Evolution of Tax Evasion

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June 2003

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December 6, 2005

Abstract

In this paper we analyze a tax evasion game with taxpayers learning by imitation. If the authority commits to a fixed auditing probability, a positive share of cheating is obtained in equilibrium. This stands in contrast to the existing literature that yields full compliance of audited taxpayers who are rational, have a lot of information and thus do not need to interact. When the authority adjusts auditing probability every period, cycling in cheating-auditing occurs. Thus, the real life phenomenon of compliance fluctuations is explained within the model rather than by exogenous parameter shifts. *JEL Classification*: C79, H26

Keywords: tax evasion, imitation, learning

1 Introduction

The magnitude and importance of shadow sector is hard to overestimate. Just to mention one case, the official estimate of informal GDP for Russia is about 1/3 of formal GDP in the recent years[1]. A fundamental aspect of the informal activities is *tax evasion*, which is usually defined as an effort to lower one's tax liability in the way prohibited by law. The paper considers exclusively this phenomenon, leaving

*I am grateful to Gregor Langus, David Perez-Castrillo, Rick van der Ploeg and Karl Schlag for valuable comments.

[1] A summary of recent attempts to estimate the size of tax evasion, avoidance and other informal activities is given in Schneider and Enste (2000). The results vary a lot with method and country considered; one common finding is that the shadow sector is growing over time.

tax avoidance and criminal activities aside. Specifically, it is devoted to the income tax evasion, which has received the most attention in the theoretical modeling of tax evasion. Such attention can be partially attributed to the existence of relatively reliable data on this matter (Tax Compliance Measurement Program (TCMP) in the US). Another reason might be tradition founded by the seminal model of Allingham and Sandmo in 1972.

A detailed survey of the models of income tax evasion can be found in Andreoni, Erard and Feinstein (1998). They identify two directions in the modeling of strategic interaction between taxpayers and tax authorities: principal-agent approach (e. g., Vasin and Vasina (2002)) and game theoretic approach (e. g., Reinganum and Wilde (1986), Erard and Feinstein (1994), Peter Bardsley (1997), Waly Wane (2000)). Both approaches treat the taxpayers as a single player maximizing expected payoff of cheating. All the literature in tax evasion we are aware of maintains the assumption of no communication among the taxpayers. In reality however a taxpayer is not an isolated decision maker, she/he lives in a society and constantly interacts with other taxpayers.

This paper is an attempt to relax the assumption of no social interaction among the taxpayers, hence its main feature is an explicit characterization of taxpayers' communication. This is achieved by using the framework of learning in games. The type of learning we use is imitation. Our taxpayers are boundedly rational, and they decide whether to cheat or not cheat depending on payoffs from cheating obtained in the previous period by themselves and by those whom they meet. This can be contrasted with more rational Bayesian updating, for instance when the agents have priors on the probability distribution of auditing intensity and learn more about this distribution through their own play and interaction with others. Our model matches a number of stylized facts about evasion.

First, in reality taxpayers possess poor knowledge of audit rules, usually overestimating the probability of audit (Andreoni et al. 1998). In the model they do not know it, but rather imitate actions of other individuals. Other possible sources of information, such as media, are not considered for the sake of simplicity. Second, another feature of reality - heterogenous information taxpayers possess - is reflected in the initial distribution of strategies between cheating and not cheating. Third, the real world tax authority is an organization that acts in a substantially different way than an individual taxpayer. This organization has resources and incentives to gather all available information, whereas every individual prefers not to incur costs of information collection. The model reflects this asymmetry directly: the tax authority is updating its belief about the distribution of the taxpayers, whereas each of the taxpayers just imitates. Fourth, tax evasion is an intertemporal decision. This is supported among others by the Engel and Hines' study (1999). In our framework the individuals have one-period memory that allows them to choose a strategy tomorrow on the basis of today's observation of the behavior of the others and their own. As the income reporting is a rare (annual) event, short memory can be a plausible assumption.

These four features are considered in a simple version of tax evasion relation with two levels of income and homogenous population of low-income taxpayers. Learning rule used by the members of this population is a simple imitation of a better-performing strategy. We consider three scenarios that vary according to the amount of information of taxpayers and their attitude towards punishment.

The main result of the model is the cycling dynamics it generates. In all scenarios considered, both the share of evading taxpayers and the auditing intensity of tax authority exhibit fluctuations giving rise to stable cycles. The system is cycling around an unstable steady state, in which the share of cheaters (people evading their tax) is the same as in the Nash equilibrium of one-shot game, whereas the auditing probability is not related to its Nash equilibrium value. This happens because in the game the cheating is effectively determined by the rationality of tax authority, whereas the intensity of auditing is actually established by the learning rule and parameters of the game.

From the dynamics generated we can see that in presence of boundedly rational agents the equilibrium play does not actually occur. Therefore, the dynamics is necessary to be taken into account in order to make accurate inference about the welfare effects of various policies. The estimation of such effects, however, requires calibration of the parameters of the model, which is a separate issue.

Our model replicates a number of stylized facts. Firstly, non-zero cheating of audited taxpayers is obtained for the commitment case, which is certainly more plausible than absolute honesty of the most of the conventional principle-agent models (for example, Sanchez and Sobel 1993, Andreoni et al.1998). Secondly, in the non-commitment case the following features of dynamics are explained: decreasing compliance (Graetz, Reinganum and Wilde 1986) and auditing probability (Dubin, Graetz and Wilde 1990, Adreoni at al. 1998, p.820) observed in the US in the second half of XX century, as well as the recent increase in auditing probability with continuing decrease of compliance (Slemrod 2004, p.1). These patterns could not be explained by the literature to date since only static models were used.

Additionally, an alternative explanation for the puzzle of too much compliance is offered. It is largely discussed in the literature that people comply much more than a simple lottery model of evasion predicts. Our results suggest that the reason might not be in the presence of intrinsically honest taxpayers[2], but in the fact that the system is far from the equilibrium. This is best illustrated in the commitment case: if the share of cheating taxpayers is below its equilibrium value, it stays so forever, and it looks as if taxpayers were cheating too little.

The rest of the paper is organized in the following way. Section 2 outlines the simple static game which is then played repeatedly. Dynamics of such play is analyzed in section 3, where different learning rules are considered. Underlying the baseline average payoff rule is the norm of high tolerance of evasion and the assumption that taxpayers share among themselves a lot of information about evasion. The effective punishment rule is less favorable to the cheating, but keeps high informational requirement. The popularity rule is least encouraging for evasion, it also assumes minimum information communicated between taxpayers. Concluding section stresses limitations of the model and outlines its possible extensions and applications. All propositions in the paper bear a letter corresponding to a learning rule: "A" for the average payoff, "E" for effective punishment, and "M" for m agents meeting.

2 Classical game theory: a simple model

As a starting point for modeling dynamics we take a simple one-shot game of tax evasion, based on Graetz, Reinganum and Wilde (1986). Intrinsically honest taxpayers (who can not evade for moral reasons) are eliminated from that model, as their presence does not change the results in the given setup. The timing is as follows:

1. The nature chooses income for each individual from two levels, high H with probability γ and low L with probability $1-\gamma$;

2. Taxpayers report their income, choosing whether to evade or not;

[2] This is how the puzzle is usually resolved in the literature (for references see, e.g., Slemrod (2000)).

3. Tax agency decides whether to audit or not.

It is obvious, that low income people never choose to evade, because they are audited for sure, if they report anything lower than L. At the same time, the high income people can evade, since with a report L tax agency does not know, whether it faces a truthful report by the lower income people, or cheating from the higher income ones. Then the game simplifies to the one between higher income people and tax agency:

	audit	*not audit*
cheat	$(1-t)H - st(H-L),\ tH + st(H-L) - c$	$H - tL,\ tL$
not cheat	$(1-t)H,\ tH - c$	$(1-t)H,\ tH$

where t is an income tax rate, s is a surcharge rate, determining fine for the given amount of tax evaded, c is audit cost; all of them are assumed to be constant and exogenously given for the tax-raising body[3].

Then the tax authority maximizes its expected revenue choosing the probability to audit p given probability of cheating q:

$$p(\frac{q\gamma}{q\gamma + 1 - \gamma}(tH + st(H-L)) + \frac{1-\gamma}{q\gamma + 1 - \gamma}tL - c) + (1-p)tL \tag{1}$$

which is linear in probability due to linearity of audit cost function. The multipliers for the payoffs are probabilities to come across high income cheaters $(q\gamma)$ or low income honest taxpayers $(1-\gamma)$ given that only low income reports are audited.

First order condition holds with equality for the value of

$$q = \frac{1-\gamma}{\gamma}\frac{c}{t(1+s)(H-L) - c}$$

This is the value of cheating probability in the unique mixed strategies Nash equilibrium.

A high income individual maximizes its expected payoff given probability of audit:

$$pq\left((1-t)H - st(H-L)\right) + (1-p)q(H - tL) + (1-q)(1-t)H \tag{2}$$

[3]Endogenous determination of tax and penalty rate is an interesting task, but it constitutes the problem of a government rather than a tax authority. Moreover, it has been largely discussed in the literature, see, for example, Cowell (1990).

which is also linear in probability because of the nature of expected utility. The equilibrium value of p is $\frac{1}{1+s}$.

Simple comparative statics shows that auditing probability is decreasing with fine; the extent of evasion is increasing with costs of auditing and decreasing with fine, tax rate, income differential and share of high income people. Among others, we implicitly assumed here linear tax and penalty schemes, risk neutral individuals, and linear cost function for the tax authority. Even with these strong assumptions, considering the dynamics generated by the game allows to get some non-trivial insights of what could be going on in reality.

3 Dynamics

Now consider the game presented in the previous section played every period from 0 to infinity. As before, the populations of high income and low income taxpayers are infinite size with measures of γ and $1-\gamma$ respectively, and this is a common knowledge. The proportion q_τ of high income population is cheating by reporting low income at time τ, the agency is auditing the low income reports with probability p_τ.

Since we consider two aggregated levels of income, it is plausible to assume that the people know the income of those whom they interact with. In reality the precise amount of income is not known, but whether a given person has high or low income is easily guessed from observable by other taxpayers characteristics. But the authority does not observe these characteristics, so the type of income is private knowledge of the agents who meet. On the other hand, the probability of auditing p is a private knowledge of the authority. Between the rounds the tax agency updates its belief about the distribution of taxpayers between cheating and not cheating, the high income agents are learning whose strategy performs better.

