Disruptive Innovation by Heterogeneous Incumbents and Economic Growth: When do incumbents switch to new technology?

Ohki, Kazuyoshi

31 October 2019

Disruptive Innovation by Heterogeneous Incumbents and Economic Growth:

When do incumbents switch to new technology? *

Kazuyoshi Ohki†

October, 30, 2019

Abstract

In this paper, we construct a tractable endogenous growth model to examine heterogeneous incumbents' current technology-switching behavior. Then, we examine the effects of policies such as a subsidy for innovation by incumbents, a subsidy for innovation by entrants, and the extension of patent length. Our setting suggests interesting and counterintuitive results. High quality incumbents tend to be less likely to conduct innovation, which is inconsistent with Schumpeter's hypothesis. A subsidy for innovation by entrants decreases the average quality of differentiated goods. Moreover, it may decrease the growth rate of the economy if the positive spillover of innovation from average quality production is adequately large. Aggregate innovation can be small even when the population size is large if the barriers to entry are extremely high.

Keywords: Economic Growth, R&D, Firm-Heterogeneity, Innovation by Incumbents, IPR Policy

JEL classification: O31, O32, O33, O34, O41

*The author is grateful to Tatsuro Iwaisako and the participants of the RoMacS Workshop in Kanazawa for their valuable comments. The author also gratefully acknowledges financial support from a Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Young Scientists (No.19K13646). Any remaining errors are the responsibility of the author.

†Faculty of Economics and Management, Institute of Human and Social Sciences, Kanazawa University, Kakuma, Kanazawa 920-1192 JAPAN. emai:kazuyoshi.ohki@gmail.com

1 Introduction

Since the eighteenth century, the world economy has grown through the accumulation of capital stock, population growth, and technological improvements. Technological improvements, which are the result of research and development (R&D) activities, have perhaps played the most important role for economic development in recent years. Many researchers have examined the effects of R&D activities via an endogenous R&D-based growth model. First, Romer (1990), Segerstrom, Anant and Dinopoulos (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992) constructed very simple R&D-based growth models that focused solely on innovation by entrants (or followers).[1] The weak point of these models is that they cannot examine the innovative activity of incumbents (or leaders).[2] Second, Thompson and Waldo (1994), Aghion, Harris, and Vickers (1997), Peretto (1998), Segerstrom and Zornierek (1999), Aghion, Harris, Howitt, and Vickers (2001), Etro (2004), Segerstrom (2007), Ledezma (2013), and Kiedaisch (2015) construct advanced R&D-based growth models that can examine the innovative activity of incumbents and leaders that have technological superiority or first move advantage. These models still have weak points in that they cannot capture the heterogeneity of innovative activity by incumbents or they restrict their analysis to a particular industry.[3] Recently, Klette and Kortum (2004), Acemoglu and Akcigit (2012), Denicolo and Zanchettin (2012), Acemoglu and Cao (2015), Akcigit and Kerr (2018), Parello (2019), and Iwaisako and Ohki (2019) attempted to construct more advanced R&D-based growth models that can examine the innovative activity of both heterogeneous incumbents and entrants (or both heterogeneous leaders and followers) in many industries. The present paper examines a series of these studies.[4]

The present paper considers innovation by both heterogeneous incumbents and entrants. Incumbents are heterogeneous in the quality of their invented differentiated goods and efficiency of production, which are drawn from an exogenous distribution when they invent new goods. The contribution of our paper is that we consider the incumbents' technology-switching behavior. We assume that the quality of their invented differentiated goods declines exogenously with the passage of time, which decreases their instan-

[1] Followers are defined as firms once having state-of-the-art technology that have been leapfrogged by another firm.

[2] Bartelsman and Doms (2000) report that 75 percent of total factor productivity growth results from R&D activities by incumbent firms.

[3] In the real economy, there are many industries, so firm size and strategic incentives differ, and industries can act heterogeneously with the same economic policy.

[4] Some studies examine economic growth or welfare under firm heterogeneity, without, however, examining the endogenous innovative activity of heterogeneous incumbents (or leaders): Melitz (2003), Minniti, Parello, and Segerstrom (2013), Chu, Cozzi, Furukawa, and Liao (2017), and Chu, Cozzi, Fan, Furukawa, and Liao (2019).

taneous profit. Then, incumbents decide to wash their hands of current technology and switch to new technology. The timing of the switch is determined endogenously. When they switch to new technology, they disrupt their current position to obtain a new sequential profit flow. In this paper, we call this incumbent behavior "disruptive innovation", which is defined by Christensen (1997). Using this model, we examine how a subsidy (tax) for innovation by entrants, a subsidy (tax) for innovation by incumbents, and the extension (shortening) of patent length affect the growth rate of an economy and the average quality of differentiated goods.

The main results of our analysis are as follows. First, incumbents with high-quality products tend not to conduct disruptive innovation. This result is inconsistent with Schumpeter's hypothesis; however, it is consistent with the finding of Christensen (1997), who observed this in, for example, the hard-disk and excavating equipment industries. From this point of view, the present paper examines the relationship between economic growth and R&D activity, analytically capturing the exciting property discovered by Christensen (1997). [5] Second, in our basic model, a subsidy for innovation by entrants, a subsidy for innovation by incumbents, and the extension of patent length increase the growth rate of the economy. This result is consistent with the idea that generous treatment of firms stimulates the incentive to entry, which is pointed out in many related studies. Third, a subsidy for innovation by entrants decreases the average quality of differentiated goods, while that for innovation by incumbents and an extension of patent length increase it. In our model, average quality is an increasing function of disruptive innovation by incumbents. It is then a natural result that a subsidy for innovation by incumbents increases the average quality of differentiated goods because it stimulates the incentive to conduct disruptive innovation. Extension of patent length also stimulates disruptive innovation by incumbents because, as noted above, it stimulates innovation by entrants, which decreases the instantaneous profit of incumbents. A decrement of instantaneous profit decreases the incentive to put off disruptive innovation, thereby stimulating the incentive to conduct disruptive innovation. [6] A subsidy for innovation by entrants has two opposite effects on the incentive to conduct disruptive innovation. On one hand, it decreases the instantaneous profit of incumbents, which increases the incentive to engage in disruptive innovation. On the other hand, it decreases the expected benefit from disruptive innovation, which increases the incentive to put

[5] In the economic context, except for our paper, Igami (2017) examines the theory of Christensen (1997), focusing on the evolution of market structure using the empirical industrial organization literature.

[6] This result is interesting in that the extension of patent length makes incumbents producing at not so high quality switch their current technology earlier.

off disruptive innovation. We show analytically that the latter effect always dominates the former, and that a subsidy for innovation by entrants decreases the average quality of differentiated goods. Fourth, if the positive spillover of innovation from average quality is sufficiently large, a subsidy for innovation by entrants does decrease the growth rate of the economy. This counterintuitive result comes from our original framework in which heterogeneous incumbents conduct disruptive innovation. A subsidy for innovation by entrants decreases average quality, which decreases the expected benefit of conducting disruptive innovation. This effect decreases the value of incumbents given the growth rate of the economy, and then the growth rate of the economy must decrease to satisfy the free entry condition. In section 3.2, we show analytically that this phenomenon dominates the effect that a subsidy for innovation by entrants decreases the cost of entry when the positive spillover of innovation from average quality is sufficiently large. Fifth, if there is no negative externality of innovation from market size, and if barriers to entry are extremely high, then aggregate innovation is small when population size is large. This counterintuitive result also comes from our new framework in which heterogeneous incumbents conduct disruptive innovation. When barriers to entry are extremely high, the number of entrants goes to zero. Then, innovation is conducted only by incumbents. Large population size makes instantaneous profit high if there is no negative externality of innovation from market size, which increases the incentive to put off switching from current technology. As shown in section 3.3, this effect decreases aggregate disruptive innovation. [7]

Recently, the definition of innovation has become increasingly diverse, and some studies examine diverse innovation simultaneously.[8] On one hand, innovation to create new products and capture market leadership is defined as "exploration innovation", "product innovation", and "external innovation". On the other hand, innovation to improve product lines that they are currently serving is defined as "exploitation innovation", "process innovation", "internal innovation", and "incremental innovation".[9] In this context, our model is interpreted as follows. On one hand, entrants conduct innovation to capture market leadership; this innovation is categorized as " exploration innovation", "product innovation", and

[7] Although in the present paper, disruptive innovation does not affect the growth rate of the economy directly, this result may be one of the explanations of the scale effect puzzle: many related papers construct endogenous growth models showing that the growth rate of the economy is high when population size is large; however, Jones (1995) pointed out that empirical studies do not support this result. By changing our setting with felicity, one can construct a model in which disruptive innovation by incumbents affects the growth rate of the economy directly, and that may show that large population size discourages innovation where barriers to entry are high, which offsets the effect that large population size encourages economic growth through innovation by entrants.

[8] See Acemoglu and Cao (2015) and Akcigit and Kerr (2018).

[9] Some papers define "radical innovation" or "drastic innovation" as innovation in which the degree of progress is large or invention is epoch-making.

"external innovation". On the other hand, incumbents conduct innovation to switch from the aged product lines they are currently serving to a new one; this activity cannot be categorized under the above definition. Following Christensen (1997), we define this activity as "disruptive innovation".

Most related literature on the innovative activity of both heterogeneous incumbents and entrants examines "exploitation innovation", "process innovation", "internal innovation", and "incremental innovation" by incumbents, and "exploration innovation", "product innovation", and "external innovation" by entrants or incumbents. One exception is Iwaisako and Ohki (2019). [10] They unintentionally examine the "disruptive innovation" by leaders using the extended quality ladder model. In their model, the quality advantage over followers is caused by the exogenous distribution, and then heterogeneity exists between leaders. Their unique setting is that leaders have the opportunity to redraw the quality advantage, and then leaders abandon their current position and innovate to earn a larger profit flow. [11] They find a negative relationship between the quality of the invented good and leaders' motivation to conduct innovation, which is also found in the present paper. This result is contrary to Schumpeter's hypothesis that large firms tend to conduct R&D activity proportionally more than smaller ones, which is supported by Acemoglu and Cao (2015) and Akcigit and Kerr (2018) in the theoretical context by analyzing "incremental innovation" by incumbents.[12] However, this difference is natural because the definition of innovation in their model is clearly different, and our result is consistent with Christensen's idea that large firms tend to conduct large "incremental innovation" but much less "disruptive innovation". In the near future, progressive technology, such as solid-state batteries and artificial intelligence, will play a role in reconstructing existing markets, and the presence of "disruptive innovation" is expected to rise. We think both "incremental innovation" and "disruptive innovation" play important roles in the development of the economy, so the relationships between these two types of innovation and economic

[10]The model constructed in the present paper and that constructed by Iwaisako and Ohki (2019) are completely different except on one point: both models consider "disruptive innovation". The present model is based on the variety expansion model; however, Iwaisako and Ohki (2019) is based on the quality ladder model. On that point, their model cannot examine the effect of extending patent length. Additionally, they cannot capture the relative obsolescence effect, which plays a crucial role in section 3.2 in our paper, unless generalizing the constant elasticity of substitution (CES) utility function. The present model also constructs a general equilibrium; however, Iwaisako and Ohki (2019) use a quasi-linear utility function. Although this assumption makes analysis easy, it restricts the range of analysis, and their model cannot examine the resource effect as examined in section 3.3. The reason they choose such a setting in spite of these weak points is that their model can capture the intensity of innovative activity, and the probability of successful innovation is determined endogenously, which makes distribution of incumbents important. (In the present model, the probability of success is assumed to be one, and key endogenous variables do not depend on the distribution of incumbents.) In their model, growth rate of the economy is affected through the change of the endogenous distribution of incumbents, the "distribution effect", which is never obtained from the present or related papers.

[11]Because they do not consider absolute or relative obsolescence, and their setting permits followers to leapfrog, the leaders' motivation for innovation is simply earning higher profits. The important point is that leaders abandon their current position, and this setting shows the typical property of "disruptive innovation".

[12]Cohen and Levin (1989) and Cohen (1995) provide a survey related to Schumpeter's hypothesis.

growth models have to be examined carefully. This paper is the first step in this research plan.

The remainder of our paper is structured as follows. In section 2, we construct a general equilibrium model without specifying R&D technology. As for the homogeneous growth model, the properties of the long-run equilibrium crucially depend on the specification of R&D technology. In section 3.1, we construct the simplest model in which the degree of innovation by incumbents and entrants are independent of labor market conditions. In section 3.2, we generalize the positive externality from the average quality of differentiated goods, which is strongly affected by incumbents' innovation. In section 3.3, we remove our assumption imposed only for simplicity, and conduct a numerical analysis using a model in which the degree of innovation by incumbents and entrants is dependent on labor market conditions. In section 3.4, we remove the positive externality from the number of differentiated goods, and examine the steady-state equilibrium where the number of differentiated goods is determined endogenously instead of the growth rate.

2 Model

We construct an infinite representative agent model where the productivity of production and the quality of differentiated goods, which change as an exogenous law of motion, differ between incumbents. Using this model, we analyze the relationship between incumbents' technology-switching behavior, new inventions by entrants, and economic growth. We normalize the wage rate as the numeraire, $w(t) = 1$. We focus on the balanced growth path (BGP) equilibrium, where all variables grow at a constant rate.

2.1 Households

There are $L(t)$ households at time t that grow at exogenous rate g_L. Household members live forever and are endowed with one unit of labor, which is supplied inelastically. Each household maximizes its discounted utility:

$$U = \int_0^\infty e^{-\rho t} u(t) dt, \tag{1}$$

where ρ is a subjective discount rate and $u(t)$ represents instantaneous utility from consumption at time t. Consumption goods are provided by one industry. Within that industry, there is a continuum of horizontally differentiated goods, and preferences are expressed in the form of a CES in accordance with

Dixit and Stiglitz (1977):

$$u(t) = \left[\int_0^{n(t)} q(j,t)\, x(j,t)^{\frac{\sigma-1}{\sigma}} dj \right]^{\frac{\sigma}{\sigma-1}}, \tag{2}$$

where $q(j,t)$ and $d(j,t)$ denote the quality and consumption volume of incumbent j at time t, respectively, and $\sigma > 1$ is the elasticity of substitution of differentiated goods.

Solving the static utility maximization problem of the household, we now derive the per capita demand for each differentiated good:

$$x(j,t) = \left[\frac{q(j,t)}{p(j,t)} \right]^{\sigma} P(t)^{\sigma-1} E(t), \tag{3}$$

where $P(t) = \left[\int_0^{n(t)} q(j,t)^{\sigma} p(j,t)^{-(\sigma-1)} dj \right]^{\frac{-1}{\sigma-1}}$ denotes the price index of differentiated goods at time t, and $E(t)$ is the per capita expenditure at time t.

Given (3), inter-temporal utility maximization yields $\frac{\dot{E}(t)}{E(t)} = r(t) - \rho$, where $r(t)$ is the interest rate at time t. In the BGP equilibrium, the growth rate of expenditure g_E is constant. Thus, the interest rate is also constant, and expressed as:

$$r = \rho + g_E. \tag{4}$$

2.2 Production

Each differentiated good is produced by incumbents. They have heterogeneous production technologies and quality for the goods they produce. Incumbents having productivity ϕ must hire $\frac{1}{\phi}x$ units of labor to produce x units of goods, and there is no fixed cost to produce differentiated goods. The instantaneous profit of incumbent j is expressed as $\pi(j,t) = x(j,t)\, L(t)\, (p(j,t) - 1)$, where $p(j,t)$ is the price of incumbent j. As incumbents supply goods monopolistically, they choose the optimal price $p(j,t) = \frac{\sigma}{\sigma-1} \frac{1}{\phi(j,t)}$. Using (3), we obtain the incumbents' quantity of production having productivity ϕ and quality q as:

$$x(j,t)\, L(t) = [q(j,t)\, \phi(j,t)]^{\sigma} \left(\frac{\sigma-1}{\sigma} \right)^{\sigma} P(t)^{\sigma-1} E(t)\, L(t). \tag{5}$$

We define the composite factor of the incumbents' property (hereafter referred to as adjusted quality) as:

$$\theta(j,t) \equiv q(j,t)^{\sigma}\, \phi(j,t)^{\sigma-1}. \tag{6}$$

Because all incumbents with the same adjusted quality behave symmetrically, we index incumbents hereafter by θ instead of j. Then, the labor demand, $(x/\phi)\,L$, and profit, π, of incumbents having adjusted quality θ are expressed as:

$$\frac{x(\theta,t)\,L(t)}{\phi} = \theta(t)\,\frac{\sigma-1}{\sigma}\,\frac{E(t)L(t)}{Q(t)}, \tag{7}$$

$$\pi(\theta,t) = \theta(t)\,\frac{1}{\sigma}\,\frac{E(t)L(t)}{Q(t)}, \tag{8}$$

where $Q(t) \equiv n(t)\,\Theta(t)$ expresses the improvement level of differentiated goods in this industry, which is composed of the number of differentiated goods, $n(t)$, and average quality, $\Theta(t) = \int_{\theta_{\min}}^{\theta_{\max}} \theta\mu(\theta,t)\,d\theta$, at time t. Here, $\mu(\theta,t)$ is the distribution of incumbents having adjusted quality θ at time t.[13]

2.3 Entry

Entrants must hire $s_F C_F(t)$ units of labor to invent new differentiated goods at time t, and draw an initial adjusted quality θ_0 from a given distribution $G(\theta_0)$. Here, s_F is the subsidy (tax) for innovation by entrants when $s_F < 1$ ($s_F > 1$). We specify the assumed Pareto distribution as:[14]

$$G(\theta_0) = 1 - \theta_0^{-k} \quad 1 \le \theta_0 < \infty \ , \tag{9}$$

where the expected value of θ_0 is given by $k/(k-1)$, which decreases in $k > 1$. After the invention, entrants enter the market and become incumbents. Invented differentiated goods are protected by a perfect but finite intellectual property rights policy, and then incumbents can supply goods monopolistically without risk of imitation until their patent protection expires. They can obtain a monopoly profit throughout $\bar{T}$ periods at most, where $\bar{T}$ is patent length. Initial instantaneous profits of an incumbent entering at time s and drawing θ_0 is expressed as:

$$\pi(\theta_0,0,s) = \frac{\theta_0}{\sigma}\,\frac{E(s)L(s)}{Q(s)}. \tag{10}$$

[13] Although $\mu(\theta,t)$ must be determined endogenously, the equilibrium of our economy does not depend on $\mu(\theta,t)$. Thus, we do not pay attention to $\mu(\theta,t)$.

[14] In general, the cumulative distribution function of the Pareto distribution is expressed as $G(\theta_0) = 1 - a^k\theta_0^{-k} \quad a \le \theta_0 < \infty$. In this paper, we simplify $a = 1$. Then, the density function is expressed as $g(\theta_0) = k\theta_0^{-(k+1)}$.

