Clique games: a family of games with coincidence between the nucleolus and the Shapley value

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Clique games: a family of games with coincidence between the nucleolus and the Shapley value*

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Abstract

We introduce a new family of cooperative games for which there is coincidence between the nucleolus and the Shapley value. These so-called clique games are such that agents are divided into cliques, with the value created by a coalition linearly increasing with the number of agents belonging to the same clique. Agents can belong to multiple cliques, but for a pair of cliques, at most a single agent belong to their intersection. Finally, if two agents do not belong to the same clique, there is at most one way to link the two agents through a chain of agents, with any two non-adjacent agents in the chain belonging to disjoint sets of cliques. We provide multiple examples for clique games. Graph-induced games, either when the graph indicates cooperation possibilities or impossibilities, provide us with opportunities to confirm existing results or discover new ones. A particular focus are the minimum cost spanning tree problems. Our result allows us to obtain new coincidence results between the nucleolus and the Shapley value, as well as other cost sharing methods for the minimum cost spanning tree problem.

Keywords: nucleolus; Shapley value; clique; minimum cost spanning tree

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1 Introduction

The Shapley value (Shapley, 1953) and the (pre)nucleolus (Schmeidler, 1969) are two well known values for cooperative games. The Shapley value is an average of the contributions of an agent, while the prenucleolus is the value that minimizes the dissatisfaction of the worst-off coalitions. The nucleolus differs from the prenucleolus by only taking into account individually rational imputations.

Coincidence between the Shapley value and the (pre)nucleolus is uncommon and, in general, difficult to check without computing both values. Recently, Yokote et al. (2017) provide a sufficient and necessary condition for this coincidence to hold, but it requires the computation of both the Shapley value and of a parametric family of sets, for which the computation mimics that of the (pre)nucleolus.[1] This characterization can be applied in order to identify the coincidence in some particular classes of games, such as airport games (Littlechild and Owen, 1973), bidder collusion games (Graham et al., 1990) and polluted river games (Ni and Wang, 2007). Csóka and Herings (2017) also find coincidence is some three-agent games based on bankruptcy problems. As discussed by Kar et al. (2009), for general coalitional form games we have coincidence if the game only has two agents or if all agents are symmetric within the normalized game. Some other games have also been proposed (Deng and Papadimitriou, 1994; van den Nouweland et al., 1996), all having in common that the value of a coalition is equal to the sum of the values created by the pairs composing that coalition. The coincidence persists in games that satisfy the so-called PS property (Kar et al., 2009). These games are such that the contributions of agent i to any coalition and its complement sum up to an agent-specific constant. A particular instance of such games is studied by Chun et al. (2016).

González-Díaz and Sánchez-Rodríguez (2014) also study the coincidence from a geometric point of view. Instead of providing classes of games where both values coincide, they study the properties that lead to this result in some

[1] Additionally, the condition also requires to check whether the sets in this parametric family are balanced.

already existing classes, as for example PS-games. A similar, yet different problem, is the invariance of the payoff assigned by an allocation rule to a specific player in two related games. See Béal et al. (2015) for the case of the Shapley value.

In this paper, we present another family of games, called *clique games*, in which the Shapley value and the nucleolus coincide. The family can be described as follows: the set of agents is divided into cliques that cover it. A coalition creates value when it contains many agents belonging to the same clique, with the value increasing linearly with the number of agents in the same clique. Agents may belong to more than one clique, but the intersection of two cliques contains at most one agent. Finally, if two agents are not in the same clique, there exists at most one way to "connect" them through a chain of connected cliques.

The family of clique games has a non-empty intersection with PS-games, but some clique games are not PS-games, and some PS-games are not clique games. A clique game is convex, and hence its Shapley value is the average of the extreme points in its core. We thus obtain a link between three crucial concepts of cooperative game theory: the nucleolus, the core, and the Shapley value.

Naturally, graph-induced games provide a fertile ground to apply our result. We first consider the graph-restricted cooperative games introduced by Myerson (1977). In these games, a coalitional value function is accompanied by a graph that summarizes the cooperation possibilities: a coalition S cannot fully cooperate if some of its members have no path between them that uses only the vertices of agents in S. When we consider a symmetric coalitional value function, assigning shares of the value created among agents is akin to defining centrality measures (Gomez et al., 2003). We show that when the coalitional value function increases linearly with the number of agents in a coalition (starting with the second one) we obtain coincidence of the Shapley value (known as the Myerson value in this context) and the nucleolus for a family of graphs.

Another graph-induced game that we study is the minimum coloring game (Deng et al., 1999), in which the graph represents conflicts between pairs of agents. We wish to assign agents to facilities, but cannot assign agents that are in conflict to the same facility. As facilities all have a cost of one, we wish to minimize the number of facilities used. Okamoto (2008) noticed a coincidence between the Shapley value and the nucleolus for a particular family of graphs. We explain this coincidence by the fact that the graphs induce clique games.

Our third example is the one we mainly focus on: the minimum cost spanning tree (*mcst*) problem (Bird, 1976). This well-studied problem has agents connecting to a source through a network, with the cost of an edge being a fixed amount that is paid if the edge is used, regardless of the number of users of the edge. Any such problem has a non-empty core even though it may not be convex. Moreover, its Shapley value is not always in its core (Dutta and Kar, 2004).

Nevertheless, Bergantiños and Vidal-Puga (2007a) and Trudeau (2012) propose Shapley value-based solutions that are in the core, by first modifying the costs of the edges. For any pair of nodes in the network, Bergantiños and Vidal-Puga (2007a) look at the paths going from one to the other and ranks them according to their most expensive edge. The edge between the pair of nodes is then assigned the cost of that cheapest most expensive edge, allowing to obtain the so-called irreducible *mcst* problem. The Shapley value of that problem yields the so-called folk solution. The solution proposed by Trudeau (2012) is similar, but looks at cycles instead of paths, yielding a cycle-complete *mcst* problem and the cycle-complete solution. Bergantiños and Vidal-Puga (2007b) also provide another Shapley value-based definition of the folk solution, by defining a cost game assuming that any coalition can connect either to the source or to any other node.

We identify *mcst* problems that generate clique games. In particular, it turns out that if we consider elementary *mcst* problems (in which all edges have a cost of 0 or 1), which form a basis for all *mcst* problems, the subset of cycle-complete problems (which include irreducible problems) generates clique games. Our result on clique games then applies, yielding that the nucleolus coincides with the cycle-complete solution for cycle-complete problems and with the folk solution for irreducible problems.

We can extend the coincidence one step further: for all elementary *mcst* problems, the folk (cycle-complete) solution coincides with the nucleolus and the permutation-weighted average of the extreme points of the core of the public (private) *mcst* game.

The paper is divided as follows: preliminary definitions are in Section 2. Section 3 describes and illustrates clique games. Section 4 contains the coincidence results. Applications to graph-induced games are discussed in Section 5. The application and extension of the results to *mcst* problems are described in Section 6.

