Economic Dynamics with Renewable Resources and Pollution

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Abstract

This article considers a two-sector economy with externalities. In particular,the analysis involves an industrial sector whose production activities have negative effects on the regeneration of a natural resource in the other sector. Without the usual convexity or the super-modularity structure, we prove that the economy evolves to increase the *net gain of stock*, and establish the conditions ensuring the convergence of the economy in the long run.

Keywords. Ramsey model, two-sector model, renewable resources, pollution.

Contents

1 Introduction

The role of natural resources in the economy is an important question and hence there is a large existing literature on this subject. Intriguingly, natural resources might have opposing effects on the economy. In particular, an abundant resource may help a country avoid the poverty trap (see Le Van & al [11]), but may slow the growth rate of the economy in the long run (see Rodiguez & Sachs [13], Elisson & Turnovsky [6]).

Another feature which is carefully studied in the literature is the effect of externalities between the production sector and that of the natural resource, such as the case between an industrial sector and forestry or fishery. The natural resource may have positive effects on the productivity of the production sector, or provide an additional source of income. By contrast, pollution from industrial activities may have negative impacts on the regeneration renewable resources.

Such a situation has been analyzed by Beltratti & al [4], and Ayong Le Kama [3]. These authors present the renewable resource stock as a consumption good as well as an input for production. The renewable capacity of the resource is impaired by pollution from the industrial activities to produce the final good. Under suitable conditions, the existence of a stationary state and its local stability are proved.

This approach is very appealing, but as Wirl [18] has observed, there is always room for limit cycles. Multiple long run outcomes exist and are separated by a threshold, even under a sufficiently concave structure.

The purpose of our article is to study a two-sector economy with renewable resource under discrete time configuration. We propose a new approach to the problem and specify conditions ensuring the convergence of the economy in long term. Our approach can apply not only to the work of Beltratti & al [4] and Ayong Le Kama [3], but also for other multisector models.

We consider a two-sector economy with an industrial sector and an exploitation sector of a renewable resource.[1] This resource constitutes one source of income, the other coming from the industrial sector. The price of the exploited resource is assumed to be given by an international market. There is one aggregate consumer who lives infinitely. She allocates total incomes between consumption and capital investment to maximize intertemporal utility. The income from the renewable resources exploitation can be used to acquire physical capital and promote a more rapid development of the country as well as a higher consumption level of the aggregate agent.

This problem is challenging since we cannot follow the traditional techniques in the dynamic programming literature to study the longterm behavior of the economy. Usually, as well presented in Stokey & Lucas (with Prescott) [16], or Dana & Le Van [9], an analysis of the Euler equations provides us with information on the optimal choice of investment and exploitation. In this article, such a technique is inapplicable since we cannot be sure that the optimal choice belongs to the interior of the domain of definition. Moreover, the presence of two control variables rules out super-modularity[2].

To overcome this difficulty, we introduce and analyze the *net gain of stock*, which is similar to the *net gain of investment* concept presented in the analysis of Majumdar & Nermuth [12], Dechert & Nishimura [5], Mitra & Ray [15] or Kamihigashi & Roy [8]. As Kamihigashi & Roy [8], we prove that the economy evolves to increase the value of the *net gain of stock* in some day in the future. This property has an important implication. It ensures that in the long term, the economy becomes very close to the set of steady states. If this set is a singleton, the optimal path converges to its unique point in the long run. In this article we specify the conditions for the uniqueness of the steady states. It is interesting and surprising to see that the notion of *net gain of investment* can illuminate our understanding of economic dynamics.

[1] Though there is only one type of renewable resource in the model, it can be understood as a metaphor for any kind of renewable resource.

[2] For the definition and a detailed survey about the super-modularity economy, see the works of Amir [1] and [2].

The article is organized as follows. Section 2 considers the problem without externality of the production sector on the renewable resource. Our main results are found in Section 3, where industrial activities reduce the regeneration of the renewable resource. Section 3 also characterizes conditions for the uniqueness of steady state and hence for the convergence of the economy in the long run. All proofs are given in the appendix.

2 Model without emission

2.1 Fundamentals

We consider a country which has two sectors, one is industrial production, and the other, the exploitation of renewable resources, for example forestry or fishery. In this section, we assume that the pollution has no effect on renewable resources.

The industrial sector is characterized by a production function f, satisfying usual conditions in literature, such as monotonicity, concavity or Inada. For the exploitation sector, the regeneration function η of renewable resource is supposed to depend in only the stock of fish. The price of this resource is supposed to be given by international market and denoted by θ.

At the beginning period of time t, the economy posses a stock of capital k_t and renewable resource y_t, which generate an output from production $f(k_t)$ and a stock $\eta(y_t)$. The planer chooses to exploit a quantity of resource x_t. With the revenue $R_t = f(k_t) + \theta x_t$, she/he consumes c_t and invests in physical capital for the following date a quantity k_{t+1}. With discount factor $\beta \in (0, 1)$, she/he maximizes the inter-temporal sum of utilities $\sum_{t=0}^{\infty} \beta^t u(c_t)$, under the constraints:

$$c_t + k_{t+1} \leq f(k_t) + \theta x_t,$$

$$x_t \leq \eta(y_t),$$

$$y_{t+1} \leq \eta(y_t) - x_t.$$

Observe that as we can extract the renewable resource in order to increase the investment for tomorrow, the capital stock k_{t+1} can be greater than the production $f(k_t)$. The exploitation obviously cannot overcome the total resource. For given (k_0, y_0), by replacing x_t as $\eta(y_t) - y_{t+1}$, we can re-write the problem as:

$$
\begin{aligned}
v(k_0, y_0) = \max \sum_{t=0}^{\infty} \beta^t u(c_t), \\
& c_t + k_{t+1} + \theta y_{t+1} \leq f(k_t) + \theta\eta(y_t), \\
& y_{t+1} \leq \eta(y_t), \\
& c_t, k_t, y_t \geq 0 \text{ for any t.}
\end{aligned}
$$

We first assume some standard conditions on utility function, production function and regeneration function.

Assumption H1. i) *The utility function u is strictly concave, strictly increasing, continuously differentiable in $\mathbb{R}_+$ satisfying Inada condition $u'(0) = +\infty$.*

ii) *The function f is strictly concave, strictly increasing, continuously differentiable in $\mathbb{R}_+$ and $f(0) = 0, f(+\infty) < 1, f'(0) = +\infty$.*

iii) *The growth function of fish is strictly concave, strictly increasing, continuously differentiable and $\eta(0) = 0, \eta(+\infty) < 1, \eta'(0) = \infty$.*

iv) *For any $(k, y) \in \mathbb{R}^2_+$, there exists a feasible sequence $\{(k_t, y_t)\}_{t=0}^{\infty}$ such that*

$$\sum_{t=0}^{\infty} \beta^t u\big(f(k_t) + \theta\eta(y_t) - k_{t+1} - y_{t+1}\big) > -\infty.$$

These conditions are usual in literature. They ensure that the set of feasible path $\Pi(k_0, y_0)$ is compact in the product topology and the value function v is upper semi-continuous. This ensures the existence of optimal path. Since there is no externality effect, the strict concavity ensures that the optimal path is unique. Moreover, we can establish the Bellman functional equation, which has v as a solution. For the details, see Dana & Le Van [9] or Le Van & Morhaim [10].

For each $(k, y) \in \mathbb{R}^2_+$, define

$$\Gamma(k, y) = \{(k', y') \in \mathbb{R}^2_+ \text{ such that } k' + \theta y' \leq f(k) + \theta \eta(y) \text{ and } y' \leq \eta(y)\}.$$

First, following the same analysis line presented in Stockey & Lucas (with Prescott) [16], we prove that this correspondence has non-empty, convex compact value. Moreover, the value function is a solution of Bellman functional equation (for the case where the utility function is bounded from below, it is the unique solution). The optimal policy function is well defined and continuous.

PROPOSITION 2.1. *Assume* ***H1***.

i) *The correspondence* Γ *is continuous on* $\mathbb{R}^2_+$ *and convex, compact-valued.*

ii) *The value function* v *satisfied the Bellmann functional equation:*

$$v(k, y) = \max_{(k', y') \in \Gamma(k, y)} \left[u\big(f(k) + \theta \eta(y) - k' - \theta y'\big) + \beta v(k', y') \right].$$

Moreover, if the utility function u *is bounded from below,* v *is the unique solution.*

iii) *There exists optimal policy function* φ *such that*

$$\varphi(k, y) = \operatorname*{argmax}_{(k', y') \in \Gamma(k, y)} \left[u\big(f(k) + \theta \eta(y) - k' - \theta y'\big) + \beta v(k', y') \right].$$

iv) *The feasible sequence* $\{(k_t, y_t)\}_{t=0}^{\infty}$ *is optimal if and only if for any* t,

$$(k_{t+1}, y_{t+1}) = \varphi(k_t, y_t).$$

v) *Assume that* $k_0 > 0$ *and* $y_0 > 0$. *Denote by* $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ *the optimal sequence. For any* $t \geq 0$ *we have* $k_t^* > 0$, *and* $y_t^* > 0$.

Denote by (k^s, y^s) the stocks such that

$$f'(k^s) = \frac{1}{\beta} \text{ and } \eta'(y^s) = \frac{1}{\beta}.$$

As the convexity structure is well established, we can verify that (k^s, y^s) is the unique steady state of the problem.