We assume that the adjusted quality of each incumbent grows at the exogenous rate g_{θ_0}.[15] The number of households grows at exogenous growth rate g_L, and the improvement level of differentiated goods, Q, and aggregate expenditure, E, grow at endogenous growth rates g_Q and g_E, respectively. Then, an instantaneous profit of τ times passed incumbents entering at time s, and drawing θ_0 is expressed as:

$$\pi(\theta_0, \tau, s) = \exp\left[-\left[g_Q - g_\theta - g_E - g_L\right]\tau\right] \frac{\theta_0}{\sigma} \frac{E(s)L(s)}{Q(s)}. \tag{11}$$

2.4 Disruptive Innovation

From (11), instantaneous profits decrease with decreases in own adjusted quality, θ_0, which captures absolute obsolescence or saturation of demand. It also decreases as the improvement level of differentiated goods, Q, increases, which captures the effect of relative obsolescence. In this paper, we consider that incumbents have the opportunity to wash their hands of current and switch to new technology by conducting R&D activity. We call this activity disruptive innovation. Incumbents must hire $s_D C_D(t)$ units of labor to conduct disruptive innovation, and then invent a new differentiated good and reenter the market at time t. They draw an initial adjusted quality θ_0 from a given distribution $G(\theta_0)$. Here, s_D is the subsidy (tax) for innovation by incumbents when $s_D < 1$ ($s_D > 1$). We assume that $C_D(t)$ is strictly smaller than $C_F(t)$.[16]

The value of incumbents entering at time s consists of aggregate discounted instantaneous profits from time s to the timing of their technology switch and expected discounted net benefit from disruptive innovation, which is the difference between the expected initial value of incumbents and the cost of technology-switching. Then, the initial values of incumbents entering at time s and drawing θ_0 is expressed as:

$$V(\theta_0, 0, s) = \int_0^{\tilde{T}} \exp[-r\tau]\,\pi(\theta_0, \tau, s)\,d\tau + \exp\left[-r\tilde{T}\right]\left[EV\left(s+\tilde{T}\right) - s_D C_D\left(s+\tilde{T}\right)\right]. \tag{12}$$

where $EV(t) = \int_{\theta_0^{\min}}^{\theta_0^{\max}} V(\theta_0, 0, t)\,dG(\theta_0)$ is the expected initial value of incumbents entering at time t, and $\tilde{T}$ is the timing of conducting disruptive innovation, which is determined to satisfy the following

[15] When incumbents conduct disruptive innovation, the growth rate of adjusted quality of each incumbent, g_{θ_0}, must be negative to satisfy a sufficient condition, which is discussed later. In this paper, we assume that g_{θ_0} is constant; thus, g_{θ_0} must be negative.

[16] This assumption means that incumbents have cost advantages over entrants, and the related literature provides several plausible interpretations of the assumption; for example, because only incumbents have state-of-the-art technology in their industries, they have better experience, ability, or knowledge of R&D activities in their industries.

condition:

$$\pi\left(\theta_0, \tilde{T}, s\right) = [r - g_C]\, exp\left[g_C \tilde{T}\right] [EV(s) - s_D C_D(s)], \tag{13}$$

where g_C is the growth rate of innovation cost.[17] This is a necessary condition with respect to $\tilde{T}$ to maximize (12), and expresses the trade-off between marginal benefit and marginal loss from putting off the disruptive innovation. When incumbents put off the disruptive innovation, they can obtain additional instantaneous profits; however, the expected net benefit from disruptive innovation decreases because the cumulative discount rate of expected net benefit from disruptive innovation increases. If the marginal benefit from delaying disruptive innovation is larger (smaller) than that of the marginal loss, incumbents move the timing of disruptive innovation later (forward). Thus, the timing of disruptive innovation is determined to satisfy that the marginal benefit from delaying the disruptive innovation is equal to that of the marginal loss. A sufficient condition with respect to $\tilde{T}$ to maximize (12) is expressed as:

$$-[g_Q - g_\theta - g_E] - g_C < 0. \tag{14}$$

This condition implies that the marginal loss from delaying the disruptive innovation has to increase more than that of marginal benefit when incumbents conduct disruptive innovation. Equation (13) determines the interior solution of $\tilde{T}$ as a function of θ_0, which corresponds to one-to-one. From (8) and (13), we can recognize the positive relationship between θ_0 and $\tilde{T}$, and then incumbents drawing high θ_0 tend to engage in disruptive innovation later.

In addition to the case of the interior solution, we have to consider the case of a corner solution. On one hand, for incumbents drawing low θ_0 at time s, the marginal benefit from delaying the disruptive innovation can be strictly less than that of the marginal loss even at the moment when they enter the market:

$$\pi(\theta_0, 0, s) - [r - g_C][EV(s) - s_D C_D(s)] < 0.$$

Incumbents drawing such a low θ_0 decide to conduct disruptive innovation immediately, rather than

[17] We assume, for simplicity, that the growth rate of EV equals that of C_D, $g_{EV} = g_{C_D} \equiv g_C$, which implies a growth rate of net benefit from disruptive innovation equal to the growth rate of innovation cost, $\frac{d[EV(t) - s_D C_D(t)]/dt}{[EV(t) - s_D C_D(t)]} = g_C$. This assumption is satisfied when cost of disruptive innovation by incumbents and the cost of innovation by entrants grow at same rate, which can be confirmed later.

producing differentiated goods. We define the critical level of initial adjusted quality in terms of whether the incumbent produces as θ_0^{c1}, which satisfies the following condition:

$$\pi\left(\theta_0^{c1}, 0, s\right) - [r - g_C]\left[EV(s) - s_D C_D(s)\right] = 0. \tag{15}$$

On the other hand, for incumbents drawing a high θ_0 at time s, the marginal benefit from delaying the disruptive innovation can be strictly greater than that of the marginal loss even at time $s + \bar{T}$:

$$\pi\left(\theta_0, \bar{T}, s\right) - [r - g_C]\, exp\left[g_C \bar{T}\right]\left[EV(s) - s_D C_D(s)\right] > 0.$$

Incumbents drawing such a high θ_0 conduct no disruptive innovation before their patent protection expires. We assume, for simplicity, that incumbents drawing such a high θ_0 switch their technology immediately after their patent protection expires. We define the critical level of initial adjusted quality in terms of whether the incumbent conducts disruptive innovation before their patent protection expires as θ_0^{c2}, which satisfies the following condition:

$$\pi\left(\theta_0^{c2}, \bar{T}, s\right) - [r - g_C]\, exp\left[g_C \bar{T}\right]\left[EV(s) - s_D C_D(s)\right] = 0. \tag{16}$$

Thus, we can write the timing of disruptive innovation as a function of initial adjusted quality:

$$\begin{array}{lcll} \tilde{T} = & 0 & for & \theta_0 \leq \theta_0^{c1} \\ \tilde{T} = & \tilde{T}(\theta_0) & for & \theta_0^{c1} < \theta_0 < \theta_0^{c2} \\ \tilde{T} = & \bar{T} & for & \theta_0 \geq \theta_0^{c2} \end{array} \tag{17}$$

where $\frac{\partial \tilde{T}(\theta_0)}{\partial \theta_0} > 0$ is satisfied.

From (11), (13), (15) and (16), we can express θ_0^{c2} and $\tilde{T}(\theta_0)$ as a function of θ_0^{c1}:

$$\begin{array}{lcl} \theta_0^{c2} & = & \exp\left[\left[-g_\theta\right]\bar{T}\right]\theta_0^{c1} \\ \tilde{T}(\theta_0) & = & \frac{1}{-g_\theta}\ln\left[\frac{\theta_0}{\theta_0^{c1}}\right] \end{array}. \tag{18}$$

From (12), we can express the initial values of an incumbent entering at time s and drawing θ_0 as:

$$\begin{aligned}
V_0(\theta_0,0,s) &= [EV(s) - s_D C_D(s)] && for \quad \theta_0 \leq \theta_0^{c1} \\
V_1(\theta_0,0,s) &= \int_0^{\tilde{T}(\theta_0)} exp[-r\tau]\,\pi(\theta_0,\tau,s)\,d\tau + \exp\left[-[r-g_C]\,\tilde{T}(\theta_0)\right][EV(s) - s_D C_D(s)] && for \quad \theta_0^{c1} < \theta_0 \leq \theta_0^{c2} \\
V_2(\theta_0,0,s) &= \int_0^{\bar{T}} exp[-r\tau]\,\pi(\theta_0,\tau,s)\,d\tau + \exp\left[-[r-g_C]\,\bar{T}\right][EV(s) - s_D C_D(s)] && for \quad \theta_0^{c2} < \theta_0
\end{aligned} \tag{19}$$

where $\tilde{T}$, θ_0^{c1}, and θ_0^{c2} satisfy (13) and (18), respectively. The first term represents the gain from the monopoly profit and the second represents the gain from disruptive innovation.[18]

2.5 Free Entry Condition

When we calculate an expected initial value of incumbents, we have to consider the magnitude of the correlation between the minimum initial adjusted quality, $\theta_0^{\min}$, and the critical level of initial adjusted quality, θ_0^{c1}, θ_0^{c2}. First, if $\theta_0^{\min} \leq \theta_0^{c1}$ is satisfied, we obtain an equilibrium that incumbents drawing $\theta_0 < \theta_0^{c1}$ conduct disruptive innovation immediately, incumbents drawing $\theta_0^{c1} < \theta_0 < \theta_0^{c2}$ conduct disruptive innovation after $\tilde{T}(\theta_0)$ periods have passed from entry, and incumbents drawing $\theta_0^{c2} \leq \theta_0$ conduct disruptive innovation after $\bar{T}$ periods have passed from the entry. Second, if $\theta_0^{c1} < \theta_0^{\min} \leq \theta_0^{c2}$ is satisfied, we obtain an equilibrium that incumbents drawing $\theta_0^{\min} \leq \theta_0 < \theta_0^{c2}$ conduct disruptive innovation after $\tilde{T}(\theta_0)$ periods have passed from entry, and incumbents drawing $\theta_0^{c2} \leq \theta_0$ conduct disruptive innovation after $\bar{T}$ periods have passed.[19] Third, if $\theta_0^{c2} < \theta_0^{\min}$ is satisfied, we obtain an equilibrium in which all incumbents conduct disruptive innovation after $\bar{T}$ periods have passed from entry.[20] Then, we can express an expected initial value of incumbents as:

$$\begin{aligned}
EV &= \int_{\theta_0^{\min}}^{\theta_0^{c1}} V_0(\theta_0,0,s)\,dG(\theta_0) + \int_{\theta_0^{c1}}^{\theta_0^{c2}} V_1(\theta_0,0,s)\,dG(\theta_0) + \int_{\theta_0^{c2}}^{\theta_0^{\max}} V_2(\theta_0,0,s)\,dG(\theta_0) && if \quad \theta_0^{\min} \leq \theta_0^{c1} \\
EV &= \int_{\theta_0^{\min}}^{\theta_0^{c2}} V_1(\theta_0,0,s)\,dG(\theta_0) + \int_{\theta_0^{c2}}^{\theta_0^{\max}} V_2(\theta_0,0,s)\,dG(\theta_0) && if \quad \theta_0^{c1} < \theta_0^{\min} \leq \theta_0^{c2} \\
EV &= \int_{\theta_0^{\min}}^{\theta_0^{\max}} V_2(\theta_0,0,s)\,dG(\theta_0) && if \quad \theta_0^{c2} < \theta_0^{\min}
\end{aligned} \tag{20}$$

[18] Incumbents drawing $\theta_0 \leq \theta_0^{c1}$ conduct disruptive innovation and gain no monopolistic profit.
[19] In this case, there are no incumbents conducting disruptive innovation at the moment when they invent.
[20] In this case, there are no incumbents conducting disruptive innovation before their patent protection expires.

The free entry condition is satisfied in each period. The expected gain from entry at time t equals to zero, and we obtain the following condition:

$$\begin{aligned}
&\int_{\theta_0^{\min}}^{\theta_0^{c1}} V_0(\theta_0,0,t)\,dG(\theta_0) + \int_{\theta_0^{c1}}^{\theta_0^{c2}} V_1(\theta_0,0,t)\,dG(\theta_0) + \int_{\theta_0^{c2}}^{\theta_0^{\max}} V_2(\theta_0,0,t)\,dG(\theta_0) \leq s_F C_F(t) \quad &&if \quad \theta_0^{\min} \leq \theta_0^{c1} \\
&\int_{\theta_0^{\min}}^{\theta_0^{c2}} V_1(\theta_0,0,t)\,dG(\theta_0) + \int_{\theta_0^{c2}}^{\theta_0^{\max}} V_2(\theta_0,0,t)\,dG(\theta_0) \leq s_F C_F(t) \quad &&if \quad \theta_0^{c1} < \theta_0^{\min} \leq \theta_0^{c2} \\
&\int_{\theta_0^{\min}}^{\theta_0^{\max}} V_2(\theta_0,0,t)\,dG(\theta_0) \leq s_F C_F(t) \quad &&if \quad \theta_0^{c2} < \theta_0^{\min}
\end{aligned} \tag{21}$$

When an equilibrium value of g_n is positive, (21) is satisfied with equality. When an equilibrium value of g_n is negative, the expected gain from entry is too small for entrants to enter the market. Then, there is no new entry, and the number of differentiated goods does not increase.[21] Thus, we obtain an interior equilibrium in which the number of differentiated goods remains constant, $g_n = 0$, if the expected gain from entry is strictly negative, even when $g_n = 0$.

2.6 Labor Market

From (7), aggregate labor demand for production at time t is:

$$n(t) \int_{\theta_0^{\min}}^{\theta_0^{\max}} \frac{x(\theta,t)\,L(t)}{\phi} \mu(\theta,t)\,d\theta = \frac{\sigma-1}{\sigma} E(t)\,L(t)\,. \tag{22}$$

Because $C_F(t)$ units of labor are required to invent one new differentiated good, $n(t)\,g_n C_F(t)$ units of labor are required to invent $\dot{n}(t)$ new differentiated goods, where $g_n \equiv \dot{n}(t)/n(t)$ is the growth rate of the number of differentiated goods. Aggregate labor demand for innovation by entrants at time t is expressed as:

$$I_F(t) = n(t)\,g_n C_F(t)\,. \tag{23}$$

[21] Note that this does not mean that incumbents exit the market.

Because $C_D(t)$ units of labor are required to conduct disruptive innovation, aggregate labor demand for innovation by incumbents at time t is expressed as:

$$I_D(t) = \nu n(t) C_D, \tag{24}$$

where ν is the ratio of the number of differentiated goods conducting disruptive innovation to the aggregate number of differentiated goods, which is expressed as:[22]

$$\begin{aligned}
\nu &= \frac{\left[\theta_0^{c1}\right]^k [g_n + k[-g_\theta]]}{1-\exp\left[-[g_n+k[-g_\theta]]\bar{T}\right]} - g_n && if \quad \theta_0^{\min} \leq \theta_0^{c1} \\
\nu &= \frac{\left[\left[\theta_0^{c1}\right]^{\frac{g_n}{[-g_\theta]}} \frac{k[-g_\theta]}{g_n+k[-g_\theta]} + \left[\theta_0^{c1}\right]^{-k} \exp\left[-[g_n+k[-g_\theta]]\bar{T}\right] \frac{g_n}{g_n+k[-g_\theta]}\right]}{1-\left[\left[\theta_0^{c1}\right]^{\frac{g_n}{[-g_\theta]}} \frac{k[-g_\theta]}{g_n+k[-g_\theta]} + \left[\theta_0^{c1}\right]^{-k} \exp\left[-[g_n+k[-g_\theta]]\bar{T}\right] \frac{g_n}{g_n+k[-g_\theta]}\right]} g_n && if \quad \theta_0^{c1} < \theta_0^{\min} \leq \theta_0^{c2} \quad , \\
\nu &= \frac{\exp\left[-g_n\bar{T}\right]}{1-\exp\left[-g_n\bar{T}\right]} g_n && if \quad \theta_0^{c2} < \theta_0^{\min}
\end{aligned} \tag{25}$$

Labor supply is exogenously given, so the labor market clearing condition is:

$$\frac{\sigma-1}{\sigma} E(t) L(t) + \nu n(t) C_D(t) + g_n n(t) C_F(t) = L(t). \tag{26}$$

2.7 Average Quality of Differentiated Goods

The average quality of differentiated goods, Θ, is calculated as the sum of the quality of differentiated goods divided by the number of differentiated goods produced at time t:[23]

$$\begin{aligned}
\Theta &= \frac{k}{k-1}\left[\theta_0^{c1}\right] && if \quad \theta_0^{\min} \leq \theta_0^{c1} \\
\Theta &= \frac{\frac{[-g_\theta]}{g_n+[-g_\theta]}\left[1-\left[\theta_0^{c1}\right]^{\frac{g_n+[-g_\theta]}{[-g_\theta]}}\right] + \frac{1}{k-1}\left[1-\left[\theta_0^{c1}\right]^{-[k-1]} \exp\left[-[g_n+k[-g_\theta]]\bar{T}\right]\right]}{\frac{[-g_\theta]}{g_n}\left[1-\left[\theta_0^{c1}\right]^{\frac{g_n}{[-g_\theta]}}\right] + \frac{1}{k}\left[1-\left[\theta_0^{c1}\right]^{-k} \exp\left[-[g_n+k[-g_\theta]]\bar{T}\right]\right]} && if \quad \theta_0^{c1} < \theta_0^{\min} < \theta_0^{c2} \quad . \\
\Theta &= \frac{k}{k-1}\frac{g_n}{g_n+[-g_\theta]}\frac{\left[1-\exp\left[-[g_n+[-g_\theta]]\bar{T}\right]\right]}{\left[1-\exp\left[-[g_n]\bar{T}\right]\right]} && if \quad \theta_0^{c2} < \theta_0^{\min}
\end{aligned} \tag{27}$$

If $\theta_0^{\min} \leq \theta_0^{c1}$ is satisfied, Θ is an increasing function of θ_0^{c1} and independent from g_n and g_θ. First, from (18), high θ_0^{c1} leads to low $\tilde{T}$ for any incumbents, which means that incumbents drawing $\theta_0 \leq \theta_0^{c2}$ make disruptive innovation early. Because of absolute obsolescence, this effect makes average quality high. Second, the average quality of any vintage is identical: $\frac{\int_{\theta_0(\bar{T})}^{\infty} \exp\left[g_\theta \tilde{T}\right] k\theta_0^{-k} d\theta_0}{\int_{\theta_0(\bar{T})}^{\infty} k\theta_0^{-(k+1)} d\theta_0} = \frac{k}{k-1}\theta_0^{c1}$. The reason for

[22] In appendix A, we derive (25).
[23] In appendix B, we derive (27).

this is on one hand, the adjusted quality of any incumbent decreases at the rate of g_θ. Then, the average quality of old differentiated goods tends to be low, and this effect becomes large when the absolute value of g_θ is high. On the other hand, incumbents producing low-quality goods conduct disruptive innovation earlier, and differentiated goods having low adjusted quality goods go out of production over time. This raises the average quality, and the effect becomes large when the absolute value of g_θ is high because a high absolute value of g_θ decreases the incentive to put off disruptive innovation. These two effects offset completely, irrespective of the value of g_θ. Then, the ratio of old to new differentiated goods does not matter when $\theta_0^{\min} \leq \theta_0^{c1}$ is satisfied.

If $\theta_0^{c2} < \theta_0^{\min}$ is satisfied, Θ is an increasing function of g_n, a decreasing function of the absolute value of g_θ, and independent of θ_0^{c1}. First, no incumbents conduct disruptive innovation before their patent protection expires when $\theta_0^{c2} < \theta_0^{\min}$ is satisfied. Then, the adjusted quality of all incumbent decreases at the rate g_θ, and the average quality of old differentiated goods tends to be low. When the absolute value of g_θ is high, the average quality of old differentiated goods rapidly renders them obsolete. Then, the total average quality is low with a high absolute value of g_θ. When $g_n > 0$, the number of old differentiated goods, which are on average of low quality, is lower than that of new differentiated goods, which are on average of high quality. This effect becomes large when g_n is large, and then, the total average quality is high with high g_n. Second, even an incumbent drawing the lowest initial adjusted quality does not conduct disruptive innovation until their patent protection expires when $\theta_0^{c2} < \theta_0^{\min}$ is satisfied. Then, $\tilde{T} = \bar{T}$ is satisfied for any incumbent, and Θ is not affected by θ_0^{c1}.

