value of total potential losses, $\mathbb{E}(L^{(m)})$. For our long trading position and under the condition $y_t < VaR^{(m)}_{(1-p)}$, we have $y_t < 0$ and $VaR^{(m)}_{(1-p)} < 0$, as well. So, for $y_t < VaR^{(m)}_{(1-p)}$, we have $|y_t - VaR^{(m)}_{(1-p)}| = -(y_t - VaR^{(m)}_{(1-p)})$. Hence:

$$\mathbb{E}(L^{(m)}) = \sum_{t=1}^{T}\sum_{j=1}^{\tau} E\big(|y_t - VaR^{(m)}_{(1-p)}| \times I_{\{y_t < VaR^{(m)}_{(1-p)}\}}\big) = -\sum_{t=1}^{T}\sum_{j=1}^{\tau} p\mathbb{E}\big(y_t - VaR^{(m)}_{(1-p)} | y_t < VaR^{(m)}_{(1-p)}\big). \tag{3.3}$$

Note also that $\mathbb{E}(I_{\{y_t < VaR^{(m)}_{(1-p)}\}}) = p$. As $\mathbb{E}(y_t | y_t < VaR^{(m)}_{(1-p)}) = ES^{(m)}_{(1-p)}$ and $\mathbb{E}(VaR^{(m)}_{(1-p)} | y_t < VaR^{(m)}_{(1-p)}) = VaR^{(m)}_{(1-p)}$, we can show that:

[10] For a short trading position, the 95% quantile point of the empirical distribution would have been considered.

$$\mathbb{E}(L^{(m)}) = -T\tau p(ES^{(m)}_{(1-p)} - VaR^{(m)}_{(1-p)}). \tag{3.4}$$

Under the data generated process, we can prove that[11],

$$VaR^{(m)}_{(1-p)} = \Phi_{(1-p)}\left(\mathbb{E}(y_t) = 0, V(y_t) = \left(\frac{1}{P_1}\sqrt{\frac{m}{h^2}}\right)^2\right) \tag{3.5}$$

and

$$ES^{(m)}_{(1-p)} = \frac{\sigma}{1-p}\varphi\Big(\Phi_{(1-p)}(0,1)\Big), \tag{3.6}$$

where φ denotes the standard normal probability density function and $\sigma = \frac{1}{P_1}\sqrt{\frac{m}{h^2}}$. Hence, the $VaR^{(m)}_{(1-p)}$ measure equals to the $(1-p)$ percentile point of the inverse cumulative normal distribution with zero mean and $\frac{1}{P_1}\sqrt{\frac{m}{h^2}}$ standard deviation (we should note to the reader that $m = \tau^{-1}$ is the intraday sampling (trading) frequency).

So, as presented in Figure 3, the total potential losses per trading frequency m are affected not only by the number of observations, $T\tau$, but also from the distance between the two risk measures; i.e. the expected shortfall and the VaR, i.e. $(ES^{(m)}_{(1-p)} - VaR^{(m)}_{(1-p)})$. Therefore, as the trading frequency increases, *(i)* the number of trades increases exponentially (see the effect of size in Figure 3), while *(ii)* the distance between the two risk measures decreases (see the expected shortfall and value-at-risk in Figure 3). Indicatively, some estimates of (3.4) are $\mathbb{E}(L^{(1)}) = 2.089$ and $\mathbb{E}(L^{(120)}) = 0.1907$, for $ES^{(1)}_{(5\%)} - VaR^{(1)}_{(5\%)} = 4.2E-5$ and $ES^{(120)}_{(5\%)} - VaR^{(120)}_{(5\%)} = 4.6E-4$.

[Insert Figure 3 around here]

Thus, both analytical and simulated evidence show that the total potential losses per trading frequency m are affected positively by the number of trades and negatively by the distance between the two risk measures. Having established the aforementioned effects, in the paragraphs that follow, we investigate whether there are any additional effects from kurtosis.