2 Preliminaries

Let $N = \{1, \cdots, n\}$ be a set of agents. A *transferable utility game* (*TU game*, for short) is a pair (N, v) where v is a real-valued function defined on all subsets $S \subseteq N$ satisfying $v(\emptyset) = 0$. Given $i \in N$ and $S \subseteq N \setminus \{i\}$, the contribution of agent i to S is defined as

$$\Delta_i^v(S) = v(S \cup \{i\}) - v(S).$$

A game is *convex* if $\Delta_i^v(S) \leq \Delta_i^v(T)$ for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$.

A *value* is a function that associates with each TU game (N, v) an *allocation* $x \in \mathbb{R}^N$. Two well-known values are the Shapley value (Shapley, 1953) and the (pre)nucleolus (Schmeidler, 1969).

The *Shapley value* of the game (N, v) is the allocation $Sh(v)$ defined as

$$Sh_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} \Delta_i^v(P_i(\pi))$$

for all $i \in N$, where Π is the set of all orderings of N and $P_i(\pi)$ is the set of predecessors of agent i in π, i.e. $P_i(\pi) = \{j : \pi(j) < \pi(i)\}$.

The *excess* of a coalition S in a TU game (N, v) with respect to an allocation x is defined as $e(S, x, v) = \sum_{i \in N} x_i - v(S)$. The vector $\theta(x)$ is constructed by rearranging the 2^n excesses in (weakly) increasing order. If $x, y \in \mathbb{R}^N$ are two allocations, then $\theta(x) >_L \theta(y)$ means that $\theta(x)$ is lexicographically larger than $\theta(y)$. As usual, we write $\theta(x) \geq_L \theta(y)$ to indicate that either $\theta(x) >_L \theta(y)$ or $x = y$.

The *nucleolus* of the game (N, v) is the set

$$Nu(v) = \{x \in X : \theta(x) \geq_L \theta(y) \forall y \in X\}$$

where $X = \left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \forall i \in N\right\}$ is the set of individually rational allocations. When $X \neq \emptyset$, as it is the case for the TU games we study here, it is well-known that $Nu(v)$ is a singleton, whose unique element we denote, with some abuse of notation, also as $Nu(v)$.

By contrast, the *prenucleolus* of the game (N, v) is the set

$$Pre(v) = \left\{x \in X^0 : \theta(x) \geq_L \theta(y) \forall y \in X^0\right\}$$

where $X^0 = \left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N)\right\}$ is the set of allocations. Whenever the prenucleolus is individually rational, which will be the case in all games that we consider, it coincides with the nucleolus. Therefore, from now on, we focus exclusively on the nucleolus.

The *core* is the set of allocations such that no coalition is assigned less than its stand-alone value. Formally,

$$Core(v) = \left\{x \in X^0 : \sum_{i \in S} x_i \geq v(S) \forall S \subset N\right\}.$$

When $Core(v) \neq \emptyset$, for each $\pi \in \Pi$, let $y^\pi \in Core(v)$ be the allocation that lexicographically maximizes the individual allocations with respect to the order given by the permutation. The *permutation-weighted average of extreme points of the core* is the average of these allocations:

$$\bar{y}(v) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} y^\pi(v).$$

If the game is convex, $\bar{y}$ is the Shapley value. This average is also closely related to the "selective value" (Vidal-Puga, 2004) and the "Alexia value" (Tijs, 2005). All of these values coincide for the minimum cost spanning tree problem studied in Section 5.

On some occasions, we work with transferable cost games (N, C), where C is a real-valued function defined on all subsets $S \subseteq N$ satisfying $C(\emptyset) = 0$. We then define v^C as follows: For all $S \subseteq N$, $v^C(S) = \sum_{i \in S} C(\{i\}) - C(S)$. An allocation x for the cost game C is equivalent to an allocation x^C for the TU game v^C if $x_i^C = C(\{i\}) - x_i$ for all $i \in N$. We then say that $x \in Nu(C)$ iff $x^C \in Nu\left(v^C\right)$. We say that $x \in Core(C)$ iff $x^C \in Core(v^C)$. Finally, we say that C is concave iff $-C$ is convex. It is straightforward to check that C is concave iff v^C is convex.

3 Clique games

We say that $\mathcal{Q} = \left\{Q^1, \ldots, Q^K\right\}$ is a cover of N if $Q^k \subseteq N$ for $k = 1, ..., K$ and $\cup_{k=1}^K Q^k = N$. For a cover $\mathcal{Q} = \left\{Q^1, \ldots, Q^K\right\}$ and each $Q^k \in \mathcal{Q}$, the interior of Q^k, $Int\left(Q^k\right)$, is the set of agents who only belong to Q^k, i.e.

$$Int\left(Q^k\right) = \left\{i \in Q^k : i \notin Q^l \forall l \neq k\right\}.$$

A *path* between Q^k and Q^l is a sequence $P^{kl} = \left\{Q^{k_1}, \ldots, Q^{k_M}\right\}$ such that $Q^{k_1} = Q^k$, $Q^{k_M} = Q^l$ and $\left|Q^{k_m} \cap Q^{k_{m+1}}\right| = 1$ for all $m = 1, \ldots, M-1$. Analogously, *a path between Q^k and Q^l through agent i* is a path $P_i^{kl} = \left\{Q^{k_1}, \ldots, Q^{k_M}\right\}$ such that $Q^{k_1} \cap Q^{k_2} = \{i\}$. The set of agents connected to Q^k via a path through agent $i \in Q^k$ is denoted as

$$N_{k,i}^P = \left\{j \in N : \exists l, P_i^{kl} \text{ such that } j \in Q^l\right\}.$$

Example 1 *Let $\mathcal{Q} = \{Q^1, Q^2, Q^3\}$ with $Q^1 = \{1, 2\}$, $Q^2 = \{2, 3, 4\}$ and $Q^3 = \{4, 5, 6\}$ (see Figure 1).*

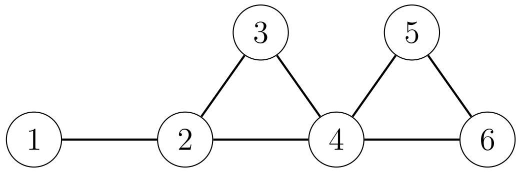

Figure 1: Example of a cover represented as cliques.