2.2 Local and global dynamics

2.2.1 Local dynamics

The difficulty in analysing this problem is that, though the Inada conditions are satisfied, we can not exclude the possibility that for some date t, there is not fishing activity, which is equivalent to $y^*_{t+1} = \eta(y^*_t)$. This prevents us to apply directly the well-known results in dynamic programming theory to study the long term behaviour of the economy. Moreover, the lack of the super-modularity does not allow us to apply the same approach as Amir [1].

In order to overcome this difficulty, we will first study the long term behaviour for the case the economy begins sufficiently "near" the steady state.

Consider now the following modified problem.

For each $z > 0$, define

$$F(z) = \max_{k+\theta y = z} \left(f(k) + \theta\eta(y) \right).$$

Following Rockafellar [14], Lemma 2.1 is verified. This ensures the concavity of F, and hence the modified problem satisfies the well known properties in literature.

Lemma **2.1.** *The function F is strictly concave. Moreover, with*

$$(k^z, y^z) = \operatorname*{argmax}_{k+\theta y = z} \left(f(k) + \theta\eta(y) \right),$$

we have $0 < k^z < z$ and $0 < y^z < \dfrac{z}{\theta}$. The derivatives satisfy $f'(k^z) = \eta'(y^z) = F'(z)$.

Define $S = f(k_0) + \theta\eta(y_0)$ and $z_0 = F^{-1}(S)$[3]. Consider the maximization modified

[3] This consideration is necessary, since (k_0, y_0) may not belong to $\operatorname{argmax}_{k+\theta y = z_0} \left(f(k) + \theta\eta(y) \right)$.

problem with given z_0:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t),$$
$$c_t + z_{t+1} \leq F(z_t) \text{ for } t \geq 0.$$

As usual properties are satisfied, the modified problem has unique optimal path, which converges monotonically to the steady state z^s, solution to $F'(z) = \frac{1}{\beta}$. We verify easily that $z^s = k^s + \theta y^s$. For the optimal $\{z_t^*\}_{t=0}^{\infty}$ of the modified problem, define the corresponding path $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$, with

$$(\tilde{k}_t, \tilde{y}_t) = \underset{k+\theta y = z_t^*}{\operatorname{argmax}} \big(f(k) + \theta\eta(y)\big).$$

The main difficulty is that, the corresponding path $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ may not satisfy the constraint $\tilde{y}_{t+1} \leq \eta(\tilde{y}_t)$. If the initial economy begins near the steady state (k^s, y^s), this constraint is satisfied and the sequence $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ is solution of the initial problem.

Lemma **2.2.** *Assume **H1**. The modified problem has unique solution. Moreover,*

i) *Consider the solution $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ of the initial problem. Define*

$$z_0 = F^{-1}(f(k_0) + \theta\eta(y_0))$$
$$z_t^* = k_t^* + \theta y_t^*.$$

If for any $t \geq 0$, $0 < y_{t+1}^* < \eta(y_t^*)$, then the sequence $\{z_t^*\}_{t=0}^{\infty}$ is solution of the modified problem.

ii) *Consider the solution $\{\tilde{z}_t\}_{t=0}^{\infty}$ of the modified problem. For any $t \geq 1$, define*

$$(\tilde{k}_t, \tilde{y}_t) = \underset{k+\theta y = \tilde{z}_t}{\operatorname{argmax}} \left(f(k) + \theta\eta(y)\right).$$

If for any $t \geq 0$, $0 < \tilde{y}_{t+1} \leq \eta(\tilde{y}_t)$, then $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ is solution of the initial problem.

The analysis of the modified problem give us the possibility to study local dynamic properties of the initial problem in a neighborhood of its steady state. Using the results in [16], the convergence follows a geometrical speed.

PROPOSITION 2.2. *Assume **H1**. Denote by z^s the steady state of the modified problem and (k^s, y^s) the steady state of the inital problem.*

We have:

i) *The point (k^s, y^s) satisfies*

$$(k^s, y^s) = \operatorname*{argmax}_{k+\theta y = z^s} \left(f(k) + \theta\eta(y) \right).$$

ii) *There exists a neighborhood $\mathscr{V}$ of (k^s, y^s) such that for any $(k_0, y_0) \in \mathscr{V}$, the optimal sequence $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ which begins from (k_0, y_0) converges to (k^s, y^s).*

2.2.2 Global dynamics

For the general case, the analysis becomes more complicated, since in some date t we can have $y_{t+1}^* = \eta(y_t^*)$. We can not sure that (k_t^*, y_t^*) maximizes $f(k) + \theta\eta(y)$ under the constraint $k + \theta y = z_t^*$. It is possible that the solutions of two maximization problems are not the same. To overcome this difficulty, first we will prove that for T sufficiently big, the constraints do not bind for $t \geq T$. Precisely, we have $0 < y_{t+1}^* < \eta(y_t^*)$ for any $t \geq T$.

We consider here the important notion, called *net gain of stock*. For each $(k, y) \in \mathbb{R}_+^2$, define

$$\Psi(k, y) = \beta\big(f(k) + \theta\eta(y)\big) - (k + \theta y).$$

This notion is presented first and carefully analysed by Majumdar & Nermuth [12], Dechert & Nishimura [5] and Mitra & Ray [15] to study the properties of steady states. Kamihigashi Roy in [7], [8] prove that the economy always evolves in order to increase in the future the value of *gain function*, otherwise we are at steady state. This is an important property which gives us deep understanding about economic dynamics. Following their spirit, we also consider the *net gain of stock* and prove that this value will increase in the future. This allows us to prove the convergence in the long term of the economy.

The idea runs as follows. Suppose that the economy begins with a state which is not stable. The value of *net gain of stock* must always increase in the future, hence there must have some period t which has the state goes very "near" to the steady state. Proposition 2.2 ensures that from that period, the optimal path converges rapidly to the steady state (k^s, y^s). Observe also that (k^s, y^s) is the maximizer of $\psi(k, y)$.

Consider first Lemma 2.3, which has an easy proof, using the concavity of two functions f and η.

LEMMA **2.3.** *Assume* ***H1****. The steady state is the only solution which maximizes* Ψ*:*

$$\underset{(k,y)\in\mathbb{R}_+^2}{\operatorname{argmax}}\, \Psi(k, y) = \{(k^s, y^s)\}.$$

The Lemma 2.4 is the most important intermediary result in the establishment of the long term behaviour of optimal path. It states that though the sequence of $\{\Psi(k_t^*, y_t^*)\}_{t=0}^{\infty}$ can be non-monotonic, there exists some day in the future the value of *net gain of stock* increases.

LEMMA **2.4.** *Assume* ***H1****. Consider the initial state* (k_0, y_0) *such that* $y_0 < \eta(y_0)$*. There is exactly one of the following statement is true:*

i) *For any* t*,* $k_t^* = k_0$ *and* $y_t^* = y_0$*.*

ii) *There exists some* $t > 0$ *such that*

$$\Psi(k_t^*, y_t^*) > \Psi(k_0, y_0).$$

With Lemma 2.4, for any initial state which is not steady, the value of the *net* *gain of stock* will increase in some day in future. Taking the sub-sequence whose the value *net gain stock* converges to the maximum one of the optimal state. If this subsequence converges to another state which is not stable, then the economy beginning from this state must increases the *net gain of stock* in the future, which leads us to a contradiction.

Proposition 2.3 states the convergence in the long term of the optimal path.

PROPOSITION 2.3. *Assume* ***H1***. *For any* $(k_0, y_0) \in \mathbb{R}^2_+$, *the optimal path beginning from* (k_0, y_0) *converges to* (k^s, y^s).

Let us now illustrate the existence of a unique steady state and global convergence to this steady state. For simplicity, suppose that the utility function verifies constant intertemporal elasticity of substitution (CIES), the production functions in both the fishery and final good sectors are Cobb-Douglas. The parameters chosen for this simulation are listed in Table 2.2.2.

Parameter	**Value**
θ	1
β	0.98
α_k	0.67
α_y	0.8
A (TFP in final good sector)	2
B (TFP in fishery sector)	1
k_0 (Initial stock of physical capital)	$1.5k^s$
y_0 (Initial stock of fish)	$0.2y^s$

Table 1: Parameters used for the simulated optimal paths under no emission

Notice that even though changing the intertemporal elasticity of substitution (IES) alters neither the steady state values nor the global convergence result, it affects the speed of convergence. In particular, the smaller the IES, the slower the the optimal sequences converge to their corresponding steady states, as shown in Fig.1 and Fig.2.

figures/noem.pdf

Figure 1: Optimal paths under no emission with constant intertemporal elasticity of substitution equal to 1 (logarithmic uitlity)

figures/noem2.pdf

Figure 2: Optimal paths under no emission with constant intertemporal elasticity of substitution equal to 0.1

3 Model with emission and externalities

3.1 Fundamentals

The analysis in Section 2 do not consider the effect of the production sector on the capacity of regeneration of the renewable resources. In reality, the industry sector causes significant pollution to the environment, especially to the fishery sector. The larger scale of production, the more pollution it causes, and that will reduce the capacity to regenerate of renewable resource. In order to facilitate the exposition, we assume that the industrial activities (which represented by E_t), depends linearly in the capital stock: $E_t = \alpha k_t$, with α a positive parameter capturing the negative effect of production process. We suppose that the regeneration of renewable resources depends negatively with the level of industrial activities E_t. The growth rate of fish at the end of period t is affected by pollution due to the production from period 0 to period t, means it is a function of y_t and E_t, we denote it is $\eta(y_t, E_t)$.