3.2.2. Assessing the impact of heavy tails on potential losses

We now turn to investigating the effect of heavy tails on potential losses; in particular, we assess the impact of excess kurtosis. To this end, we consider a heavy tail data generating process, as follows:

$$P_t = P_{t-1} + z_t, \tag{3.7}$$

for $z_t \overset{i.i.d.}{\sim} t(0, 1/h^2, v)$, for $v \geq 2$ and initial price of P_1=\$1000. The probability density function for Student t is considered as: $\varphi(0, 1/h^2, v) = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})\sqrt{\frac{\pi v}{h^2}}}\left(1 + \frac{z_t^2 h^2}{v}\right)^{-\frac{v+1}{2}}$.

Figure 4 shows the values of the sample kurtosis and the potential losses $L^{(m)}$, $\bar{L}^{(m)}$ for the simulated log-returns, from the Student t random walk data generated process under various degrees of freedom, $v = 3, 4, 5, 10, 20, 30, 100$ and $h = 10$; clearly, the population kurtosis does not exist for the first two values.

[Insert Figure 4 around here]

[11] The proof is available upon request.

We note that, for the near-Gaussian cases $v = 30, 100$, the kurtosis remains constant across various values of trading frequency, and, as expected, the potential losses $L^{(m)}$ and $\bar{L}^{(m)}$ are almost identical to those found in the previous section. As is natural to expect, as the degree of freedom v decreases, the sample kurtosis increases; for example, for $v = 10$, the kurtosis of log-returns at 120-minutes trading frequency is around 3.1, whereas for $v = 3$ the kurtosis of log-returns at 120-minutes trading frequency is around 4.5. However, Figure 4 shows a remarkable feature of the sample kurtosis. Heuristically, when $v \leq 4$, the sample kurtosis should pass to infinity as the sample size $T\tau \to \infty$. By the Marcinkiewicz-Zygmund Strong Law of Large Numbers, the sample kurtosis should pass to infinity at a rate given by (modulo some slowly varying sequence) $O\left((T\tau)^{\frac{4-v}{v}}\right)$.
For example, when $v = 3$, the sample kurtosis should diverge to infinity as fast as (approximately) $O\left((T\tau)^{1/3}\right)$. Based on these heuristic considerations, when e.g. $v = 3$, one could expect the sample kurtosis for the 1-minute trading frequency to be larger by a factor $120^{1/3}$ than the sample kurtosis calculated for the 120-minutes trading frequency, keeping T constant. On the contrary, from Figure 4, it is apparent that the sample kurtosis is bigger than the theory would predict. For example, for $v = 3$, the 1-minute sampling frequency log-returns have kurtosis of 150, which is approximately 30 times as much as in the 120-minute sampling frequency case, whereas the theory would predict an increase of approximately 5 times only.

Turning now to the relationship between kurtosis and potential losses at the highest trading frequency of 1-minute, we document that a kurtosis of 4 (as derived for $v = 10$ degrees of freedom) implies potential losses of $L^{(1)} = 2.78$ and $\bar{L}^{(1)} = 0.080$, whereas a kurtosis of 150 (as derived for $v = 3$) increases the potential losses to $L^{(1)} = 7.58$ and $\bar{L}^{(1)} = 0.218$. Therefore, it is obvious that the potential losses are heavily affected by the presence of kurtosis.
Both simulated and analytical results strengthen our findings, from the real data, that the potential losses are dependent on the trading frequency, where higher trading frequency leads to higher potential losses. Most importantly, though, we observe the effect of *superkurtosis*, which is also evident in the real data, to influence radically the potential losses. Put it differently, the shape of the potential losses from the simulated data do resemble those of the actual data. Thus, as the sampling (trading) frequency increases, the effect of *superkurtosis* becomes more material and the potential losses increase substantially.

We should highlight that we have further experimented with the previous simulation for $P_1 = 1$ in order to mimic the exchange rate values, as well as, using a conditionally heteroscedastic data generating process and the results remain robust. For brevity we do not show the results here but rather they are available upon request.