Irrespective of the rule, at time τ there are the following types of high income taxpayers: (i) honest, comprising proportion $1-q_\tau$ of population and receiving payoff $(1-t)H$; (ii) caught cheating, $q_\tau p_\tau$ of population with payoff $(1-t)H - st(H-L)$; (iii) not caught cheating, $q_\tau(1-p_\tau)$ of population with payoff $H - tL$.

Note that the payoff when not cheating is bigger than when cheating and caught, but smaller than when cheating and not caught. The tax agency is maximizing its long-run expected revenue by choosing auditing probability for all periods (commitment), or its expected revenue in the next period by choosing auditing probability for the next period (no commitment).

As it has already been mentioned, we can not take a ready aggregate dynamics for the population of taxpayers because of the asymmetric nature of the players: in the no commitment case the tax authority is using miopic best response, whereas taxpayers imitate each other. Without such asymmetry, our game resembles emulation dynamics as it is defined by Fudenberg and Levine (1998), which is known to converge to replicator dynamics under some assumptions. However, these assumptions are not satisfied in our setup: most strikingly, each individual communicates with more than one other. Due to this feature, even payoff monotonicity of the aggregate dynamics can not be established. Thus, we have to derive aggregate dynamics every time from the elementary imitation rules.

In order to proceed we have to specify these learning rules.

3.1 Meeting two others: Average payoff principle

An agent A meets agents B and C, each of them has played strategies s_A, s_B, s_C. If $s_A \neq s_B = s_C$, and average payoff of B and C is greater than the payoff of A, he/she switches to their strategy. If $s_C = s_A \neq s_B$, A switches in case the payoff of B is greater. The average of caught and not caught payoffs is bigger than the payoff of honest $((1-t)H - st(H-L) + H - tL > 2(1-t)H)$ for plausible value of fine $s < 1$.

As a result, an honest taxpayer (recall that we have $1-q$ of them) switches to cheating, if it faces either two non-caught cheaters (this happens with probability $((1-p)\,q)^2)$ or a non-caught cheater and an honest taxpayer $((1-p)\,q\,(1-q))$. It remains honest with a complementary probability $(1-((1-p)\,q)^2-(1-p)\,q\,(1-q))$. A caught taxpayer (there are pq) switches to honest, if it observes either two honest taxpayers $((1-q)^2)$ or an honest and a caught taxpayer $(2pq(1-q))$. A not caught taxpayer never switches.

Summing up, between rounds $(1-q+qp)^2$ of honest agents remain honest, and $(1-q)(1-q+2qp)$ of caught taxpayers become honest. Thus,

$$1-q_{\tau+1} = (1-q_\tau)\left[(1-q_\tau+q_\tau p_\tau)^2 + p_\tau q_\tau(1-q_\tau+2p_\tau q_\tau)\right] \tag{3}$$

This equation defines the aggregate dynamics of the population we were interested in. As can be seen, the proportion of honest taxpayers tomorrow is completely determined by the proportion of honest taxpayers today and the probability of auditing today[4].

[4] Notice that it has nothing in common with discrete time approximations of replicator dynamics

We want to see what features this dynamics possess, namely, we want to know whether the proportion of honest taxpayers is shrinking or expanding as time passes. For this purpose it proves useful to define a threshold level of cheating $\bar{q}$ as

$$\bar{q} := \frac{2-3p}{1-3p+3p^2}$$

The following proposition shows that the evolution of the cheating share crucially depend on the level of auditing probability. We drop the time subscript τ for convinience wherever all the variables belong to the same period.

Proposition 1A Consider the dynamics of the share of evaders q for the average payoff principle defined by (3). The share of honest taxpayers in the next period is lower than that in this period if auditing probability in this period is $p \in [0, \frac{1}{\sqrt{3}})$. It is higher in the next period if $p \in (\frac{2}{3}, 1]$. For $p \in (\frac{1}{\sqrt{3}}, \frac{2}{3})$ it is lower, if $q < \bar{q}$ and higher, if $q > \bar{q}$.

All proofs are left to the appendix. Unexpectedly, in the small middle interval, change in the proportion of honest taxpayers is negatively related to the number of the honest taxpayers. This "anti-scale" effect is explained by the high enough detection probability, for which the caught cheaters contribute more to the increase of proportion of the honest, than the honest themselves.

Further we consider two cases for the behavior of tax authority. If it is unable to announce its auditing probability and keep it forever, we are in the "game theoretic" framework, and the our dynamics has two dimensions: already derived one for q and another one for p. We start, however, with a more simple case, when the auditors can credibly commit to a certain constant in time strategy (probability), and hence the dynamics is collapsing to one dimension.

3.1.1 Commitment

Assume that the authority commits to a certain auditing probability p once and forever (this corresponds to the principle-agent framework defined by Andreoni et al, 1998). Letting $q_0 \neq 0$ and $q_0 \neq 1$ we obtain from the proposition 1A the following corollary:

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or any other well-known dynamics, as was expected.

Corollary In the evasion game with the average payoff dynamics defined by (3)are $\lim_{\tau\to\infty} q_\tau = 1$ when $p \in \left[0, \frac{1}{\sqrt{3}}\right]$, $\lim_{\tau\to\infty} q_\tau = 0$ when $p \in \left[\frac{2}{3}, 1\right]$, and $\lim_{\tau\to\infty} q_\tau = \bar{q}$ when $p \in \left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$.

Let us define a function $\hat{q}(p) : [0,1] \to [0,1]$ that maps a set of possible auditing probabilities into a set of long-run outcomes of q. As the corollary states, this function has a following form:

$$\hat{q}(p) = \begin{cases} 1, & p \in \left[0, \frac{1}{\sqrt{3}}\right] \\ \bar{q}, & p \in \left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right) \\ 0, & p \in \left[\frac{2}{3}, 1\right] \end{cases}$$

The authority chooses p to maximize its payoff

$$\begin{gathered} \gamma(1-\hat{q}(p))tH + p(\hat{q}(p)\gamma(tH + st(H-L)) + (1-\gamma)tL) \\ -cp(\hat{q}(p)\gamma + 1 - \gamma) + (1-p)(\hat{q}(p)\gamma + 1 - \gamma)tL \end{gathered} \tag{4}$$

Let us call the probability that maximizes thsis expression the optimal auditing probability p^*. If $p^* \in \left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$, it satisfies the first order condition

$$(1+s)(\bar{q} + p\bar{q}') - \bar{q}' = \mu\left[1 - \gamma + \gamma(\bar{q} + p\bar{q}')\right] \tag{5}$$

where $\mu := \frac{c}{t(H-L)}$.

Note that $p^* \notin \left(0, \frac{1}{\sqrt{3}}\right)$ and $p^* \notin \left(\frac{2}{3}, 1\right)$ because for constant q the objective function is linear in p. Furthermore, $p = 1$ is never optimal because the objective function is non-increasing on the interval $\left(\frac{2}{3}, 1\right)$. Hence, the only two possibilities for optimal p are $p^* = 0$ and p^* given by the first order conditions. This is true for all learning rules considered in this paper, since the argument does not use any properties of $q(p)$ apart from differentiability. In the following proposition we look at a condition, which is satisfied whenever the optimal p is positive (this will be called interior solution henceforth).

Proposition 2A With the average payoff learning rule, the necessary condition for interior solution is

$$\frac{-1/p^* + (2 - 3p^*)\,s + 2}{1 - 3p^{*2} + \frac{1}{\gamma}\left(1 - 3p^* + 3p^{*2}\right)} > \mu$$

Notice that p^* in the proposition is by itself depending on the paprameters on the model. Since we know, however, that it is limited by the interval $\left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$ in the case of interior solution, we can still infer from this proposition the direction of parameter changes that would bring about interior solution.

Namely, for high values of c and low values of γ, s, t, H (and hence high rhs and low lhs) "no auditing - all cheating" equilibrium is chosen. Notice that at $p = \frac{1}{\sqrt{3}}$ this condition translates into $\gamma(1+s) > \mu$; for $p = \frac{2}{3}$ into $4.5\frac{\gamma}{1-\gamma} > \mu$. From both inequalities it can be seen that the interior solution does not arise, if γ is small relative to μ. This is easy to interpret: with small share of high income people (γ) and low benefit from auditing (stH) it is better not to audit anybody, given that it is costly (c).

The comparative statics for the interior solution[5] gives the following relations (they hold for other learning rules considered as well):

$$\frac{dp^*}{ds} = \frac{q + p^*\bar{q}'}{\bar{q}'' - (1+s-\mu)\left(2\bar{q}' + p^*\bar{q}''\right)}; \tag{6}$$

$$\frac{dp^*}{d\mu} = -\frac{\bar{q} + p^*\bar{q}' + \frac{1}{\gamma} - 1}{\bar{q}'' - (1+s-\mu)\left(2\bar{q}' + p^*\bar{q}''\right)}; \tag{7}$$

$$\frac{dp^*}{d\gamma} = \frac{\mu/\gamma^2}{\bar{q}'' - (1+s-\mu)\left(2\bar{q}' + p^*\bar{q}''\right)}. \tag{8}$$

where the notation is $\bar{q} = \bar{q}(p^*), \bar{q}' = \bar{q}'(p^*), \bar{q}'' = \bar{q}''(p^*)$. We keep this notation henceforth.

The sign of any of these derivatives is ambiguous, and first we derive the conditions for these derivatives to have expected signs. By 'expected' we mean that the parameter changes act in the same direction for the cases of continuous adjustment in the interior and discontinuous jump on the border. For example, since increase in γ brings about, ceteris paribus, the interior solution to be more likely, we expect it also to increase auditing probability in this solution, that is, we expect $\frac{dp^*}{d\gamma} > 0$. Similarly, we expect $\frac{dp^*}{ds} > 0$ and $\frac{dp^*}{d\mu} < 0$.