If $\theta_0^{c1} < \theta_0^{\min} < \theta_0^{c2}$ is satisfied, Θ is an increasing function of θ_0^{c1} and g_n, and a decreasing function of the absolute value of g_θ. The reason that Θ is an increasing function of θ_0^{c1} is the same in the case of $\theta_0^{\min} \leq \theta_0^{c1}$. The reason that Θ is an increasing function of g_n and a decreasing function of the absolute value of g_θ is as follows: when $\theta_0^{c1} < \theta_0^{\min} < \theta_0^{c2}$ is satisfied, there are some periods during which even an incumbent drawing the lowest initial adjusted quality does not conduct disruptive innovation. Then, the average quality of old differentiated goods is lower than that of new differentiated goods. Thus, the total average quality is high with high g_n or a low absolute value of g_θ, which is similar to the case of $\theta_0^{c2} < \theta_0^{\min}$.

Regardless of the difference between θ_0^{c1} and $\theta_0^{\min}$, Θ is a decreasing function of k, and is constant as long as θ_0^{c1} is constant.[24] Because the improvement level of differentiated goods is $Q(t) \equiv n(t)\Theta$, the

[24] Note that the expected value of θ_0 is a decreasing function of k.

growth rate of improvement level of differentiated goods is expressed as:

$$g_Q = g_n. \tag{28}$$

2.8 Growth Rate

We call the growth rate of this economy the growth rate of instantaneous utility. Substituting (5) and (6) into (2) yields:

$$u(t) = Q(t)^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma} E(t). \tag{29}$$

Then, the growth rate of this economy is expressed as

$$g_u = \frac{1}{\sigma-1} g_Q + g_E. \tag{30}$$

3 Equilibrium

In the previous section, we constructed a closed model; however, we have yet to specify the innovation technology. The property of the long-run equilibrium is crucially dependent on how we specify the innovation technology; therefore, we derive several types of equilibriums by specifying several forms of innovation technology.

3.1 The Case Where $C_F(t) = \frac{E(t)L(t)}{Q(t)} c_F$ and $C_D(t) = \frac{E(t)L(t)}{Q(t)} c_D$

In this section, we specify the innovation technology as:

$$C_F(t) = \tfrac{E(t)L(t)}{Q(t)} c_F \quad C_D(t) = \tfrac{E(t)L(t)}{Q(t)} c_D \ . \tag{31}$$

These technology functions contain a positive externality from the improvement level of differentiated goods, and a negative externality from market scale.[25] By specifying this formula, we can derive the simplest BGP equilibrium, where the equilibrium value of θ_0^{c1}, θ_0^{c2}, $\tilde{T}(\theta_0)$, g_Q, and Θ, which are important

[25] When the improvement level of differentiated goods is high, much knowledge is accumulated, and then, innovation cost tends to be low. When the market scale is large, the costs of marketing their differentiated goods or research are high. Then, the innovation cost, as an initial investment cost of releasing a new differentiated good, tends to be high.

endogenous variables in this paper, are determined independent of the labor market clearing condition. Another merit for this specification is that g_Q is independent of population; thus, there is no scale effect.

From (26) and (31), the labor market clearing condition is expressed as:

$$E(t)\left[\frac{\sigma-1}{\sigma}+\nu\frac{c_D}{\Theta}+g_Q\frac{c_F}{\Theta}\right]=1. \tag{32}$$

From (32), in the BGP equilibrium, $E(t)$ must be constant, and we obtain:

$$g_E=0. \tag{33}$$

From (31) and (33), the growth rate of the innovation cost is:

$$g_C=g_L-g_Q. \tag{34}$$

From (14), (33), and (34), the growth rate of the adjusted quality has to be negative:

$$g_\theta<0. \tag{35}$$

Substituting (4), (11), (21), (34) and (31) into (13) and (18) yields:

$$\begin{aligned}\theta_0^{c1}(g_Q) &= \sigma\left[\rho-g_L+g_Q\right]\left[s_Fc_F-s_Dc_D\right]\\ \theta_0^{c2}(g_Q) &= \exp\left[\left[-g_\theta\right]\bar{T}\right]\theta_0^{c1}(g_Q)\\ \tilde{T}(\theta_0,g_Q) &= \tfrac{1}{-g_\theta}\ln\left[\tfrac{\theta_0}{\theta_0^{c1}(g_Q)}\right]\end{aligned}. \tag{36}$$

Equation (36) determines the equilibrium value of θ_0^{c1}, θ_0^{c2}, and $\tilde{T}$ with given g_Q. We can confirm $\frac{\partial\tilde{T}(\theta_0,g_Q)}{\partial\theta_0}>0$ and $\frac{\partial^2\tilde{T}(\theta_0,g_Q)}{\partial\theta_0\partial\theta_0}<0$, and we get following proposition.

Proposition 3.1. *An incumbent drawing a high initial adjusted quality and earning a large profit tends to invest in disruptive innovation (switch from current technology) later. When patent length is finite, some incumbents drawing a high initial adjusted quality conduct no disruptive innovation before their patent protection expires.*

This proposition is consistent with the finding of Iwaisako and Ohki (2019), who also examined

disruptive innovation using a quality ladder model. The result of this proposition is caused by the weak-form Arrow effect: incumbents with a high adjusted quality lose much of their value when they wash their hands of current technology; thus, incumbents with a high adjusted quality tend not to conduct disruptive innovation.[26]

We also confirm $\frac{\partial \theta_0^{c1}(g_Q)}{\partial g_Q} > 0$, $\frac{\partial \theta_0^{c2}(g_Q)}{\partial g_Q} > 0$ and $\frac{\partial \tilde{T}(\theta_0, g_Q)}{\partial g_Q} < 0$, and interpret these results as follows. A marginal increment of g_Q decreases the growth rate of the expected net benefit from disruptive innovation, which increases the marginal loss from putting off disruptive innovation. Then, incumbents move the timing of disruptive innovation forward.[27]We depict the relationship between $\tilde{T}$ and θ_0 in Figure 3.1. The horizontal axis represents θ_0 and the vertical axis represents $\tilde{T}$. The timing of disruptive innovation, $\tilde{T}$, slopes upward in θ_0 with given g_Q, and shifts downward when g_Q increases.

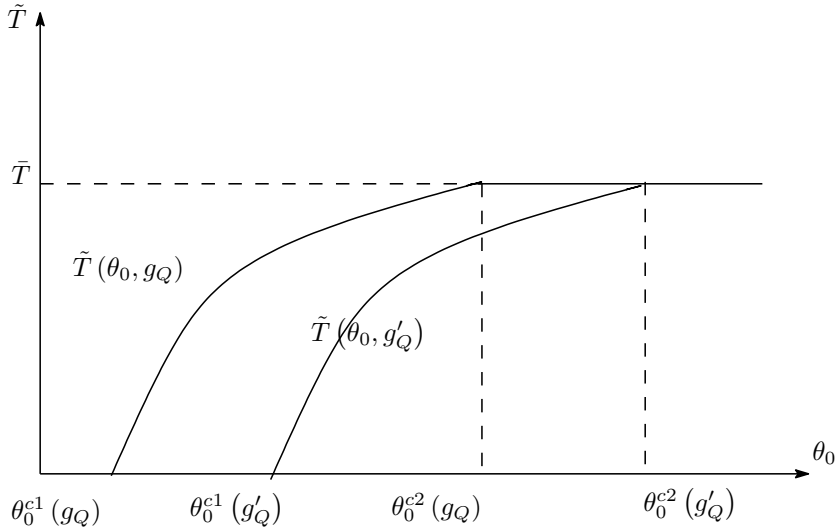

Figure 3.1. Relationship between $\tilde{T}$ and θ_0 in Figure 3.1. Here, $g_Q < g'_Q$.

Moreover, we can confirm $\frac{\partial \theta_0^{c1}(g_Q)}{\partial s_D} < 0$, $\frac{\partial \theta_0^{c2}(g_Q)}{\partial s_D} < 0$, $\frac{\partial \tilde{T}(\theta_0, g_Q)}{\partial s_D} > 0$, $\frac{\partial \theta_0^{c1}(g_Q)}{\partial s_F} > 0$, $\frac{\partial \theta_0^{c2}(g_Q)}{\partial s_F} > 0$, $\frac{\partial \tilde{T}(\theta_0, g_Q)}{\partial s_F} < 0$, $\frac{\partial \theta_0^{c1}(g_Q)}{\partial \bar{T}} = 0$, $\frac{\partial \theta_0^{c2}(g_Q)}{\partial \bar{T}} > 0$ and $\frac{\partial \tilde{T}(\theta_0, g_Q)}{\partial \bar{T}} = 0$. We can interpret these results as follows. First, on one hand, a subsidy (tax) for disruptive innovation by incumbents decreases (increases) the cost of disruptive innovation, which then increases (decreases) the expected discounted net benefit from disruptive innovation with given g_Q. On the other hand, a subsidy (tax) for innovation by entrants stimulates (mitigates) competition, which decreases (increases) the expected initial value of incumbents, and the expected discounted net benefit from disruptive innovation decreases (increases) with given g_Q.

[26]In the context of empirical studies, as in Igami (2017), this effect is called "cannibalization".

[27]Strictly speaking, there are two other effects from a marginal increment of g_Q. One is a marginal increment of g_Q that decreases instantaneous profit, which decreases the marginal benefit from putting off the disruptive innovation. The other is a marginal increment of g_Q that decreases the level of expected net benefit from disruptive innovation, which in turn decreases the marginal loss from putting off disruptive innovation. We can easily confirm that these two opposite effects offset each other; then, the only effect that remains is a marginal increment of g_Q that decreases the growth rate of the expected net benefit from disruptive innovation.

The decrement (increment) of expected discounted net benefit from disruptive innovation decreases (increases) the marginal loss from putting off the disruptive innovation, and then the timing of disruptive innovation moves later (earlier). Second, a change of patent length has no effect on the marginal benefit from putting off radical innovation or on the marginal loss of that with given g_Q, and then a change of patent length does not affect the timing of disruptive innovation as long as g_Q does not change.[28]

Substituting (4), (11) and (31) into (19) yields:

$$\begin{array}{lll} V_0(\theta_0,0,s) = & \frac{EL(s)}{Q(s)}X_0 & for \quad \theta_0 \leq \theta_0^{c1}(g_Q) \\ V_1(\theta_0,0,s) = & \frac{EL(s)}{Q(s)}X_1(\theta_0,g_Q) & for \quad \theta_0^{c1}(g_Q) < \theta_0 < \theta_0^{c2}(g_Q) \\ V_2(\theta_0,0,s) = & \frac{EL(s)}{Q(s)}X_2(\theta_0,g_Q) & for \quad \theta_0 \geq \theta_0^{c2}(g_Q) \end{array}, \tag{37}$$

and

$$\begin{array}{lll} X_0 & = & [s_F c_F - s_D c_D] \\ X_1(\theta_0,g_Q) & = & \left[\begin{array}{l} \frac{\theta_0}{\sigma}\frac{1-\exp\left[-[\rho-g_L+g_Q-g_\theta]\tilde{T}(\theta_0,g_Q)\right]}{\rho-g_L+g_Q-g_\theta} \\ +\exp\left[-[\rho-g_L+g_Q]\tilde{T}(\theta_0,g_Q)\right][s_F c_F - s_D c_D] \end{array}\right] \\ X_2(\theta_0,g_Q) & = & \frac{\theta_0}{\sigma}\frac{1-\exp\left[-[\rho-g_L+g_Q-g_\theta]\bar{T}\right]}{\rho-g_L+g_Q-g_\theta} + \exp\left[-[\rho-g_L+g_Q]\bar{T}\right][s_F c_F - s_D c_D] \end{array},$$

where θ_0^{c1}, θ_0^{c2}, and $\tilde{T}(\theta_0,g_Q)$ satisfy (36). From these equations, we can obtain some properties with respect to the value of incumbents. First, we can confirm $\frac{\partial X_0}{\partial \theta_0}=0$, $\frac{\partial X_0}{\partial g_Q}=0$, $\frac{\partial^2 X_0}{\partial \theta_0 \partial g_Q}=0$, $\frac{\partial X_0}{\partial s_D}<0$, $\frac{\partial X_0}{\partial s_F}>0$, $\frac{\partial^2 X_0}{\partial \theta_0 \partial s_D}=0$, $\frac{\partial^2 X_0}{\partial \theta_0 \partial s_F}=0$, and $\frac{\partial X_0}{\partial \bar{T}}=0$, and interpret these results as follows. All incumbents drawing $\theta_0 \leq \theta_0^{c1}$ decide not to produce differentiated goods and conduct disruptive innovation immediately. Thus, there is no difference between incumbents drawing $\theta_0 \leq \theta_0^{c1}$ (: $\frac{\partial X_0}{\partial \theta_0}=0$, $\frac{\partial^2 X_0}{\partial \theta_0 \partial g_Q}=0$, $\frac{\partial^2 X_0}{\partial \theta_0 \partial s_D}=0$). Because g_Q does not affect the expected net benefit from disruptive innovation as long as the timing of disruptive innovation is immediately after their entry, it does not affect the value of incumbents drawing $\theta_0 \leq \theta_0^{c1}$ (: $\frac{\partial X_0}{\partial g_Q}=0$). A subsidy (tax) for disruptive innovation by incumbents and a tax (subsidy) for innovation by entrants increases (decreases) the expected net benefit from disruptive innovation. Thus, a marginal decrement (increment) of s_D and a marginal increment (decrement) of s_F increases the value of incumbents drawing $\theta_0 \leq \theta_0^{c1}$ (: $\frac{\partial X_0}{\partial s_D}<0$, $\frac{\partial X_0}{\partial s_F}>0$). Because incumbents drawing $\theta_0 \leq \theta_0^{c1}$ conduct disruptive innovation before patent protection expires, a marginal change of patent length has no direct

[28] The reason $\frac{\partial \theta_0^{c2}}{\partial \bar{T}}>0$ is trivial is as follows. We suppose that the incumbent drawing θ_{0a} conducts disruptive innovation as soon as $\bar{T}$ time has passed after entry. In this case, $\theta_{0a}=\theta_a^{c2}$ is satisfied. When patent length extends $\bar{T}'$ from $\bar{T}$, what happens? Because $\frac{\partial \tilde{T}(\theta_0,g_Q)}{\partial \bar{T}}=0$, the incumbent drawing θ_{0a} continues to conduct disruptive innovation when $\bar{T}$ time has passed after entry. That is, an incumbent drawing θ_{0a} conducts disruptive innovation before patent protection expires, $\theta_{0a}<\theta_a^{c2'}$.

effect on their value (: $\frac{\partial X_0}{\partial T} = 0$).

Second, we can confirm $\frac{\partial X_1(\theta_0, g_Q)}{\partial \theta_0} > 0$, $\frac{\partial X_1(\theta_0, g_Q)}{\partial \tilde{T}} = 0$, $\frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} < 0$, $\frac{\partial^2 X_1(\theta_0, g_Q)}{\partial \theta_0 \partial g_Q} < 0$, $\frac{\partial X_1(\theta_0, g_Q)}{\partial s_D} < 0$, $\frac{\partial X_1(\theta_0, g_Q)}{\partial s_F} > 0$, $\frac{\partial^2 X_1(\theta_0, g_Q)}{\partial \theta_0 \partial s_D} > 0$, $\frac{\partial^2 X_1(\theta_0, g_Q)}{\partial \theta_0 \partial s_F} < 0$, and $\frac{\partial X_1(\theta_0, g_Q)}{\partial T} = 0$, and we interpret these results as follows. Incumbents drawing $\theta_0^{c1} \leq \theta_0 \leq \theta_0^{c2}$ enjoy monopoly profit until they conduct disruptive innovation, which is optimally chosen to maximize their value. A marginal increment of θ_0 increases discounted instantaneous monopoly profit for any period, which increases the value of incumbents drawing $\theta_0^{c1} \leq \theta_0 \leq \theta_0^{c2}$ (: $\frac{\partial X_1(\theta_0, g_Q)}{\partial \theta_0} > 0$). A marginal increment of $\tilde{T}$ extends the period enjoying monopoly profit, which increases the value of incumbents. However, a marginal increment of $\tilde{T}$ decreases the expected discounted net benefit from disruptive innovation. From (13), $\tilde{T}$ is determined to offset these two opposite effects. Therefore, a marginal increment of $\tilde{T}$ has no effect on X_1 (: $\frac{\partial X_1(\theta_0, g_Q)}{\partial \tilde{T}} = 0$). A high growth rate of Q decreases the value of incumbents through two channels. On one hand, it makes the relative obsolescence rate high, and then, the discounted instantaneous monopoly profit decreases. On the other hand, it makes the growth rate of the expected discounted net benefit from disruptive innovation low, and then, the expected discounted net benefit from disruptive innovation decreases. Thus, through these tow same direction effects, a marginal increment of g_Q decreases the value of incumbents drawing $\theta_0^{c1} \leq \theta_0 \leq \theta_0^{c2}$ (: $\frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} < 0$). A marginal increment of g_Q, on one hand, decreases the discounted instantaneous monopoly profit of incumbents drawing high θ_0, larger relative to incumbents drawing low θ_0. A marginal increment of g_Q, on the other hand, decreases the discounted net benefit from the disruptive innovation of incumbents drawing high θ_0, larger relative to incumbents drawing low θ_0. Thus, a marginal increment of g_Q decreases the value of incumbents drawing high θ_0, larger relative to incumbents drawing low θ_0 (: $\frac{\partial^2 X_1(\theta_0, g_Q)}{\partial \theta_0 \partial g_Q} < 0$). For the same reason as X_0, a marginal decrement (increment) of s_D and a marginal increment (decrement) of s_F increases the value of incumbents (: $\frac{\partial X_1(\theta_0, g_Q)}{\partial s_D} < 0$, $\frac{\partial X_1(\theta_0, g_Q)}{\partial s_F} > 0$). A marginal decrement (increment) of s_D and a marginal increment (decrement) of s_F increases (decreases) the expected discounted net benefit from disruptive innovation, which increases (decreases) the value of incumbents drawing high θ_0, smaller relative to incumbents drawing low θ_0 because incumbents drawing high θ_0 tend to set the timing of disruptive innovation later, and then obtain a smaller benefit from disruptive innovation relative to incumbents drawing low θ_0 (: $\frac{\partial^2 X_1(\theta_0, g_Q)}{\partial \theta_0 \partial s_D} > 0$, $\frac{\partial^2 X_1(\theta_0, g_Q)}{\partial \theta_0 \partial s_F} < 0$). For the same reason as X_0, a marginal change of patent length has no direct effect on their value (: $\frac{\partial X_1(\theta_0, g_Q)}{\partial T} = 0$).