4. Conclusions

It is common practice for risk metrics users to assume that the moments of asset returns exist and thus their observed values can be used to adequately capture the potential losses from asset trading, using metrics such as the VaR or ES. However, we do not know whether this assumption holds under UHFT and if so, what the likely consequences are.
In this paper, we studied the potential losses arising from UHFT, as these are assessed by a risk management metric, such as VaR or ES, which are based on the assumption that the higher moments of asset returns exist, i.e. they are finite. Using both real data and a simulation exercise, we (rather unsurprisingly) found that, when high frequency trading is implemented (daily trading), VaR estimates deliver as expected, with potential losses being in line with what the theory predicts. Results, conversely, were far more striking when ultra-high frequency trading (intraday) was considered. Indeed, we found that in such a case the potential losses are much bigger than what the theory would predict. In order to provide some explanation of this stylised fact, we tried to disentangle

its possible causes. We found that there are two concurring features that can explain our findings. Firstly, the sheer sample size: as the sampling frequency increases, so does the number of datapoints; our results show that this increases the potential loss. Secondly, ultra-high frequency data are more likely to have heavy tails, as our empirical analysis shows. In particular, the fourth moment of high frequency data is invariably found not to exist.

This is not merely at odds with the VaR or ES frameworks, which postulate the existence of these moments: in our simulations we find that the sample kurtosis computed using datasets whose population fourth moment does not exist tends to be higher than what the theory would predict, which in turn, exponentially increases the potential losses. Put it differently, moving from high frequency trading to UHFT, asset returns have infinite high order moments (due to heavy-tailed distributions), which leads to massively higher potential losses, rendering the survival of a given investment at high risk. We call this discrepancy between the theoretical behaviour and the actual magnitude of the sample kurtosis *superkurtosis*, and we argue that it is one component of the heavy potential losses which, as a stylised fact, are encountered when trading at ultra-high frequency under the (implicit in the use of VaR or ES) assumption that high order moments do exist when in fact they do not.

Our findings have profound implications for the use and regulation of UHFT: employing traditional risk measures for market participants who engage with UHFT imposes serious threats to the stability of the financial markets, given that capital ratios may be severely underestimated. Indeed, based on our analysis, it follows that the hidden risks from UHFT from the foreign exchange market alone in the financial system are immense and leverage must be reduced substantially (e.g. by at least 45 times) so that risk metrics like VaR or ES match the Central Bank's capital adequacy preset levels for financial stability reasons.

In light of our findings, we conclude that it would be very important to accumulate additional evidence in this line of research, by replicating our results for other asset classes, as well as, portfolios of assets. These extensions are currently under investigation by the authors.

References

Beddington, J., C. Furse, P. Bond, D. Cliff, C. Goodhart, K. Houstoun, O. Linton, and J.-P. Zigrand (2012). Foresight: the future of computer trading in financial markets: final project report.

Bradley, B. O. and M. S. Taqqu (2003). Financial risk and heavy tails. In *Handbook of heavy tailed distributions in finance*, pp. 35–103. North-Holland.

Horváth, L. and L. Trapani (2016). Statistical inference in a random coefficient panel model. *Journal of Econometrics 193*(1), 54–75.

Kirilenko, A., A. S. Kyle, M. Samadi, and T. Tuzun (2017). The flash crash: High-frequency trading in an electronic market. *The Journal of Finance 72*(3), 967–998.

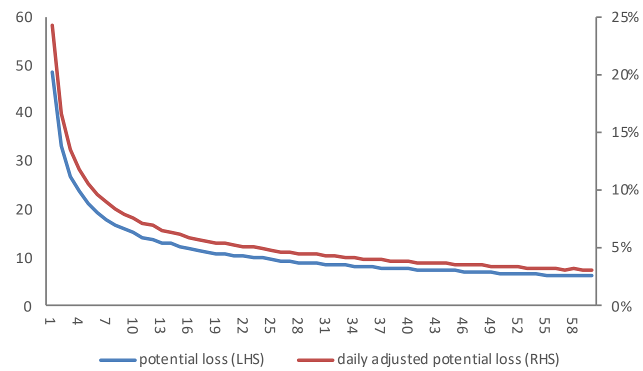
Kirilenko, A. A. and A. W. Lo (2013). Moore's law versus murphy's law: Algorithmic trading and its discontents. *Journal of Economic Perspectives 27*(2), 51–72.

Trapani, L. (2016). Testing for (in) finite moments. *Journal of Econometrics 191*(1), 57–68.