Assumption H2. *Assume conditions* (*i*), (*ii*), (*iv*) *and* (*v*) *in* ***H1***. *Moreover,*

i) *The function η is continuous, strictly increasing in respect to the first argument and strictly decreasing in respect to the second one.*

ii) $\eta(0, E) = 0$.

iii) *For any $E > 0$, $\eta'(0, E) = +\infty$ and $\eta'(\infty, E) < 1$.*

Since the function η is decreasing in respect to E, it is unrealistic to impose the concavity on η. We face a situation where the model does not satisfy neither convexity structure or super-modularity. The compactness remains verified, and hence solution always exists. We cannot exclude the possibility that there exist multiple optimal paths beginning from the same initial state.

The model now becomes:

$$v(k_0, y_0) = \max \sum_{t=0}^{\infty} \beta^t u(c_t),$$

$$c_t + k_{t+1} + \theta y_{t+1} \leq f(k_t) + \theta \eta(y_t, \alpha k_t),$$

$$y_{t+1} \leq \eta(y_t, \alpha k_t),$$

$$c_t, k_t, y_t \geq 0 \text{ for any t.}$$

Following the spirit as in the previous case, we firstly establish the basic properties: For each $(k, y) \in \mathbb{R}^2_+$, define

$$\Gamma(k, y) = \{(k', y') \in \mathbb{R}^2_+ \text{ such that } k' + \theta y' \leq f(k) + \theta\eta(y, \alpha k) \text{ and } y' \leq \eta(y, \alpha k)\}.$$

PROPOSITION 3.1. *Assume* ***H2****.*

i) *The correspondence* Γ *is continuous on* $\mathbb{R}^2_+$ *and convex, compact-valued.*

ii) *The value function* v *satisfied the Bellmann functional equation:*

$$v(k, y) = \max_{(k', y') \in \Gamma(k,y)} \left[u\big(f(k) + \theta\eta(y, \alpha k) - k' - \theta y'\big) + \beta v(k', y') \right].$$

Moreover, if the utility function u *is bounded from below,* v *is the unique solution.*

iii) *There exists optimal policy correspondence* φ *which is upper semi-continuous:*

$$\varphi(k, y) = \operatorname*{argmax}_{(k', y') \in \Gamma(k,y)} \left[u\big(f(k) + \theta\eta(y, \alpha k) - k' - \theta y'\big) + \beta v(k', y') \right].$$

iv) *The feasible sequence* $\{(k_t, y_t)\}_{t=0}^{\infty}$ *is optimal if and only if for any* t*,*

$$(k_{t+1}, y_{t+1}) \in \varphi(k_t, y_t).$$

v) *Assume that* $k_0 > 0$ *and* $y_0 > 0$*. Denote by* $\{(k^*_t, y^*_t)\}_{t=0}^{\infty}$ *the optimal sequence. For any* $t \geq 0$ *we have* $k^*_t > 0$*, and* $y^*_t > 0$*.*

3.2 Long-term dynamical analysis

3.2.1 Existence of steady states

Without the convexity structure, the existence, and the unicity of steady state are not ensured. We will try first describe some properties of long term behaviour of the economy. As in the previous section, define the function of *net gain investment.*

$$\Psi^e(k,y) = \max_{(k,y)\in\mathbb{R}^2_+} \left[\beta\left(f(k) + \theta\eta(y,\alpha k)\right) - (k+\theta y)\right].$$

By the compactness of the model, the set of argmax is no empty. Define

$$S^m = \underset{(k,y)\in\mathbb{R}^2_+}{\operatorname{argmax}} \left[\beta\left(f(k) + \theta\eta(y,\alpha k)\right) - (k+\theta y)\right].$$

By the continuity of η, it easy to verify that $S^m \neq \emptyset$ and for any $(k,y) \in S^m$, the constant sequence $\{k_t, y_t\}_{t=0}^{\infty}$ with $(k_t, y_t) = (k,y)\ \forall\ t$, is feasible. Hence steady state exists. For any initial state which is not steady state, the value of *net gain of stock* will increase in the future.

Proposition 3.2. *Assume **H2**.*

i) *Steady state exists.*

ii) *Either the initial state (k_0, y_0) is steady state, or for any optimal path $\{k_t^*, y_y^*\}_{t=0}^{\infty}$ beginning from (k_0, y_0), there is some $t \geq 0$ such that*

$$\Psi^e(k_t^*, y_t^*) > \Psi^e(k_0, y_0).$$

As in Section 2, Proposition 3.2 allows us to prove that any optimal sequence must goes very "near" the set of steady state(s) in some day in future. It this set contains unique point, this state must be an absorb point, in the sense that beginning in a neighborhood of this, there exists always one optimal path converging to it. The arguments as in Section 2 proves that beginning from every initial state, there exists always one optimal converging to the steady state.

Observe that though the possibility of multiple optimal paths can not be excluded, the set initial states which generate multiple optimal paths has zero measure (see Decher & Nishimura [5]). So we can *almost sure* that the economy always converges in the long term.

3.2.2 Long-term dynamics

Denote by η_1 and η_2 respectively the partial derivatives of function η in respect to the first and the second arguments.

Assumption H3. *Assume that the following system has unique solution:*

$$\begin{aligned} f'(k) + \theta\alpha\eta_2(y, \alpha k) &= \frac{1}{\beta}, \\ \eta_1(y, \alpha k) &= \frac{1}{\beta}. \end{aligned}$$

Since this system of equations is necessary condition for steady state, the assumption **H3** ensures the unicity of this.

First, we give an analysis for the dynamic which begins near the steady state. Define

$$G(z) = \max_{k+\theta y = z} \big[f(k) + \theta\eta(y, \alpha k)\big].$$

The function G is strictly increasing and differentiable. By **H3**, there exists unique solution to $G'(z) = \frac{1}{\beta}$. By Inada conditions, $G'(0) = \infty$ and $G'(\infty) < 1$. This implies that $G'(z) > \frac{1}{\beta}$ for $0 < z < z^s$ and $G'(z) < \frac{1}{\beta}$ for $z > z^s$[4] .

Consider the following modified problem for each given z_0,

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$c_t + z_{t+1} \leq G(z_t) \text{ for any } t \geq 0.$$

[4]Since G is differentiable, its derivatives function satisfies also the famous *Bosano - Cauchy* property, which states that if $G'(z) > \frac{1}{\beta}$ and $G'(z') < \frac{1}{\beta}$, then there exists some $\tilde{z}$ between z and z' such that $G'(\tilde{z}) = \frac{1}{\beta}$. Hence we do not have to require the continuity of G'.

LEMMA **3.1.** *Assume **H2** and **H3**. For any initial state z_0, every optimal path of the modified problem converges monotonically to the unique steady state z^s.*

Similarly to Section 2, Lemma 3.1 allows us to describe the behaviour of optimal once initial state is found sufficiently near the steady state (k^s, y^s). From that result, we establish our main result of this section: for any initial state, there exists an optimal path which converges to the steady state. The idea is that any optimal path must goes "close" to the steady state, and from that new position, there is a path which converges monotonically to (k^s, y^s).

PROPOSITION 3.3. *Assume **H2, H3**.*

i) *There exists a neighborhood $\mathcal{V}$ of (k^s, y^s) such that for any $(k_0, y_0) \in \mathcal{V}$, there exists an optimal path beginning from (k_0, y_0) which converges to (k^s, y^s).*

ii) *For any (k_0, y_0), there exists an optimal path beginning from (k_0, y_0) which converges to (k^s, y^s).*

3.2.3 UNIQUENESS OF THE STEADY STATE AND LONG-TERM CONVERGENCE

In this section, we give some added characterization for the regeneration function η, which imply the uniqueness of the steady state.

The concavity of η is unrealistic. We can assume that η is concave in respect to the second argument. Since for any y, $\eta(y, \cdot)$ is strictly decreasing, this implies that for high value of k, we obtain negative value of renewable resource, which is absurd. Moreover, since the condition $\lim_{k\to\infty} \eta(y, k) = 0$ is natural, it is better to assume that $\eta(y, \cdot)$ is convex in respect to the second argument. Assume that the regeneration function η is separable:

$$\eta(y, \alpha k) = g(y)h(\alpha k).$$

Assumption H4. i) *The function g is strictly increasing, strictly concave, satisfying $g'(0) = \infty$ and $g'(\infty) < 1$.*

ii) *The function h is strictly decreasing and convex.*

Observe that the function G is increasing but its concavity is not ensured. We will add a mild condition assuring that the unicity of steady state of the modified problem, and the concavity of G.

Define by k^m the solution to $f(k) = k$ and y^m the solution to $g(y) = y$. Let $z^m = k^m + \theta y^m$.

Assumption H5. *For any $0 \leq k \leq z \leq z^m$, we have*

i)

$$f''(k)+\frac{1}{\theta}g''\left(\frac{z-k}{\theta}\right)h(\alpha k)-2\alpha g'\left(\frac{z-k}{\theta}\right)h'(\alpha k)+\alpha^2\theta g\left(\frac{z-k}{\theta}\right)h''(\alpha k)<0.$$

ii)

$$\frac{1}{\theta}g''\left(\frac{z-k}{\theta}\right)h(\alpha k)-\alpha g'\left(\frac{z-k}{\theta}\right)h'(\alpha k)<0.$$

Under **H5**, the function G is strictly concave. There is unique solution to $G'(z) = \frac{1}{\beta}$, and hence **H3** is satisfied.

PROPOSITION 3.4. *Assume **H2**, **H4**, and **H5**. The steady state (k^s, y^s) is unique and for any (k_0, y_0), there exists an optimal path beginning from (k_0, y_0) which converges to (k^s, y^s).*

Observe that for any functions f, g and h, for α or θ sufficiently small, the assumption **H5** is verified, and the economy converges in the long term.