In this case, $P_2^{13} = \{Q^1, Q^2, Q^3\}$ *is a path between* Q^1 *and* Q^3 *through agent* 2*. The other paths are* $P_2^{12} = \{Q^1, Q^2\}$*,* $P_2^{21} = \{Q^2, Q^1\}$*,* $P_4^{23} = \{Q^2, Q^3\}$*,* $P_4^{32} = \{Q^3, Q^2\}$*, and* $P_4^{31} = \{Q^3, Q^2, Q^1\}$*. Moreover,* $N_{1,1}^P = \emptyset$*,* $N_{2,2}^P = \{1, 2\}$*,* $N_{1,2}^P = \{2, 3, 4, 5, 6\}$*,* $N_{2,4}^P = \{4, 5, 6\}$*, and so on.*

A game $\left(N, v^{\mathcal{Q}}\right)$ is a *clique game* if there exists $\mathcal{Q} = \left\{Q^1, \ldots, Q^K\right\}$ that covers N, $\{v_i\}_{i \in N} \subset \mathbb{R}_+$ and $\{v_Q\}_{Q \in \mathcal{Q}} \subset \mathbb{R}_+$ such that:

i) for all $k \in \{1, ..., K\}$ and all $i \neq j, i, j \in Q^k$, $N_{k,i}^P \cap N_{k,j}^P = \emptyset$ (there is at most one path between any two elements of $\mathcal{Q}$),

ii) for all $S \subseteq N$,

$$v^{\mathcal{Q}}(S) = \sum_{i \in S} v_i + \sum_{Q \in \mathcal{Q}(S)} \left(|Q \cap S| - 1\right) v_Q \tag{1}$$

with $\mathcal{Q}(S) = \{Q \in \mathcal{Q} : S \cap Q \neq \emptyset\}$.

We write $\mathcal{Q}(i)$ for $\mathcal{Q}\left(\{i\}\right)$.

Let $\mathcal{C}$ be the set of all clique games.

We conclude this section by proposing an example of a clique game.

Example 2 *(Trading goods) The agent set is* $N = \{1, 2, 3, 4, 5\}$*, with* 1 *and* 2 *being producers and* 3, 4 *and* 5 *being buyers. Producer* 1 *has a capacity to produce two units at constant marginal cost* c_1 *while producer* 2 *can produce a single unit at cost* c_2*. Each buyer* i *is interested in a single unit that she values at* R_i*. We suppose that these valuations are larger than the marginal cost of the producers.*

Producers 1 *and* 2 *have exclusive territories (because of vertical restraints or collusion) and buyers* 3 *and* 4 *are on the territory of producer* 1 *and buyer*

5 on the territory of producer 2. The producers' unused capacity can be sold to external buyers at price q and buyers have the option of buying from an external supplier at price p, with $R_i > p > q > c_j$.

When a coalition forms, trades occur between buyers and sellers in the same territory, with unsatisfied demands and unsold supply resolved on the outside market. For example, coalition $\{1,2,3,5\}$ can organize trades between 1 and 3 and 2 and 5, generating a surplus of $R_3 + R_5 - c_1 - c_2$. In addition, producer 1 can sell its extra unit on the outside market, generating an additional surplus of $q - c_1$.

The game can thus be represented (see Figure 2) by a clique game, with cover $\mathcal{Q} = \{\{1,2\},\{1,3\},\{1,4\},\{2,5\}\}$ and $v_1 = 2q - 2c_1$, $v_2 = q - c_2$, $v_i = R_i - p$ for $i = 3,4,5$, $v_{\{1,2\}} = 0$ and $v_Q = r \equiv p - q$ otherwise.

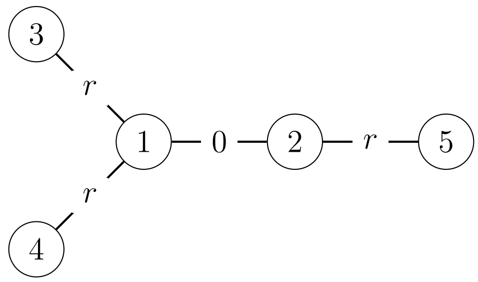

Figure 2: Clique cover of a trading goods game.

4 Coincidence between the Shapley value and the nucleolus

In this section we show that for clique games, the Shapley value and the nucleolus coincide, and we provide a closed-form expression for their value. To get to this result, we first calculate the contributions in a clique game.

Lemma 1 *Given a clique game $(N, v^{\mathcal{Q}})$, the contribution of agent $i \in N$ to $S \subseteq N \setminus \{i\}$ is*

$$\Delta_i^{v^{\mathcal{Q}}}(S) = v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(i)} v_Q.$$

Proof. By definition of a contribution,

$$\begin{aligned}
\Delta_i^{v^{\mathcal{Q}}}(S) &= v^{\mathcal{Q}}(S\cup\{i\}) - v^{\mathcal{Q}}(S)\\
&\overset{(1)}{=} v_i + \sum_{Q\in\mathcal{Q}(S\cup\{i\})} \left(|Q\cap(S\cup\{i\})|-1\right) v_Q - \sum_{Q\in\mathcal{Q}(S)} \left(|Q\cap S|-1\right) v_Q\\
&= v_i + \sum_{Q\in\mathcal{Q}(S)\cap\mathcal{Q}(i)} \left[\left(|Q\cap(S\cup\{i\})|-1\right) - \left(|Q\cap S|-1\right)\right] v_Q\\
&+ \sum_{Q\notin\mathcal{Q}(S), i\in Q} \left(|Q\cap(S\cup\{i\})|-1\right) v_Q\\
&= v_i + \sum_{Q\in\mathcal{Q}(S)\cap\mathcal{Q}(i)} \left[|Q\cap S)| - \left(|Q\cap S|-1\right)\right] v_Q\\
&+ \sum_{Q\notin\mathcal{Q}(S), i\in Q} \left(|\{i\}|-1\right) v_Q\\
&= v_i + \sum_{Q\in\mathcal{Q}(S)\cap\mathcal{Q}(i)} v_Q.
\end{aligned}$$

■

We are now ready for the main result of this section.

Theorem 1 *For all $v^{\mathcal{Q}} \in \mathcal{C}$ and all $i \in N$,*

$$Sh_i\left(v^{\mathcal{Q}}\right) = \bar{y}_i\left(v^{\mathcal{Q}}\right) = Nu_i\left(v^{\mathcal{Q}}\right) = v_i + \sum_{Q\in\mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q.$$

Proof. It is obvious from Lemma 1 that v^Q is a convex game. Thus, the Shapley value is the average of extreme points of the core (Shapley, 1971; Ichiishi, 1981) and $Sh\left(v^{\mathcal{Q}}\right) = \bar{y}\left(v^{\mathcal{Q}}\right)$. We show that for all $i \in N$,

$$Sh_i\left(v^{\mathcal{Q}}\right) = Nu_i\left(v^{\mathcal{Q}}\right) = v_i + \sum_{Q\in\mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q.$$

We suppose that for all $k \in \{1, ..., K\}$, $\bigcup_{i\in Q^k} N_{k,i}^P = N \setminus Int(Q^k)$, that is, there is a (unique) path between any two elements of $\mathcal{Q}$. Without that assumption, we can partition our agents into groups unconnected by paths, and we can compute the Shapley value and the nucleolus independently on each component of the partition.