Then, for the auditing probability to increase in the proportion of high income people we need $\bar{q}'' - (1+s-\mu)(2\bar{q}' + p^*\bar{q}'') > 0$; for it to increase in the surcharge rate we need in addition $\bar{q} + p^*\bar{q}' > 0$; to decrease in cost - tax collection ratio μ the two conditions above suffice. So, jointly for the expected signs we need

[5] Note that these conditions do not hold on the border, e.g. for $p = \frac{2}{3}$.

$$-2(1+s-\mu)\bar{q}' > \bar{q}''\left((1+s-\mu)\,p^*-1\right); \quad (9)$$
$$\bar{q} > -p^*\bar{q}'. \quad (10)$$

To elaborate more on these relations, we have to use the knowledge of the particular function $\bar{q}(p)$, which is specific to each learning principle.

Proposition 3A For the average payoff principle, the share of cheaters $\bar{q}$ in the interior steady state is a decreasing and strictly convex function of p, e.g. $\bar{q}'(p) < 0, \bar{q}''(p) > 0$.

Using this proposition, from 9 we can immediately say that under the best average principle $\frac{dp^*}{d\gamma} > 0$ for our baseline parameter values (the generalization of this result is presented in the appendix), $s = 0.8$, $t = 0.3$, $\frac{c}{H-L} = 0.06$ ($\mu = 0.2$), $\gamma = 0.5$. The fine is usually up to the amount of tax evaded, and I take 20% less than the whole. The income tax rate ranges from 0.1 to 0.5 across developed countries; the measures for both $\frac{c}{H-L}$ and γ are bound to be arbitrary, since in reality the auditing function depends on many more variables than just income, and there is a continuum of income levels rather than two. A convenient way to think of the first measure is as of what share of audited income has to be foregone for the auditing itself. Andreoni et al. (1998, p. 834) take 0.05 as an example, I think of 0.01 to 0.1 as a possible range. Finally, γ to certain extent reflects the income distribution, and 0.5 gives an extreme case where there is equal number of rich and poor.

The resulting $p^* = 0.62$ brings about $\frac{dp^*}{d\gamma} = 0.02 > 0$. This states that with increase of the share of high income taxpayers (the only ones who can cheat!) the auditing probability in steady state rises, since marginal revenue from auditing goes up, whereas marginal costs stay the same. For baseline parameter values auditing is also increasing in the cost - tax bill ratio μ ($\frac{dp^*}{d\mu} = 0.15 > 0$) and decreasing in the amount of fine ($\frac{dp^*}{ds} = -0.18 < 0$), which is contrary to what was expected. Algebraically, this happens because $\bar{q} < -p^*\bar{q}'$ is always satisfied (the proof is in the appendix).

Intuitively, faced with higher fine or lower auditing costs, the taxpayers will cheat less in steady state, hence there is no need for the tax authority to commit to a higher auditing probability. In fact, this stems from the strong asymmetry in the behavior of tax authority and individuals: the authority is very "smart" in the sense that it can predict the level to which the cheating converges for given auditing probability; the individuals are, to the opposite, very naive, since they just imitate a strategy with higher payoff.

To get a quantitative feeling about the influence of parameters, I plot the auditing probability as a function s for the baseline parameter values $t = 0.3$, $c/H = 0.06$, $\gamma = 0.5$.

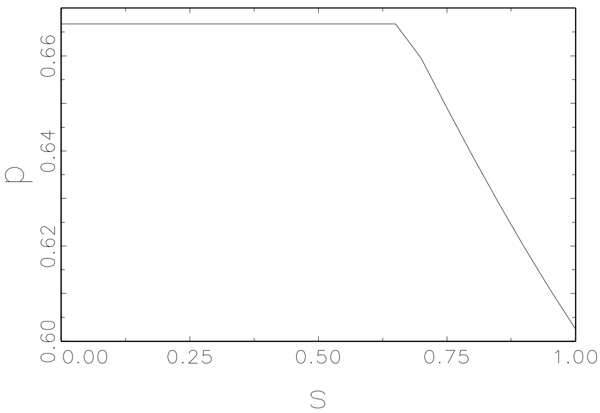


Figure 1. Optimal auditing probability depending on s, baseline parameter values ($\mu = 0.2$).

Comparison with classical game theory The solution obtained can be compared with the Stackelberg-like equilibrium of the classical evasion game, when the tax authority moves first (much weaker asymmetry). Recall, that in this setup $q = 1$ if $p < \frac{1}{1+s}$, $q = 0$ if $p > \frac{1}{1+s}$, and undetermined for the equality. Since auditing is costly, the authority will choose either $p = 0$, $q = 1$, or $p = \frac{1}{1+s}$, $q = 0$. The latter is preferred whenever the auditing is not too costly, namely $c < \frac{\gamma}{1-\gamma}(1+s)t(H-L)$, or, in other terms, $\mu < \frac{\gamma}{1-\gamma}(1+s)$ (analogous to the expression in the dynamic version). Comparative statics is trivial in this setup: zero cheating result is independent of parameter changes as long as they do not violate rather mild condition of relatively not too expensive auditing. Auditing probability is decreasing in the surcharge rate, just as in the previous model. The solution of the static model is discrete, and the probability of audit jumps to zero for high enough μ or low s.

The prediction of the dynamic model appears to be more plausible, since non-zero cheating is not observed in reality. As it is known from the literature, the result of zero cheating in commitment case generalizes for more complicated models with continuum of taxpayers and presence of intrinsically honest taxpayers. Moreover, the commitment models are usually criticized on the basis of this unrealistic prediction.

The model presented eliminates this fault, and allows us to reconsider the view of commitment as something implausible. Then it just boils down to the classical case of dynamic inconsistency, and the willingness to commit is equivalent to the planning horizon of the authorities.

The comparison of the payoffs of tax authority in evolutionary and classical settings is ambiguous ($R(q,p)$ is the tax revenue):

$$R(0, \frac{1}{1+s}) = \gamma tH + (1-\gamma)tL - \frac{1-\gamma}{1+s}$$

$$R(\bar{q}(p), p) = \gamma \frac{3p^2-1}{1-3p+3p^2} tH + p(\frac{2-3p}{1-3p+3p^2}\gamma(tH + st(H-L)) +$$

$$+ (1-\gamma)tL) - cp(\frac{1-3p^2}{1-3p+3p^2}\gamma + 1) + (1-p)(\frac{1-3p^2}{1-3p+3p^2}\gamma + 1)tL$$

For the parameter values chosen ($\gamma = 0.5$, $\mu = \frac{1}{3}$), it is better-off with imitating taxpayers for the magnitude of fine smaller than 0.5 and worse off for the magnitude larger than 0.5. This is quite intuitive, since low (high) values of s result in large (small) auditing probability of static no cheating equilibrium; auditing, in turn, is costly to implement. In dynamic setting the auditing probability for given parameter values hits the upper bound of $\frac{2}{3}$, and hence is independent of the surcharge rate, except for the values of s close to 1. Consequently, "static" revenue is increasing with the fine, whereas the "dynamic" is staying constant.

For the high values of γ the picture remains the same, except that now for very large values of fine the "dynamic" revenue rises so much that it exceeds the "static" one. Finally, with decrease in μ the solution with p strictly less than $\frac{2}{3}$ is obtained for larger and larger set of s values, approaching $s \in (\frac{1}{2}, 1]$. Correspondingly, the superiority of "static" revenue is preserved only at $s = \frac{1}{2}$ in the limit (μ close to 0).

An average taxpayer with high income in Stackelberg setting can only cheat or not cheat with probability one; in the dynamic case there is a possibility of a mixed equilibrium:

$$I(0, \frac{1}{1+s}) = (1-t)H$$

$$I(\bar{q}(p), p) = (1-\bar{q}(p))(1-t)H + p\bar{q}(p)((1-t)H - st(H-L)) + (1-p)\bar{q}(p)(H - tL)$$

This brings about higher "dynamic" payoff for the individual, if $p < \frac{1}{1+s}$, and lower payoff otherwise[6]. This simple result is straightforward: in the classical setup

[6] Evaluating $I(\bar{q}(p), p) - I(0, \frac{1}{1+s})$, we get expression the sign of which depends only on the sign

the equilibrium payoff of taxpayers does not depend on the auditing probability or the magnitude of fine. Hence, the expected payoff in the dynamic model is greater, if the audit probability is lower than in the static model, and visa versa. Note that this does not depend on the learning rule.

Convergence We talk here about dynamic model without explicitly considering the dynamics itself. It is important that even in steady state the picture is very different from Nash (Stackelberg) equilibrium. The question of how long it takes to converge to a steady state escaped our attention so far. As could be expected, the speed of convergence depends on the particularity of the imitation rule. In the present case, the learning procedure is in a sense favorable to the cheaters: it takes a long time to approach no cheating equilibrium, and relatively short time - all cheating one. Starting from the middle ($q = \frac{1}{2}$), getting as close as 0.001 to the steady state takes 597 periods for honesty case and only 14 periods for cheating case.

This result was obtained analytically by iterating the function $q_{\tau+1}(q_\tau, p)$ respective number of times. For the honesty case then $p = \frac{2}{3}$, $q_{597} = 0.001$; for the case of cheating $p = \frac{1}{\sqrt{3}}$, $q_{14} = 0.999$, whereas $q_1 = 0.5$ in both cases.

3.1.2 No commitment

The authority decides on the optimal auditing rule in every period, assuming that the distribution of the taxpayers has not changed from the last period $q_{\tau+1} = q_\tau$ (myopic best response). Then the payoff of the authority is

$$\begin{gathered}\gamma(1-q)tH + p(q\gamma(tH + st(H-L)) + \ (1-\gamma)tL) \\ -c(p(q\gamma + 1 - \gamma)) + (1-p)(q\gamma + 1 - \gamma)tL\end{gathered} \tag{11}$$

Further I consider a linear cost function for the sake of tractability, so that $c(p(q\gamma + 1 - \gamma)) = cp(q\gamma + 1 - \gamma)$.

Then the best response strategy is

$$BR\left(q_\tau\right) = \begin{cases} 0, \text{ if } c > \bar{c} := \frac{q_\tau \gamma t(1+s)(H-L)}{1-\gamma+\gamma q_\tau} \\ 1, \text{ if } c < \bar{c} \end{cases}$$

As the tax authority is very unlikely to jump from not auditing anybody to auditing everybody and back, we explicitly augment the choice of tax agency with inertia

of $1 - p - ps$.

variable:

$$p_{\tau+1} = \alpha BR(q_\tau) + (1-\alpha)p_\tau \tag{12}$$

where α determines speed of adjustment; BR is the best response function, which is defined above as revenue maximizing p given the belief about the distribution of taxpayers. With $\alpha \to 1$, we are back to the case of jumping from 0 to 1 probability; with $\alpha \to 0$, the probability of audit stays very close to an initial level forever. The dynamics is best seen on the picture.