In the case the inequality in part (i) of assumption **H5** is satisfied without the presence of $f''(k)$, we get the implication (i) implies (ii) and Corollary 3.1.

COROLLARY 3.1. *Assume **H2**, **H4**. Assume that for any $0 \leq k \leq z \leq z^m$, we have*

$$\frac{1}{\theta}g''\left(\frac{z-k}{\theta}\right)h(\alpha k)-2\alpha g'\left(\frac{z-k}{\theta}\right)h'(\alpha k)+\alpha^2\theta g\left(\frac{z-k}{\theta}\right)h''(\alpha k)<0.$$

The steady state (k^s, y^s) is unique and for any (k_0, y_0), there exists an optimal path beginning from (k_0, y_0) which converges to (k^s, y^s).

Proof. Since $f''(k) \leq 0$ for any k, the condition (i) in assumption **H5** is satisfied. Moreover, since $\alpha g'\left(\frac{z-k}{\theta}\right) h'(\alpha k)$ and $\alpha^2 g\left(\frac{z-k}{\theta}\right)$ are positive, the assumption in the statement of this corollary implies the satisfaction of the condition (ii) in **H5**. The assumption **H5** is hence satisfied. Applying directly Proposition 3.3, the proof is completed. QED

For a more precise function h, $k(\alpha k) = e^{-\gamma\alpha k}$, condition in **H5** can be reduced to a simple condition being imposed on g.

Assumption H6. *For any $0 \leq y \leq y^m$, we have*

$$\frac{1}{\theta} g''(y) + 2\alpha\gamma g'(y) + \alpha^2\gamma^2\theta g(y) < 0.$$

Under assumption **H6**, it is easy to verify that the conditions in **H5** is satisfied. The Proposition 3.5 is direct consequence of Proposition 3.2.

PROPOSITION 3.5. *Consider the case $\eta(y, \alpha k) = g(y)e^{-\gamma\alpha k}$. Assume **H2**, **H4**, and **H6**. The steady state (k^s, y^s) is unique and for any (k_0, y_0), there exists an optimal path beginning from (k_0, y_0) which converges to (k^s, y^s).*

Assumption **H5** may raise concern that we can only obtain a good description for the long term behaviour in the case α, ou θ small, which means that the convergence is only ensured when the renewable resource has little importance in the economy. The following provides a response for this inquietude. For the case of Cobb-Douglass function, for any value of parameters, the convergence of the economy is satisfied.

PROPOSITION 3.6. *Assume that $f(k) = Ak^{\alpha_k}$, $g(y) = By^{\alpha_y}$ and $h(E) = e^{-\gamma E}$. Then the assumption **H3** is satisfied. For any initial state (k_0, y_0), there exists an optimal path which converges to the unique steady state (k^s, y^s).*

SIMULATION exponential

For simplicity assume that the utility function is logarithmic. The following parameter values are used for the numerical exercise.

Parameter	**Value**
γ	0.5
β	0.98
α_k	0.67
α_y	0.8
α (Emission coefficient)	0.2
A (TFP in final good sector)	2
B (TFP in fishery sector)	1
k_0 (Initial stock of physical capital)	$2k^s$
y_0 (Initial stock of fish)	$0.2y^s$

Table 2: Parameters used for the numerical simulation under exponential emission

We simulated the optimal paths of consumption, fish and physical capital for two different values of fish price in Fig.3 and Fig.4. In both cases we start with an initial physical capital stock greater than the steady state and an initial fish stock lower than the steady state by the same fraction for convenient comparability. Observe that the higher the price of fish, the greater the steady state values of consumption and fish stock, and the smaller the steady state value of physical capital. The convergence speed also appears to be slower when the fish price is higher.

SIMULATION nonexponential

We again assume logarithmic utility, Cobb-Douglas production for this simulation. All parameters are as given in Table 3.2.3 for the exponential emission case, except that here we set $\theta = 1$ and simulated the optimal paths for $\zeta = 0.5$ and $\zeta = 10$ in Fig.5 and Fig.6, respectively. Recall ζ represents the impact of pollution on fishery (while α reflects the intensity of industrial pollution).

A few comments are in order. First, the impact of pollution on fishery has a mild negative effect on steadystate consumption and positive effect on steadystate physical capital. Second the impact of pollution on steadystate fish stock is dramatic: when ζ is sufficiently large, the stock of fish is depleted at the steady state.

figures/emexp2.pdf

Figure 3: Optimal paths under exponential emission with low fish price $\theta = 1$

4 Conclusion

In this article, in a configuration where the usual structures as convexity or supermodularity are not satisfied, we develop a new method to analyse the long term dynamic of the economy and we prove that under suitable conditions, generally the economy converges in long term. The simulations suggest that in the first periods, the economy may exhibit some fluctuations but rapidly, it can return to convergence with high speed to the steady state.

figures/emexp1.pdf

Figure 4: Optimal paths under exponential emission with high fish price $\theta = 50$

5 Appendix

5.1 Proof of Proposition 2.1

We only need to prove that the correspondence Γ is continuous:

* Γ is lower hemi continuous:

Denote $x = (k, y)$ and $x_n = (k_n, y_n) \to x$ and $z = (k', y') \in \Gamma(x)$. Therefore $\lim_{n\to\infty} f(k_n) = f(k); \lim_{n\to\infty} \eta(y_n) \to \eta(y)$.

- Consider the case $\begin{cases} 0 \leq y' < \eta(y) \\ 0 \leq k' + \theta y' < f(k) + \theta\eta(y). \end{cases}$

figures/emnonexp1.pdf

Figure 5: Optimal paths under non-exponential emission with $\zeta = 0.5$

We can find $\epsilon > 0$ and N such that $\forall n > N$ then:

$$\begin{cases} 0 \le y' \le \eta(y) - \epsilon \le \eta(y_n) \\ 0 \le k' + \theta y' \le f(k) - \epsilon + \theta(\eta(y) - \epsilon) \le f(k_n) + \theta\eta(y_n). \end{cases}$$

Hence, we can choose $z_n = (k', y')\forall n > N$ then $z_n \to z$ and $z_n \in \Gamma(x_n)$.

- Consider the case $\begin{cases} y' = \eta(y) \\ k' + \theta y' = f(k) + \theta\eta(y). \end{cases}$
  We can chose $z_n = (f(k_n), \eta(y_n))$ then $z_n \to z$ and $z_n \in \Gamma(x_n)$.
- Consider the case $\begin{cases} y' = \eta(y) \\ 0 \le k' + \theta y' < f(k) + \theta\eta(y) \end{cases}$
  there exists $\epsilon > 0 : k' < f(k) - \epsilon \le f(k_n)$. Hence can chose $z_n = (k', \eta(y_n))$ then $z_n \to z$ and $z_n \in \Gamma(x_n)$.

figures/emnonexp4.pdf

Figure 6: Optimal paths under non-exponential emission wih $\zeta = 10$

- If $\begin{cases} 0 \leq y' < \eta(y) \\ 0 \leq k' + \theta y' = f(k) + \theta\eta(y) \end{cases}$ means $k' = f(k) + \theta(\eta(y) - y')$ there exists $\{\epsilon_n\}$ such that $\eta(y) - \epsilon_n < \eta(y_n)$ and $\epsilon_n \to 0$ then We can chose $z_n = (f(k_n) + \theta(\eta(y) - y' - \epsilon), y')$ then $z_n \to z$ and $z_n \in \Gamma(x_n)$.

* Γ is upper hemi continuous:

Consider a sequence $x_n = (k_n, y_n) \to x = (k, y)$ and sequence $z_n = (k'_n, y'_n) \in \Gamma(x_n)$. Hence there exists a subsequence (k'_{nj}, y'_{nj}) converges to (k', y'). Since $z_{nj} \in \Gamma(x_{nj})$ then

$$k'_{nj} + \theta y'_{nj} \leq f(k_{nj}) + \theta\eta(y_{nj});$$
$$y'_{nj} \leq \eta(y_{nj})$$

Hence, it is obviously that:

$$k' + \theta y' \leq f(k) + \theta\eta(y);$$
$$y' \leq \eta(y)$$

That means $(k', y') \in \Gamma(k, y)$.

The proof for the rest of the Proposition can be found in Le Van & Morhaim [10]. Remark that for the case the utility function is bounded from below, the unicity properties can be ensured using fixed-point techniques presented in Stockey & Lucas (with Prescott) [16].

5.2 Proof of Lemma 2.2

The uniqueness of solution of problem (Q) is assured by the convex structure of the model. From the strictly concavity of f and η, the function F is strictly concave.

i) Consider solution of problem (P), $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ satisfying for any t, $k_t^* > 0$ and $0 < y_{t+1}^* < \eta(y_t^*)$. By Euler equations, we have $f'(k_t^*) = \eta'(y_t^*)$. Since f et η are concave functions, this implies

$$(k_t^*, y_t^*) = \underset{k+\theta y = z}{\operatorname{argmax}} \left(f(k) + \theta\eta(y)\right).$$

Hence we have for any t, $F'(z_t^*) = f'(k_t^*) = \eta'(y_t^*)$, or the sequence $\{z_t^*\}_{t=0}^{\infty}$ satisfies Euler equation: for any t,

$$u'(c_t^*) = \beta u'(c_{t+1}^*) F'(z_{t+1}^*).$$

From the transverlity condition of problem (P), we have:

$$\lim_{t\to\infty} \beta^t u'(c_t^*) z_{t+1}^* = \lim_{t\to\infty} \beta^t u'(c_t^*) \left(k_{t+1}^* + \theta\eta(y_{t+1}^*)\right)$$
$$= 0.$$

Hence the transversality condition is satisfied. The sequence $\{z_t^*\}_{t=0}^{\infty}$ is solution of problem (Q).

ii) Consider solution $\{\tilde{z}_t\}_{t=0}^{\infty}$ of problem (Q) and $(\tilde{k}_t, \tilde{y}_t) = \text{argmax}_{k+\theta y=\tilde{z}_t}\left(f(k) + \theta\eta(y)\right)$. If for any t, $\tilde{k}_t > 0$ and $0 < \tilde{y}_{t+1} < \tilde{\eta}(y_t)$, then $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ is a feasible sequence of problem (P).