Finally, we can confirm $\frac{\partial X_2(\theta_0,g_Q)}{\partial \theta_0} > 0$, $\frac{\partial X_2(\theta_0,g_Q)}{\partial g_Q} < 0$, $\frac{\partial^2 X_2(\theta_0,g_Q)}{\partial \theta_0 \partial g_Q} < 0$, $\frac{\partial X_2(\theta_0,g_Q)}{\partial s_D} < 0$, $\frac{\partial X_2(\theta_0,g_Q)}{\partial s_F} > 0$, $\frac{\partial^2 X_2(\theta_0,g_Q)}{\partial \theta_0 \partial s_D} = 0$, $\frac{\partial^2 X_2(\theta_0,g_Q)}{\partial \theta_0 \partial s_F} = 0$, and $\frac{\partial X_2(\theta_0,g_Q)}{\partial \bar{T}} > 0$, and interpret these results as follows. Incumbents drawing $\theta_0 > \theta_0^{c2}$ enjoy monopoly profits until their patent protection expires. They conduct innovation immediately after their patent protection expires, irrespective of θ_0. A marginal increment of θ_0 increases the discounted instantaneous monopoly profit for any period, and then increases the incumbents' value drawing $\theta_0 > \theta_0^{c2}$ (: $\frac{\partial X_2(\theta_0,g_Q)}{\partial \theta_0} > 0$). For the same reason as X_1, a marginal increment of g_Q decreases the instantaneous profit and the value of incumbents drawing high θ_0, larger relative to incumbents drawing low θ_0 (: $\frac{\partial X_2(\theta_0,g_Q)}{\partial g_Q} < 0$, $\frac{\partial^2 X_2(\theta_0,g_Q)}{\partial \theta_0 \partial g_Q} < 0$). For the same reason as X_0 and X_1, a marginal decrement (increment) of s_D and a marginal increment (decrement) of s_F increase the value of incumbents (: $\frac{\partial X_2(\theta_0,g_Q)}{\partial s_D} < 0$, $\frac{\partial X_2(\theta_0,g_Q)}{\partial s_F} > 0$). Because incumbents drawing $\theta_0 > \theta_0^{c2}$ conduct disruptive innovation after $\bar{T}$ time has passed from the entry, irrespective of θ_0, there is no difference in expected discounted net benefit from disruptive innovation between incumbents drawing $\theta_0 > \theta_0^{c2}$. Thus, a marginal change of s_D and s_D has the same effect on the value of incumbents drawing $\theta_0 > \theta_0^{c2}$ (: $\frac{\partial^2 X_2(\theta_0,g_Q)}{\partial \theta_0 \partial s_D} = 0$, $\frac{\partial^2 X_2(\theta_0,g_Q)}{\partial \theta_0 \partial s_F} = 0$). Because a marginal increment of $\bar{T}$ extends periods enjoying monopoly profit, it increases the value of incumbents drawing $\theta_0 > \theta_0^{c2}$ (: $\frac{\partial X_2(\theta_0,g_Q)}{\partial \bar{T}} > 0$). [29]

Substituting (31) and (37) into (21), we obtain:

$$\begin{aligned}
&\int_{\theta_0^{\min}}^{\theta_0^{c1}} X_0 dG(\theta_0) + \int_{\theta_0^{c1}}^{\theta_0^{c2}} X_1 dG(\theta_0) + \int_{\theta_0^{c2}}^{\theta_0^{\max}} X_2 dG(\theta_0) = s_F c_F \quad if \quad \theta_0^{\min} \leq \theta_0^{c1} \\
&\int_{\theta_0^{\min}}^{\theta_0^{c2}} X_1 dG(\theta_0) + \int_{\theta_0^{c2}}^{\theta_0^{\max}} X_2 dG(\theta_0) = s_F c_F \quad if \quad \theta_0^{c1} < \theta_0^{\min} \leq \theta_0^{c2} \\
&\int_{\theta_0^{\min}}^{\theta_0^{\max}} X_2 dG(\theta_0) = s_F c_F \quad if \quad \theta_0^{c2} < \theta_0^{\min}
\end{aligned} . \tag{38}$$

Equation (38) determines the equilibrium value of g_Q.

In the following, we investigate the properties of the equilibrium value of g_Q. Because $\frac{\partial X_0}{\partial g_Q} = 0$, $\frac{\partial X_1}{\partial g_Q} < 0$, $\frac{\partial X_2}{\partial g_Q} < 0$, $\frac{\partial X_0}{\partial s_D} < 0$, $\frac{\partial X_1}{\partial s_D} < 0$ and $\frac{\partial X_2}{\partial s_D} < 0$ are satisfied, we can obtain:

$$\frac{dg_Q}{ds_D} = -\frac{\int_{\theta_0^{\min}}^{\theta_0^{c1}(g_Q)} \frac{\partial X_0}{\partial s_D} dG(\theta_0) + \int_{\theta_0^{c1}(g_Q)}^{\theta_0^{c2}(g_Q)} \frac{\partial X_1(\theta_0,g_Q)}{\partial s_D} dG(\theta_0) + \int_{\theta_0^{c2}(g_Q)}^{\theta_0^{\max}} \frac{\partial X_2(\theta_0,g_Q)}{\partial s_D} dG(\theta_0)}{\int_{\theta_0^{c1}(g_Q)}^{\theta_0^{c2}(g_Q)} \frac{\partial X_1(\theta_0,g_Q)}{\partial g_Q} dG(\theta_0) + \int_{\theta_0^{c2}(g_Q)}^{\theta_0^{\max}} \frac{\partial X_2(\theta_0,g_Q)}{\partial g_Q} dG(\theta_0)} < 0. \tag{39}$$

[29] In appendix C, we confirm these results analytically.

This condition implies that a subsidy (tax) for disruptive innovation by incumbents increases (decreases) the growth rate of the improvement level of differentiated goods. The intuition is as follows. A subsidy (tax) for disruptive innovation by incumbents increases (decreases) the expected discounted net benefit from disruptive innovation for all incumbents, and then the expected initial value of incumbents increases with a given g_Q. An increment (decrement) of expected initial value of incumbents accelerates (decelerates) entry, which continues until the free entry condition is satisfied. An acceleration (deceleration) of entry raises (lowers) the growth rate of the number of differentiated goods g_n, which raises (lowers) the growth rate of the improvement level of differentiated goods g_Q.

Because $\frac{\partial X_0}{\partial g_Q} = 0$, $\frac{\partial X_1}{\partial g_Q} < 0$, $\frac{\partial X_2}{\partial g_Q} < 0$, $c_F = \frac{\partial X_0}{\partial s_F} \geq \frac{\partial X_1(\theta_0, g_Q)}{\partial s_F} \geq \frac{\partial X_2(\theta_0, g_Q)}{\partial s_F}$, and $\bar{T} > 0$ are satisfied, we can obtain:[30]

$$\frac{dg_Q}{ds_F} = \frac{c_F - \int_{\theta_0^{\min}}^{\theta_0^{c1}(g_Q)} \frac{\partial X_0}{\partial s_F} dG(\theta_0) - \int_{\theta_0^{c1}(g_Q)}^{\theta_0^{c2}(g_Q)} \frac{\partial X_1(\theta_0, g_Q)}{\partial s_F} dG(\theta_0) - \int_{\theta_0^{c2}(g_Q)}^{\theta_0^{\max}} \frac{\partial X_2(\theta_0, g_Q)}{\partial s_F} dG(\theta_0)}{\int_{\theta_0^{c1}(g_Q)}^{\theta_0^{c2}(g_Q)} \frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} dG(\theta_0) + \int_{\theta_0^{c2}(g_Q)}^{\theta_0^{\max}} \frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q} dG(\theta_0)} < 0. \tag{40}$$

This condition implies that a subsidy (tax) for innovation by entrants increases (decreases) the growth rate of the improvement level of differentiated goods. The intuition is as follows. A subsidy (tax) for innovation by entrants affects equation (38) through two channels. On one hand, it decreases (increases) the expected discounted net benefit from disruptive innovation for all incumbents, and then the expected initial value of incumbents, EV, decreases (increases) with a given g_Q. On the other hand, it decreases (increases) the cost of initial investment for entrants, $s_F C_F$. Because the latter effect dominates the former, the net expected benefit from entry, $EV - s_F C_F$, increases (decreases) with a given g_Q. This accelerates (decelerates) entry, which continues until the free entry condition is satisfied. An acceleration (deceleration) of entry raises (lowers) the growth rate of the number of differentiated goods g_n, which raises (lowers) the growth rate of the improvement level of differentiated goods g_Q.

[30] When $\bar{T} > 0$, $c_F > \frac{\partial X_1(\theta_0, g_Q)}{\partial s_F}$ and $c_F > \frac{\partial X_2(\theta_0, g_Q)}{\partial s_F}$ are satisfied. Then, the numerator of this equation is unambiguously positive.

Because $\frac{\partial X_0}{\partial g_Q} = 0$, $\frac{\partial X_1}{\partial g_Q} < 0$, $\frac{\partial X_2}{\partial g_Q} < 0$, $\frac{\partial X_0}{\partial \bar{T}} = 0$, $\frac{\partial X_1}{\partial \bar{T}} = 0$, and $\frac{\partial X_2}{\partial \bar{T}} > 0$ are satisfied, we can obtain:

$$\frac{dg_Q}{d\bar{T}} = \frac{-\int_{\theta_0^{c2}(g_Q)}^{\theta_0^{\max}} \frac{\partial X_2(\theta_0, g_Q)}{\partial \bar{T}} dG(\theta_0)}{\int_{\theta_0^{c1}(g_Q)}^{\theta_0^{c2}(g_Q)} \frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} dG(\theta_0) + \int_{\theta_0^{c2}(g_Q)}^{\theta_0^{\max}} \frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q} dG(\theta_0)} > 0. \tag{41}$$

This condition implies that extending (shortening) the patent length increases (decreases) the growth rate of the improvement level of differentiated goods. The intuition is as follows. Extending (shortening) the patent length extends (shortens) the period incumbents drawing $\theta_0 > \theta_0^{c2}$ enjoy monopoly profit, and the value of incumbents drawing $\theta_0 > \theta_0^{c2}$ increases (decreases), which increases (decreases) the expected initial value of incumbents with a given g_Q. An increment (decrement) of expected initial value of incumbents accelerates (decelerates) entry, which continues until the free entry condition is satisfied. An acceleration (deceleration) of entry raises (lowers) the growth rate of the number of differentiated goods g_n, which raises (lowers) the growth rate of the improvement level of differentiated goods g_Q.

From (39), (40), and (41), we can write the equilibrium value of g_Q, which is determined by (38), as a function of policy variables:

$$g_Q^* = g_Q\left(s_D, s_F, \bar{T}\right), \tag{42}$$

where $\frac{\partial g_Q\left(s_D, s_F, \bar{T}\right)}{\partial s_D} < 0$, $\frac{\partial g_Q\left(s_D, s_F, \bar{T}\right)}{\partial s_F} < 0$, and $\frac{\partial g_Q\left(s_D, s_F, \bar{T}\right)}{\partial \bar{T}} > 0$.

Substituting (33) and (42) into (30), we can confirm that the growth rate of this economy increases proportionally with g_Q, and is expressed as:

$$g_u^* = \frac{1}{\sigma - 1} g_Q^*. \tag{43}$$

Substituting (42) into (36), we obtain the equilibrium value of θ_0^{c1}, θ_0^{c2}, and $\tilde{T}(\theta_0)$. In the following, we investigate the property of the equilibrium value of θ_0^{c1}, θ_0^{c2}, and $\tilde{T}(\theta_0)$.

Differentiating (36) with s_D, we can obtain:[31]

$$\begin{array}{rcll} \frac{d\theta_0^{c1}(g_Q)}{ds_D} & = & \sigma\frac{\partial g_Q\left(s_D,s_F,\bar{T}\right)}{\partial s_D}\left[s_F c_F - s_D c_D\right] - \sigma\left[r+g_Q\right]c_D & < \; 0 \\ \frac{d\theta_0^{c2}(g_Q)}{ds_D} & = & \exp\left[\left[-g_\theta\right]\bar{T}\right]\frac{d\theta_0^{c1}(g_Q)}{ds_D} & < \; 0 \\ \frac{d\tilde{T}(\theta_0,g_Q)}{ds_D} & = & -\frac{1}{[-g_\theta]\theta_0^{c1}}\frac{d\theta_0^{c1}(g_Q)}{ds_D} & > \; 0 \end{array} \cdot \tag{44}$$

This condition implies that a subsidy (tax) for disruptive innovation by incumbents moves the timing of disruptive innovation by each incumbent earlier (later). The intuition is as follows. On one hand, a subsidy (tax) for disruptive innovation by incumbents increases (decreases) the expected discounted net benefit from disruptive innovation, and then the marginal loss from putting off disruptive innovation increases (decreases). On the other hand, a subsidy (tax) for disruptive innovation by incumbents stimulates (mitigates) innovation by entrants, which decreases (increases) the growth rate of the expected net benefit from disruptive innovation, g_C, and then the marginal loss from putting off disruptive innovation increases (decreases). Because these two effects are in same direction, we can obtain the above unambiguous result.

As shown in Appendix D, we obtain:[32]

$$\begin{array}{rcll} \frac{d\theta_0^{c1}(g_Q)}{ds_F} & = & \sigma\frac{\partial g_Q\left(s_D,s_F,\bar{T}\right)}{\partial s_F}\left[s_F c_F - s_D c_D\right] + \sigma\left[r+g_Q\right]c_F & > \; 0 \\ \frac{d\theta_0^{c2}(g_Q)}{ds_F} & = & \exp\left[\left[-g_\theta\right]\bar{T}\right]\frac{d\theta_0^{c1}(g_Q)}{ds_F} & > \; 0 \\ \frac{d\tilde{T}(\theta_0,g_Q)}{ds_F} & = & -\frac{1}{[-g_\theta]\theta_0^{c1}}\frac{d\theta_0^{c1}(g_Q)}{ds_F} & < \; 0 \end{array} \cdot \tag{45}$$

This condition implies that a subsidy (tax) for innovation by entrants moves the timing of disruptive innovation by each incumbent later (earlier). The intuition is as follows. On one hand, a subsidy (tax) for innovation by entrants decreases (increases) the expected discounted net benefit from disruptive innovation, and thus, the marginal loss from putting off disruptive innovation decreases (increases). On the other hand, a subsidy (tax) for innovation by entrants stimulates (mitigates) innovation by entrants, which decreases (increases) the growth rate of the expected net benefit from disruptive innovation, g_C, and thus, the marginal loss from putting off disruptive innovation increases (decreases). Although these two effects are in opposite directions, the former effect dominates the latter; thus, we can obtain an

[31] Since $\frac{\partial g_Q\left(s_D,s_F,\bar{T}\right)}{\partial s_D} < 0$, $\frac{d\theta_0^{c1}(g_Q)}{ds_D}$ is unambiguously negative. Because $\theta_0^{c2}\left(g_Q\right)$ is an increasing function of $\theta_0^{c1}\left(g_Q\right)$ and $\tilde{T}\left(\theta_0, g_Q\right)$ is a decreasing function of $\theta_0^{c1}\left(g_Q\right)$, $\frac{d\theta_0^{c2}(g_Q)}{ds_D}$ is negative and $\frac{d\tilde{T}(\theta_0,g_Q)}{ds_D}$ is positive.

[32] Since $\frac{\partial\theta_0^{c1}(g_Q)}{\partial g_Q} < 0$, the sign of $\frac{d\theta_0^{c1}(g_Q)}{ds_F}$ seems to be ambiguous. However, we can obtain an unambiguous result by calculating in detail.

unambiguous result. [33]

Because $\frac{\partial g_Q\left(s_F,s_D,\bar{T}\right)}{\partial \bar{T}} > 0$, we can obtain:

$$\begin{array}{rclcl} \frac{d\theta_0^{c1}(g_Q)}{d\bar{T}} & = & \sigma \frac{\partial g_Q\left(s_D,s_F,\bar{T}\right)}{\partial \bar{T}} \left[s_F c_F - s_D c_D\right] & > & 0 \\ \frac{d\theta_0^{c2}(g_Q)}{d\bar{T}} & = & \exp\left[\left[-g_\theta\right]\bar{T}\right] \frac{d\theta_0^{c1}(g_Q)}{d\bar{T}} & > & 0 \\ \frac{d\tilde{T}(\theta_0,g_Q)}{d\bar{T}} & = & -\frac{1}{[-g_\theta]\theta_0^{c1}} \frac{d\theta_0^{c1}(g_Q)}{d\bar{T}} & < & 0 \end{array} . \tag{46}$$

This condition implies that extending (shortening) the patent length moves the timing of disruptive innovation by each incumbent forward (later). This result is somewhat counterintuitive because a longer patent length makes incumbents decide front-loading switch of their current technology. However, the interpretation is simple: extending the patent length stimulates innovation by entrants, which decreases the growth rate of the expected net benefit from disruptive innovation, g_C, and then the marginal loss from putting off disruptive innovation increases.[34]

From (44), (45), and (46), we can write the equilibrium value of θ_0^{c1} as a function of the policy variable:

$$\theta_0^{c1*} = \theta_0^{c1}\left(s_D, s_F, \bar{T}\right), \tag{47}$$

where $\frac{\partial \theta_0^{c1}\left(s_D,s_F,\bar{T}\right)}{\partial s_D} < 0$, $\frac{\partial \theta_0^{c1}\left(s_D,s_F,\bar{T}\right)}{\partial s_F} > 0$ and $\frac{\partial \theta_0^{c1}\left(s_D,s_F,\bar{T}\right)}{\partial \bar{T}} > 0$. [35]

We summarize the above discussion in the following proposition:

Proposition 3.2. *When we specify the innovation technology as $C_F(t) = \frac{E(t)L}{Q(t)} c_F$, and $C_D(t) = \frac{E(t)L}{Q(t)} c_D$, we obtain the following results.*

- *A subsidy (tax) for disruptive innovation by incumbents stimulates (mitigates) innovation by entrants, and then increases (decreases) the growth rate of the economy.*
- *A subsidy (tax) for disruptive innovation by incumbents moves the timing of disruptive innovation by each incumbent earlier (later), and the number of incumbents conducting disruptive innovation*

[33] This result is consistent with the result of Iwaisako and Ohki (2019), in which a subsidy (tax) for innovation by followers makes innovation by each individual leader small. They interpreted this result as follows. On one hand, a subsidy for followers decreases leaders' expected discounted net benefit from innovation. On the other hand, a subsidy for innovation by followers increases replacement risk, which decreases current value, and then the Arrow effect weakens. Although these two effects are in opposite directions, the former effect dominates the latter; thus, they show that the effect of a subsidy for innovation by followers on the "individual effect" is unambiguously negative.

[34] This result does not mean that all incumbents move the timing of disruptive innovation forward. Incumbents drawing initial adjusted quality $\theta_0 > \theta_0^{c2}$ conduct disruptive innovation when their patent protection expires. Then, their disruptive innovation is moved later.

[35] We can write the equilibrium value of θ_0^{c2} and $\tilde{T}\left(\theta_0, g_Q\right)$ as a function of $\theta_0^{c1}\left(s_D, s_F, \bar{T}\right)$: $\theta_0^{c2*} = \exp\left[\left[-g_\theta\right]\bar{T}\right]\theta_0^{c1*}$, and $\tilde{T}^* = \frac{1}{-g_\theta} \ln\left[\frac{\theta_0}{\theta_0^{c1*}}\right]$.

before their patent protection expires increases (decreases).

- *A subsidy (tax) for innovation by entrants stimulates (mitigates) innovation by entrants, and then increases (decreases) the growth rate of the economy.*

- *A subsidy (tax) for innovation by entrants moves the timing of disruptive innovation by each incumbent later (earlier), and the number of incumbents conducting disruptive innovation before their patent protection expires decreases (increases).*

- *Extension (shortening) of patent length stimulates (mitigates) innovation by entrants, and then increases (decreases) the growth rate of the economy.*

- *Extending (shortening) the patent length moves the timing of disruptive innovation by each incumbent earlier (later), and increases (decreases) the number of incumbents conducting disruptive innovation before their patent protection expires.*

Next, we derive the equilibrium values of the average quality of differentiated goods, Θ. From the first row of (27) and (47), if $\theta_0^{c1} \geq \theta_0^{min}$ is satisfied, we obtain the equilibrium value of the average quality of differentiated goods as:

$$\Theta^* = \frac{k}{k-1}\theta_0^{c1*}. \tag{48}$$

, which is an increasing function of the expectation of initial adjusted quality, $k/(k-1)$, and is an increasing function of the equilibrium value of the critical level of initial adjusted quality in terms of whether the incumbent conducts production, θ_0^{c1}. [36] The intuition is as follows. A high expectation of initial adjusted quality makes adjusted quality with given g_θ high at any time. High θ_0^{c1}, which means low $\tilde{T}$, denotes that incumbents conduct disruptive innovation earlier, and then incumbents having low adjusted quality tend to exit the market earlier. Thus, high adjusted quality differentiated goods tend to remain, which raises the average quality of remaining differentiated goods.