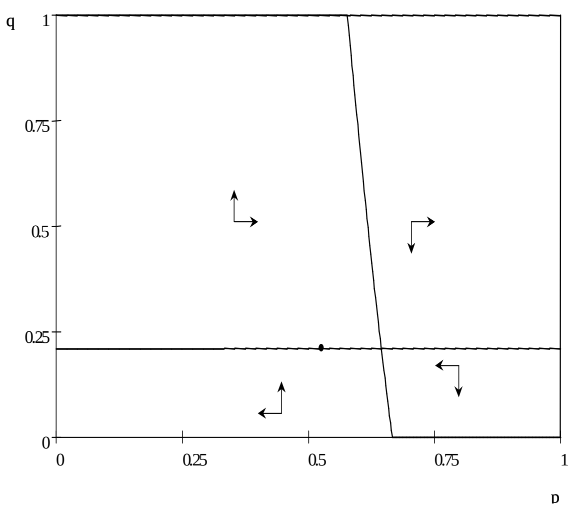


Figure 2. Phase diagram

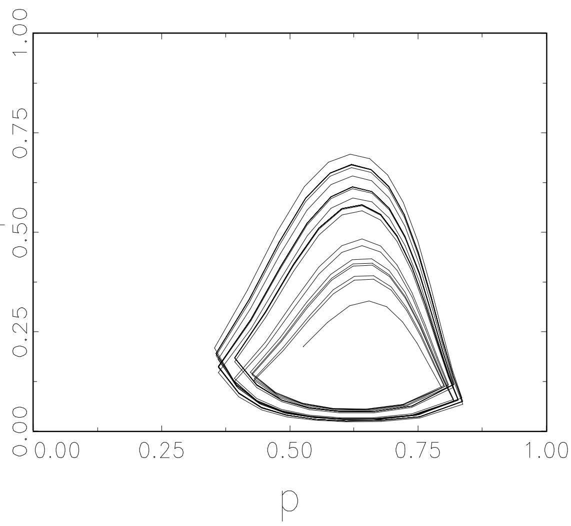


Figure 3. Discrete time dynamics, μ_1= 0.125

Let μ_1 be the level of cheating that induces switch of best response from zero to one or back:

$$\mu_1 := \frac{1-\gamma}{\gamma} \frac{c}{t(1+s)(H-L)-c}$$

It is interesting whether the dynamics we are considering brings about convergence of the system to a steady state with

$$\begin{aligned} q^{ss} &= \mu_1 \\ p^{ss} &= \frac{\mu_1 - 1 + \sqrt{-\frac{1}{3}(\mu_1^2 - 2\mu_1 - 3)}}{2\mu_1} \end{aligned}$$

or the cycling around this point is possible. Simulation results show that in discrete time setup the cycles are observed, whereas in the continuous time the system converges. The latter fact is also shown analytically in the appendix.

Comparative static result for the steady state is possible to obtain because all the parameters are indexed to single μ_1, which is bounded by unit interval.

$$\frac{dp^{ss}}{d\mu_1} = \frac{1}{2\mu_1^2}\left(1 - \left(\frac{1}{\mu_1^2} + \frac{2}{3\mu_1} - \frac{1}{3}\right)^{-\frac{1}{2}}\left(\frac{1}{\mu_1} + \frac{1}{3}\right)\right),$$

which is always negative. Hence, probability of audit in steady state is decreasing in costs of auditing and increasing in the share of high income taxpayers, the tax rate, the magnitude of fine, and the income differential. Compared to the Nash equilibrium, where probability to audit only depends on the surcharge rate, our result looks more plausible.

Still, for all parameters but s and γ the effects are the opposite of those in the commitment model. Whereas it is an open question what horizon a particular tax authority has, we can compare predictions of the two models by their conformability with stylized facts. First, it is almost uniformly accepted that evasion is increasing in the tax rate (See, for example, Clotfelter (1983), Poterba (1987), Giles and Caragata (1999)), so here the commitment model seems to make a better job. Second, there is also a weak evidence that evasion is rising with the income (Witte and Woodbury 1985), and in this sense the long horizon authority is also superior. There is no convincing evidence on the influence of auditing costs on the auditing probability, and it is really difficult to say which model is closer to reality on this point.

Comparison with Nash

In the rest point of the evolutionary game q is the same as in the Nash equilibrium of one shot game, since it is derived from the same maximizing revenue decision of tax authority. Auditing probability p can be greater or smaller depending on the parameter values, because it is determined by the behavior of the individuals, which is modelled differently. The variation in steady state p is very small: from $\frac{1}{\sqrt{3}}$ to $\frac{2}{3}$, compared to $(\frac{1}{2}, 1)$ in static case for $s < 1$. Hence, the difference in p for these two models is primarily dependent on s: for large values of fine Nash equilibrium gives less intensive auditing, and for small fines our model results in lower auditing.

As for the payoffs, since q is set so that the tax authority is indifferent between auditing and not auditing, its revenue is exactly the same in static and dynamic setups. With fixed q the payoff of the average high income taxpayer is unambiguously decreasing with p, so that for high penalties the average income is larger in static model.

Note on the dynamics feature

It is worth noting that the south-west and north-west parts of the picture is consistent with stylized facts presented in the introduction: both audit probability and the proportion of honest taxpayers decrease (second part of XXth century), and then eventually audit probability starts increasing, while non-compliance is still increasing (recent years). According to this explanation, the observed behavior is out-of-equilibrium adjustment, and sooner or later the tax evasion will have to go down.

The south-west of generated dynamics also produces values of non-compliance significantly lower than the Nash equilibrium. This can be taken as an alternative explanation to the puzzle of too high compliance, usually resolved by introduction of intrinsically honest taxpayers (Andreoni et al. 1998, Slemrod 2002).

To see whether this kind of dynamics is not idiosyncratic for the learning rule under consideration, let us proceed to the other specifications of interactions between the taxpayers.

3.2 Meeting two others: Effective punishment principle

An agent meets two others, and the following (more favorable to the flourishing of honesty) procedure, in which we go even further away from the expected utility maximizing agents, takes place. If 3 honest people meet, they will all stay honest for the next round. If 2 honest and 1 caught cheater (or 1 honest and 2 caught cheaters) meet, they will all play honest next time. If 2 honest and 1 not caught cheater (or 1 honest and 2 not caught cheaters) meet, they are all cheating next time. If 3 not caught people meet, they play cheat next time. If 2 caught cheaters and 1 not caught cheater (or 2 not caught cheaters and 1 caught cheater), they all play honest in the next round. If all three types meet (or 3 caught people), they play honest next round.

The first four rules are standard; the last two result from the assumption that to observe punished people (or to be punished) is enough to deter one from cheating for the next year, and that the cheaters are aware of the option to be honest.

Between rounds $q\,(1-p)\,(2\,(1-q)+q(1-p))$ of honest taxpayers switch to cheating, $(1-q+(1-p)\,q)^2$ of not caught taxpayers continue cheating, caught taxpayers do not cheat in the next round. The derivation of the respective probabilities is left to the appendix. As a result, the law of motion for q is given by

$$q_{\tau+1} = q_\tau\,(1-p_\tau)\left(3\,(1-q_\tau)^2 + 3\,(1-q_\tau)\,(1-p_\tau)\,q_\tau + (1-p_\tau)^2\,q_\tau^2\right) \qquad (13)$$

This is aggregate population dynamics, notice that it does not differ qualitatively from the previous learning rule. We show this in the following proposition, first defining a threshold share of cheating as

$$\bar{q} := \frac{3(1-p^2) - \sqrt{1+12p-18p^2+8p^3-3p^4}}{2(1-p^3)}$$

Proposition 1E Consider the law of motion of q for the effective punishment principle given by (13). If $p \in [0, \frac{2}{3})$ in this period, then the share of cheating taxpayers increases in the next period for $q < \bar{q}$ and decreases for $q > \bar{q}$. If $p \geq \frac{2}{3}$, cheating in the next period decreases.

Further we do not make special subsections for the cases of commitment and absence of it, as well as for comparison with Nash equilibrium. The logic of the exposition is the same as for the average payoff learning principle, and the results are similar.

We start with the **commitment** case: the dynamic converges to $p^* \in \left[0, \frac{2}{3}\right]$ compared with the same as before Stackelberg outcome. The comparative statics is very much the same (recall that the results (6)-(8) hold for all the rules considered) as for the previous learning rule, since the relation between p and q is still negative. The effective punishment rule is contributing more to the honest reporting, and it is of no surprise that the optimal probability of auditing is lower here for the same parameter values:

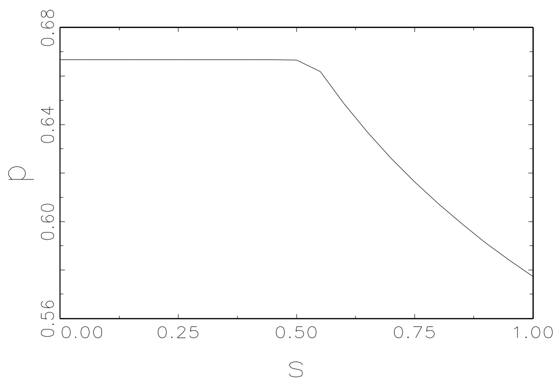


Figure 4. Optimal auditing probability depending on s, baseline parameter values.

However, the following proposition establishes an important result of similarity of the two learning rules. Namely, the steady state values of the variables are affected in the same manner by small parameter changes.

Proposition 2E Effective punishment principle results in the same comparative statics as the average payoff principle, i.e.

$$\frac{dp^*}{d\gamma} > 0, \frac{dp^*}{d\mu} > 0, \frac{dp^*}{ds} < 0.$$

From this we can conclude that the one-dimentional dynamics generated by two rules do not qualitatively differ. Comparison with the static equilibrium is exactly the same as before. The payoffs of the tax authority for the effective punishment are increasing in the magnitude of fine slower than for the best average. As a result, the interval of s for which the tax revenue of static game exceeds that of dynamic is larger for the effective punishment rule, holding all the parameters constant. The result for the payoffs of individuals does not change, as it does not depend on the learning principle.