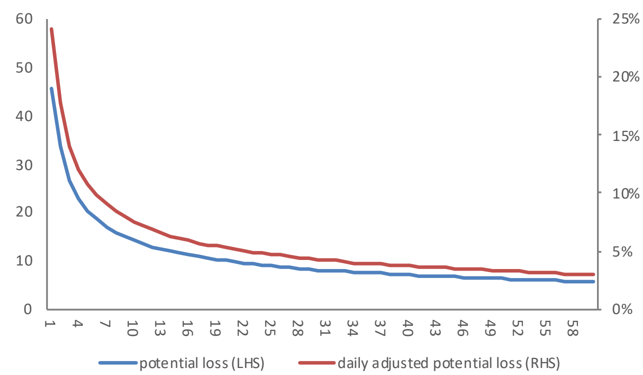
Yadav, Y. (2015). How algorithmic trading undermines efficiency in capital markets. *Vanderbilt Law Review 68*, 1607.

FIG 1. *Trader's potential capital losses, on the three exchange rates, above the anticipated losses from the VaR measure.*

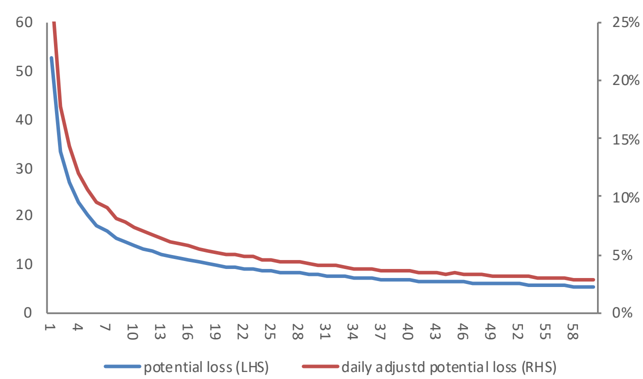
Euro/Dollar



Pound/Dollar

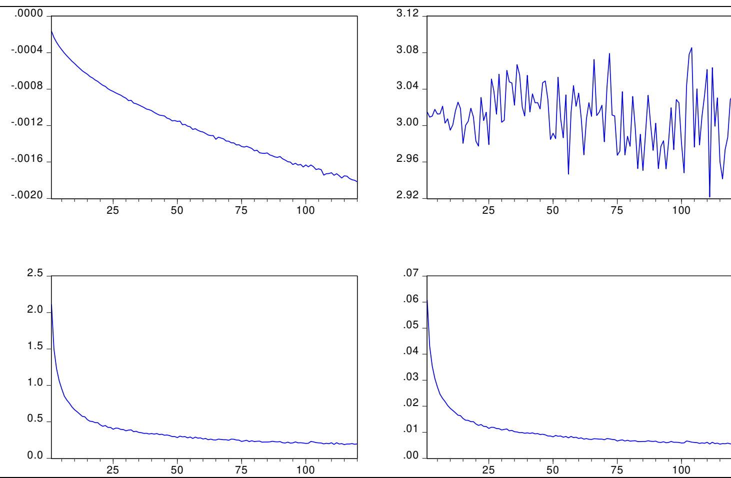


Canadian/Dollar



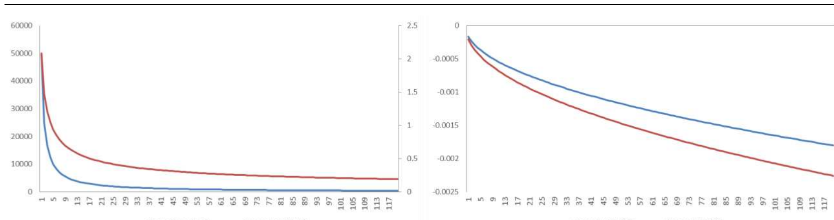
Note: The x-axis denotes the 1 to 60 minute intraday sampling (trading) frequencies. The values in the y-axis refer to number of times, whereas in the z-axis refers to percentages.