Because we have already confirmed $\frac{\partial \theta_0^{c1*}}{\partial s_D} < 0$, $\frac{\partial \theta_0^{c1*}}{\partial s_F} > 0$, and $\frac{\partial \theta_0^{c1*}}{\partial T} > 0$, we can confirm $\frac{\partial \Theta^*}{\partial s_D} < 0$, $\frac{\partial \Theta^*}{\partial s_F} > 0$, and $\frac{\partial \Theta^*}{\partial T} > 0$. Then, we obtain following proposition:

[36] If $\theta_0^{c1} \geq \theta_0^{min}$ is not satisfied, the equilibrium values of the average quality of differentiated goods is determined from the second or third row of (27). We refer to this case when we conduct numerical analysis.

Proposition 3.3. *When we specify the innovation technology as* $C_F(t) = \frac{E(t)L}{Q(t)}c_F$, *and* $C_D(t) = \frac{E(t)L}{Q(t)}c_D$, *and when the average quality of differentiated goods is sufficiently large, we get the following results.*

- *A subsidy (tax) for disruptive innovation by incumbents stimulates (mitigates) disruptive innovation by incumbents, and then increases (decreases) the average quality of differentiated goods.*
- *A subsidy (tax) for innovation by entrants mitigates (stimulates) disruptive innovation by incumbents, and then decreases (increases) the average quality of differentiated goods.*
- *Extending (shortening) patent length stimulates (mitigates) disruptive innovation by incumbents, and then increases (decreases) the average quality of differentiated goods.*

Substituting the equilibrium value of g_Q, ν, and Θ into (32), we obtain the equilibrium value of consumption per capita, E. From (32), consumption per capita is affected by policy variables through three channels. First, consumption per capita is a decreasing function of g_Q. An increment of g_Q requires resources devoted to innovation activity by entrants, which decreases resources devoted to production activity. Second, consumption per capita is a decreasing function of ν. An increment of ν requires resources devoted to innovation activity by incumbents, which decreases resources devoted to production activity. Third, consumption per capita is an increasing function of Θ. An increment of Θ strengthens the positive externality of innovation activity, which requires fewer resources with a given level of innovation activity by entrants and incumbents. Then, resources devoted to production activity increase with an increment of Θ. From these three effects, it is difficult to confirm analytically the effects of policy variables on consumption per capita. In the following, we conduct a numerical analysis to confirm the effects of policy variables visually.

Following precedent studies, we set parameter values $\rho = 0.04$, $g_L = 0.01$, and $\sigma = 6$.[37] In addition, we set $c_D = 1$ as the standard relative to c_F.

In Figure 1, we confirm the effect of changing policy variables with different efficiency of innovation by entrants, c_F. We examine the case where $c_F = 3$, $c_F = 5$, and $c_F = 7$ while setting $g_\theta = -0.02$, $k = 2$. [38] The first row of Figure 1 expresses the relationship between the growth rate of Q, g_Q, and

[37] From Jones and Williams (2000), we interpret the interest rate as the expected internal rate of return to R&D projects, and we set $\rho = 0.04$. From the World Development Indicators (World Bank 2007), the average annual rate of population growth in the US between 1975 and 2007 was around 1%, so we set $g_L = 0.01$. From Basu (1996) and Norrbin (1993), the average price markup over marginal cost, $\frac{\sigma}{\sigma-1}$, has been estimated as ranging between 1.05 and 1.4, so we set $\sigma = 6$, which

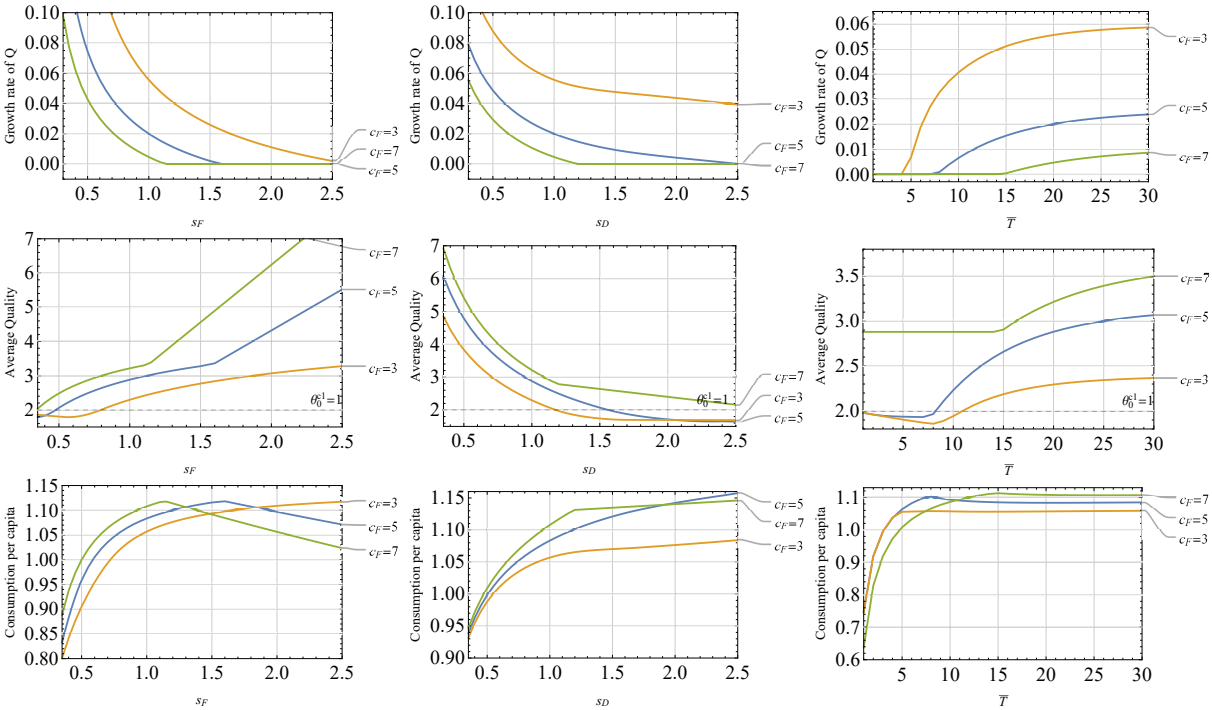

Figure 1: A numerical analysis changing c_F

policy variables. First, we can confirm that g_Q is a decreasing function of s_F and s_D, and an increasing function of $\bar{T}$, which is consistent with proposition 3.2. Second, g_Q increases rapidly when s_F and s_D are very low; however, g_Q increases slowly when $\bar{T}$ is very large. [39] Third, for any policy variable, the growth rate of Q is large when c_F is low. This means that a high c_F makes it expensive to enter the market, which discourages entry. Discouraging entry reduces the growth rate of the small number of differentiated goods, which reduces g_Q.

The second row of Figure 1 expresses the relationship between the average quality of differentiated goods, Θ, and policy variables. The broken line of $\theta_0^{c1} = 1$ helps us confirm whether the equilibrium is $\theta_0^{c1} \geq \theta_0^{min}$ or $\theta_0^{c1} < \theta_0^{min}$. [40] If the average quality is greater (smaller) than the broken line, $\theta_0^{c1} \geq \theta_0^{min}$ ($\theta_0^{c1} < \theta_0^{min}$) is satisfied. First, if $\theta_0^{c1} \geq \theta_0^{min}$ is satisfied, we can confirm that the average quality, Θ, is a decreasing function of s_D and an increasing function of s_F. This result is consistent with proposition

means the markup is 1.2.

[38] The first column examines the effect of a subsidy (tax) on innovation for entrants, the second examines the effect of a subsidy (tax) on disruptive innovation for incumbents, and the third examines the effect of patent length.

[39] A subsidy for innovation for incumbents and entrants increases the expected profit from entry, and thus, the expected value of incumbents must decrease through an increment of g_Q to satisfy the free entry condition. Because $\frac{\partial X_1}{\partial g_Q} < 0$, $\frac{\partial X_2}{\partial g_Q} < 0$, $\frac{\partial}{\partial g_Q}\frac{\partial X_1}{\partial g_Q} > 0$, and $\frac{\partial}{\partial g_Q}\frac{\partial X_2}{\partial g_Q} > 0$ are satisfied, the expected value of incumbents decreases less when g_Q is large than when g_Q is small. When s_F and s_D are very low, g_Q is large, and then g_Q must increase more to satisfy the free entry condition. High $\bar{T}$ increases the expected value of incumbents with given g_Q through extending the duration of instantaneous profit. This benefit is enjoyed only by incumbents drawing a high initial adjusted quality. Incumbents enjoying this effect decrease when $\bar{T}$ is very high, and the effect that high $\bar{T}$ enlarges the expected value of incumbents is reduced when $\bar{T}$ is very high.

[40] Since we set $k = 2$, $\theta_0^{c1} = 1$ is satisfied when $\Theta = 2$, from (48).

3.3. Second, if $\theta_0^{c1} \geq \theta_0^{min}$ is satisfied, Θ, is an increasing function of $\bar{T}$ when $g_Q > 0$ is satisfied, however, is independent of $\bar{T}$ when $g_Q = 0$. An increment of $\bar{T}$ makes disruptive innovation active through a increment of g_Q, and the this effect does not work as long as $g_Q = 0$. Third, if $\theta_0^{c1} < \theta_0^{min}$ is satisfied, Θ can increase despite a small s_F. This is because the average quality of any vintage decreases over time if $\theta_0^{c1} < \theta_0^{min}$. Small s_F leads to high g_Q, which increases the ratio of new differentiated goods, which have a high quality on average. If this effect dominates the effect that small s_F leads to small disruptive innovation, the average quality can be raised with a small s_F. We can confirm this case when s_F is low in the second column of the left panel. Fourth, if $\theta_0^{c2} < \theta_0^{min}$ is satisfied, Θ can increase with a small $\bar{T}$. When $\bar{T}$ is small, θ_0^{c2} becomes small, and tends to be less than θ_0^{min}. In this case, all incumbents conduct no disruptive innovation until their patent protection expires, and then a small $\bar{T}$ makes the timing of disruptive innovation for all incumbents earlier. An earlier technological switch for all incumbents raises the average quality of differentiated goods. We can confirm this case when $\bar{T}$ is low in the second column of the right panel. Fifth, if $\theta_0^{c1} \geq \theta_0^{min}$ is satisfied, Θ is large when c_F is high for any policy variables. On one hand, a high c_F makes the expected return to disruptive innovation high and stimulates disruptive innovation. On the other hand, a high c_F leads to a small g_Q, which discourages disruptive innovation. These two effects are in the opposite direction; however, our numerical result shows that the former effect dominates the latter. Sixth, if $\theta_0^{c1} < \theta_0^{min}$ is satisfied, Θ can be greater when c_F is small. When $\theta_0^{c1} < \theta_0^{min}$, the average quality is an increasing function of g_Q. Then, the effect that a high c_F leads to a small g_Q, which decreases Θ with a given level of disruptive innovation, is added. If this additive effect dominates the effect that a high c_F leads to active disruptive innovation with a given g_Q, Θ is large with a low c_F. We can confirm this case when s_F is low in the second column of the left panel. Seventh, if $\theta_0^{c2} < \theta_0^{min}$ and $g_Q = 0$ are satisfied, Θ is independent of c_F. If $\theta_0^{c2} < \theta_0^{min}$ is satisfied, the effect that a high c_F leads to active disruptive innovation does not work. Moreover, if $g_Q = 0$ is satisfied, the effect that a high c_F leads to a small g_Q, which decreases Θ, does not work. We can confirm this case when s_D is high in the second column of the center panel. [41]

The third row of Figure 1 expresses the relationship between consumption per capita, E, and policy variables. First, if $g_Q > 0$ is satisfied, E is an increasing function of s_F. An increment of s_F decreases g_Q and increases ν and Θ.[42] On one hand, a decrement of g_Q and an increment of Θ leads to consumption

[41] Using L'Hopital's rule in the third row of (27) when $g_n = 0$, $\Theta = \frac{k}{k-1}\frac{1}{\bar{T}}\frac{1-\exp[-g_\theta]\bar{T}}{[-g_\theta]}$ is satisfied. We can easily confirm that Θ is independent of c_F.

[42] If $\theta_0^{c1} < \theta_0^{min}$ is satisfied, an increment of s_F can decrease Θ, however, this effect is negligibly small.

per capita being large, while on the other hand, an increment of ν reduces consumption per capita. Our numerical result shows that the former effect dominates the latter. Second, if $g_Q = 0$ is satisfied, E is a decreasing function of s_F . In this case, the effect that an increment of s_F decreases g_Q does not work. Then, the effect that an increment of s_F increases Θ cannot dominate the effect that an increment of s_F increases ν, as shown in our numerical result.[43] Third, if $g_Q > 0$ is satisfied, E is an increasing function of s_D . An increment of s_D decreases g_Q, ν and Θ. On the one hand, a decrement of g_Q and ν leads to large consumption per capita, on the other hand, a decrement of Θ leads to small consumption per capita. Our numerical result shows that the former effect dominates the latter. Fourth, even if $g_Q = 0$ is satisfied, E is an increasing function of s_D. In this case, the effect that an increment of s_D decreases g_Q does not work; however, only the effect that an increment of s_D decreases ν dominates the effect that an increment of s_F decreases Θ, as shown in our numerical result. Fifth, if $g_Q > 0$ is satisfied, E temporarily decreases with high $\bar{T}$ when g_Q increases rapidly; however, E shifts from decreasing to increasing when the increment of g_Q becomes small, and E converges to a finite value. On one hand, high $\bar{T}$ leads to large g_Q and large disruptive innovation by incumbents drawing $\theta_0 < \theta_0^{c2}$, which makes consumption per capita small. On the other hand, high $\bar{T}$ leads to large Θ, and high $\bar{T}$ makes incumbents drawing $\theta_0 > \theta_0^{c2}$ put off disruptive innovation, which leads to large consumption per capita. Our numerical result shows that the former effects dominate the latter when g_Q increases rapidly; however, the former effects become weak when the increment of g_Q becomes small, and then the latter effects dominate the former. We also confirm that these two effects are balanced, and that E does not change when $\bar{T}$ is very large. Sixth, if $g_Q = 0$ is satisfied, E is an increasing function of $\bar{T}$. If $g_Q = 0$ is satisfied, the effect that high $\bar{T}$ leads to large innovation by both incumbents and entrants does not work. Then, E increases with high $\bar{T}$ through the effect that high $\bar{T}$ moves the timing of disruptive innovation for incumbents drawing $\theta_0 > \theta_0^{c2}$ later. Seventh, if $g_Q > 0$ is satisfied, E is large when c_F is high for any policy variable. High c_F requires large resources with a given g_Q, which directly makes consumption per capita small. When c_F is high, for any policy variable, g_Q is small, and ν and Θ are large. On one hand, small g_Q and large Θ make consumption per capita large. On the other hand, large ν and high c_F itself make consumption per capita small. Our numerical result shows that the former effect dominates the latter. Eighth, if $g_Q = 0$ is satisfied, E is small when c_F is high. In this case, the effect that high c_F leads to small g_Q and the

[43] When $g_Q = 0$, the labor market clearing condition is expressed as $E = \left[\frac{\sigma-1}{\sigma} + \frac{k-1}{k}\frac{[\theta_0^{c1}]^{k-1}[-kg_\theta]}{1-\exp[-[-kg_\theta]\bar{T}]}c_D\right]^{-1}$. From this function, E is a decreasing function of θ_0^{c1}.

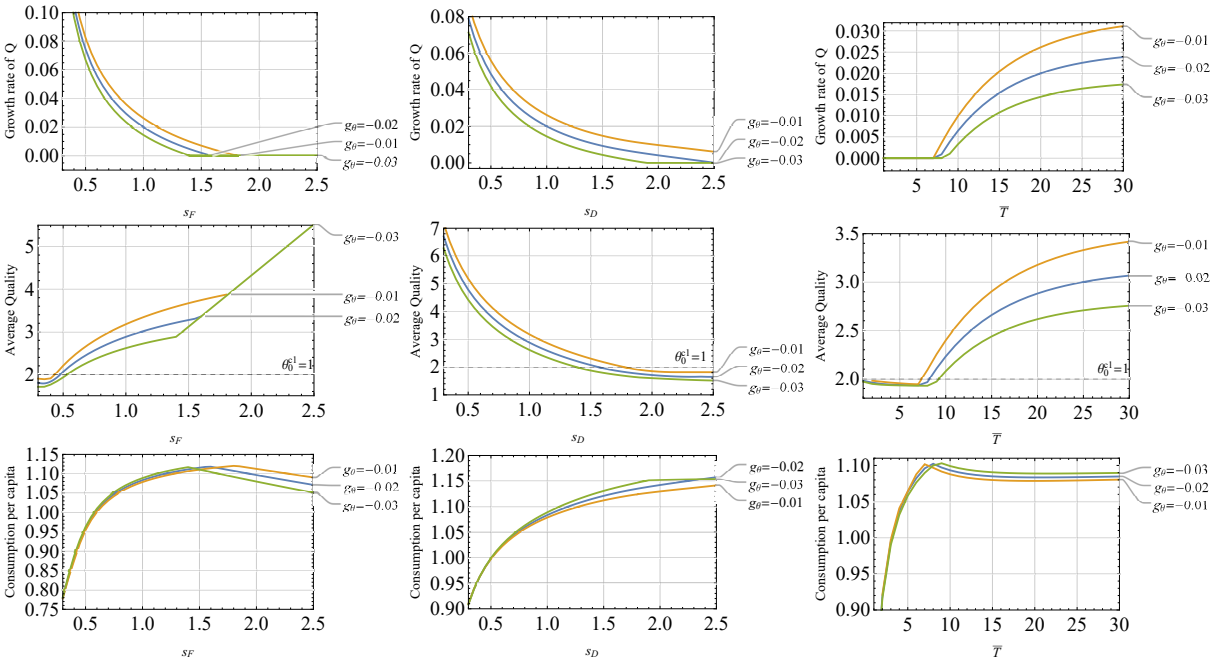

Figure 2: A numerical analysis of changing g_θ

direct resource effect of c_F do not work. Our numerical result shows that the effect that high c_F leads to large Θ cannot dominate the effect that high c_F makes large ν.