Convergence features are not altered either: to reach no cheating state from the middle takes 594 periods now (compared with 597 before); to get to all cheating takes 3 periods (14 before). The latter, however, can not be compared directly, as for the best average all cheating was attainable at $p \in \left[0, \frac{1}{\sqrt{3}}\right]$ and computed for $p = \frac{1}{\sqrt{3}}$; for the present rule it can only happen for $p = 0$.

In the **no commitment** case the system converges (in continuous time) to the steady state with lower probability of auditing than with the previous rule. The discrete time cycling has very small amplitude, so that steady state is actually a very good approximation in this case. The steady state value for q certainly remains the same, as it does not depend on the learning principle and actually coincide with the Nash equilibrium value. The steady state value of p comes from (13) as a solution of the third-order polynomial $-\mu_1^2 p^3 + 3\mu_1 p^2 - 3p + \mu_1^2 - 3\mu_1 + 2 = 0$. From the phase portrait it is clear that this value is lower than for the best average principle.

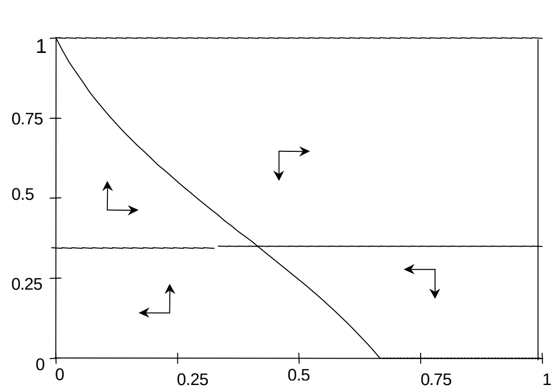


Figure 5. Phase diagram

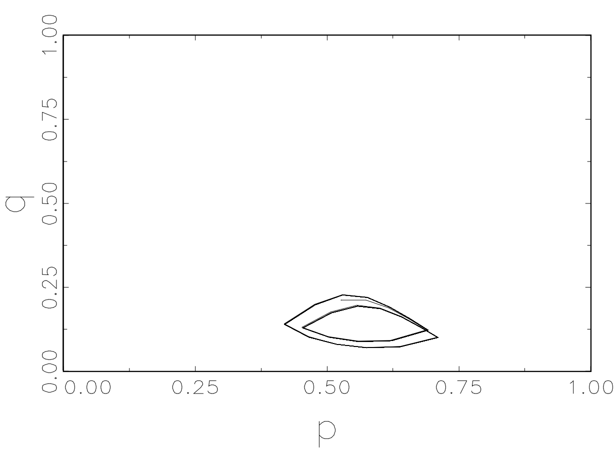


Figure 6. Discrete time dynamics

For the baseline parameter values ($\mu_1 = 0.07$) the steady state value of p is equal to 0.625 (compared with 0.659 for the previous rule). The values of $\frac{dq}{dx}$, where x is any parameter of the model (s, γ, t, H, L) are completely unchanged, and the sign of $\frac{dp^{ss}}{dx} = \frac{dp^{ss}}{dq^{ss}} \frac{dq^{ss}}{dx}$ is unchanged, since $\frac{dp^{ss}}{dq}$ is non-positive for both rules, as we confirm in the following proposition.

Proposition 3E For the effective punishment principle, the share of cheaters in interior steady state $\bar{q}$ is a decreasing function of auditing probability p, e.g. $\bar{q}'(p) \leq 0$.

As far as the comparison with the Nash equilibrium of the static game is concerned, everything said for the best average rule remains valid. Among others, the "dynamic" auditing probability is normally smaller than the "static" one ($p \in \left(0, \frac{2}{3}\right)$ in dynamic case and $p \in \left(\frac{1}{2}, 1\right)$ in the static case). Finally, the dynamic feature of cycling is also very similar and is consistent with the evidence.

In total, all the main conclusions of best average imitation are preserved under the effective punishment principle. There is more averse attitude towards the risk of being punished embodied in this rule. This results in lower cheating in commitment case, and lower auditing probability for both cases.

3.3 Meeting m others: Popularity principle

When m others are observed (and m is substantially larger than 2), we can specify an imitation rule that requires minimal information about the individual, and namely only whether he/she was caught cheating. Assume that the availability of this information is assured by the tax authority for the purpose of deterring the others. This seems plausible in the self-employment sector, especially for the professionals like doctors, auditors, etc. Let k^* be the maximal number of observed caught individuals that does not induce switching to honesty. The rule is then the following. For a not caught taxpayer: if more than k^* caught individuals are observed, play honest in the next round, if less or equal - play cheat; for a caught taxpayer: play honest.

The probability to observe less or k^* caught individuals is defined by

$$\Pr(k \leq k^*) = \sum_{i=0}^{k^*} \binom{m}{i} (pq)^i (1 - pq)^{m-i}$$

Then the cheating is evolving according to

$$q_{\tau+1} = (1 - q_\tau p_\tau) \Pr(k \leq k^*)$$

The problem with this dynamic is that once the systems comes close to extreme values of q (0 or 1), it is jumping between "almost all cheating" and "almost all honest" states in every period. This problem obviously states from an 'epidemic' nature of the specified principle: once there are very many cheaters, almost everybody meets a caught cheater, and then all those switch to playing honest. But once almost everybody is playing honest, almost nobody meets a caught cheater, and then almost everybody is playing cheat.

To make the dynamics more smooth, the usual method is to introduce some kind of inertia into the system, just like it was already done from the side of the tax authority. So, let us say that with probability β every unpunished individual changes his/her strategy according to already specified rule, and, correspondingly, with probability $1-\beta$ plays the same strategy as in the previous period. As before, punished people switch to no cheating with probability 1.

Then in every period $(1-\beta)(1-p)q+\beta q(1-p)P(k \leq k^*)$ cheaters remain cheaters plus $\beta(1-q)P(k \leq k^*)$ honest people switch to cheating. The dynamics is described by

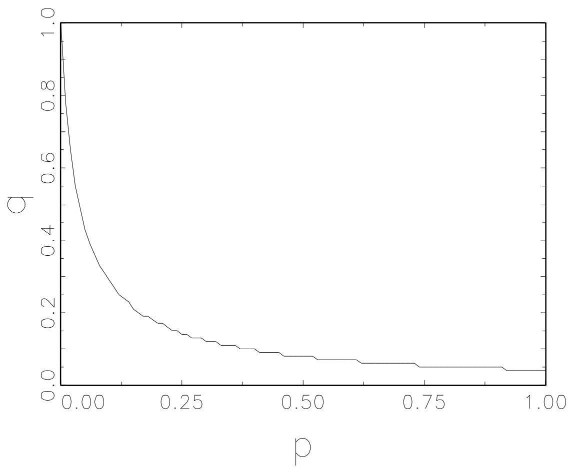
$$q_{\tau+1} = q_\tau(1-p_\tau)(1-\beta) + \beta(1-q_\tau p_\tau)P(k \leq k^*) \tag{14}$$

For small enough values of β it converges to a steady state (cycle in discrete time) rather than jumps between two extreme values. The weakness of this formulation is that steady state value of p depends on the inertia parameter, and this gives an additional 'degree of arbitrariness' to our model. For baseline parameter values p is increasing in β, which is understandable: tax authority has to control more, if larger part of individuals is reconsidering their decision at every period. Formally, in steady state $(p+\beta-p\beta)q = \beta(1-qp)P(k \leq k^*)$. Then $\frac{dp}{d\beta} = \frac{P(1-qp)+pq-q}{q-q\beta-\beta\left(\frac{\partial P}{\partial p}(1-qp)-Pq\right)}$, which has an arbitrary sign.

It is very hard to analyze the m-dynamic analytically, since for variable k^* it involves operating with sums of variable length. That is why I for the moment restrict my attention to the case where observing one caught individual is enough to deter from evasion ($k^*=0$), just like it was specified in the effective punishment rule. The dynamics is then

$$q_{\tau+1} = q_\tau(1-p_\tau)(1-\beta) + \beta(1-q_\tau p_\tau)^{m+1} \tag{15}$$

For obvious reasons the closed form solution is impossible to obtain even for this simplified problem. So, I simulate steady state for $\beta = 0.1$ and $m = 19$. Obviously, the other line is still $q = \mu_1$. In **no commitment** case we again observe small cycles around the steady state with the implications similar to the previous rules.



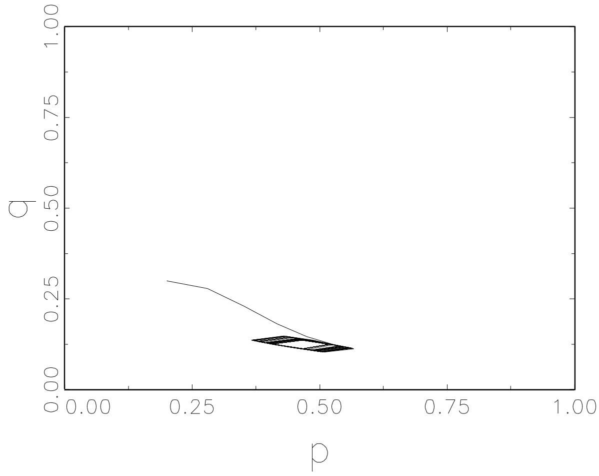


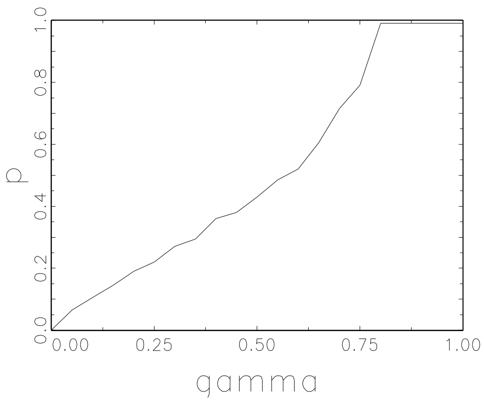
Figure 7. Steady state line for m-rule, $q_{\tau+1}= q_\tau$

Figure 8. Discrete time dynamics

Compared to the previous imitation rules, the line $q_{\tau+1} = q_\tau$ is shifted to low cheating - low auditing corner, meaning that steady state is more likely to have low probability of auditing. This comes from 2 factors: inertia in decision making β and number of people to meet m. Notice, however, that even for $p \to 1$ cheating is not eliminated completely. Indeed, for $p_\tau = 1$ $q_{\tau+1} = \beta(1 - q_\tau)^{m+1}$, so that $q = 0$ only for $\beta = 0$, which is impossible. Hence, for large auditing probabilities m-rule results in larger cheating than 2-rules. This seemingly strange result stems from poor information set of the individuals: if nobody is cheating, nobody is caught, so in the next period β of individuals will cheat.