We start with $Sh\left(v^{\mathcal{Q}}\right)$. Given $\pi \in \Pi$, under Lemma 1, the contribution of agent i to $P_i(\pi)$ is $v_i + \sum_{Q \in \mathcal{Q}(P_i(\pi)) \cap \mathcal{Q}(i)} v_Q$. For each $Q \in \mathcal{Q}(i)$, the probability that $Q \in \mathcal{Q}\left(P_i(\pi)\right) \cap \mathcal{Q}(i)$ is $\frac{|Q|-1}{|Q|}$. Summing up, we obtain the desired result.

We now focus on $Nu\left(v^{\mathcal{Q}}\right)$. Let $x \in \mathbb{R}^N$ defined as $x_i = v_i + \sum_{Q \in \mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q$ for all $i \in N$. We have that

$$\begin{aligned} e\left(S, x, v^{\mathcal{Q}}\right) &= \sum_{i \in S} v_i + \sum_{i \in S} \sum_{Q \in \mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q - \sum_{i \in S} v_i - \sum_{Q \in \mathcal{Q}(S)} \left(|Q \cap S| - 1\right) v_Q \\ &= \sum_{Q \in \mathcal{Q}(S)} \left(\frac{|Q \cap S| \left(|Q| - 1\right)}{|Q|} - \left(|Q \cap S| - 1\right) \right) v_Q \\ &= \sum_{Q \in \mathcal{Q}(S)} \frac{\left(|Q \cap S| \left(|Q| - 1\right) - |Q \cap S| - 1\right) |Q|}{|Q|} v_Q \\ &= \sum_{Q \in \mathcal{Q}(S)} \frac{|Q| - |Q \cap S|}{|Q|} v_Q \end{aligned}$$

for all $S \subset N$, $S \neq \emptyset$. Assume without loss of generality $\frac{v_{Q^1}}{|Q^1|} \leq \frac{v_{Q^2}}{|Q^2|} \leq \cdots \leq \frac{v_{Q^K}}{|Q^K|}$.

For each $i \in Q^1$, let $S_i^1 = N \setminus \left(N_{1,i}^P \cup \{i\}\right)$. Note that for all $Q \in \mathcal{Q} \setminus \{Q^1\}$, either $S_i^1 \cap Q = \emptyset$ or $S_i^1 \cap Q = Q$. In addition, $S_i^1 \cap Q^1 = Q^1 \setminus \{i\}$. Thus, $e\left(S_i^1, x, v^{\mathcal{Q}}\right) = \frac{v_{Q^1}}{|Q^1|}$. By construction, this is the lowest excess value. To see why, note that any $S \subset N$ has at least one $Q \in \mathcal{Q}\left(S\right)$ such that $|Q \cap S| < |Q|$. That generates an excess of $\frac{|Q|-|Q \cap S|}{|Q|} v_Q \geq \frac{v_Q}{|Q|} \geq \frac{v_{Q^1}}{|Q^1|}$.

For each $i \in Q^1$, let $T_i^1 = N_{1,i}^P \cup \{i\} = N \setminus S_i^1$. Take $\{T_i^1\}_{i \in Q^1}$. This is a partition of N. To see why, note that each T_i^1 is nonemtpy (because $i \in T_i^1$ for all $i \in Q^i$), their union is N (because all cliques are connected through a path), and they are pairwise disjoint (because of assumption i)). Thus, we have $|Q^1|$ coalitions whose complements have the minimal excess, with each agent belonging to exactly one of of these coalitions. Therefore, to increase the excess of one of these coalitions we would need to decrease the excess of another coalition, and the corresponding allocation could not be the nucleolus.

We repeat the process for all Q^k to obtain that

$$\sum_{j \in S_i^k} Nu_j\left(v^{\mathcal{Q}}\right) = \sum_{j \in S_i^k} x_j \tag{2}$$

for all $Q^k \in \mathcal{Q}$ and all $i \in Q^k$. In case $i \in Int(Q^k)$ for some $k \in \{1, ..., K\}$, we have $S_i^k = N \setminus \{i\}$, from where (2) and efficiency of x imply $Nu_i\left(v^{\mathcal{Q}}\right) = x_i$.

In case $\mathcal{Q} = \{Q^1\}$, we have $N = Int\left(Q^1\right)$ and hence $Nu\left(v^{\mathcal{Q}}\right) = x$. So, we assume $|\mathcal{Q}| > 1$. From condition i) in the definition of clique games, there exist some $i \in N$ and $Q^k \in \mathcal{Q}(i)$ such that $Q = Int(Q) \cup \{i\}$ for all $Q \in \mathcal{Q}(i) \setminus \{Q^k\}$. This implies that $Nu_j\left(v^{\mathcal{Q}}\right) = x_j$ for all $j \in Q^k \in \mathcal{Q}(i) \setminus \{Q^k\}$. Under (2) and the efficiency of x, we deduce $Nu_i\left(v^{\mathcal{Q}}\right) = x_i$. Repeating the reasoning, we can always find a new $i \in N$ and $Q^k \in \mathcal{Q}(i)$ such that $Nu_j\left(v^{\mathcal{Q}}\right) = x_j$ for all $j \in Q^k \in \mathcal{Q}(i) \setminus \{Q^k\}$, so that (2) and the efficiency of x imply $Nu_i\left(v^{\mathcal{Q}}\right) = x_i$, and so on until we get $Nu\left(v^{\mathcal{Q}}\right) = x$. ■

We next establish the connection between clique games and the PS-games of Kar et al. (2009). We say that a game (N, v) is a *PS-game* if there exists $a \in \mathbb{R}^N$ such that $\Delta_i^v(S) + \Delta_i^v\left(N \setminus (S \cup \{i\})\right) = a_i$ for all $i \in N$ and $S \subseteq N \setminus \{i\}$.

We show the condition needed for a clique game to be a PS-game, which illustrates that not all clique games are PS-games.

Proposition 1 *A clique game $v^{\mathcal{Q}}$ is a PS-game if and only if for all $Q \in \mathcal{Q}$ either $|Q| \leq 2$ or $v_Q = 0$.*

Proof. Under Lemma 1, for any clique game $v^{\mathcal{Q}}$, we have that

$$\Delta_i^{v^{\mathcal{Q}}}(S) = v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(i)} v_Q$$

and thus $\Delta_i^{v^{\mathcal{Q}}}(S) + \Delta_i^{v^{\mathcal{Q}}}\left(N \setminus (S \cup \{i\})\right)$

$$\begin{aligned} &= v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(i)} v_Q + v_i + \sum_{Q \in \mathcal{Q}(N \setminus (S \cup i)) \cap \mathcal{Q}(i)} v_Q \\ &= 2v_i + \sum_{Q \in \mathcal{Q}(N \setminus \{i\}) \cap \mathcal{Q}(i)} v_Q + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup \{i\})) \cap \mathcal{Q}(i)} v_Q. \end{aligned}$$

Hence, $v^{\mathcal{Q}}$ is a PS-game if and only there exists $b \in \mathbb{R}^N$ such that

$$\sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup \{i\})) \cap \mathcal{Q}(i)} v_Q = b_i$$

for all $S \subseteq N \setminus \{i\}$. In this case, $a_i = 2v_i + \sum_{Q \in \mathcal{Q}(N \setminus \{i\}) \cap \mathcal{Q}(i)} v_Q + b_i$ for all $i \in N$.