In Figure 2, we confirm the effect of changing policy variables with different growth rates of adjusted quality, g_θ. We examine the case that $g_\theta = -0.01$, $g_\theta = -0.02$, and $g_\theta = -0.03$ while setting $c_F = 5$, $k = 2$. Although a tendency of the effect of changing policy variables is similar to the above numerical analysis, there are some points worth noting. First, g_Q is small when the absolute value of g_θ is high for any policy variable. When the obsolescence rate is high, the profit of incumbents decreases rapidly, and then the expected value of incumbents with given g_Q is low, which discourage entry. Second, if $g_Q > 0$ is satisfied, Θ is small when the absolute value of g_θ is high. A high obsolescence rate makes g_Q small (g_C large), which decreases the marginal loss from putting off disruptive innovation and makes disruptive innovation small. Third, if $g_Q = 0$ and $\theta_0^{c1} \geq \theta_0^{min}$ are satisfied, Θ is not affected by g_θ. As discussed in section 2.7, the average quality of any vintage is not affected by g_θ as long as $\theta_0^{c1} \geq \theta_0^{min}$ is satisfied. We can confirm this case when s_F is high in the second column of the left panel. Fourth, if $g_Q = 0$ and $\theta_0^{c1} < \theta_0^{min}$ are satisfied, Θ is small when the absolute value of g_θ is high, which is also discussed in section 2.7. We can confirm this case when s_D is high in the second column of the center panel. Fifth, if $g_Q > 0$ is satisfied, E is large when the absolute value of g_θ is high. A high obsolescence rate makes g_Q and Θ small. From (25), a high obsolescence rate increases ν through the direct effect; however, it decreases ν

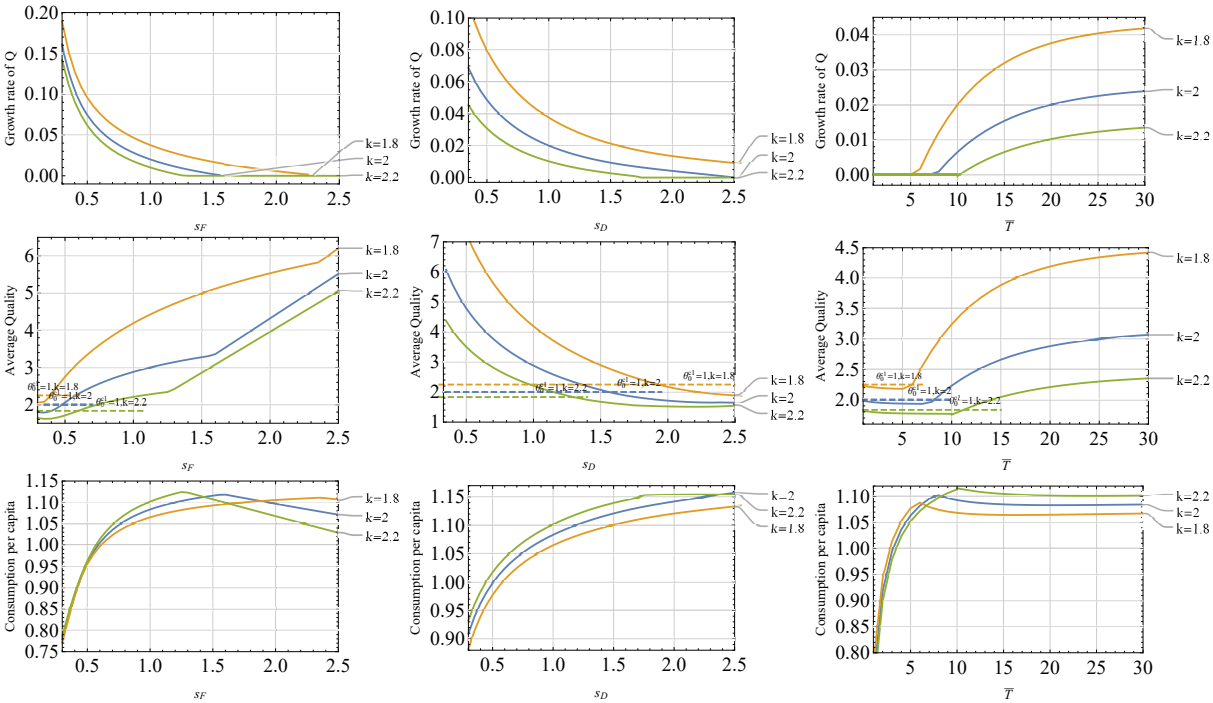

Figure 3: A numerical analysis changing k

through a decrement of g_Q. On one hand, small g_Q and the indirect effect of ν lead to large consumption per capita. On the other hand, small Θ and the direct effect of ν lead to small consumption per capita. Our numerical result shows that the former effect dominates the latter. Sixth, if $g_Q = 0$ is satisfied, E is small when the absolute value of g_θ is high. In this case, the effect that a high absolute value of g_θ leads to small g_Q and the indirect effect of ν does not work. Then, only the effect that a high absolute value of g_θ leads to small Θ and the direct effect of ν remains, which leads to consumption per capita being small.

In Figure 3, we confirm the effect of changing policy variables with a different expected value of θ_0, $\frac{k}{k-1}$. We examine the case where $k = 1.8$, $k = 2$, and $k = 2.2$ while setting $c_F = 5$, $g_\theta = -0.02$.[44] In regard to Figure 3, there are some points worth noting.[45] First, for any policy variable, g_Q is small when k is high. When k is high, the expected value of the initial adjusted quality is low, which lowers expected profit. Then, the expected value of incumbents with given g_Q is small, which discourages entry. Second, for any policy variable, Θ is small when k is high. High k makes g_Q small, which raises the incentive to put off disruptive innovation. Moreover, high k itself leads to average quality with a given level of disruptive innovation, and then Θ is small with high k even when g_Q is satisfied. Third, if $g_Q > 0$ is

[44] $\frac{k}{k-1} = 2.25$, $\frac{k}{k-1} = 2$, and $\frac{k}{k-1} = 1.83$ when $k = 1.8$, $k = 2$, and $k = 2.2$, respectively.

[45] When k is different, θ_0^{c1} takes a different value with given Θ. Then, we draw three broken lines in the second column of figure 3 to confirm whether the equilibrium is $\theta_0^{c1} \geq \theta_0^{min}$, or $\theta_0^{c1} < \theta_0^{min}$ when $k = 1.8$, $k = 2$, and $k = 2.2$ respectively.

satisfied, E is large when k is high. High k makes g_Q, ν, and Θ small. On one hand, small g_Q and ν lead to large consumption per capita. On the other hand, small Θ leads to small consumption per capita. Our numerical result shows that the former effect dominates the latter. Fourth, if $g_Q = 0$ is satisfied, E is small when k is high. When $g_Q = 0$ is satisfied, the effect that high k leads to small g_Q does not work. Our numerical result shows that the effect that high k leads to ν being small dominates the effect that high k makes Θ small.

3.2 The Case Where $C_F(t) = \frac{E(t)L(t)}{n(t)[\Theta]^\gamma} c_F$ and $C_D(t) = \frac{E(t)L(t)}{n(t)[\Theta]^\gamma} c_L$

In this section, we generalize the externality from the improvement level of differentiated goods as:

$$C_F(t) = \frac{E(t)L(t)}{n(t)[\Theta]^\gamma} c_F \quad C_D(t) = \frac{E(t)L(t)}{n(t)[\Theta]^\gamma} c_L \ . \tag{49}$$

These technology functions contain a positive externality from average quality and the number of differentiated goods, and a negative externality from market scale. We restrict the parameter $\gamma \geq 1$. [46] For simplicity, we also restrict our analysis to where $\theta_0^{c1} \geq \theta_0^{\min} = 1$ and $g_Q > 0$ are satisfied.

The labor market clearing condition is expressed as:

$$E(t)\left[\frac{\sigma - 1}{\sigma} + \frac{\nu}{[\Theta]^\gamma} c_D + \frac{g_Q}{[\Theta]^\gamma} c_F\right] = 1. \tag{50}$$

From (50), in the BGP equilibrium, $g_E = 0$ must be satisfied, and then the growth rate of innovation cost is $g_C = g_L - g_Q$.[47]

Necessary conditions for the maximization problem $\tilde{T}$ are expressed as:[48]

$$\begin{aligned} \theta_0^{c1}(g_Q) &= [\sigma[\rho - g_L + g_Q][s_F c_F - s_D c_D]]^{\frac{1}{\gamma}} \left[\frac{k-1}{k}\right]^{\frac{\gamma-1}{\gamma}} \\ \theta_0^{c2}(g_Q) &= \exp\left[[-g_\theta]\bar{T}\right] \theta_0^{c1}(g_Q) \\ \tilde{T}(\theta_0, g_Q) &= \frac{1}{-g_\theta} \ln\left[\frac{\theta_0}{\theta_0^{c1}}\right] \end{aligned} \ . \tag{51}$$

Because we generalize the externality from the improvement level of differentiated goods, the incentive for disruptive innovation is affected by the average quality of differentiated goods. On one hand, a high av-

[46] When $\gamma = 1$, this specification coincides with the previous section.
[47] $g_\theta < 0$ must be satisfied not to break the sufficient condition of the maximization problem of $\tilde{T}$.
[48] (15) is expressed as $\theta_0^{c1}(g_Q) = \sigma[\rho - g_L + g_Q][s_F c_F - s_D c_D][\Theta]^{1-\gamma}$. If $\theta_0^{c1} \geq \theta_0^{\min} = 1$ is satisfied, we obtain (51).

erage quality of differentiated goods decreases the expected net benefit from disruptive innovation, which decreases the marginal loss from putting off disruptive innovation. On the other hand, a high average quality of differentiated goods also decreases the instantaneous profit of incumbents, which decreases the marginal benefit from putting off disruptive innovation. When $\gamma = 1$, these two opposing effects offset each other completely. For $\gamma > 1$, the former effect dominates the latter, and then the high average quality of differentiated goods decreases the marginal loss from putting off disruptive innovation relative to the marginal benefit. Thus, incumbents move the timing of disruptive innovation later with given g_Q, and the average quality of differentiated goods, $\Theta = \frac{k}{k-1}\theta_0^{c1}$, is smaller than when $\gamma = 1$ with given g_Q.

The initial values of an incumbent entering at time s and drawing θ_0 is expressed as:

$$\begin{aligned} V(\theta_0, 0, s) &= \frac{EL(s)}{n(s)}\left[\frac{k}{k-1}\theta_0^{c1}(g_Q)\right]^{-\gamma} X_0 && for \quad \theta_0 \leq \theta_0^{c1} \\ V(\theta_0, 0, s) &= \frac{EL(s)}{n(s)}\left[\frac{k}{k-1}\theta_0^{c1}(g_Q)\right]^{-\gamma} X_1\left(\theta_0, g_Q, \theta_0^{c1}(g_Q)\right) && for \quad \theta_0^{c1} \leq \theta_0 \leq \theta_0^{c2} \\ V(\theta_0, 0, s) &= \frac{EL(s)}{n(s)}\left[\frac{k}{k-1}\theta_0^{c1}(g_Q)\right]^{-\gamma} X_2\left(\theta_0, g_Q, \theta_0^{c1}(g_Q)\right) && for \quad \theta_0 > \theta_0^{c2} \end{aligned}, \tag{52}$$

where

$$\begin{aligned} X_0 &= [s_F c_F - s_D c_D] \\ X_1\left(\theta_0, g_Q, \theta_0^{c1}(g_Q)\right) &= \left[\begin{array}{l} \frac{\theta_0}{\sigma}\frac{1-\exp\left[-[\rho-g_L+g_Q-g_\theta]\tilde{T}(\theta_0,g_Q)\right]}{\rho-g_L+g_Q-g_\theta}\left[\frac{k}{k-1}\theta_0^{c1}(g_Q)\right]^{\gamma-1} \\ +\exp\left[-[\rho-g_L+g_Q]\tilde{T}(\theta_0,g_Q)\right][s_F c_F - s_D c_D] \end{array}\right] \\ X_2\left(\theta_0, g_Q, \theta_0^{c1}(g_Q)\right) &= \left[\begin{array}{l} \frac{\theta_0}{\sigma}\frac{1-\exp\left[-[\rho-g_L+g_Q-g_\theta]\bar{T}\right]}{\rho-g_L+g_Q-g_\theta}\left[\frac{k}{k-1}\theta_0^{c1}(g_Q)\right]^{\gamma-1} \\ +\exp\left[-[\rho-g_L+g_Q]\bar{T}\right][s_F c_F - s_D c_D] \end{array}\right] \end{aligned}, \tag{53}$$

where θ_0^{c1}, θ_0^{c2}, and $\tilde{T}(\theta_0, g_Q)$ satisfy (51). The value of the incumbent is composed of two parts: aggregate discounted instantaneous profits and expected discounted net profit from disruptive innovation. On one hand, aggregate discounted instantaneous profits are proportional to EL/Q, while on the other hand, the expected discounted net benefit from disruptive innovation is proportional to the innovation technology, as in (31) and (49). In the model of section 3.1, for simplicity, we specify innovation technology to be proportional to EL/Q, which makes the value of the incumbent proportional to EL/Q. In the present section, we generalize the externality from the average quality of differentiated goods, and then the expected discounted net benefit from disruptive innovation is proportional to $EL\Big/\left(Q\left[\Theta\right]^{\gamma-1}\right)$. An increment of Θ decreases aggregate discounted instantaneous profits to be smaller than the expected

discounted net benefit from disruptive innovation when $\gamma > 1$. Because we multiply $\Theta^{-\gamma}$ before X_0, X_1, and X_2 to simplify the free entry condition, X_1 and X_2 are increasing functions of Θ.

The free entry condition is expressed as:

$$\int_{1}^{\theta_0^{c1}(g_Q)} X_0 dG(\theta_0) + \int_{\theta_0^{c1}(g_Q)}^{\theta_0^{c2}(g_Q)} X_1\left(\theta_0, g_Q, \theta_0^{c1}(g_Q)\right) dG(\theta_0) + \int_{\theta_0^{c2}(g_Q)}^{\infty} X_2\left(\theta_0, g_Q, \theta_0^{c1}(g_Q)\right) dG(\theta_0) = s_F c_F. \tag{54}$$

(51) and (54) determine the equilibrium value of g_Q. Substituting the equilibrium value of g_Q into (51), we obtain the equilibrium values of θ_0^{c1}, θ_0^{c2}, and $\tilde{T}$. Substituting the equilibrium value of θ_0^{c1} into (27), we obtain the equilibrium value of Θ. Substituting the equilibrium value of g_Q into (30), we obtain the equilibrium value of the growth rate of this economy, g_u. Substituting the equilibrium values of g_Q and θ_0^{c1} into (50), we obtain the equilibrium value of E.

Regarding the effect of a subsidy (tax) for innovation by entrants, we obtain the interesting result:[49]

$$\begin{aligned} \tfrac{dg_Q}{ds_F} &< 0 \quad if \quad \gamma < k \\ \tfrac{dg_Q}{ds_F} &= 0 \quad if \quad \gamma = k \\ \tfrac{dg_Q}{ds_F} &> 0 \quad if \quad \gamma > k \end{aligned} \cdot \tag{55}$$

We can interpret this result as follows. As discussed in the previous section, a decrement of s_F (the subsidy for entrants) simultaneously makes EV and the cost of innovation by entrants small; however, EV decreases less than the cost of innovation by entrants when $\gamma = 1$. Thus, g_Q must increase, which decreases EV, to satisfy the free entry condition. When $\gamma > 1$ is satisfied, X_1 and X_2 are increasing functions of Θ. A decrement of s_F decreases the expected profit from disruptive innovation with given g_Q, which mitigates disruptive innovation and makes Θ small. Then, the subsidy for entrants makes X_1 and X_2 small through a decrement of Θ with given g_Q, and this effect is strengthened with high γ. Thus, when γ is sufficiently high ($\gamma > k$), a decrement of s_F decreases EV more than the cost of innovation by entrants. Thus, g_Q must decrease to satisfy the free entry condition.

Regarding the effect of a subsidy (tax) for disruptive innovation by incumbents and the effect of extending (shortening) patent length, we obtain a result similar to that in the previous section, as well

[49] In appendix (D), we can confirm (55) analytically. The signs of $\frac{dg_Q}{ds_D}$, $\frac{dg_Q}{dT}$, $\frac{d\theta_0}{ds_F}$, $\frac{d\theta_0}{ds_D}$, and $\frac{d\theta_0}{dT}$ are the same as those in the previous section.

as the following proposition:

Proposition 3.4. *When we specify the innovation technology as* $C_F(t) = \frac{E(t)L(t)}{n(t)[\Theta]^{\gamma}} c_F$ *and* $C_D(t) = \frac{E(t)L(t)}{n(t)[\Theta]^{\gamma}} c_L$*, we get the following results.*

If $\gamma < k$ *is satisfied,*

- *A subsidy (tax) for disruptive innovation by entrants stimulates (mitigates) innovation by entrants, and then increases (decreases) the growth rate of the economy.*

If $\gamma = k$ *is satisfied,*

- *A subsidy (tax) for disruptive innovation by entrants does not affect innovation by entrants or the growth rate of the economy.*

If $\gamma > k$ *is satisfied,*

- *A subsidy (tax) for disruptive innovation by entrants mitigates (stimulates) innovation by entrants, and then decreases (increases) the growth rate of the economy.*

Regardless of the magnitude of the correlation between γ *and* k*, we obtain the following results.*

- *A subsidy (tax) for disruptive innovation by incumbents and an extension (shortening) of patent length stimulate (mitigate) innovation by entrants, and then increase (decrease) the growth rate of the economy.*

- *A subsidy (tax) for disruptive innovation by incumbents, tax (subsidy) for innovation by entrants, and extension (shortening) of patent length move the timing of disruptive innovation by each incumbent earlier (later), and the number of incumbents conducting disruptive innovation before their patent protection expires increases (decreases).*

This proposition shows that a subsidy for entrants decreases not only disruptive innovation by incumbents, but also innovation by entrants if the positive externality from the average quality is sufficiently strong, which decreases both the growth rate of the economy and the average quality of differentiated goods. Many related studies have concluded that support for entrants and the deregulation of barriers to entry stimulate economic growth.[50] The present paper considers firm heterogeneity and the timing of

[50] Regarding a welfare analysis, however, related studies have noted that there can be some cases where support for entrants and deregulation of barriers to entry harm social welfare.

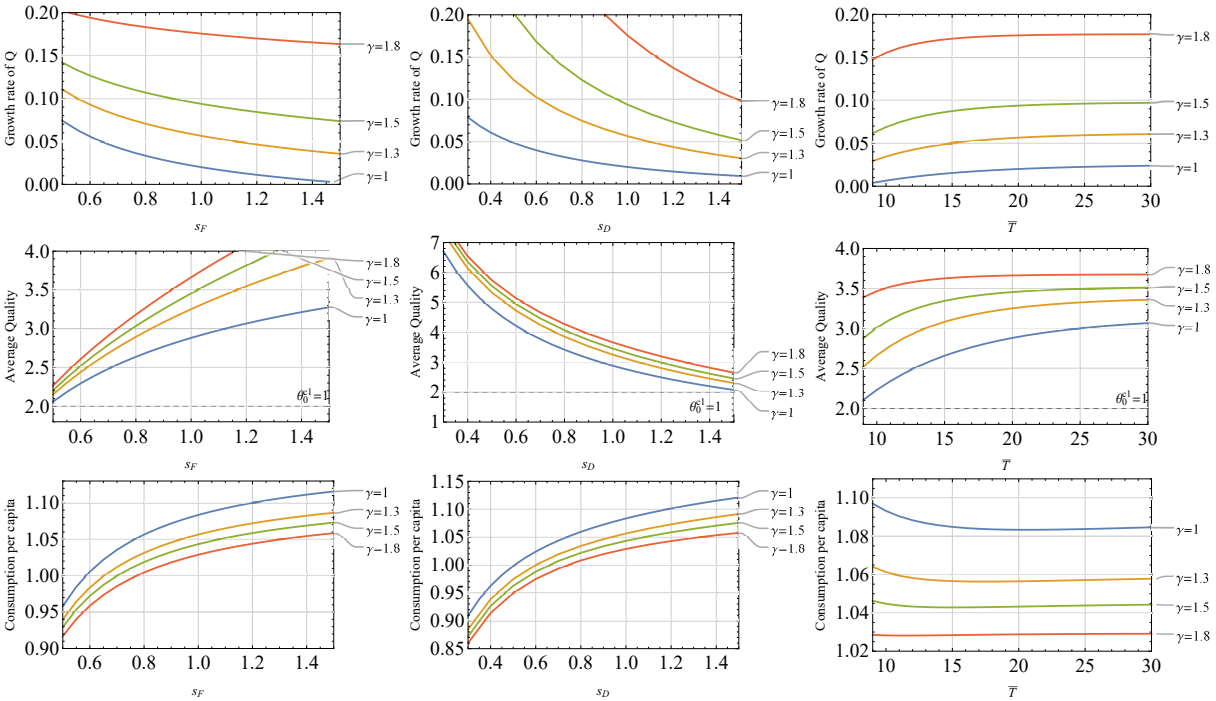

Figure 4: A numerical analysis of changing $\gamma < k$

incumbents switching their current technology, and these settings make the average quality endogenous. Because support for entrants and deregulation of barriers decrease the incentive to switch current technology, the average quality of differentiated goods decreases endogenously.[51] If the positive externality of innovation from average quality is strong, support for entrants decreases the value of incumbents more relative to entry cost. Then, the incentive to entry decreases, which decreases g_Q. This interesting result implies that deregulation does not always stimulate economic growth, and that the role of disruptive innovation is important.