In the **commitment** case, then, cheating can be decreasing or increasing depending on whether $q_\tau > \hat{q}(p^*)$ or the opposite. This is true for any value of p chosen by the tax authority. Comparative statics is again similar to the previous rules, since the relation between p and q is negative. Since honest reporting is favored even more by this m-rule, we expect optimal auditing to be lower for the same parameters. The magnitude of fine is almost irrelevant under the present imitation rule, since there is no information about payoffs, and people are deterred from evasion by observing caught cheaters regardless of financial costs of being caught.

This intuition is supported by simulation results: for our parameter values the change of surcharge rate is not changing optimal p, and the equilibrium auditing is lower than before: 0.43 compared with 0.65 and 0.64 for the first two rules (the difference between rules is increasing with the cost of auditing). Note that this stems mostly from higher number of people who interact, rather than from the different information structure of the rule. Indeed, for $m = 2$ optimal probability is 0.61, not substantially lower than for the other two rules. The auditing probability is rising with the proportion of high income taxpayers, just like in two previous cases. However, it is decreasing in the cost of auditing, and hence in μ.



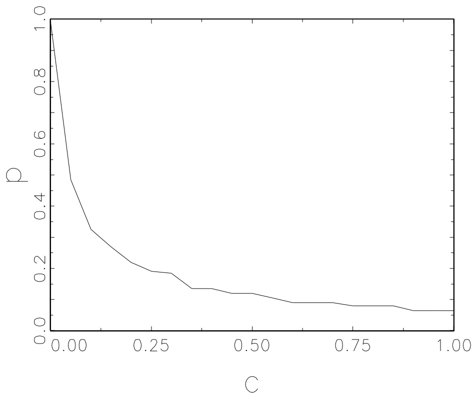


Figure 9. Optimal probability depending on γ.Figure 10. Optimal probability depending on c.

So, with the minimum information learning rule the difference in comparative statics between commitment and no commitment cases disappears. Such astounding difference in comparison with first two learning rules is fully attributed to the form of steady state relation between q and p. Recall that $\frac{dp^*}{ds}, \frac{dp^*}{d\gamma}, \frac{dp^*}{d\mu}$ vary across the rules only in $\hat{q}' = \hat{q}'(p)$ and $\hat{q}'' = \hat{q}''(p)$. Thus, the reversed result for the present rule is due to

$$\hat{q} + \hat{p}q' > 1 - \frac{1}{\gamma} \tag{16}$$

To further characterize the steady state with popularity principle, we take $m = 1$ still keeping $k^* = 0$ to arrive at the following proposition.

Proposition 1M With population dynamics of poor information (popularity) rule given by (15) and $m = 1$, the relation between share of cheaters and auditing probability $\hat{q}(p^*)$ in steady state is a non-increasing and convex function. At the same time, $\hat{q} + p\hat{q}' \geq 0$ holds.

Taking into account this proposition and (16) we have $1 - \frac{1}{\gamma} < 0 \leq \hat{q} + p^*\hat{q}'$, and hence the result of $\frac{dp^*}{d\mu} < 0$ holds for any parameter values in our example.

In total, the poor information rule shows that the unusual results for commitment case is due to high informativeness of the taxpayers about each others. When there is no information about payoffs contained in communication, the individuals abstain from evasion on even more "irrational" grounds than before. Then, for not too high auditing, more honesty results. Increasing the number of people met in this rule also brings about less cheating, because seeing more people means higher chance of observing a caught one.

The main conclusions from previous imitation rule are still valid for m people meeting. Namely, there is still non-zero cheating with commitment and the dynamics in the western part which is consistent with observations.

4 Conclusion

The model presented in the paper is designed to capture a number features of reality, which were largely neglected in the literature on tax evasion, and especially in the game-theoretic approach to the problem. These features are social interaction, poor knowledge of auditing probability, asymmetry in the behavior of two parties under consideration, and intertemporal nature of the tax evasion decision. The interaction in the model is learning each others' strategies and payoffs. This allows individuals to make decisions without acquiring information about auditing probability. Moreover, with simple imitation rules specified in the game, people also avoid costs of processing information, as they effectively know what decision to take without solving complicated maximization problems.

The model rationalizes decrease of auditing probability and compliance observed in the US over past decades as out-of-equilibrium dynamics. The same is true for the recent continuing increase of evasion along with tightening auditing. The model can also explain "too little" cheating by taxpayers: having initially overestimated auditing probability, they "undercheat" for a long time due to the inertia and imperfections of the learning rules. All these results hold with three different specifications of the learning rule (our rules differ in how much people are afraid of being caught and how much information they can learn from each other).

When we allow the tax agency to commit to a certain probability of auditing, positive cheating may arise in equilibrium. This is seems more plausible than the result obtained in the most of static commitment models. Such models usually have zero cheating of audited taxpayers in equilibrium. Moreover, the comparative statics with respect to tax rate does not contradict empirical evidence (cheating is increasing with tax), as opposed to the models in the literature. However, the model has its obvious limitations. For instance, nothing can be said about the extent of inertia in auditing decision, though this could probably be empirically testable. Without good feeling about the inertia parameter and the learning rule we can not say much about the precise form the dynamics takes.

In general, the dynamic approach to tax compliance games reopens a whole bunch of policy issues. Are the recommendations of equilibrium theory valid, if the systems never comes to equilibrium? Are some changes in the existing taxation worth undertaking, if we take into consideration not only difference in benefits between initial and final states, but also the costs of transition? Can the decision rules of the tax authorities and the learning mechanisms governing taxpayers behavior be manipulated in the way to achieve maximal social welfare?

As a building block for more general models, the evolutionary approach can be employed in the studies on how the government can ensure higher degree of trust in society (and less evasion as a result), how it can provide optimal (from the point of view of social welfare) level of public goods, how it can bring about faster growth of an economy. For this it would be necessary to consider more complicated government (and hence tax authorities) strategies, involving more than one period memory, and possibly heterogenous taxpayers.

Finally, the approach taken by no means limits us to consideration of income tax evasion. Even more interesting and exciting task would be to look at all other taxes, especially those levied on enterprises. In this case learning is probably more intensive, as well as interaction with tax authorities. Moreover, the absolute size of evasion is very likely to be higher than in case with individuals. The modeling of enterprise cheating would probably allow us to understand better how the shadow sector in general is functioning.

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5 Appendix

Proof of proposition 1A Solving 3 for the steady state, we get $q = \frac{2-3p}{1-3p+3p^2}$. Knowing that $0 \leq q \leq 1$, we obtain $\frac{1}{\sqrt{3}} \leq p \leq \frac{2}{3}$ for this steady state relation to hold. For $p < \frac{1}{\sqrt{3}}$ and $p > \frac{2}{3}$ the steady state solutions are corner ones, with $q = 1$ and $q = 0$ respectively. For the interior solution, solving 3 as inequality gives the statements of proposition.

Proof of proposition 2A In order to get interior solution, we have to get more tax revenue at $p = p^*$ than at $p = 0$. That is, the following inequality should hold:

$$\begin{aligned} \gamma(1-q)tH + p(q\gamma(tH + st(H-L)) + \ (1-\gamma)tL) \\ -c(p(q\gamma + 1 - \gamma)) + (1-p)(q\gamma + 1 - \gamma)tL > tL \end{aligned} \tag{17}$$

$$\gamma t(H-L)[1 - q + pq(1+s)] > cp(1 - \gamma + q\gamma) \tag{18}$$

$$\frac{1 - q + pq(1+s)}{p(\frac{1}{\gamma} - 1 + q\gamma)} > \mu \tag{19}$$

$$\frac{-1/p + (2-3p)s + 2}{1 - 3p^2 + \frac{1}{\gamma}(1 - 3p + 3p)} > \mu, \tag{20}$$

which is just the statement of the proposition.

Proof of proposition 3A Directly differentiating $q(p)$ gives the expression $\frac{3(3p^2-4p+1)}{(1-3p+3p^2)^2}$ with positive denominator. The nominator is negative for $p \in (\frac{1}{3}, 1)$. Since our p is defined on $\left[\frac{1}{\sqrt{3}}, \frac{2}{3}\right]$, $q' < 0$, $q(p)$ is decreasing.

Differentiating $q(p)$ twice, we get $3\left[\frac{3p^2-1}{1-3p+3p^2} + \frac{1-4p+3p^2}{1-3p+3p^2}\frac{3-6p}{(1-3p+3p^2)^2}\right]$. All the denominators are again positive; the first nominator is positive for $p > \frac{1}{\sqrt{3}}$, the second nominator is negative for $p \in (\frac{1}{3}, 1)$, the third one is negative for $p > \frac{1}{2}$. Hence, $q'' > 0$, $q(p)$ is strictly convex on the interval where it is defined.

Generalization of comparative statics results A sufficient condition for the signs of this and other derivatives to be the same as at the baseline is

$$1 + s > \mu \text{ and } p^* < \frac{1}{1 + s - \mu}$$

To see that this condition is not very restrictive, notice that first inequality certainly holds for any values of s and μ we think of as plausible. The second expression is satisfied for any p^*, if it is for $p^*_{\max} = \frac{2}{3}$. In other words, it certainly holds for any combination of s and μ such that $s - \mu < \frac{1}{2}$. Referring back to our plausible ranges, for any μ this condition holds for $s \leq \frac{1}{2}$.

Finally, for $\frac{dp}{d\mu} > 0$ we have to add $\gamma > \frac{1}{7}$ to gueranty that the term $\frac{1}{\gamma} - 1$ does not outweigh the negative $q + pq'$.