By the Lemma 2.1, for any $t \geq 0$, we have for any $t \geq 1$, $f'(\tilde{k}_t) = \eta'(\tilde{y}_t) = F'(\tilde{z}_t)$. From the Euler equations:

$$
\begin{aligned}
u'(\tilde{c}_t) &= \beta u'(\tilde{c}_{t+1}) f'(\tilde{k}_{t+1}) \\
&= \beta u'(\tilde{c}_{t+1}) \eta'(\tilde{y}_{t+1}).
\end{aligned}
$$

Observe that for any $t \geq 1$, $\tilde{k}_t \leq \tilde{z}_t$ and $\tilde{y}_t \leq \dfrac{\tilde{z}_t}{\theta}$. From the transversality condition of problem (Q):

$$
\begin{aligned}
&\lim_{t\to\infty} \beta^t u'(\tilde{c}_t)\tilde{k}_{t+1} = 0, \\
&\lim_{t\to\infty} \beta^t u'(\tilde{c}_t)\tilde{y}_{t+1} = 0.
\end{aligned}
$$

The sequence $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ satisfies Euler equations and transversality condition of problem (P), hence this sequence is the optimal problem.

5.3 Proof of Proposition 2.2

i) From Inada conditions, one has $f'(k^s) = \eta'(y^s) = F'(z^s) = \frac{1}{\beta}$. This implies $0 < y^s < \eta(y^s)$. Hence the sequence $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ with $k_t^* = k^s$ and $y_t^s = y^s$ for any t satisfies Euler equations and transversality condition for the problem (P) with initial state $(k_0, y_0) = (k^s, y^s)$.

ii) Take a neighborhood $\mathscr{V}_z$ of z^s such that if $z_0 \in \mathscr{V}_z$, the optimal sequence $\{z_t^*\}_{t=0}^{\infty}$ is subset of $\mathscr{V}_z$ and converges to z^s. Define $\tilde{\mathscr{V}}$ the set of (k_0, y_0) such that $z_0 = F^{-1}\big(f(k_0 + \theta y_0)\big)$ belongs to $\mathscr{V}_z$.

Obviously, $\tilde{\mathscr{V}}$ contrains a neighborhood $\mathscr{V}$ of (k^s, y^s). For any $(k_0, y_0) \in \mathscr{V}$, define

$z_0 = f(k_0) + \theta\eta(y_0)$. The optimal solution $\{\tilde{z}_t\}_{t=0}^{\infty}$ of problem (Q) with inital z_0 satisfies $z_t \in \mathcal{V}_z$ for any t and converges to z^s. Moreover, since $0 < y^s < \eta(y^s)$, the corresponding sequence $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ satisfies $0 < \tilde{y}_{t+1} < \eta(y_t)$ for any t and hence $f'(\tilde{k}_t) = \eta'(\tilde{y}_t) = F'(\tilde{z}_t)$. Obvisouly, this sequence satisfies transversality condition. By Lemma 2.2, the sequence $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ is solution of (P) and from the convergence of $\{\tilde{z}_t\}_{t=0}^{\infty}$ to z^s, this sequence converges to (k^s, y^s).

5.4 Proof of Lemma 2.4

First, observe that for any T,

$$
\begin{aligned}
&\sum_{t=0}^{T} \beta^t \left(f(k_t^*) + \theta\eta(y_t^*) - k_{t+1}^* - \theta y_{t+1}^*\right) \\
&= f(k_0) + \theta\eta(y_0) - k_1^* - \theta y_1^* \\
&+ \beta\left(f(k_1^*) + \theta\eta(y_1^*) - k_2^* - \theta y_2^*\right) \\
&+ \beta^2\left(f(k_2^*) + \theta\eta(y_2^*) - k_3^* - \theta y_3^*\right) + \ldots \\
&+ \beta^T \left(f(k_T^*) + \theta\eta(y_T^*) - k_{T+1}^* - \theta y_{T+1}^*\right) \\
&= f(k_0) + \theta\eta(y_0) + \beta\left(f(k_1^*) + \theta\eta(y_1^*)\right) - k_1^* - \theta y_1^* \\
&+ \beta\left(f(k_2^*) + \theta\eta(y_2^*)\right) - k_2^* - \theta y_2^* \\
&+ \ldots \\
&+ \beta^T\left(f(k_T^*) + \theta\eta(y_T^*)\right) - k_T^* - \theta y_T^* - \beta^T\left(k_{T+1}^* + \theta y_{T+1}^*\right) \\
&= f(k_0) + \theta\eta(y_0) \\
&+ \Psi(k_1^*, y_1^*) \\
&+ \beta\Psi(k_2^*, y_2^*) \\
&+ \ldots \\
&+ \Psi(k_T^*, y_T^*) \\
&- \beta^T\left(k_{T+1}^* + \theta y_{T+1}^*\right) \\
&= f(k_0) + \theta\eta(y_0) + \sum_{t=0}^{T} \beta^t \Psi(k_{t+1}^*, y_{t+1}^*) - \beta^T\left(f(k_{T+1}^*) + \theta\eta(y_{T+1}^*)\right).
\end{aligned}
$$

Let T converges to infinity, we get

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t c_t^* &= \lim_{T\to\infty}\left[k_0+\theta\eta(y_0)+\sum_{t=0}^{T}\beta^t\Psi(k_{t+1}^*,y_{t+1}^*)-\beta^T\left(f(k_{T+1}^*)+\theta\eta(y_{T+1}^*)\right)\right]\\ &= f(k_0)+\theta\eta(y_0)+\sum_{t=0}^{\infty}\beta^t\Psi(k_{t+1}^*,y_{t+1}^*).\end{aligned}$$

Assume that for any $t\geq 0$, we have $\Psi(k_t^*,y_t^*)\leq\Psi(k_0,y_0)$. This implies

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t c_t^* &= f(k_0)+\theta\eta(y_0)+\sum_{t=0}^{\infty}\beta^t\Psi(k_t^*,y_t^*)\\ &\leq f(k_0)+\theta\eta(y_0)+\frac{\Psi(k_0,y_0)}{1-\beta}\\ &= f(k_0)+\theta\eta(y_0)+\frac{\beta\left(f(k_0)+\theta\eta(y_0)\right)-k_0-y_0}{1-\beta}\\ &= \frac{f(k_0)-k_0+\theta(\eta(y_0)-y_0)}{1-\beta}.\end{aligned}$$

Hence by the concavity of u:

$$\begin{aligned}u\big(f(k_0)-k_0+\theta(\eta(y_0)-y_0)\big) &\geq u\left((1-\beta)\sum_{t=0}^{\infty}\beta^t c_t^*\right)\\ &\geq (1-\beta)\sum_{t=0}^{\infty}\beta^t u(c_t^*).\end{aligned}$$

Since $y_0<\eta(y_0)$, the sequence $\{k_t,y_t\}_{t=0}^{\infty}$ such that $k_t=k_0$, $y_t=y_0$ for any t is feasible. Moreover, by the concavity of u,

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t u\big(f(k_t)-k_{t+1}+\theta(\eta(y_t)-y_{t+1})\big) &= \sum_{t=0}^{\infty}\beta^t u\big(f(k_0)-k_0+\theta(\eta(y_0)-y_0)\big)\\ &= \frac{u\big(f(k_0)-k_0+\theta(\eta(y_0)-y_0)\big)}{1-\beta}\\ &\geq \sum_{t=0}^{\infty}\beta^t u(c_t^*).\end{aligned}$$

Since $\{k_t^*,y_t^*\}_{t=0}^{\infty}$ is the unique optimal path, this implies $k_t^*=k_0$ and $y_t^*=y_0$ for any $t\geq 0$. The optimal sequence is constant.

For the case the optimal sequence is not constant, the above arguments imply the existence of t such that $\Psi(k_t^*, y_t^*) > \Psi(k_0, y_0)$.

5.5 Proof of Proposition 2.3

The proof is divided in some intermediary steps.

i) There exists T such that $y_t^* < \eta(y_t^*)$ for any $t \geq T$.

ii) There exists T such that $f'(k_T^*) > \eta'(y_T^*)$.

iii) The equality $\sup_{t \geq 0} \Psi(k_t^*, y_t^*) = \Psi(k^s, y^s)$.

iv) The convergence of the optimal path.

(i) Consider $(k_0, y_0) \in \mathbb{R}_+^2$. Denote by $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ the optimal path beginning from (k_0, y_0). Without loss of generality, we can assume that $y_0 < \eta(y_0)$. Indeed, we will prove the existence of T such that $y_T^* < \eta(y_T^*)$.