Next, we conduct a numerical analysis focused on γ.[52] In Figure 4, we confirm the effect of changing policy variables with different $\gamma < k$. We examine the case where $\gamma = 1$, $\gamma = 1.3$, $\gamma = 1.5$, and $\gamma = 1.8$ while setting $c_F = 5$, $g_\theta = -0.02$, $k = 2$, and $\sigma = 16$.[53] Regarding Figure 4, there are some points worth noting. First, for any policy variables, g_Q is large when γ is high. When γ is high, the cost of innovation decreases more than the expected value of incumbents with given g_Q, which encourages entry. Then, g_Q must be large with high γ to satisfy the free entry condition. Second,for any policy variables, Θ is large when γ is high. This result is somewhat counterintuitive because from (51), Θ is small with high γ with given g_Q. However, high γ leads to large g_Q, which increases the marginal loss from putting off disruptive

[51]Note that a decrement of s_F itself decreases the expected benefit from disruptive innovation with given g_Q.
[52]Note that we restrict our analysis to where $\theta_0^{c1} \geq \theta_0^{\min} = 1$ and $g_Q > 0$ are satisfied.
[53]We set a high $\sigma = 16$ to obtain a plausible growth rate. When $\sigma = 16$, the markup is 1.06.

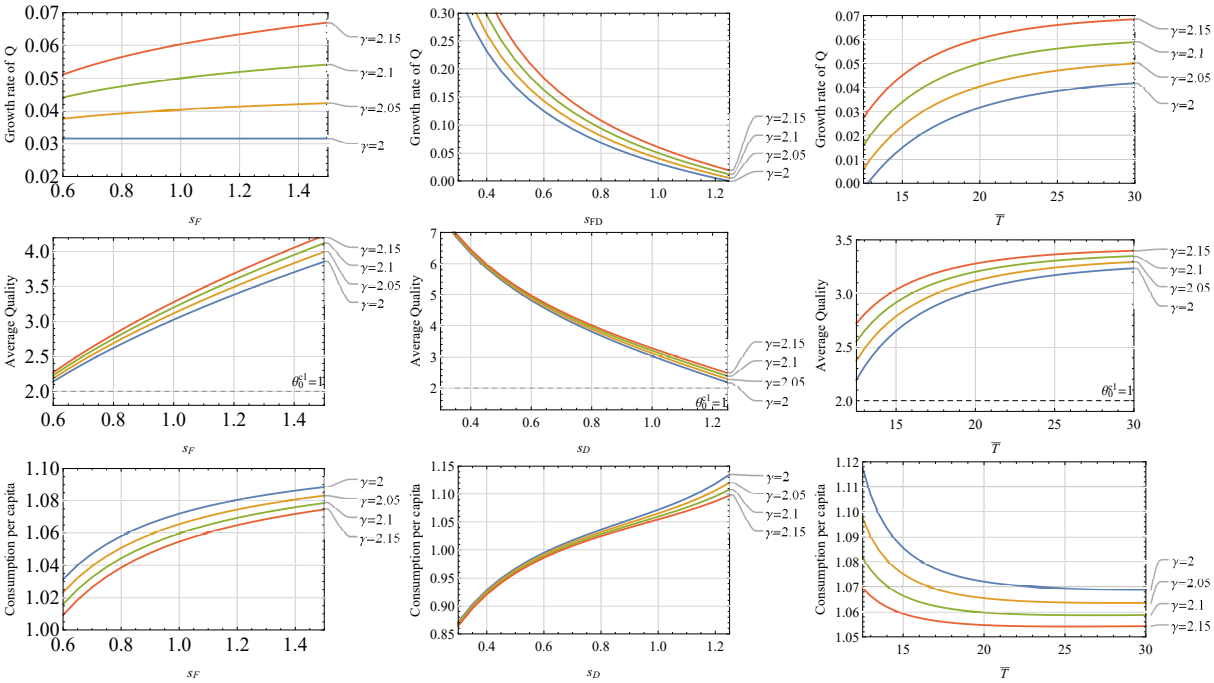

Figure 5: A numerical analysis changing $\gamma \geq k$

innovation and makes disruptive innovation large. Our numerical result shows that this effect dominates the former. Third, for any policy variables, E is small when γ is high. High γ leads to large g_Q, ν, and Θ. On one hand, large Θ leads to consumption per capita being large; on the other hand, large g_Q and ν lead to consumption per capita being small. Our numerical result shows that the latter effect dominates the former.

In Figure 5, we confirm the effect of changing policy variables with different $\gamma \geq k$. We examine the case where $\gamma = 2$, $\gamma = 2.05$, $\gamma = 2.1$, and $\gamma = 2.15$ while setting $c_F = 5$, $g_\theta = -0.02$, $k = 2$, and $\sigma = 16$. Regarding Figure 5, there are some points worth noting. First, g_Q is independent of s_F when $\gamma = k$. Second, g_Q is an increasing function of s_F when $\gamma > k$ is satisfied. These features are consistent with proposition 3.4. Third, we can confirm that this other feature is similar to Figure 4.

3.3 The Case Where $C_F(t) = \frac{1}{Q(t)} c_F$ and $C_D(t) = \frac{1}{Q(t)} c_D$

In the previous two sections, we consider the negative externality of innovation caused by market size. This setting enables us to analyze g_Q and Θ without considering the labor market clearing condition. However, in this setting, we cannot examine how population size affects key endogenous variables like g_Q, Θ, and E. In this section, we eliminate this useful assumption and specify the innovation technology

as:

$$C_F(t) = \frac{1}{Q(t)} c_F \quad C_D(t) = \frac{1}{Q(t)} c_D \quad . \tag{56}$$

These technology functions contain a positive spillover from the improvement level of differentiated goods; however, there is no negative externality from market scale. With this specification, the labor market clearing condition affects the equilibrium value of θ_0^{c1}, θ_0^{c2}, $\tilde{T}(\theta_0)$, g_Q, andΘ; thus, these values are dependent on population size. In this section, we restrict our analysis to where $\theta_0^{c1} \geq \theta_0^{\min} = 1$, and $g_L = 0$ are satisfied.[54]

The labor market clearing condition is expressed as:

$$E(t) L \frac{\sigma - 1}{\sigma} + \frac{\nu}{\Theta} c_D + \frac{g_Q}{\Theta} c_F = L. \tag{57}$$

From (57), in the BGP equilibrium, $g_E = 0$ must be satisfied, and then the growth rate of innovation cost is $g_C = -g_Q$.[55]

The necessary conditions of the maximization problem of $\tilde{T}$ are expressed as:

$$\begin{aligned} \theta_0^{c1}(g_Q, E) &= \tfrac{1}{EL} \sigma [\rho + g_Q] [s_F c_F - s_D c_D] \\ \theta_0^{c2}(g_Q, E) &= \exp\left[[-g_\theta] \bar{T}\right] \theta_0^{c1}(g_Q) \quad . \\ \tilde{T}(\theta_0, g_Q, E) &= \tfrac{1}{-g_\theta} \ln\left[\tfrac{\theta_0}{\theta_0^{c1}}\right] \end{aligned} \tag{58}$$

Market scale increases the instantaneous profit of incumbents, which increases the marginal benefit from putting off disruptive innovation. In the case of section 3.1, market scale also increases the expected net profit from disruptive innovation, which increases the marginal loss from putting off disruptive innovation; these two opposing effects offset each other completely. In this section, the absence of a negative externality from market scale makes the timing of disruptive innovation depend on market scale, EL. A large EL only increases the marginal benefit from putting off disruptive innovation, and then incumbents move the timing of disruptive innovation later. When $EL > 1$ ($EL < 1$), this effect is greater (smaller) than in the case of section3.1. Thus, incumbents move the timing of disruptive innovation later (earlier) with given g_Q, and the average quality of differentiated goods, $\Theta = \frac{k}{k-1}\theta_0^{c1}$, is smaller (greater) than the

[54] If these restrictions are not satisfied, analysis becomes far too complex and beyond the scope of the present paper.
[55] $g_\theta < 0$ must be satisfied to avoid breaking the sufficient condition of the maximization problem of $\tilde{T}$.

case of section3.1 with given g_Q.

The initial values of an incumbent entering at time s and drawing θ_0 is expressed as:

$$\begin{aligned} V(\theta_0,0,s) &= \tfrac{1}{Q(s)}X_0 && for && \theta_0 \leq \theta_0^{c1}(g_Q,E) \\ V(\theta_0,0,s) &= \tfrac{1}{Q(s)}X_1(\theta_0,g_Q,E) && for && \theta_0^{c1}(g_Q,E) \leq \theta_0 \leq \theta_0^{c2}(g_Q,E) \\ V(\theta_0,0,s) &= \tfrac{1}{Q(s)}X_2(\theta_0,g_Q,E) && for && \theta_0 > \theta_0^{c2}(g_Q,E) \end{aligned} , \tag{59}$$

where

$$\begin{aligned} X_0 &= [s_F c_F - s_D c_D] \\ X_1(\theta_0,g_Q,E) &= \tfrac{\theta_0 EL}{\sigma}\tfrac{1-\exp\left[-[\rho+g_Q-g_\theta]\tilde{T}(\theta_0,g_Q)\right]}{\rho+g_Q-g_\theta} + \exp\left[-[\rho+g_Q]\tilde{T}(\theta_0,g_Q)\right][s_F c_F - s_D c_D] \\ X_2(\theta_0,g_Q,E) &= \tfrac{\theta_0 EL}{\sigma}\tfrac{1-\exp\left[-[\rho+g_Q-g_\theta]\bar{T}\right]}{\rho+g_Q-g_\theta} + \exp\left[-[\rho+g_Q]\bar{T}\right][s_F c_F - s_D c_D] \end{aligned} . \tag{60}$$

where θ_0^{c1}, θ_0^{c2}, and $\tilde{T}(\theta_0,g_Q)$ satisfy (58). The value of an incumbent is composed of two parts: aggregate discounted instantaneous profits and expected discounted net benefits from disruptive innovation. On one hand, aggregate discounted instantaneous profits are proportional to EL/Q. On the other hand, expected discounted net benefits from disruptive innovation are proportional to $1/Q$. An increment of EL increases aggregate discounted instantaneous profits; however, it does not increase the expected discounted net benefits from disruptive innovation. Because we multiply $1/Q(s)$ before X_0, X_1 and X_2 to simplify the free entry condition, X_1 and X_2 are increasing functions of EL.

The free entry condition is expressed as:

$$\int_1^{\theta_0^{c1}(g_Q,E)} X_0 dG(\theta_0) + \int_{\theta_0^{c1}(g_Q,E)}^{\theta_0^{c2}(g_Q,E)} X_1(\theta_0,g_Q,E)\,dG(\theta_0) + \int_{\theta_0^{c2}(g_Q,E)}^{\infty} X_2(\theta_0,g_Q,E)\,dG(\theta_0) = s_F c_F. \tag{61}$$

From (57), (58), and (61), the equilibrium values of g_Q and E are determined. Unlike the case of section 3.1, this model captures the resource effect; however, it generates a scale effect. Substituting the equilibrium value of g_Q and E into (58), we obtain the equilibrium values of θ_0^{c1}, θ_0^{c2}, and $\tilde{T}$. Substituting the equilibrium value of θ_0^{c1} into (27), we obtain the equilibrium value of Θ. Substituting the equilibrium value of g_Q into (30), we obtain the equilibrium value of the growth rate of this economy.

Next, we conduct a numerical analysis with this equilibrium. In Figure 6, we confirm the effect of changing policy variables with different population size, L. We examine the case $L = 0.8$, $L = 0.9$, $L = 1$,

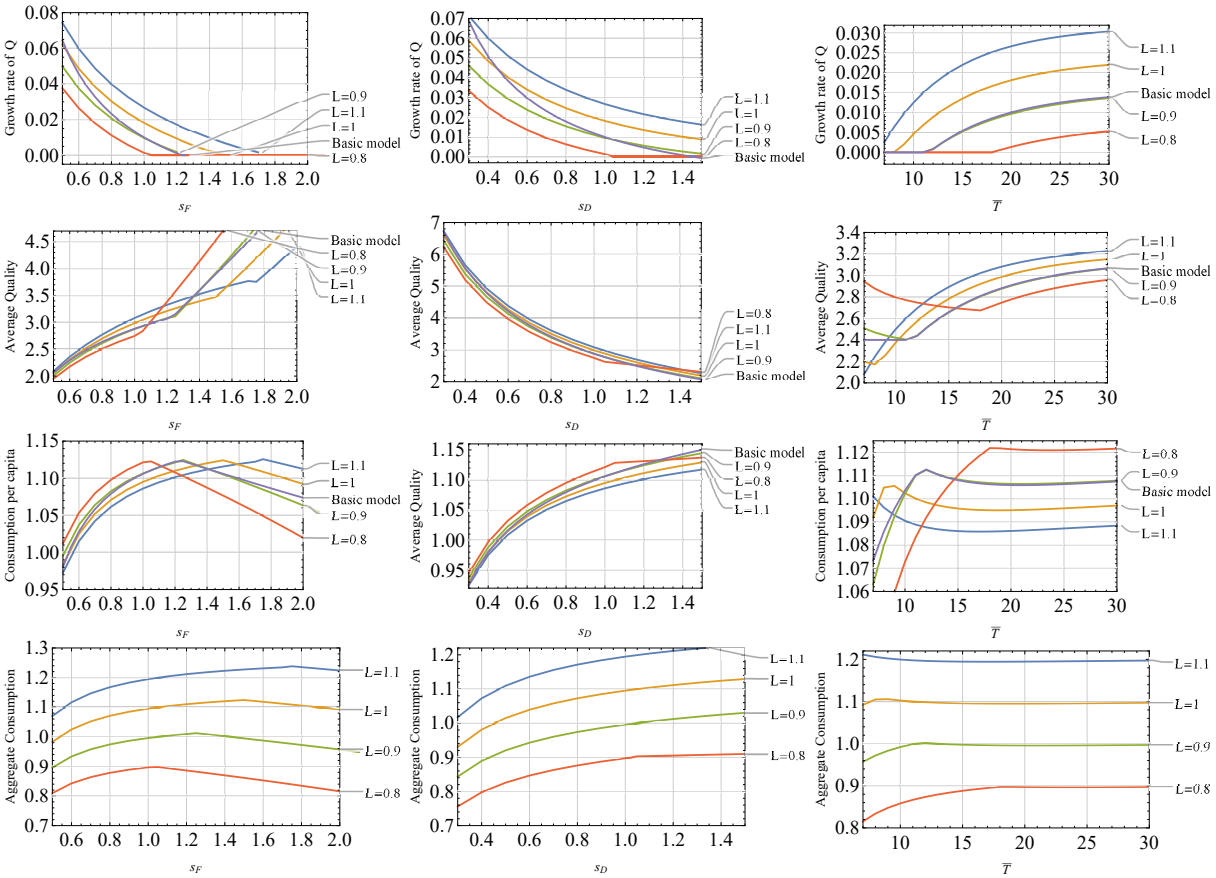

Figure 6: A numerical analysis where $C_F(t) = \frac{1}{Q(t)} c_F$ and $C_D(t) = \frac{1}{Q(t)} c_D$

and $L = 1.1$ while setting $c_F = 5$, $g_\theta = -0.02$, $k = 2$. The curve tagged "basic model" expresses the relationship between key endogenous and policy variables using the model constructed in section 3.1, where the parameter is set as $c_F = 5$, $g_\theta = -0.02$, $k = 2$, and $g_L = 0$.

Regarding Figure 6, there are some points worth noting. First, for any policy variables, g_Q is large when L is high. When L is high, the instantaneous profit of incumbents is high, which increases the expected value of incumbents with given g_Q. Then, g_Q must be large with high L to satisfy the free entry condition.[56] Second, the elasticity of g_Q with respect to s_F and s_D is smaller when there is no negative externality from market scale compared with the case of section 3.1. Low s_F and s_D stimulate innovation by entrants, which lowers E. In the case of section 3.1, a decrement of E affects the incentive for innovation by entrants through two channels. On one hand, a decrement of E decreases the instantaneous profit of incumbents, which lowers the incentive for innovation by entrants. On the other hand, a decrement of E decreases the cost of innovation by entrants, which raises the incentive for innovation by entrants. We constructed a model in section 3.1, in which these two effects offset each other completely. However,

[56] In the model of the previous two sections, high L also increases the cost of innovation by entrants, which offsets completely the effect that high L increases the expected value of incumbents.

in the present section, we remove this useful assumption, and the latter effect does not work. Thus, because of this resource effect, the subsidy for innovation by incumbents and entrants stimulates growth weaker than in the case of section 3.1. [57] Third, as suggested in our analytical section, we can confirm that g_Q is greater (smaller) than in the case of section 3.1 when aggregate consumption, EL, is greater (smaller) than 1. [58] Fourth, if $g_Q > 0$ is satisfied, Θ is large when L is high. This result is somewhat counterintuitive because from (58), Θ is small with high L with given g_Q. However, high L leads to large g_Q, which increases the marginal loss from putting off disruptive innovation and makes disruptive innovation large. Our numerical result shows that this effect dominates the former. Fifth, if $g_Q = 0$ is satisfied, Θ is small when L is high. In this case, the effect that high L leads to large g_Q does not work, and then only the effect that high L leads to small Θ with given g_Q remains. Sixth, if $g_Q > 0$ is satisfied, E is small when L is high. High L leads to large g_Q, ν, and Θ if $g_Q > 0$ is satisfied. On one hand, large Θ makes consumption per capita large, while on the other hand, large g_Q and ν make consumption per capita small. Our numerical result shows that the latter effect dominates the former. Seventh, if $g_Q = 0$ is satisfied, E is large when L is high. In this case, the effect that high L leads to small g_Q does not work. Our numerical result shows that the effect that high L leads to small ν dominates the effect that L leads to small Θ.

Many growth models derive a positive relationship between population size and the degree of innovation (economic growth rate). However, as pointed out by Jones (1995), a positive relationship between population size and economic growth rate is not observed in the real economy. Many papers attempt to remove this irrelevant property in various ways. The result obtained in this section has a potential to offer one new solution to the scale effect. We extend the instantaneous utility function as used in Ohki (2015):

$$u(t) = \left[\left[\int_0^{n_1} q_1(j,t) \left(x_1(j,t)\right)^{\frac{\sigma-1}{\sigma}} dj \right]^{\frac{\sigma}{\sigma-1}} \right]^{\alpha} \left[\left[\int_0^{n_2} q_2(j,t) \left(x_1(j,t)\right)^{\frac{\sigma-1}{\sigma}} dj \right]^{\frac{\sigma}{\sigma-1}} \right]^{1-\alpha} \tag{62}$$

where $q_i(j,t)$ and $d_i(j,t)$ denote the quality and consumption volume of incumbent j in industry i at time t, respectively, and α denotes the market share of industry 1. We assume that barriers to entry in

[57] On one hand, the benefit of the setting of the present section is that we can capture the resource effect and examine the effect of population size. On the other hand, the disadvantage of the setting of the present section is that we cannot conduct analytical analysis, and that numerical calculation takes a long time. Because the difference between the two models is qualitatively small, we think that the assumption used in section 3.1 has sufficient validity.