Proof of the statement $\bar{q} < -p^*\bar{q}'$ Plugging in expressions for q and q' into the inequality claimed, we get

$$\begin{aligned} \frac{(2-3p)\left(1-3p+3p^2\right)}{3\left(1-4p+3p^2\right)} &< -p \\ (2-3p)\left(1-3p+3p^2\right) &< -3p\left(1-4p+3p^2\right) \\ 3p^2-6p+2 &< 0, \end{aligned}$$

which holds for $p \in (1-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}})$, and hence for the relevant for us interval $\left[\frac{1}{\sqrt{3}}, \frac{2}{3}\right]$.

Stability of the steady state in continuous time To investigate stability of the steady state analytically, we have to make two approximations. First, consider the system in continuous time: this makes sense, if we imagine that both the tax authority and individuals update their evasion or auditing decisions every day, rather than fixing it once for a whole year. We can rewrite our system of equations as

$$\begin{aligned} q_{\tau+\Delta} &= q_\tau + \Delta f(q_\tau, p_\tau), \\ p_{\tau+\Delta} &= p_\tau + \Delta g(q_\tau, p_\tau); \end{aligned}$$

and letting Δ be very small ($\frac{1}{365}$, if we think of daily updating), in the limit we obtain

$$\begin{aligned} \dot{q} &= f(q,p), \\ \dot{p} &= g(q,p); \end{aligned}$$

Explicitly,

$$\begin{aligned} \dot{q} &= 2q - 3q^2 + q^3 - 3pq + 6pq^2 - 3pq^3 - 3p^2q^2 + 3p^2q^3, \\ \dot{p} &= \alpha\left(BR(q) - p\right); \end{aligned}$$

The stability matrix of this system is $\begin{pmatrix} \frac{\partial \dot{q}}{\partial q} & \frac{\partial \dot{q}}{\partial p} \\ \frac{\partial \dot{p}}{\partial q} & \frac{\partial \dot{p}}{\partial p} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, where

$$
\begin{aligned}
a_{11}x &= 2-6q-3p+12pq+3q^2-6p^2q-9pq^2+9p^2q^2, \\
a_{12} &= -3q+6q^2-6pq^2-3q^3+6pq^3, \\
a_{21} &= \alpha BR'(q), \\
a_{22} &= -\alpha.
\end{aligned}
$$

The problem with this formulation is that the best response function is not continuous at the point of steady state, so we can not compute $BR'(q_{ss})$. To go around it, we can make the second approximation: instead of the discontinuous best response we take a continuous function $ABR(q) = \Phi\left(\frac{\bar{c}(q)-c}{\sigma}\right)$, which approaches $BR(q) = \begin{cases} 0, \text{ if } \bar{c} < c \\ 1, \text{ if } \bar{c} > c \end{cases}$ with $\sigma \to 0$. Conventionally, Φ is cumulative distribution function of a standard normal random variable. Then $ABR'(q) = \phi\left(\frac{\bar{c}(q)-c}{\sigma}\right)\frac{\bar{c}'(q)}{\sigma}$. Recalling the expression for $\bar{c}(q)$ and evaluating at steady state ($\bar{c}(q_{ss}) = c$), we get

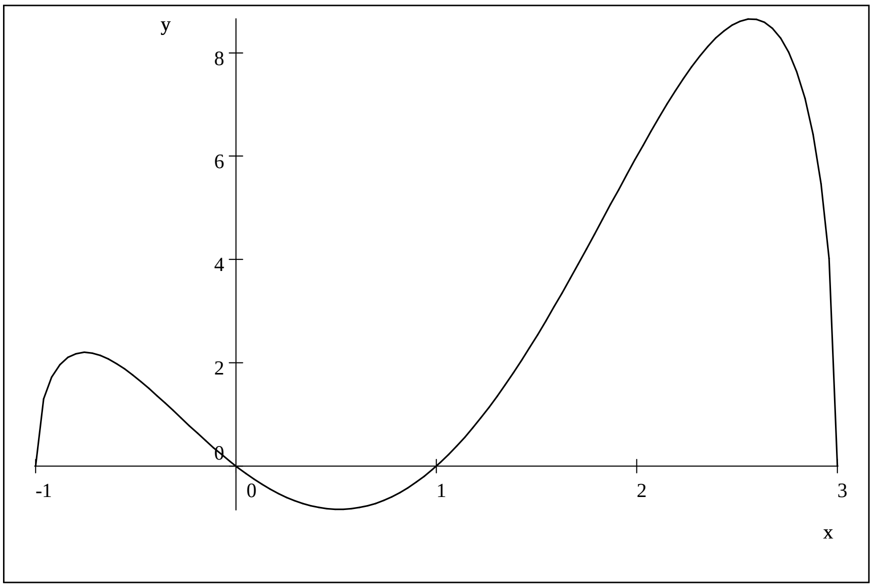
$$
ABR'(q_{ss}) = \phi(0)\frac{(1-\gamma)c}{\sigma q(1-\gamma+q\gamma)} \Rightarrow a_{21} \approx \frac{\alpha(1-\gamma)c}{\sqrt{2\pi}\sigma^2 q(1-\gamma+q\gamma)}.
$$

Note that we can make a_{21} (since it is positive) arbitrary large by making σ small enough and thus getting better approximation of initial best response function.

Now we are ready to adress the question of stability of the steady state. If the real parts of both eigenvalues of the stability matrix are negative, the steady state is stable (see, for example, Hirsch and Smale (1974). The eigenvalues of our system are

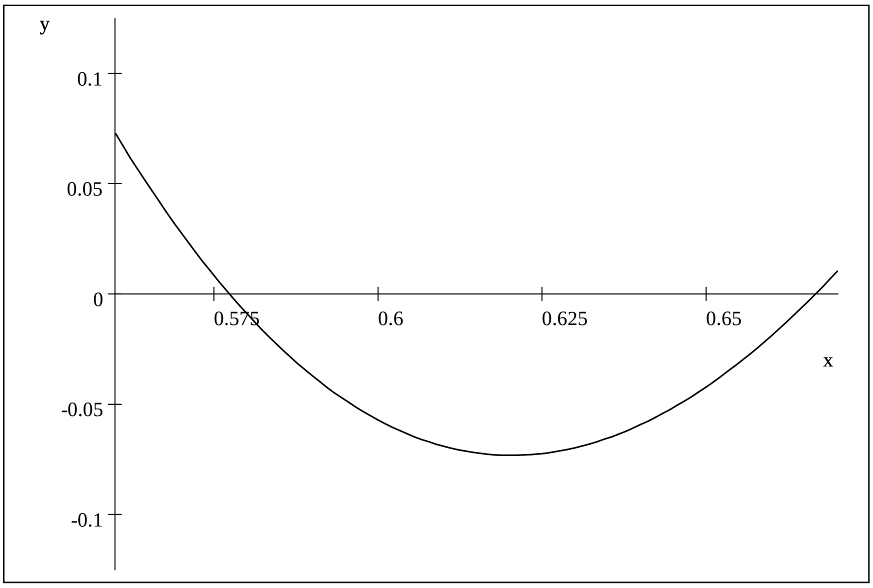
$$
\lambda_{1,2} = \frac{1}{2}\left(a_{11}+a_{22} \pm \sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}\right).
$$

First we show that the real parts of the two eigenvalues are identical. This is equivalent to showing that the square root is an imaginary number, or that the expression under the square root is negative. Indeed, since $a_{11}-a_{22}$ is bounded and a_{21} is arbitrary large, the square root is imaginary, if $a_{12} < 0$. Let us plot it as a function of q, using the fact that in steady state $p = \frac{3q-3+\sqrt{9(1-q)^2-12q(q-2)}}{6q}$:



As q is determined on the interval $[0,1]$, $a_{12} \leq 0$ (with equality in the corners), so that the real parts of the both eigenvalues are identical and equal to $a_{11} + a_{22}$.

It is left to determine the sign of a_{11}, since $a_{22} = -\alpha$. We plot it as a function of p, using the fact that in steady state $q = \frac{2-3p}{1-3p+3p^2}$:



As p is determined on the interval $\left[\frac{1}{\sqrt{3}}, \frac{2}{3}\right]$, and is zero at the corners, $a_{11} < 0$

for all interior points. Hence, both eigenvalues have negative real parts[7] - our steady state is stable in continuous time.

Probabilities of switching Recall that there are $(1-q)$ honest taxpayers, each of which meet another honest and not caught cheater with probability $2(1-q)(1-p)q$, or two not caught cheaters with probability $(1-p)^2 q^2$. Summing this gives probability of switching.

There are also $(1-p)q$ not caught cheaters. $(1-q)^2$ of them meet 2 honest people, $2(1-q)(1-p)q$ meet another not caught and an honest, and $(1-p)^2 q^2$ meet 2 not caught cheaters. The sum is probability to remain a cheater.

Proof of proposition 1E Solving (13) for the steady state, we get $q = \frac{3(1-p^2)-\sqrt{1+12p-18p^2+8p^3-3p^4}}{2(1-p^3)}$. Knowing that $0 \leq q \leq 1$, we obtain $p \leq \frac{2}{3}$ for this steady state relation to hold. For $p > \frac{2}{3}$ the steady state solution is corner with $q = 0$. For the interior solution, solving (13) as inequality gives the statements of proposition.

Proof of proposition 2E We have to get explicit expressions for q' and q'', plug them into the formulas (6)-(8) and evaluate those at the baseline parameter values (for them $p^* \approx \frac{2}{3}$). Thus, $q'=\frac{p^2}{4p^6-8p^3+4}\left(18-6\sqrt{12p-18p^2+8p^3-3p^4+1}-18p^2\right)-\frac{1}{2p^3-2}\left(\frac{18p-12p^2+6p^3-6}{\sqrt{12p-18p^2+8p^3-3p^4+1}}-6p\right)$; $q''=\frac{p}{4p^6-8p^3+4}\left(36-12\sqrt{12p-18p^2+8p^3-3p^4+1}-36p^2\right)+3\frac{p^4}{6p^3-6p^6+2p^9-2}\left(18p^2+6\sqrt{12p-18p^2+8p^3-3p^4+1}-18\right)+\frac{p^2}{4p^6-8p^3+4}\left(6\frac{18p-12p^2+6p^3-6}{\sqrt{12p-18p^2+8p^3-3p^4+1}}-36p\right)+\frac{p^2}{4p^6-8p^3+4}\left(-36p-3\frac{24p^2-36p-12p^3+12}{\sqrt{12p-18p^2+8p^3-3p^4+1}}\right)+\frac{1}{2p^3-2}\left(\frac{24p-18p^2-18}{\sqrt{12p-18p^2+8p^3-3p^4+1}}+\frac{}{\sqrt{12p-18p^2+8p^3-3p^4+1}+12p\sqrt{}}\right.$
Plugging in parameters, we get $\frac{dp}{ds} = -0.258\,26 < 0$, $\frac{dp}{d\mu} = 0.043 > 0$, $\frac{dp}{d\gamma} = 0.08$ $6 > 0$.