Suppose the contrary, then for any $t \geq 0$ we have

$$y_{t+1}^* \leq \eta(y_t^*) \leq y_t^*.$$

The sequence $\{y_t^*\}_{t=0}^{\infty}$ is decreasing and hence converges to some y^* satisfying

$$y^* \leq \eta(y^*) \leq y^*,$$

which implies that $y^* = \eta(y^*) = \overline{y}$. We will prove the existence of some T such that

$$f'(k_T^*) > \eta'(y_T^*).$$

Indeed, suppose the contrary. This implies

$$\limsup_{t\to\infty} f'(k_t^*) \le \eta'(\overline{y})$$

$$< 1.$$

By the Euler equations $u'(c_t^*) = \beta u'(c_{t+1}^*) f'(k_{t+1}^*)$, there exists T sufficiently big such that for any $t \ge T$, $u'(c_t^*) \le u'(c_{t+1}^*)$, or the sequence $\{c_t^*\}_{t=T}^\infty$ is decreasing and converges to c^*.

The convergence of sequences $\{c_t^*\}_{t=T}^\infty$ and $\{y_t^*\}_{t=0}^\infty$ implies the convergence of $\{k_t^*\}_{t=0}^\infty$:

$$\lim_{t\to\infty} k_t^* = k^*.$$

From the Euler equations, we deduce that either $c^* = 0$, or $f'(k^*) = \frac{1}{\beta}$. The hypothesis that $f'(k^*) = \frac{1}{\beta}$, which is bigger than 1, leads us to a contradiction. Hence $c^* = 0$. Since $\lim_{t\to\infty} y_t^* = \overline{y}$, we have $\lim_{t\to\infty} k_t^* = \overline{k}$, the solution to $f(k) = k$. By the continuity of the optimal policy function, we have the conclusion that the consumption level at initial state $(\overline{k}, \overline{y})$ is $c^* = 0$: a contradiction.

(ii) Hence there exists some T such that

$$f'(k_{T+1}^*) > \eta'(y_{T+1}^*).$$

Fix $\epsilon > 0$ sufficiently such that:

$$f(k_{T+1}^* + \epsilon) + \theta\eta\left(y_{T+1}^* - \frac{\epsilon}{\theta}\right) > f(k_{T+1}^*) + \theta\eta\left(y_{T+1}^*\right).$$

Consider the sequence $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$ defined as

$$\begin{aligned}
\hat{y}_t &= y_t^* \text{ for any } 0 \le t \le T,\\
\hat{y}_{T+1} &= y_{T+1}^* - \frac{\epsilon}{\theta},\\
\hat{y}_{t+1} &= y_{t+1}^* \text{ for any } t \ge T,\\
\hat{k}_t &= k_t^* \text{ for any } 0 \le t \le T,\\
\hat{k}_{T+1} &= k_{T+1}^* + \epsilon,\\
\hat{k}_t &= k_t^* \text{ for any} t \ge T+2.
\end{aligned}$$

We can verify that the sequence $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$ is feasible. We have

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t u(\hat{c}_t) - \sum_{t=0}^{\infty} \beta^t u(c_t^*) &= \beta^{T+1}\left(u(\hat{c}_{T+1}) - u(c_{T+1}^*)\right)\\
&= \beta^{T+1} u\left(f(k_{T+1}^* + \epsilon) + \theta\eta\left(y_{T+1}^* - \frac{\epsilon}{\theta}\right) - k_{T+2}^* - \theta\eta(y_{T+2}^*)\right)\\
&- \beta^{T+1} u\left(f(k_{T+1}^*) + \theta\eta\left(y_{T+1}^*\right) - k_{T+2}^* - \theta\eta(y_{T+2}^*)\right)\\
&> 0,
\end{aligned}$$

a contradiction. This contradiction comes from the hypothesis that for any t, $y_t^* \ge \eta(y_t^*)$.

Then there exists some T such that $y_T^* < \eta(y_T^*)$. Hence $y_T^* < \overline{y}$. By induction we have for any $t \ge T$, $y_t^* < \overline{y}$ and hence $y_t^* < \eta(y_t^*)$. Without loss of generality, we can assume that the economy begins with $y_0 < \eta(y_0)$ and this property is satisfied by any $t \ge 0$.

(iii) Consider the subsequence $\{(k_{t_n}^*, y_{t_n}^*)\}_{n=0}^{\infty}$ such that

$$\lim_{n\to\infty} \Psi\left(k_{t_n}^*, y_{t_n}^*\right) = \sup_{t\ge 0} \Psi(k_t^*, y_t^*).$$

Recall that $\sup_{t\ge0} \Psi(k_t^*, y_t^*) \le \Psi(k^s, y^s)$. Suppose that this inequality is strict.

Since the sequence $\{(k_{t_n}^*, y_{t_n}^*)\}_{n=0}^{\infty}$ is bounded, without loss of generality, we can assume that

$$\lim_{n\to\infty} k^*_{t_n} = k^*,$$
$$\lim_{n\to\infty} y^*_{t_n} = y^*.$$

Since $\sup_{t\geq 0} \Psi(k^*_t, y^*_t) < \Psi(k^s, y^s)$, we have $\Psi(k^*, y^*) < \Psi(k^s, y^s)$ and (k^*, y^*) is not steady state. Moreover $y^* \leq \eta(y^*)$. Let $\{\tilde{k}_t, \tilde{y}_t\}_{t=0}^{\infty}$ the optimal path beginning from (k^*, y^*). By Lemma 2.4, there exists T such that

$$\Psi(\tilde{k}_T, \tilde{y}_T) > \Psi(k^*, y^*).$$

By the continuity of the problem, there is a neighborhood $\mathscr{V}$ of (k^*, y^*) such that for any $(k'_0, y'_0) \in \mathscr{V}$, the optimal path $\{k'_t, y'_t\}_{t=0}^{\infty}$ satisfies

$$\Psi(k'_T, y'_T) > \Psi(k^*, y^*).$$

Since the sequence $\{(k^*_{t_n}, y^*_{t_n})\}_{n=0}^{\infty}$ converges to (k^*, y^*), there is n sufficiently big such that $(k^*_{t_n}, y^*_{t_n}) \in \mathscr{V}$. We have

$$\begin{aligned}\Psi\left(k^*_{t_n+T}, y^*_{t_n+T}\right) &> \Psi(k^*, y^*)\\ &= \sup_{t\geq 0} \Psi(k^*_t, y^*_t),\end{aligned}$$

a contradiction. This contradiction comes from the hypothesis that $\sup_{t\geq 0} \Psi(k^*_t, y^*_t) < \Psi(k^s, y^s)$.

(iv) Hence $\sup_{t\geq 0} \Psi(k^*_t, y^*_t) = \Psi(k^s, y^s)$. Hence for any neighborhood $\mathscr{V}$ of (k^s, y^s), there is some t such that $(k^*_t, y^*_t) \in \mathscr{V}$. Using Proposition 2.2, we have

$$\lim_{t\to\infty} k^*_t = k^s,$$
$$\lim_{t\to\infty} y^*_t = y^s.$$

5.6 Proof of Proposition 3.1

We just need to check the upper semi-continuity property of the corresspondence Γ.

* Γ is lower hemi continuous:

Denote $x = (k, y)$ and $x_n = (k_n, y_n) \to x$ and $z = (k', y') \in \Gamma(x)$. Therefore $\lim_{n\to\infty} f(k_n) = f(k); \lim_{n\to\infty} \eta(y_n, \alpha k_n) \to \eta(y, \alpha k)$.

- Consider the case $\begin{cases} 0 \le y' < \eta(y, \alpha k) \\ 0 \le k' + \theta y' < f(k) + \theta\eta(y, \alpha k). \end{cases}$
  We can find $\epsilon > 0$ and N such that $\forall n > N$ then:
  $$\begin{cases} 0 \le y' \le \eta(y, \alpha k) - \epsilon \le \eta(y_n, \alpha k_n) \\ 0 \le k' + \theta y' \le f(k) - \epsilon + \theta(\eta(y, \alpha k) - \epsilon) \le f(k_n) + \theta\eta(y_n, \alpha k_n). \end{cases}$$
  Hence, we can choose $z_n = (k', y') \forall n > N$ then $z_n \to z$ and $z_n \in \Gamma(x_n)$.

- Consider the case $\begin{cases} y' = \eta(y, \alpha k) \\ k' + \theta y' = f(k) + \theta\eta(y, \alpha k). \end{cases}$
  We can chose $z_n = (f(k_n), \eta(y_n, \alpha k_n))$ then $z_n \to z$ and $z_n \in \Gamma(x_n)$.

- Consider the case $\begin{cases} y' = \eta(y, \alpha k) \\ 0 \le k' + \theta y' < f(k) + \theta\eta(y, \alpha k) \end{cases}$
  there exists $\epsilon > 0 : k' < f(k) - \epsilon \le f(k_n)$. Hence can chose $z_n = (k', \eta(y_n, \alpha k_n))$ then $z_n \to z$ and $z_n \in \Gamma(x_n)$.

- If $\begin{cases} 0 \le y' < \eta(y, \alpha k) \\ 0 \le k' + \theta y' = f(k) + \theta\eta(y, \alpha k) \end{cases}$ means $k' = f(k) + \theta(\eta(y, \alpha k) - y')$
  there exists $\{\epsilon_n\}$ such that $\eta(y, \alpha k) - \epsilon_n < \eta(y_n, \alpha k_n)$ and $\epsilon_n \to 0$ then we can chose $z_n = (f(k_n) + \theta(\eta(y, \alpha k) - y' - \epsilon), y')$ then $z_n \to z$ and $z_n \in \Gamma(x_n)$.