FIG 2. *The Value-at-Risk, kurtosis and potential losses for the random walk data generated process.*



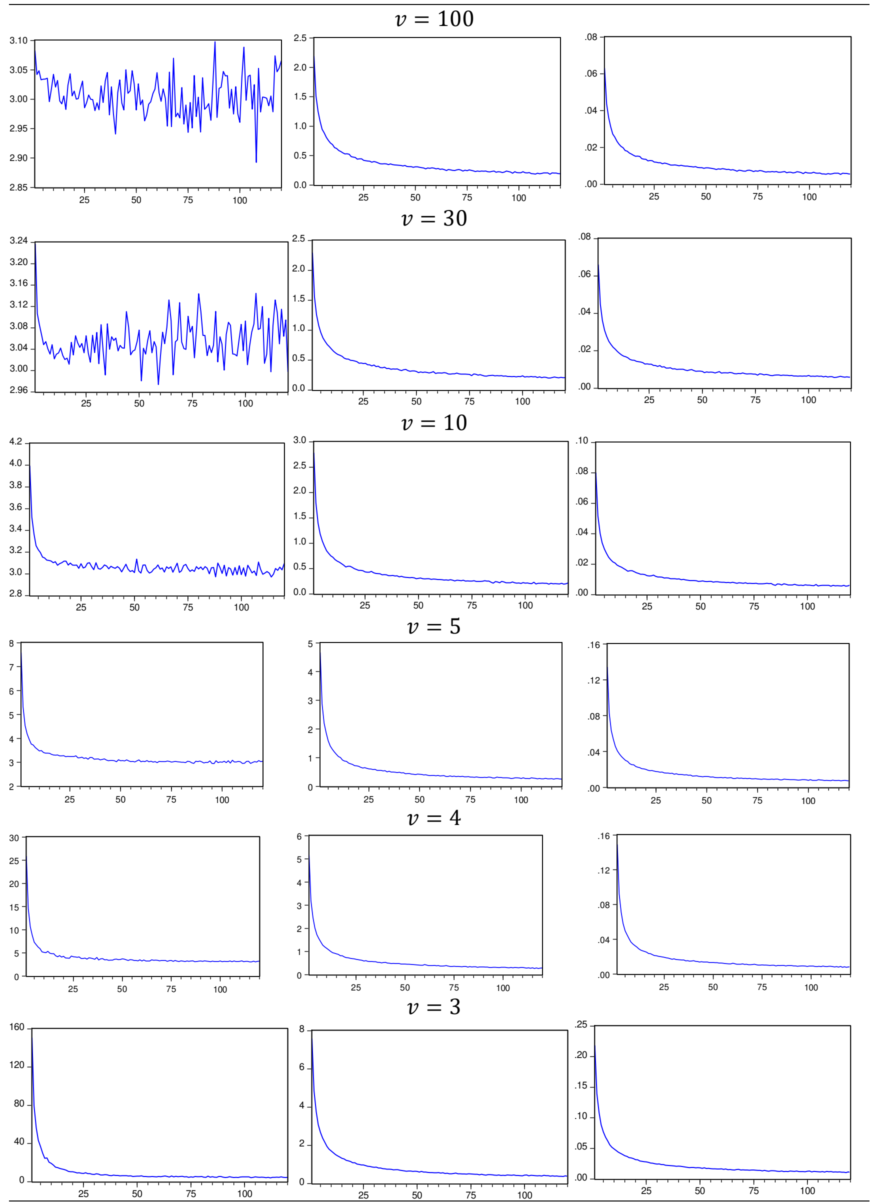
Note: The $VaR^{(m)}_{(95\%)}$ (upper left), the kurtosis of y_t (upper right) and the potential losses $L^{(m)}$ (lower left), $\bar{L}^{(m)}$ (lower right) for the simulated log-returns $y_t = log(P_t/P_{t-1})$, for $P_t = P_{t-1} + z_t$, $z_t \overset{i.i.d.}{\sim} N(0, 1/h^2)$, $P_1 = \$1000$ and $h = 10$. The x-axis denotes the 1 to 120 minute intraday sampling (trading) frequencies.