Fix $i \in N$. Let $S \subseteq N \setminus \{i\}$ and $Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup \{i\})) \cap \mathcal{Q}(i)$. If $|Q| \leq 2$ then $|Q \cap S| \leq 1$ (because $i \in Q$ and $i \notin S$). Since $Q \in \mathcal{Q}(S)$, we deduce $Q \cap S = \{j\}$ for some $j \neq i$. Thus, $Q = \{i, j\} \subseteq S \cup \{i\}$, which contradicts that $Q \in \mathcal{Q}(N \setminus (S \cup \{i\}))$. Hence, $\mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup i)) \cap \mathcal{Q}(i) \subseteq \{Q \in \mathcal{Q} : |Q| > 2\}$.

From this, we deduce that if $v_Q = 0$ for all $Q \in \mathcal{Q}$ such that $|Q| > 2$, then $b_i = 0$ for all $i \in N$.

Suppose now that there exists $Q \in \mathcal{Q}$ such that $|Q| > 2$ and $v_Q > 0$. Fix $i \in Q$. If $S = \emptyset$ we obtain $\sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup i)) \cap \mathcal{Q}(i)} v_Q = 0$ (as $\mathcal{Q}(S) = \emptyset$). If $S = \{j\}$, $j \in Q \setminus \{i\}$, we obtain $\sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup i)) \cap \mathcal{Q}(i)} v_Q \geq v_Q > 0$ (as $Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup \{i\})) \cap \mathcal{Q}(i))$. ■

Moreover, not all PS games are clique games, as the next example shows:

Example 3 (Example 3.12 in Kar et al. (2009)) *We consider the TU game (N, v) with $N = \{1, 2, 3, 4\}$ and such that $v(S) = 0$ if $|S| = 1$, 1 if $|S| = 2$, $\frac{3}{2}$ if $|S| = 3$, and 3 if $S = N$. This is a PS game with $\Delta_i^v(S) + \Delta_i^v(N \setminus (S \cup \{i\})) = \frac{3}{2}$ for all i and S. However, it is not a clique game. To see this, notice that $v(S) = 0$ if $|S| = 1$ implies that $v_i = 0$ for all $i \in N$. Then, $v(S) = 1$ if $|S| = 2$ implies that any pair i, j belong to some clique Q with $v_Q = 1$. The no-cycle condition of clique games (condition i)) leaves us with a single candidate for the set of cliques: $\mathcal{Q} = \{N\}$. But then $v(S) = |S| - 1$ for all S, which is different from the PS-game for $|S| = 3$.*

5 Graph-induced games

We apply our result to two families of games that represent cooperation possibilities or impossibilities on a graph. We need the following graph theory definitions.

We now interpret N as a set of vertices, and say that a graph on N is a set of unordered pairs of distinct members of N. Let G^N be the complete graph: $G^N = \{(i,j) \,|i,j \in N,\ i \neq j\}$. A graph G is a subset of G^N. For any $S \subseteq N$, $G[S]$ is the subgraph of G induced by S.

Suppose $S \subseteq N$, $G \subseteq G^N$, $i,j \in N$. We say that i and j are connected in S by G iff there is a path in G which goes from i to j and stays within S. That is, there is some $k \geq 1$ and a sequence $i^0, i^1, ..., i^k$ such that $i^0 = i$, $i^k = j$ and $(i^{l-1}, i^l) \in G[S]$ for all $l = 1, ..., k$. A coalition T is connected in S by G if i and j are connected in S by G, for all $i,j \in T$. If S is connected by S in G, we simply say that S is connected by G. We say that a connected coalition T in S is maximal if there does not exist a coalition $T' \supset T$ that is connected in S by G.

We say that a graph G is a clique graph if there exists Q, a cover of N with $Q = \left\{Q^1, ..., Q^K\right\}$ such that for all $k = 1, ..., K$, Q^k is connected by G. In addition, a clique graph G is said to be acyclical if for all i,j such that there does not exists $Q^k \supseteq \{i,j\}$, there exists at most a single path between them: there does not exist S such that i and j are connected in both S and $N \backslash S$ by G.

We say that a clique graph is disjoint if for all $i \in N$, $|\mathcal{Q}(i)| = 1$. It is obvious that a disjoint clique graph is also acyclical.

Let $G^C = G^N \backslash G$ be the complement of graph G.

5.1 Graph-restricted cooperative game

We consider the graph-restricted cooperation game introduced by Myerson (1977), in which a coalition of agents can cooperate together only if its members are connected.

More precisely, every graph G partitions every coalition S into a set of maximal connected subcoalitions, P_S in a natural way.

Let $V \in \mathbb{R}_+^{2^N}$ be a coalition function. A graph restricted problem is

(G, V). The graph restricted game V_G is then defined by

$$V_G(S) = \sum_{T \in P_S} V(T).$$

In words, V represents the values obtained by each coalition if all of its members can cooperate. But, in practice, cooperation within a coalition might reduce to cooperation among subcoalitions, so that a coalition S fails to extract all of its potential value $V(S)$.

While any value function can be used, symmetric functions eliminate the differences coming from the game V, allowing to focus on the graph, thus defining a centrality measure (Gomez et al., 2003). Gonzalez-Aranguena et al. (2017) use a family of symmetric functions V^k that forms a basis for all symmetric functions: for $k = 1, ..., n$, $S \subseteq N$ and $a > 0$

$$V^k(S) = \begin{cases} 0 \text{ if } |S| = 0, ..., k-1; \\ \binom{|S|}{k} a \text{ otherwise.} \end{cases}$$

For coalition S the function V^k assigns a value of a for every group of k members it contains. Given that these games are meant to represent communication opportunities between members, it is natural to make sure that a single agent cannot generate value by himself (Gonzalez-Aranguena et al., 2017). It is the case for all functions V^k except V^1, that we zero-normalize: $V^{1^*}(\emptyset) = 0$ and $V^{1^*}(S) = V^1(S) - a$ for all $\emptyset \neq S \subseteq N$. Then, in V^{1^*}, a coalition receives a value of a for each of its members, starting with the second one.

Theorem 2 *Let G be an acyclical clique graph. Then, for all $i \in S$,*

$$Sh_i(V_G^{1^*}) = Nu_i(V_G^{1^*}) = \sum_{Q \in \mathcal{Q}(i)} \frac{|Q| - 1}{|Q|} a.$$

Proof. It is not difficult to see that the combination of the acyclical clique graph and the properties of V^{1^*} gives rise to a clique game, in which $v_i = 0$ for all $i \in N$ and $v_Q = a$ for all cliques Q. The result then follows from

Theorem 1. ■

Both the Shapley value (known as the Myerson value in this context) and the nucleolus (Montero, 2013) have been proposed as centrality or power indexes. The result above allows to understand when they coincide.