*Γ is upper hemi continuous: Consider a sequence $x_n = (k_n, y_n)$ which converges to $x = (k, y)$ and a sequence $z_n = (k'_n, y'_n) \in \Gamma(x_n)$. Hence there exists a subsequence

(k'_{nj}, y'_{nj}) converges to (k', y'). Since $z_{nj} \in \Gamma(x_{nj})$ then

$$k'_{nj} + \theta y'_{nj} \leq f(k_{nj}) + \theta\eta(y_{nj}, \alpha k_{nj});$$
$$y'_{nj} \leq \eta(y_{nj}, \alpha k_{nj})$$

Hence, it is obviously that:

$$k' + \theta y' \leq f(k) + \theta\eta(y, \alpha k);$$
$$y' \leq \eta(y, \alpha k)$$

That means $(k', y') \in \Gamma(k, y)$.

The proof for the rest of the Proposition can be found in Le Van & Morhaim [10]. Remark that for the case the utility function is bounded from below, the unicity properties can be ensured using fixed-point techniques presented in Stockey & Lucas (with Prescott) [16].

5.7 Proof of Proposition 3.2

(i) Fix any $(k_0, y_0) \in S^m$. First we prove that the constant sequence beginning from (k_0, y_0) is feasible. Indeed, we have only to prove that $y_0 \leq \eta(y_0, k_0)$. Suppose the contrary, $\eta(y_0, k_0) < y_0$. Since $\eta_1(0, k_0) = \infty$, there exists y sufficiently small such that $y < \eta(y, k_0)$. It is easy to verify that $\psi(y, k_0) > \psi(y_0, k_0)$: a contradiction.

Consider an optimal path $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ beginning from (k_0, y_0). By the choice of (k_0, y_0), for any t we have $\Psi^e(k_t^*, y_t^*) \leq \Psi^e(k_0, y_0)$. Using the same arguments as in the proof of Proposition 2.3, we have

$$\sum_{t=0}^{\infty} \beta^t u\left(f(k_0) + \theta\eta(y_0, \alpha k_0) - k_0 - \theta y_0\right) \geq \sum_{t=0}^{\infty} \beta^t u(c_t^*),$$

which implies that the constant sequence $\{(k_0, y_0)\}_{t=0}^{\infty}$ is also an optimal path beginning from (k_0, y_0). Hence (k_0, y_0) is a steady state of the economy.

(ii) We follow the same line of arguments of the proof of Proposition 2.3. Fix (k_0, y_0) and an optimal path $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ beginning from (k_0, y_0). Using the same arguments, we can consider without losing the generality that $y_0 \leq \eta(y_0, \alpha k_0)$. We have

$$\sum_{t=0}^{\infty} \beta^t c_t^* = f(k_0) + \theta\eta(y_0, k_0) + \sum_{t=0}^{\infty} \beta^t \Psi^e(k_t^*, y_t^*).$$

Assue that for any $t \geq 0$, $\Psi^e(k_t^*, y_t^*) \leq \Psi^e(k_0, y_0)$. Then

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) \leq \frac{u\left(f(k_0) + \theta\eta(y_0, k_0) - k_0 - \theta y_0\right)}{1-\beta},$$

which implies that the sequences $\{c_t^*\}_{t=0}^{\infty}$ and $\{\Psi^e(k_t^*, y_t^*)\}_{t=0}^{\infty}$ are constant. Hence for any t,

$$f(k_t^*) + \theta\eta(y_t^*, \alpha k_t^*) - k_{t+1}^* - \theta y_{t+1}^* = f(k_0) + \theta\eta(y_0, k_0) - k_1^* - \theta y_1^*,$$
$$\beta\big(f(k_t^*) + \theta\eta(y_t^*, \alpha k_t^*)\big) - k_t^* - \theta y_t^* = \beta\big(f(k_0) + \theta\eta(y_0, k_0)\big) - k_0^* - \theta y_0^*.$$

Let

$$\Delta = -\beta(k_1^* + \theta y_1^*) + (k_0 + \theta y_0).$$

For any t, we have

$$\begin{aligned}
k_t^* + \theta y_t^* &= \beta(k_{t+1}^* + \theta y_{t+1}^*) + \Delta \\
&= \beta^2(k_{t+2}^* + \theta y_{t+2}^*) + \beta\Delta + \Delta \\
&= \cdots \\
&= \beta^T(k_{t+T}^* + \theta y_{t+T}^*) + \Delta \sum_{s=0}^{T-1} \beta^s.
\end{aligned}$$

Let T converges to infinity, we get for any t,

$$k_t^* + \theta y_t^* = \frac{\Delta}{1-\beta}$$
$$= \frac{(k_0 + \theta y_0) - \beta(k_1^* + \theta y_1^*)}{1-\beta}.$$

Hence for any $t \geq 0$ we have

$$k_t^* + \theta y_t^* = k_0 + \theta y_0.$$

Since the consumption sequence is constant, we get for any $t \geq 0$,

$$f(k_t^*) + \theta\eta(y_t^*, \alpha k_t^*) = f(k_0) + \theta\eta(y_0, \alpha k_0).$$

These two equalities prove that (y_0, k_0) is one steady state.

The conclusion that (k_0, y_0) belongs to the set of steady states comes from the hypothesis that $\Psi^e(k_t^*, y_y^*) \leq \Psi^e(k_0, y_0)$ for any $t \geq 0$. Hence if (k_0, y_0) is not a steady state, there exists t such that

$$\Psi^e(k_t^*, y_t^*) > \Psi^e(k_0, y_0).$$

5.8 Proof of Lemma 3.1

By **H3**, the unicity of steady state is ensured. For each $0 \leq z' \leq G(z)$, define $V(z, z') = u(G(z) - z')$. By the concavity of u and the monotonicity of G, this indirect utility function satisfies the *super-modularity* (see Amir [1]). Every optimal path of the modified problem is hence monotonic. We will prove the following claim: for any initial state $z_0 > 0$, every optimal path beginning from z_0 converges monotonically to z^s. Precisely, let $\{z_t^*\}_{t=0}^{\infty}$ an optimal path beginning from z_0. If $z_0 \leq z^s$ then this path is increasing and converges to z^s. Otherwise, if $z_0 \geq z^s$, this path is decreasing and converges to z^s.

Indeed, consider the case $0 < z_0 < z^s$. Assume that the sequence $\{z_t^*\}_{t=0}^{\infty}$ is strictly decreasing. For fixed $z < z^s$, consider the following function with variable z' belonging to $[0, z]$:

$$w(z') = u\left(G(z) - z'\right) + \frac{\beta}{1-\beta} u\left(G(z') - z'\right).$$

We have, by the concavity of u:

$$\begin{aligned}
w'(z') &= -u'\left(G(z) - z'\right) + \frac{\beta}{1-\beta} u'\left(G(z') - z'\right)\left(G'(z') - 1\right) \\
&\geq -u'\left(G(z') - z'\right) + \frac{\beta}{1-\beta} u'\left(G(z') - z'\right)\left(G'(z') - 1\right) \\
&= u'\left(G(z') - z'\right) \times \frac{\beta G'(z') - 1}{1-\beta} \\
&> 0.
\end{aligned}$$

This implies that the function w is strictly increasing in $[0, z]$. hence we have

$$\begin{aligned}
\frac{u\left(G(z) - z\right)}{1-\beta} &= w(z) \\
&\geq w(z') \\
&= u\left(G(z) - z'\right) + \frac{\beta}{1-\beta} u\left(G(z') - z'\right),
\end{aligned}$$

for any $0 \leq z' \leq z$.

The hypothesis such that $\{z_t^*\}_{t=0}^{\infty}$ is decreasing implies

$$\begin{aligned}
\frac{u\left(G(z_0) - z_0\right)}{1-\beta} &\geq u\left(G(z_0) - z_1^*\right) + \beta \frac{u\left(G(z_1^*) - z_1\right)}{1-\beta} \\
&\geq u\left(G(z_0) - z_1^*\right) + \beta u\left(G(z_1^*) - z_2^*\right) + \beta^2 \frac{u\left(G(z_2^*) - z_2^*\right)}{1-\beta} \\
&\dots \\
&\geq \sum_{t=0}^{T} \beta^t u\left(G(z_t^*) - z_{t+1}^*\right) + \beta^{T+1} \frac{u\left(G(z_{T+1}^*) - z_{T+1}^*\right)}{1-\beta} \\
&\dots \\
&\geq \sum_{t=0}^{\infty} \beta^t u\left(G(z_t^*) - z_{t+1}^*\right).
\end{aligned}$$

Hence $(z_0, z_0, \dots)$ is also an optimal path, which implies that $z_0 = z^s$: a contradiction.

Hence the sequence $\{z_t^*\}_{t=0}^{\infty}$ is increasing and converges to z^s. For $z_0 > z^s$, using the same arguments, we prove that any optimal path beginning from z_0 is decreasing and converges to z^s.

5.9 Proof of Proposition 3.3

(i) The proof follows the same arguments as Section 2. We know that for any z_0, the optimal path of the modified problem converges monotonically to the steady state z^s. We have $z^s = k^s + \theta y^s$.

For optimal path of the modified problem $\{z_t^*\}_{t=0}^{\infty}$ the optimal path beginning from $z_0 = k_0 + \theta y_0$, define (k_t^*, y_t^*) as

$$(k_t^*, y_y^*) = \underset{k+\theta y = z_t^*}{\operatorname{argmax}} \big[f(k) + \theta \eta(y, \alpha k) \big].$$

Since $y^s < g(y^s)h(k^s)$, for (k_0, y_0) belonging to a neighborhoood of (k^s, y^s), the corresponding sequence $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ satisfied $y_{t+1}^* < g(y_t^*)h(k_t^*)$ for any $t \geq 0$. This implies the sequence $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ is feasible and hence it is an optimal path of the initial problem. This sequence converges to (k^s, y^s).