FIG 3. *The total potential losses, the number of observations ($T\tau$), the expected shortfall ($ES^{(m)}_{(1-p)}$), and the Value-at-Risk ($VaR^{(m)}_{(1-p)}$), under the random walk data generated process, per sampling frequency $m = 1, ..., 120$.*



Note: The left panel shows the ($\mathbb{E}(L^{(m)})$) and the number of observations ($T\tau$), per trading frequency $m = 1, ..., 120$, whereas the right panel shows the ES and the VaR, per trading frequency $m = 1, ..., 120$. The x-axis denotes the 1 to 120 minute intraday sampling (trading) frequencies.

Fig 4. *The total potential losses and kurtosis from the Student t random walk data generated process under various degrees of freedom.*



Note: The kurtosis of y_t (left panel), the potential losses $L^{(m)}$ (middle panel) and the $\bar{L}^{(m)}$ (right panel) from the Student t random walk data generated process under various degrees of freedom, v, $P_t = P_{t-1} + z_t$, $z_t \overset{i.i.d.}{\sim} t(0, 1/h^2, v)$, $P_1 = \$1000$, $h = 10$. The x-axis denotes the 1 to 120 minute intraday (trading) sampling frequencies.

Table 1
Descriptive statistics of the three currency pairs at different sampling (trading) frequencies

Frequency	*Mean*	*Std.Dev.*	*Skewness*	*Kurtosis*
		EUR/USD		
1m	1.34E-08	0.000187	1.775107	748.3108
2m	2.68E-08	0.000261	1.362421	408.7735
5m	6.70E-08	0.000405	0.894493	171.4304
10m	1.34E-07	0.000565	0.645733	92.2061
15m	2.01E-07	0.000686	0.584515	67.0804
20m	2.68E-07	0.000790	0.412722	51.7350
30m	4.01E-07	0.000958	0.429589	38.8754
60m	8.02E-07	0.001355	0.267877	22.9572
Daily	8.99E-05	0.006458	0.060827	4.3828
		GBP/USD		
1m	-6.52E-09	0.000174	-0.350810	243.3383
2m	-1.30E-08	0.000243	-0.332122	135.5416
5m	-3.25E-08	0.000375	-0.194385	66.0113
10m	-6.49E-08	0.000522	-0.225726	42.2837
15m	-9.76E-08	0.000633	-0.119137	33.5736
20m	-1.30E-07	0.000724	-0.190707	28.6625
30m	-1.95E-07	0.000880	-0.198243	24.7113
60m	-3.93E-07	0.001231	-0.185203	16.4620
Daily	6.22E-05	0.005629	-0.269195	5.12008
		CAD/USD		
1m	1.63E-08	0.000182	-0.009072	81.9100
2m	3.25E-08	0.000252	0.005984	57.4756
5m	8.14E-08	0.000386	-0.051981	34.5443
10m	1.63E-07	0.000534	-0.042633	26.1680
15m	2.44E-07	0.000647	-0.035875	23.4246
20m	3.25E-07	0.000740	-0.078372	20.7301
30m	4.87E-07	0.000897	0.038314	18.3294
60m	9.71E-07	0.001257	0.108383	17.0792
Daily	5.09E-05	0.005742	-0.193507	5.8329

Table 2
Tests for moments existence for the EUR/USD currency pair at different sampling (trading) frequencies

Frequency	**Size**	μ_2		μ_3		μ_4	
		μ_2	*p-value*	μ_3	*p-value*	μ_4	*p-value*
1m	$5,350,729$	0.122	0.00	2313	0.95	1.78×10^6	0.66
2m	$2,675,323$	0.245	0.00	1635	0.28	8.91×10^5	0.67
5m	$1,070,149$	0.613	0.00	1034	0.42	3.56×10^5	0.75
10m	$535,033$	1.223	0.00	731	0.79	1.78×10^5	0.85
15m	$356,719$	1.839	0.00	597	0.36	1.18×10^5	0.71
20m	$269,464$	2.434	0.00	519	0.57	8.98×10^4	0.80
30m	$178,318$	3.679	0.00	422	0.62	5.94×10^4	0.84
60m	$95,140$	6.896	0.05	308	0.56	3.17×10^4	0.90
Daily	$4,035$	0.042	0.00	1.56	0.00	0.99	0.00