5.2 Minimum coloring game

We now consider a model in which the graph represents conflict situations.

Agents have to be placed in facilities, but there is potential conflict between individuals. If two agents are in conflict, they cannot be put in the same facility. We are trying to find the minimum number of facilities needed to locate all individuals in N. We assume that facilities cost 1 unit each.

Conflicts are represented in graph G: If $(i, j) \in G$, then agents i and j are in conflict. The problem is known as the minimum coloring game, as we attempt to assign colors to all vertices (agents) in the graph, with different colors for agents in conflict, all while using the minimum number of colors (Deng et al., 1999).

For all $S \subseteq N$, let $c_G^{MC}(S)$ be the minimum number of facilities to locate all members of S, given conflict subgraph $G[S]$.

Theorem 3 *(Okamoto, 2008) If G^C is a disjoint clique graph, then, for all $i \in N$,*

$$Sh_i(c_G^{MC}) = Nu_i(c_G^{MC}) = \frac{1}{|Q_i|}$$

where Q_i is the only clique that agent i belongs to.

While Okamoto (2008) notices the coincidence between the Shapley value and the nucleolus, no explanation is provided. It is obvious that if G^C is a disjoint clique graph, we obtain (once we transform the game into a value game) a clique game in which $v_i = 0$ for all $i \in N$ and $v_Q = 1$ for all cliques Q. Therefore, we now have an explanation on why we have this coincidence. We can also be pessimistic about coincidence on a larger set of minimum coloring games, as if an agent belongs to multiple cliques, we do not obtain the linear form of value creation needed for a clique game.

6 Minimum cost spanning tree problems

In this section we describe the *minimum cost spanning tree* (*mcst*) problem, showing that an important subset of such games are also clique games. In turn, this allows us to link the nucleolus to some well-known cost sharing solutions for *mcst* problems.

6.1 The problem

We assume that the agents in N need to be connected to a source, denoted by 0. Let $N_0 = N \cup \{0\}$. For any set Z, define Z^p as the set of all non-ordered pairs (i,j) of elements of Z. In our context, any element (i,j) of Z^p represents the edge between nodes i and j. Let $c = (c_e)_{e \in N_0^p}$ be a vector in $\mathbb{R}_+^{N_0^p}$ with $N_0^p = (N_0)^p$ and $c_e \in \mathbb{R}_+$ representing the cost of edge e. Let Γ be the set of all cost vectors. Since c assigns cost to all edges, we often abuse language and call c a cost matrix. A *mcst* problem is a triple $(0, N, c)$. Since 0 and N do not change, we omit them in the following and simply identify a *mcst* problem $(0, N, c)$ by its cost matrix c.

A cycle p_{ll} is a set of $K \geq 3$ edges (i_{k-1}, i_k), with $k \in \{1, \ldots, K\}$ and such that $i_0 = i_K = l$ and $i_1, \ldots, i_{K-1}$ distinct and different from l. A *path* p_{lm} between l and m is a set of K edges (i_{k-1}, i_k), with $k \in \{1, \ldots, K\}$, containing no cycle and such that $i_0 = l$ and $i_K = m$. Let $P_{lm}(N_0)$ be the set of all paths between nodes l and m.

A *spanning tree* is a non-orientated graph without cycles that connects all elements of N_0. A spanning tree t is identified by the set of its edges.

We call *mcst* a spanning tree that has minimal cost. Note that a *mcst* might not be unique. Let $C(N, c)$ be the minimal cost of a *mcst*. Let c^S be the restriction of the cost matrix c to $S_0 \subseteq N_0$. Let $C(S, c)$ be the cost of the *mcst* of the problem (S, c^S). We say that C is the stand-alone cost function associated with c.

For any cost matrix c, the associated cost game is given by (N, C) with $C(S) = C(S, c)$ for all $S \subseteq N$. We then write, with some abuse of notation, (N, c) instead of (N, C) and say that it is a *mcst* game.

A variant of the *mcst* problem, called the public *mcst* problem, allows any coalition to use all nodes, including those belonging to agents outside of the coalition, to connect to the source. The *public cost function associated with* c is defined as

$$C^{Pub}(S,c)=\min_{T\subseteq N\setminus S} C(S\cup T,c)$$

for all $S\subseteq N$. By contrast, we sometimes call (N,c) the *private cost function associated with* c and the *mcst* problem the *private mcst problem*.

Abusing language slightly, we use the term *mcst game* to designate the cost game generated by a *mcst* problem.

6.2 The irreducible and cycle-complete cost matrices

The Shapley value of a *mcst* game is not always in the core, given that the *mcst* game is typically not concave. The following two modifications to the problem allow to transform the game into a concave one.

From any cost matrix c, we define the irreducible cost matrix c^* as follows:

$$c^*_{ij}=\min_{p_{ij}\in P_{ij}(N_0)}\max_{e\in p_{ij}} c_e$$

for all $i,j\in N_0$.

From any cost matrix c, we define the cycle-complete cost matrix c^{**} as follows:

$$c^{**}_{ij}=\max_{k\in N\setminus\{i,j\}}\left(c^{N\setminus\{k\}}\right)^*_{ij}$$

for all $i,j\in N_0$, and

$$c^{**}_{0i}=\max_{k\in N\setminus\{i\}}\left(c^{N\setminus\{k\}}\right)^*_{0i}$$

for all $i\in N$, where $\left(c^{N\setminus\{k\}}\right)^*$ indicates the matrix that is first restricted to agents in $N\setminus\{k\}$ before being transformed into an irreducible matrix.

The cycle complete matrix can also be defined using cycles (Trudeau, 2012): for edge (i,j), we look at cycles that go through agents i and j. If there is one such cycle such that its most expensive edge is cheaper than a direct connection on edge (i,j), we assign this cost to edge (i,j).

Let C^* be the characteristic cost function associated with the mcst problem (N, c^*). Let C^{**} be the characteristic cost function associated with the *mcst* problem (N, c^{**}). The Shapley values of C^* and C^{**} are respectively called the folk ($y^f(c)$) and cycle-complete ($y^{cc}(c)$) solutions.

When a coalition builds its *mcst*, it connects one member of each clique to the source. The condition that agents of different cliques have no benefit in cooperating allows for the value creation to occur only when members of the same clique cooperate. The conditions on the cost of edges between members of a clique and between them and the source guarantee that the value creation increases by a constant for each additional member of a clique that cooperates, starting with the second one.

6.3 Minimum cost spanning tree games and clique games

For a *mcst* game to also be a clique game, we need members of a clique to have the same cost to connect to each other (condition a below), members of different cliques to have no gain in cooperating (condition b) and within a clique, there exists either one or two different costs to connect to the source. In that second case, a single agent has the low cost (condition c).