(ii) Fix (k_0, y_0) and an optimal path $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ beginning from (k_0, y_0). Take the sub-sequence $\{(k_{t_n}^*, y_{t_n}^*)\}_{n=0}^{\infty}$ such that

$$\lim_{n \to \infty} \Psi^e(k_{t_n}^*, y_{t_n}^*) = \sup_{t \geq 0} \Psi^e(k_t^*, y_t^*).$$

Without loss of generality, we can assume that this sub-sequence converges:

$$\lim_{n \to \infty} (k_{t_n}^*, y_{t_n}^*) = (\tilde{k}, \tilde{y}).$$

We state that $(\tilde{k}, \tilde{y}) = (k^s, y^s)$.

Assume the contrary. Consider the "sequence of sequences" $\{\mathbf{k}_{t_n}\}_{n=0}^{\infty}$, where for each n, $\mathbf{k}_{t_n} = \{(k^*_{t_n+t}, y^*_{t_n+t}\}_{t=0}^{\infty}$. By the compactness of the set of feasible sequences, we can assume that the sequence of sequences $\{\mathbf{k}_{t_n}\}_{n=0}^{\infty}$ converges to $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$, which is also feasible.

Since $\lim_{n\to\infty}(k^*_{t_n}, y^*_{t_n}) = (\tilde{k}, \tilde{y})$, the sequence $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ is an optimal path beginning from $(\tilde{k}, \tilde{y})$. By Proposition 3.2, there is some T such that

$$\Psi^e(\tilde{k}_T, \tilde{y}_T) > \Psi^e(\tilde{k}, \tilde{y}).$$

Hence for n sufficiently big, we have

$$\begin{aligned}\Psi^e(k^*_{t_n+T}, y^*_{t_n+T}) &> \Psi^e(\tilde{k}, \tilde{y}) \\ &= \sup_{t\geq 0} \Psi^e(k^*_t, y^*_t),\end{aligned}$$

a contradiction.

This contradiction comes from the hypothesis such that $(\tilde{k}, \tilde{y})$ is not steady state. By the uniqueness of steady state, we have

$$\lim_{n\to\infty} (k^*_{t_n}, y^*_{t_n}) = (k^s, y^s).$$

By the part (i), this implies that for some n sufficiently big, the point $(k^*_{t_n}, y^*_{t_n})$ belongs to the neighborhood $\mathscr{V}$ of (k^s, y^s) and there exists an optimal path $\{(k'_{t_n+t}, y'_{t_n+t})\}_{t=0}^{\infty}$ beginning from $(k^*_{t_n}, y^*_{t_n})$ which converges to (k^s, y^s). Define the sequence $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$ as

$$(\hat{k}_t, \hat{y}_t) = \begin{cases} (k^*_t, y^*_t) \text{ for } 0 \leq t \leq t_n, \\ (k'_t, y'_t) \text{ for } t \geq t_n. \end{cases}$$

The sequence $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$ is an optimal path beginning from (k_0, y_0) which converges to (k^s, y^s).

5.10 Proof of Proposition 3.4

We prove that the function G is strictly concave, hence solution to function $G'(z) = \frac{1}{\beta}$, and assumption **H3** is satisfied.

Precisely,

i) For each z, there exists unique $(k(z), y(z))$ which maximizes $f(k)+\theta g(y)h(\alpha k)$ under constraint $k + \theta y \leq z$.

ii) The function $k(z)$ is increasing in respect to z.

iii) The function G is strictly concave and there exists unique steady z^s, which is solution to $G'(z) = \frac{1}{\beta}$.

(i) For $z \geq 0$, we must find k which maximizes

$$\zeta(k) = f(k) + \theta g\left(\frac{z-k}{\theta}\right) h(\alpha k).$$

We have

$$\zeta''(k) = f''(k)+\frac{1}{\theta}g''\left(\frac{z-k}{\theta}\right)h(\alpha k)-2\alpha g'\left(\frac{z-k}{\theta}\right)h'(\alpha k)+\alpha^2\theta g\left(\frac{z-k}{\theta}\right)h''(\alpha k).$$

The assumption **H5** implies that ζ is strictly concave. Hence there exists unique $k(z) \in [0, z]$ maximizing $\zeta(k)$.

(ii) It is easy to verify that for $z > 0$, we have $0 < k(z) < z$. The value $k(z)$ is hence solution to

$$f'(k) - g'\left(\frac{z-k}{\theta}\right)h(\alpha k) + \theta\alpha g\left(\frac{z-k}{\theta}\right)h'(\alpha k) = 0.$$

By the implicit theorem, we get

$$\begin{aligned} k'(z) &= -\frac{-\frac{1}{\theta}g''\left(\frac{z-k}{\theta}\right)h(\alpha k) + \alpha g'\left(\frac{z-k}{\theta}\right)h'(\alpha k)}{f''(k) + \frac{1}{\theta}g''\left(\frac{z-k}{\theta}\right)h(\alpha k) - 2\alpha g'\left(\frac{z-k}{\theta}\right)h'(\alpha k) + \alpha^2\theta g\left(\frac{z-k}{\theta}\right)h''(\alpha k)} \\ &> 0, \end{aligned}$$

since the nominator is positive and the denominator is negative.

(iii) For any $z \geq 0$,

$$\begin{aligned}
G'(z) &= f'(k(z))k'(z) + g'\left(\frac{z-k(z)}{\theta}\right)(1-k'(z))h(\alpha k(z)) + \alpha g\left(\frac{z-k(z)}{\theta}\right)h'(\alpha k(z))k'(z) \\
&= g'\left(\frac{z-k(z)}{\theta}\right)h(\alpha k(z)).
\end{aligned}$$

This implies

$$\begin{aligned}
G''(z) &= \frac{1}{\theta}g''\left(\frac{z-k(z)}{\theta}\right)(1-k'(z))h(\alpha k(z)) + \alpha g'\left(\frac{z-k(z)}{\theta}\right)h'(\alpha k(z))k'(z) \\
&= \frac{1}{\theta}g''\left(\frac{z-k(z)}{\theta}\right)h(\alpha k(z)) \\
&+ k'(z)\left(-\frac{1}{\theta}g''\left(\frac{z-k(z)}{\theta}\right)h(\alpha k(z)) + \alpha g'\left(\frac{z-k(z)}{\theta}\right)h'(\alpha k(z))\right) \\
&< 0,
\end{aligned}$$

since the two terms are negative. The function G is strictly concave.

5.11 Proof of Proposition 3.6

By Proposition 3.3, we just have to prove the satisfaction of **H3**. Consider the following system

$$\begin{aligned}
f'(k) + \theta\alpha\eta_2(y, \alpha k) &= \frac{1}{\beta}, \\
\eta_1(k, \alpha k) &= \frac{1}{\beta},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\frac{\alpha_k A}{k^{1-\alpha_k}} - \theta\alpha\gamma B y^{\alpha_y} e^{-\alpha\gamma k} &= \frac{1}{\beta}, \\
\frac{\alpha_y B e^{-\alpha\gamma k}}{y^{1-\alpha_y}} &= \frac{1}{\beta}.
\end{aligned}$$

The second function implies

$$y = \left(\beta\alpha_y Be^{-\alpha\gamma k}\right)^{\frac{1}{1-\alpha_y}}.$$

Replacing y in the firt equation, we get

$$\frac{\alpha_k A}{k^{1-\alpha_k}} - \theta\alpha\gamma B\left(\beta\alpha_y Be^{-\alpha\gamma k}\right)^{\frac{\alpha_y}{1-\alpha_y}} e^{-\alpha\gamma k} = \frac{1}{\beta}.$$

We must prove that the following equation has unique solution:

$$\frac{C_1}{k^{1-\alpha_k}} - \frac{C_1}{e^{cy}} = \frac{1}{\beta},$$

where C_1, C_2 and c are positive constants.

Indeed, let $\varphi(k) = \frac{C_1}{k^{1-\alpha_k}} - \frac{C_2}{e^{cy}}$. We can verity that $\varphi(0) = \infty$ and $\phi(\infty) = 0$. The equation $\varphi(k) = \frac{1}{\beta}$ has solution. Denote by k^* its smallest one. We have

$$\varphi'(k) = -\frac{(1-\alpha)C_1}{k^{2-\alpha_k}} + \frac{cC_2}{e^{cy}}.$$

Since $\varphi(k) > 0$ for $k < k^*$, the derivative of φ at k^* is negative: $\varphi'(k^*) < 0$. We will prove that $\phi'(k) \leq 0$ for any $k > k^*$. Indeed, this is equivalent to

$$\frac{e^{cy}}{k^{2-\alpha}} > \frac{cC_2}{(1-\alpha_k)C_1}.$$

Since $\varphi'(k^*) < 0$, this inequality is verified for $k = k^*$. Define $\tilde{\varphi}(k) = \frac{e^{cy}}{k^{2-\alpha}}$. We have

$$\tilde{\varphi}'(k) = \frac{k^{1-\alpha_k} c^{cy}(ck - (2-\alpha_k))}{k^{2(2-\alpha)}}.$$

As $\tilde{\varphi}'(k^*) > 0$, for any $k > k^*$ we have $\tilde{\varphi}'(k) > 0$. Hence $\varphi'(k) < 0$ for any $k > k^*$, and the original equation has unique solution.

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