Proof of proposition 3E Taking the derivative of $q(p)$ gives

$\frac{12(1-p^3)\left(-p-\left(1+12p-18p^2+8p^3-3p^4\right)^{-\frac{1}{2}}\left(1-3p+2p^2-p^3\right)\right)+p^2\left(3(1-p^2)-\sqrt{1+12p-18p^2+8p^3-3p^4}\right)}{\left(2(1-p^3)\right)^2}$. The denominator is always positive; plotting the nominator shows that it is negative on the interval $\left[0, \frac{2}{3}\right]$. This yields the statement of proposition.

Proof of proposition 1M The function determining steady-sate realation between q and p is implicitly given by

[7] For the corners $(\frac{1}{\sqrt{3}}, 1)$ and $\left(\frac{2}{3}, 0\right)$ both eigenvalues are real, $\lambda_1 = 0$, $\lambda_2 = -\alpha$, which can be checked directly by direct substitution of p and q with these particular values. In any case, the corner solutions are not of interest to us, since they are not observed in reality.

$$q = q(1-p)(1-\beta) + \beta(1-qp)^2 \tag{A1}$$

From this we can see that on the domain $(0,1)$ and with the image $(0,1)$, $q(p)$ is twice continuously differentiable. By totally differentiating the steady state expression with respect to q and p, we get $\frac{dq}{dp} = \frac{q(2\beta pq-1-\beta)}{p+\beta+p\beta(1-2pq)}$. As $1-2pq > -1$ and $p\beta < p$, the denominator is non-negative. As $2\beta pq < 2\beta < 1+\beta$, the nominator is non-positive. Hence, q is non-increasing function of p.

Differentiating the slope of $q(p)$ with respect to p, we get a ratio with positive denominator $(p+\beta+p\beta(1-2pq))^2$ and nominator $q' \left[4qp\beta(p+\beta+p\beta-qp^2\beta) - (p+\beta+p\beta+\beta^2+p\beta^2\right.$ $4q^2p\beta(qp\beta-\beta-1) + q(2q\beta^2+2\beta+\beta^2+1)$,

which is claimed to be non-positive. To prove this, I first show that $qp \leq \frac{2}{5}$.

Solving the equation A1 for qp we get $qp = \frac{1}{2}\left(\frac{1}{\beta}+1-\sqrt{\left(\frac{1}{\beta}+1\right)^2 - 4\,(1-q)}\right)$.

To establish the inequality, then, we have to show that $\frac{1}{\beta}+1-\sqrt{\left(\frac{1}{\beta}+1\right)^2 - 4\,(1-q)} \leq \frac{4}{5}$

This expression is increasing in q, so we have to find minimal value of q, which by the first part of the present proposition is achieved at $p=1$. Plugging in this value into A1 we get $q = \beta(1-q)^2$ and hence $q = 1+\frac{1}{2\beta}-\frac{1}{2}\sqrt{\left(2+\frac{1}{\beta}\right)^2-4}$. Combining two expressions, one gets

$$\frac{1}{\beta}+1-\sqrt{\left(\frac{1}{\beta}+1\right)^2 - 4\left(\frac{1}{2\beta}-\frac{1}{2}\sqrt{\left(2+\frac{1}{\beta}\right)^2-4}\right)} \leq \frac{4}{5}$$

$$\left(\frac{1}{\beta}+\frac{1}{5}\right)^2 \leq \left(\frac{1}{\beta}+1\right)^2 - 4\left(\frac{1}{2\beta}-\frac{1}{2}\sqrt{\left(2+\frac{1}{\beta}\right)^2-4}\right)$$

$$\frac{2}{5\beta}+\frac{1}{25} \leq \frac{4}{\beta}+1-2\sqrt{\left(2+\frac{1}{\beta}\right)^2-4}$$

$$\frac{18}{5\beta}+\frac{24}{25}-2\sqrt{\left(2+\frac{1}{\beta}\right)^2-4} \geq 0$$

$$\left(\frac{9}{5\beta}+\frac{12}{25}\right)^2 \geq \left(2+\frac{1}{\beta}\right)^2-4$$

$$\frac{56}{25\beta^2}-\frac{71}{125\beta}+\frac{144}{625} \geq 0$$

The last expression is a parabola in $\frac{1}{\beta}$ as an argument; it is always above the x axis, hence the expression holds even with a strict sign. Thus, we actually showed that $qp < \frac{2}{5}$.

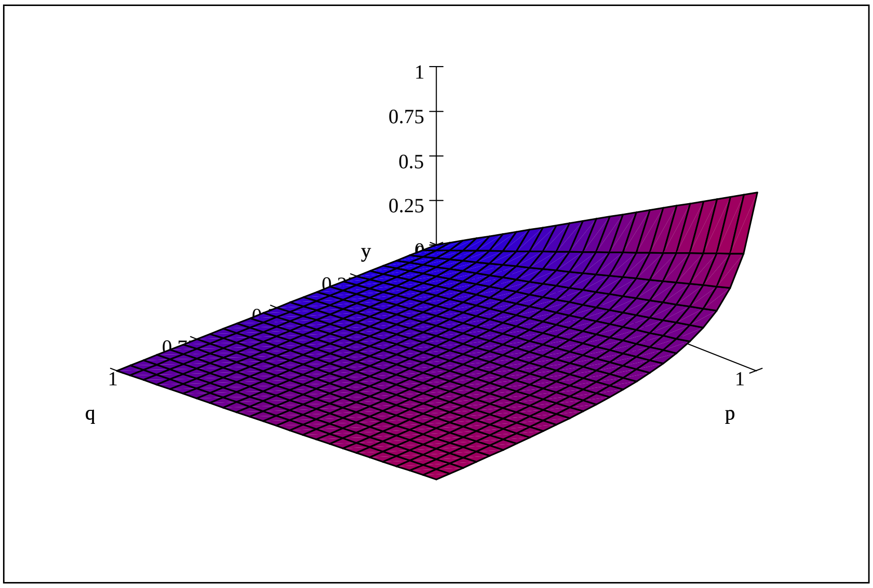
Now, to establish the sign of the second derivative, it is enough to show that

$4qp\beta(p+\beta+p\beta-qp^2\beta)-(p+\beta+p\beta+\beta^2+p\beta^2) \leq 0$ and $4q^2p\beta(qp\beta-\beta-1)+q(2q\beta^2+2\beta+\beta^2+1) \geq 0$.

First, rewrite the first condition as $p+\beta+p\beta+\beta^2+p\beta^2+4q^2p^3\beta^2 \geq 4qp^2\beta+4qp\beta^2+4qp^2\beta^2$. Notice that the right hand side is smaller than $\frac{8}{5}\beta(p+\beta+p\beta)$. Then it is enough to show that $p+\beta+4q^2p^3\beta^2 \geq \frac{3}{5}\beta(p+\beta+p\beta)$. Even without the term containing q, the inequality holds. To see that, minimize $p+\beta-\frac{3}{5}\beta(p+\beta+p\beta)$ wrt p. Since there is a global maximum at $p|0<p<1$, the minimum on this interval should be on the one of the borders. All we have to do then is to check the inequality at the borders, namely for $p=0$ and $p=1$. For the former, $\beta-\frac{3}{5}\beta^2 \geq 0$ obviously; for the latter, $1+\beta-\frac{3}{5}\beta(1+2\beta)=1+\frac{2}{5}\beta-\frac{6}{5}\beta^2$. Here, again, the global maximum is at $\beta=\frac{1}{6}$, the minimum is reached at $\beta=1$; it is $\frac{1}{5}>0$. Thus, we have proven that the first condition is satisfied.

Second, rewrite the second condition as $q(2q\beta^2+2\beta+\beta^2+1+4q^3p^2\beta^2) \geq 4q^2p\beta(\beta+1)$. This holds straightforwardly. So, $q'' \geq 0$, and hence in steady state the share of cheating as a function of auditing probability is convex.

To show that $q+pq' \geq 0$ holds we plot this expression: $y=q+p\frac{q(0.2pq-1.1)}{p+0.1+0.1p(1-2pq)}$



Proportional imitation

To apply proportional imitation rule (PIR), which was proposed and shown to be optimal by Schlag (1998), we have to modify our setup slightly. The problem with this rule is the need to know the highest and the lowest payoffs of the agents, which was assumed away so far. This does not seem to be a very strong assumption to make: people may know the payoff and still not pursue certain strategy, just because they do not know how to do it.

According to the rule, each agent meets only one other and imitate its strategy with probability proportionate to the payoff difference, if this other performed better. Recall that three payoffs of our game are $(1-t)H - st(H-L)$ if caught, $(1-t)H$ if honest, and $H - tL$ if not caught. The difference between the highest and the lowest is $(1+s)t(H-L)$, evaded tax plus a fine. Then not caught cheater never switches; honest taxpayer meeting not caught one switches with probability $\frac{1}{1+s}$; a caught cheater meeting an honest agent switches with probability $\frac{s}{1+s}$.

Then the law of motion for q is given by

$1 - q(\tau+1) = (1-q)[1 - q(1-p)\frac{1}{1+s} + qp\frac{s}{1+s}]$,

where the right hand side is again at time τ.

From this expression, the proportion of cheaters increases, if $p < \frac{1}{1+s}$, and decreases otherwise. Thus, we get the circling around $p = \frac{1}{1+s}$ and $q = q(c)$ again. Interestingly, only with proportional imitation rule the interior rest point is precisely the Nash equilibrium of the static game. It happens because in the present specification the agents possess more information (about payoffs) and have rather sophisticated learning technique.