Lemma 2 *A mcst game (N, c) is associated to a clique game if and only if there exist $\mathcal{Q}$ satisfying condition i) of clique games and $\{c_Q\}_{Q\in\mathcal{Q}} \subset \mathbb{R}$ that satisfy the following conditions:*

a) $c_{ij} = c_Q$ *for all* $Q \in \mathcal{Q}$ *and all* $i, j \in Q$*;*

b) $c_{ij} \geq \max\{c_{0i}, c_{0j}\}$ *for all* $i, j \in N$ *such that there exists no* $Q \in \mathcal{Q}$ *with* $i, j \in Q$*;*

c) for all $Q \in \mathcal{Q}$*, if* $c_{0,Q}^{min}, c_Q < c_{0,Q}^{max}$ *then* $|\arg\max_{j\in Q} c_{0j}| = |Q| - 1$*, where* $c_{0,Q}^{max} = \max_{j\in Q} c_{0j}$ *and* $c_{0,Q}^{min} = \min_{j\in Q} c_{0j}$*.*

Proof. Note first that condition b can be replaced by:

b') $c_{ij} = \max\{c_{0i}, c_{0j}\}$ for all $i, j \in N$ such that there exists no $Q \in \mathcal{Q}$ with $i, j \in Q$.

To see why, notice that an edge (i, j) with $c_{ij} > \max\{c_{0i}, c_{0j}\}$ is irrelevant in the sense that it does not affect the cost function C. Hence, the associated game (N, v^C) does not change if we reduce c_{ij} until equality holds. We then assume that c has no irrelevant edges. This also implies that $c_Q \leq \max\{c_{0i}, c_{0j}\}$ for all $Q \in \mathcal{Q}$ and all $i, j \in Q$.

We first show that the conditions generate a clique game. Suppose that we want to connect members of S to the source. Conditions a and b' make it never optimal to directly connect members of different cliques. The combination of the three conditions makes it always better to connect members of the same clique to each other. Let $\{S_1, S_2, \ldots, S_K\}$ be a partition of S such that if $i, j \in S_k$, then there exists a path between nodes i and j for which the most expensive edge is c_Q, for some $Q \in \mathcal{Q}$. Then, the cost of coalition S is

$$C(S, c) = \sum_{k=1}^{K} \min_{i \in S_k} c_{0i} + \sum_{Q \in \mathcal{Q}(S)} (|Q \cap S| - 1)\, c_Q.$$

Under condition c, if members of a clique have different costs to connect to the source, then all but one have the same high cost $c_{0,Q}^{max}$. We can thus simplify the cost of coalition S to

$$C(S, c) = \sum_{i \in S} c_{0i} + \sum_{Q \in \mathcal{Q}(S)} (|Q \cap S| - 1)\, (c_Q - c_{0,Q}^{max}).$$

We then have that v^C is such that

$$v^C(S, c) = \sum_{Q \in \mathcal{Q}(S)} (|Q \cap S| - 1)\, (c_{0,Q}^{max} - c_Q)$$

which corresponds to a clique game with $v_i = 0$ for all $i \in N$ and $v_Q = c_{0,Q}^{max} - c_Q$.

We next show that these conditions are necessary. Without condition a, there exist $i, j, k \in Q$ such that $c_{ij} \neq c_Q$ but $c_{ik} = c_Q$. Then, $C(\{i, j\}, c) = \min\{c_{0i}, c_{0j}\} + c_{ij}$, $C(\{i, k\}, c) = \min\{c_{0i}, c_{0k}\} + c_Q$ and C is no longer a clique game. Without condition b, there exist i, j belonging to different cliques such that $c_{ij} < \max\{c_{0i}, c_{0j}\}$. Then $C(\{i, j\}, c) = \min\{c_{0i}, c_{0j}\} + c_{ij}$ and C is no longer a clique game.

Without condition c, there exists a clique Q containing $m \geq 3$ agents and such that $|\arg\max_{j \in Q} c_{0j}| < m - 1$. There are thus at least two agents, say i and j, with $c_{0,Q}^{min} \equiv c_{0i} \leq c_{0j} < c_{0,Q}^{max}$. Then, $C(\{i,j\}, c) = c_0 i + c_Q < c_{0i} + c_{0j} + (c_Q - c_{0,Q}^{max})$ and C is no longer a clique game. ■

Let Γ^c be the set of matrices generating clique *mcst* problems.

Consider the subset of *mcst* problems known as elementary *mcst* (*emcst*) problems: for any $i, j \in N_0$, $c_{ij} \in \{0, 1\}$. Let Γ^e be the set of elementary cost problems.

It turns out that the intersection of clique and elementary *mcst* problems is the set of elementary cycle-complete problems,

Lemma 3 $\Gamma^c \cap \Gamma^e = \Gamma^{ecc}$*, the set of elementary cycle-complete problems.*

Proof. "$\supseteq$" We need to show that elementary and cycle-complete *mcst* games are clique games. By definition, there exists a cover $\mathcal{Q}$ of N that satisfies condition i) of clique games and such that $c_{ij} = 0$ if $i, j \in Q$ and $c_{ij} = 1$ otherwise. Thus, $c_Q = 0$ for all $Q \in \mathcal{Q}$ and conditions a) and b) of Lemma 2 are satisfied. Elementary cycle-complete matrices are such that for each $Q \in \mathcal{Q}$, either all members of Q have a cost of zero to connect to the source, all members of Q have a cost of one to connect to the source, or a single agent in Q has a cost of zero, with others having a cost of one to connect to the source. Otherwise, if agents i and j have a cost of zero, but not k, there are multiple paths of cost zero between the source and k. From this, condition c) of Lemma 2 only applies when a single agent in Q has a cost of zero, with others having a cost of one to connect to the source, so that $|\arg\max_{j \in Q} c_{0j}| = |\{j \in Q : c_{j0} = 1\}| = |Q| - 1$.

"$\subseteq$" Let $c \in \Gamma^c \cap \Gamma^e$. Assume c is not cycle-complete. Then, for some $i, j \in N_0$, we have that $c_{ij} = 1$ but there exist two distinct free paths between them. If $i, j \in N$, we cannot build $\mathcal{Q}$ that satisfies condition i) of clique games and conditions a) and b) in Lemma 2. If $j = 0$, we can assume that each node k in these paths but two (one in each path) satisfy $c_{k0} = 1$. Let i^0 and i^1 be the nodes with $c_{i^0 0} = c_{i^1 0} = 0$. We also assume that $c_{\alpha\beta} = 0$ for all

$\alpha, \beta \in N$ in the path (otherwise, we would be in the previous case). We have the following possibilities:

1. Both paths are contained in the same clique $Q \in \mathcal{Q}$. Then, condition c) in Lemma 2 implies $|\arg\max_{j \in Q} c_{0j}| = |Q| - 1$ and hence all nodes in Q but one should have cost 1 to the source. But there are two nodes (i^0 and i^1) with cost zero to the source, which is a contradiction.

2. There exist two consecutive nodes $\alpha, \beta \in N$ that belong to different cliques. Since $c_{\alpha\beta} = 0$ and $\max\{c_{\alpha 0}, c_{\beta 0}\} = 1$, condition b) in Lemma 2 does not hold, which is a contradiction.

3. There exists a path of at least two cliques between i^0 and i^1. Clearly, each of these cliques should have at least two consecutive nodes. Moreover, condition i) of clique games implies that i^0 and i^1 belong to different cliques. Thus, there exist $j^0 \in N$ consecutive node to i^0 and such that $i^0, j^0 \in Q^0$ and $i^1 \in Q^1$ with Q^0, Q^1 different cliques. Condition b) in Lemma 2 implies that $0 = c_{j^0 i^1} \geq \max\{c_{0j^0}, c_{0i^1}\} = 1$, which is a contradiction.

■

We then have that, in any *mcst* problem whose cost matrix is elementary and cycle complete, the (pre)nucleolus, the Shapley value, the permutation-weighted average of the extreme points of the core and the cycle-complete rule coincide. Formally:

Theorem 4 *For all $c \in \Gamma^{ecc}$, $Nu(C) = Sh(C) = \bar{y}(C) = y^{cc}(c)$.*

Proof. The coincidence between the nucleolus, the Shapley value and the permutation-weighted average of the extreme points of the core is obtained as a corollary of Theorem 1 and Lemma 3. Coincidence with the cycle-complete solution is by definition. ■

In addition, as long as the cost matrix is elementary, the (pre)nucleolus, the permutation-weighted average of the extreme points of the core, and the cycle-complete rule coincide. Formally:

Theorem 5 *For all $c \in \Gamma^e$, $Nu(C) = \bar{y}(C) = y^{cc}(c)$.*

Proof. Coincidence between the cycle-complete solution and $\bar{y}$ is shown in Trudeau and Vidal-Puga (2017). We show the coincidence between the nucleolus and the cycle-complete solution. It is immediate that $C^{**} \leq C$. We show that if $C^{**}(S) < C(S)$, then the excess of coalition S is ignored in the calculation of $Nu(C^{**})$.

As shown in Trudeau and Vidal-Puga (2017), there exists $T \subseteq N \setminus S$ such that $C^{**}(S) = C(S \cup T) + C(N \setminus T) - C(N) < C(S)$. This can we rewritten as

$$\sum_{i \in S \cup T} C(i) - C(S \cup T) + \sum_{i \in N \setminus T} C(i) - C(N \setminus T) - \sum_{i \in N} C(i) + C(N) > \sum_{i \in S} C(i) - C(S)$$

$$v^C(S \cup T) + v^C(N \setminus T) - v^C(N) > v^c(S)$$

$$x(S \cup T) - v^C(S \cup T) + x(N \setminus T) - v^C(N \setminus T) - x(N) + v^C(N) < x(S) - v^C(S)$$

$$e(S \cup T, x, v^C) + e(N \setminus T, x, v^C) - e(N, x, v^C) < e(S, x, v^C)$$

$$e(S \cup T, x, v^C) + e(N \setminus T, x, v^C) < e(S, x, v^C).$$

Therefore, the excess of S is not taken into account when we find $Nu(C)$. We also have that

$$C^{**}(S \cup T) + C^{**}(N \setminus T) - C^{**}(N) \leq C(S \cup T) + C(N \setminus T) - C(N) = C^{**}(S)$$

leading to conclude, in the same manner as above, that

$$e(S \cup T, x, C^{**}) + e(N \setminus T, x, C^{**}) \leq e(S, x, C^{**})$$

and thus that the excess of S is not taken into account when we find $Nu(C^{**})$. Therefore, $Nu(C)$ and $Nu(C^{**})$ depend on the same excesses, and $Nu(C) = Nu(C^{**})$. Since $Nu(C^{**}) = y^{cc}(c)$, we also have $Nu(C) = y^{cc}(c)$. ■

If we look at public *mcst* games instead of private *mcst* games, we obtain similar coincidence results. First, we consider the subset of elementary irreducible games, for which $C^{Pub} = C$. We have a coincidence between the (pre)nucleolus, the Shapley value, the permutation-weighted average of the extreme points of the core and the folk solution.[2]

Corollary 1 *For all elementary and irreducible matrices* c, $Nu(C) = Sh(C) = \bar{y}(C) = y^f(c)$.

For elementary *mcst* games, for which C^{Pub} is typically different from C, we obtain the following result:

Theorem 6 *For all* $c \in \Gamma^e$, $Nu\left(C^{Pub}\right) = \bar{y}\left(C^{Pub}\right) = y^f(c)$.

The proof is similar to the proof of Theorem 5 and is omitted.

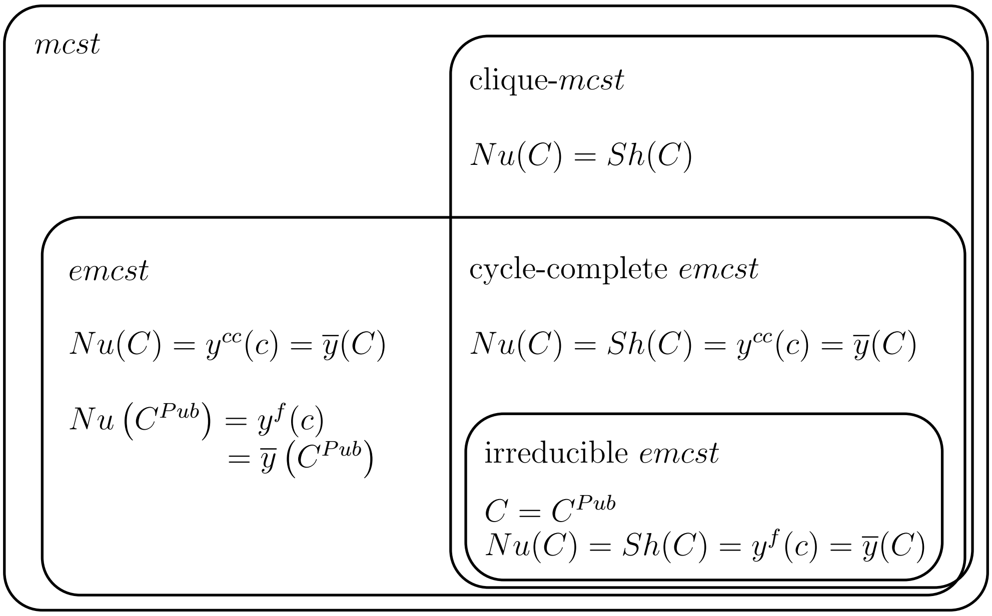

Figure 3: Summary of the results for *mcst* problems.

The results of this section are summarized in Figure 3. The set of clique-*mcst* games are those described in Subsection 5.2.

[2] A related result is provided by Subiza et al. (2016). They provide a closed-form solution for the folk solution in a class of *mcst* games that are a subset of clique games in which links between agents have a cost that is either high or low. Their result is a simplification of our closed-form expression for their family. They extend by considering games in which the set of agents can be partitioned in independent groups, such that they can all be connected separately to the source, applying their conditions on every group. One could do the same thing in our setting.

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