[58] See the first and fourth rows of Figure 6.

industry 1, c_{F1}, are too high for any entrants, $g_{n1} = 0$. In industry 2, barriers to entry are not too high, and thus, new entrants enter the market, $g_{n2} > 0$.

From the above discussion, high L allows active innovation by incumbents and entrants if new entrants enter the market ($g_n = g_Q > 0$) in the equilibrium. However, high L stifles innovation by incumbents if new entrants do not enter the market ($g_n = g_Q = 0$) in the equilibrium. If α is large, the share of industry where new entrants do not enter the market is large. Thus, if α is large, the latter effect can dominate the former, and we can show that high L leads to a small degree of aggregate innovation. Because our model is designed so that innovation by incumbents does not increase the growth rate of the economy directly, we cannot observe a negative relationship between population size and the growth rate of the economy. However, an appropriate extension of our model enables us to design a model in which innovation by incumbents raises the growth rate of the economy directly. In doing so, we will construct a new endogenous growth model that overcomes the scale effect puzzle.

3.4 The Case Where $C_F(t) = \frac{E(t)L(t)}{\Theta} c_F$ and $C_D(t) = \frac{E(t)L(t)}{\Theta} c_L$

So far, we have analyzed the economy where the number of differentiated goods grows endogenously. In the context of the endogenous growth model, especially in the homogeneous model, the economy where the number of differentiated goods is constant over the long run is also examined. In this section, we examine a model where the number of differentiated goods is constant over the long run in consideration of firm heterogeneity and disruptive innovation. We specify the innovation technology as:

$$C_F(t) = \frac{E(t)L(t)}{\Theta} c_F \quad C_D(t) = \frac{E(t)L(t)}{\Theta} c_L \ . \tag{63}$$

These technology functions contain a positive externality from the average quality of differentiated goods and a negative externality from market scale; however, there is no positive externality from the number of differentiated goods.

From (26) and (63), the labor market clearing condition is expressed as:

$$E(t)\left[\frac{\sigma - 1}{\sigma} + \frac{\nu}{\Theta} n(t) c_L + \frac{g_Q}{\Theta} n(t) c_F\right] = 1. \tag{64}$$

From (64), in the steady-state equilibrium, $g_E = 0$ and $g_n = 0$ must be satisfied; then, $g_Q = 0$, $g_C = 0$, and

$g_u = 0$ are satisfied. In this economy, the steady-state equilibrium value of n is determined endogenously instead of g_Q.

The necessary conditions of the maximization problem of $\tilde{T}$ are expressed as:[59]

$$\begin{aligned} \theta_0^{c1}(n) &= n\sigma\left[\rho - g_L\right]\left[s_F c_F - s_D c_D\right] \\ \theta_0^{c2}(n) &= \exp\left[\left[-g_\theta\right]\bar{T}\right]\theta_0^{c1}(n) \\ \tilde{T}(\theta_0, n) &= \tfrac{1}{-g_\theta}\ln\left[\tfrac{\theta_0}{\theta_0^{c1}(n)}\right] \end{aligned} \quad . \tag{65}$$

Large n decreases the instantaneous profit of incumbents, which in turn decreases the marginal benefit from putting off disruptive innovation. In the case of section 3.1, large n also decreases the expected benefit from disruptive innovation, which decreases the marginal loss from putting off disruptive innovation; these two opposite effects offset each other completely. Then, the absence of a positive externality from the number of differentiated goods makes the timing of disruptive innovation depend on n, which decreases only the marginal benefit from putting off disruptive innovation, and incumbents move the timing of disruptive innovation earlier with large n. [60]

The initial values of an incumbent entering at time s and drawing θ_0 is expressed as:

$$\begin{aligned} V(\theta_0, 0, s) &= \tfrac{EL}{\Theta}X_0 && for \quad \theta_0 \le \theta_0^{c1} \\ V(\theta_0, 0, s) &= \tfrac{EL}{\Theta}X_1(\theta_0, n) && for \quad \theta_0^{c1} \le \theta_0 \le \theta_0^{c2} \\ V(\theta_0, 0, s) &= \tfrac{EL}{\Theta}X_2(\theta_0, n) && for \quad \theta_0 > \theta_0^{c2} \end{aligned} \quad , \tag{66}$$

where

$$\begin{aligned} X_0 &= \left[s_F c_F - s_D c_D\right] \\ X_1(\theta_0, n) &= \tfrac{1-\exp\left[-[\rho-g_\theta]\tilde{T}(\theta_0,n)\right]}{\rho-g_\theta}\tfrac{\theta_0}{n\sigma} + \left[s_F c_F - s_D c_D\right]\exp\left[-\rho\tilde{T}(\theta_0, n)\right] \\ X_2(\theta_0, n) &= \tfrac{1-\exp\left[-[\rho-g_\theta]\bar{T}\right]}{\rho-g_\theta}\tfrac{\theta_0}{n\sigma} + \left[s_F c_F - s_D c_D\right]\exp\left[-\rho\bar{T}\right] \end{aligned} \quad . \tag{67}$$

where θ_0^{c1}, θ_0^{c2}, and $\tilde{T}(\theta_0, n)$ satisfy (65). The value of incumbents is composed of two parts: aggregate discounted instantaneous profits and expected discounted net benefits from disruptive innovation. On one hand, aggregate discounted instantaneous profits are proportional to EL/Q. On the other hand,

[59] $g_\theta < 0$ must be satisfied to avoid breaking the sufficient condition of the maximization problem of $\tilde{T}$.

[60] In the model of section 3.1, $g_n = 0$ is satisfied if the expected profit from the entry is too small. In this case, the number of differentiated goods is constant; however, this is not determined endogenously, rather, it is exogenously given or derived from the span when $g_n > 0$ is satisfied and the exogenous initial value of n. In the model of section 3.1, n does not affect other endogenous variables, which is clearly different from the model constructed in the present section.

expected discounted net benefits from disruptive innovation are proportional to EL/Θ. An increment to n decreases aggregate discounted instantaneous profits; however, it does not decrease expected discounted net benefits from disruptive innovation. Because we multiply EL/Θ before X_0, X_1, and X_2 to simplify the free entry condition, X_1 and X_2 are decreasing functions of n.

The free entry condition is expressed as:

$$\int_{1}^{\theta_0^{c1}(n)} X_0 dG(\theta_0) + \int_{\theta_0^{c1}(n)}^{\theta_0^{c2}(n)} X_1(\theta_0, n)\, dG(\theta_0) + \int_{\theta_0^{c2}(n)}^{\infty} X_2(\theta_0, n)\, dG(\theta_0) = s_F c_F, \tag{68}$$

where θ_0^{c1}, θ_0^{c2}, and $\tilde{T}$ satisfy (65), and X_0, X_1, and X_2 satisfy (67), which determines the equilibrium value of n. Then, we can write the equilibrium value of n as a function of policy variable:

$$n^* = n\left(s_D, s_F, \bar{T}\right), \tag{69}$$

where $\frac{\partial n\left(s_D, s_F, \bar{T}\right)}{\partial s_D} < 0$, $\frac{\partial n\left(s_D, s_F, \bar{T}\right)}{\partial s_F} < 0$, and $\frac{\partial n\left(s_D, s_F, \bar{T}\right)}{\partial \bar{T}} > 0$. Substituting (69) into (65), we obtain the equilibrium value of θ_0^{c1}.[61] Then, we can write the equilibrium value of n as a function of the policy variable:

$$\theta_0^{c1*} = \theta_0^{c1}\left(s_D, s_F, \bar{T}\right), \tag{70}$$

where $\frac{\partial \theta_0^{c1}\left(s_D, s_F, \bar{T}\right)}{\partial s_D} < 0$, $\frac{\partial \theta_0^{c1}\left(s_D, s_F, \bar{T}\right)}{\partial s_F} > 0$ and $\frac{\partial \theta_0^{c1}\left(s_D, s_F, \bar{T}\right)}{\partial \bar{T}} > 0$. Substituting (70) into (27), we obtain the equilibrium value of the average quality. Substituting (69) and (70) into (64), we obtain the equilibrium value of E.

We summarize the above discussion in the following proposition:

Proposition 3.5. *When we specify the innovation technology as $C_F(t) = \frac{E(t)L(t)}{\Theta} c_F$, and $C_D(t) = \frac{E(t)L(t)}{\Theta} c_L$, we obtain the following results.*

- *A subsidy (tax) for disruptive innovation by incumbents stimulates (mitigates) innovation by entrants in the transition process, and then increases (decreases) the number of differentiated goods in the steady state; however, it does not affect the growth rate in the steady state.*

- *A subsidy (tax) for innovation by entrants stimulates (mitigates) innovation by entrants in the*

[61] Since θ_0^{c2} and $\tilde{T}(\theta_0)$ are expressed as functions of θ_0^{c1}, we can also obtain the equilibrium value of these variables.

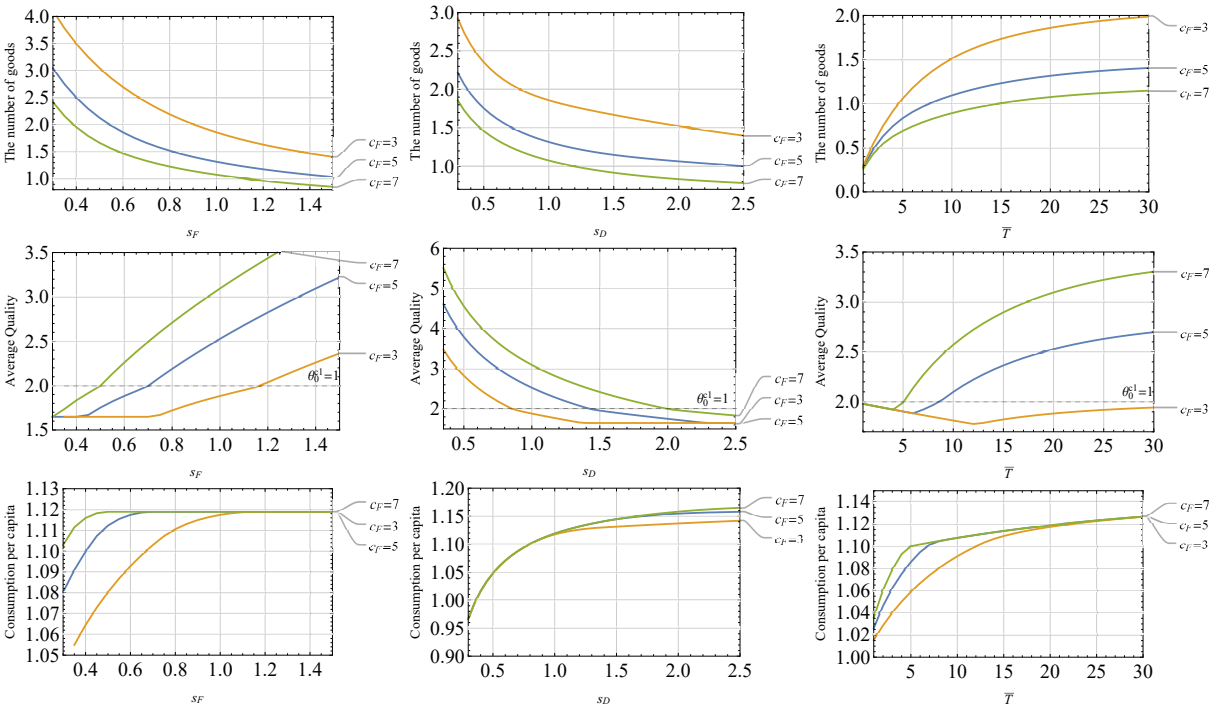

Figure 7: A numerical analysis changing c_F, where $C_F(t) = \frac{1}{Q(t)}c_F$ and $C_D(t) = \frac{1}{Q(t)}c_D$

transition process, and then increases (decreases) the number of differentiated goods in the steady state; however, it does not affect the growth rate in the steady state.

- *Extending (shortening) patent length stimulates innovation by entrants in the transition process, and then increases (decreases) the number of differentiated goods in the steady state; however, it does not affect the growth rate in the steady state.*

- *Regarding disruptive innovation by incumbents and the average quality of differentiated goods, the results obtained in this section are qualitatively the same as those in section 3.1.*

As follows, we conduct a numerical analysis. Although a numerical analysis where $C_F(t) = \frac{E(t)L(t)}{\Theta}c_F$ and $C_D(t) = \frac{E(t)L(t)}{\Theta}c_L$ is similar to the case discussed in section 3.1, there are some points to be considered. In Figure 7, we confirm the effect of changing variables with different efficiency of innovation by entrants, c_F. We examine the case where $c_F = 3$, $c_F = 5$, and $c_F = 7$ while setting $g_\theta = -0.02$, $k = 2$. Regarding Figure 7, there are some points worth noting. First, n is a decreasing function of s_F and s_D, and an increasing function of $\bar{T}$, which is consistent with proposition 3.5. Second, for any policy variable, the number of differentiated goods is large when c_F is low. This result is interpreted as follows: a high c_F imposes a high cost to enter the market, which discourages entry in the transition process. Discouragement of entry keeps the number of differentiated goods small in the steady state.

Third, if $\theta_0^{min} \leq \theta_0^{c1}$ is satisfied, E is independent of s_F. In the steady state, only disruptive innovation by incumbents is conducted. The resources devoted to disruptive innovation increases with large n and ν and small Θ. High s_F leads to small n and large ν and Θ; however, as shown in appendix E, these three effects offset each other completely. We can confirm this in the third row of the left panel when s_F is high. Fourth, if $\theta_0^{min} > \theta_0^{c1}$ is satisfied, E increases with high s_F. The effect that large ν leads to small E is stronger than the effect that large Θ leads to large E. Then, high s_F leads to small E through large ν and Θ. If $\theta_0^{min} > \theta_0^{c1}$ is satisfied, this effect is weaker than the case where $\theta_0^{min} \leq \theta_0^{c1}$ is satisfied because the incentive to conduct disruptive innovation is too small. Thus, this effect cannot dominate the effect that high s_F leads to large E through small n. Fifth, for any $\bar{T}$, E does not decrease with high $\bar{T}$. In the case of section 3.1, E temporarily decreases with high $\bar{T}$ when g_Q increases rapidly. However, in the case of the present section, our numerical result suggests that this does not happen. [62] Sixth, if $\theta_0^{min} \leq \theta_0^{c1}$ is satisfied, E is independent of c_F. High c_F leads to small n, large ν, and large Θ. As shown in appendix E, these three effects offset each other completely. Seventh, if $\theta_0^{min} > \theta_0^{c1}$ is satisfied, E increases with high c_F. This interpretation is similar to the reason E increases with high s_F if $\theta_0^{min} \leq \theta_0^{c1}$ is satisfied.

In Figure 8, we confirm the effect of changing policy variables with a different growth rate of adjusted quality, g_θ. We examine the case where $g_\theta = -0.01$, $g_\theta = -0.02$, and $g_\theta = -0.03$ while setting $c_F = 5$, $k = 2$. Regarding Figure 8, there are some points worth noting. First, for any policy variable, the number of differentiated goods is large when the absolute value of g_θ is low. We interpret this result as follows: a high absolute value of g_θ decreases instantaneous profit rapidly, which discourages entry in the transition process. Discouragement of entry means there will be a small number of differentiated goods in the steady state. Second, for any s_F, Θ is strictly high when the absolute value of g_θ is low. In the case of section 3.1, Θ takes same value when $g_Q = 0$ and $\theta_0^{min} \leq \theta_0^{c1}$ are satisfied. However, in the case of the present section, the number of differentiated goods is greater than zero, which decreases with a high absolute value of g_θ. Then, the effect that a low absolute value of g_θ leads to a large Θ through large n still works, even when s_F is very high. Third, for any variable, E is large when the absolute value of g_θ is low. In the case of section 3.1, if $g_Q > 0$ is satisfied, E is small when the absolute value of g_θ is high. In the case of section 3.1, a high absolute value of g_θ leads to large E through less innovation

[62] In the case of section 3.1, an increment of $\bar{T}$ increases g_Q rapidly, which increases the resources devoted to innovation by entrants, and then decreases the resources devoted to production. However, in the present case, innovation is conducted only by incumbents in steady-state equilibrium. Without this effect, a difference between the case of section 3.1 and the present case is created.

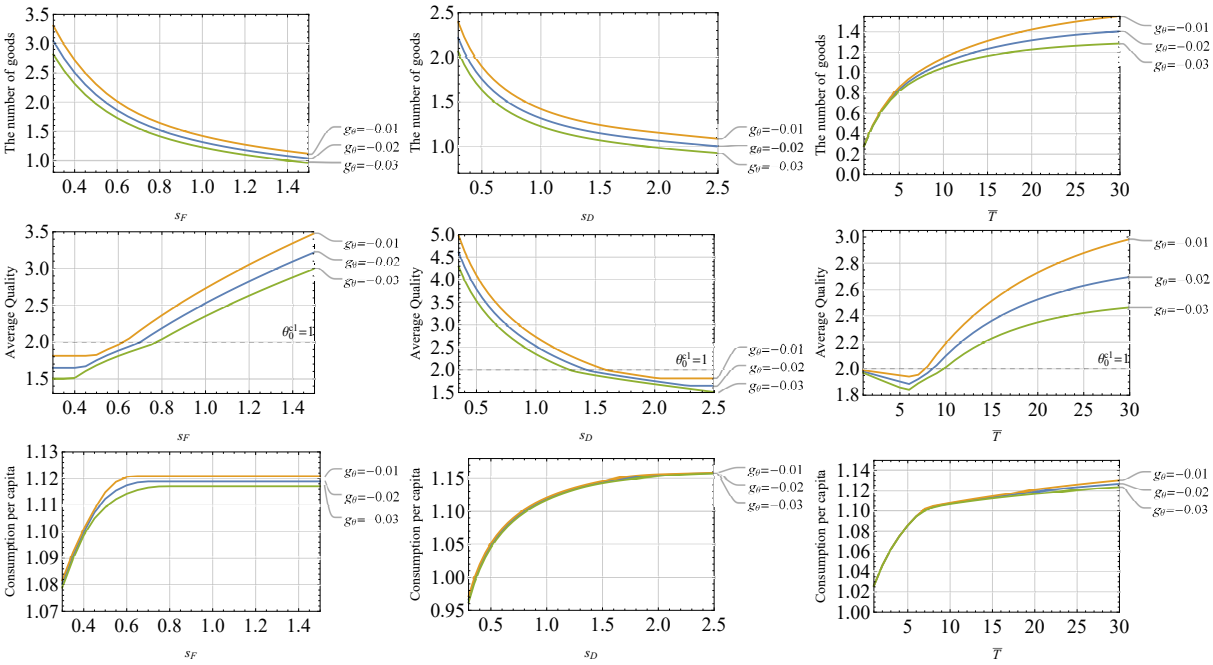

Figure 8: A numerical analysis changing g_θ, where $C_F(t) = \frac{1}{Q(t)} c_F$ and $C_D(t) = \frac{1}{Q(t)} c_D$

by incumbents. However, in the case of the present section, disruptive innovation is conducted only by incumbents in steady-state equilibrium, and then, this effect does not work.

In Figure 9, we confirm the effect of changing policy variables with different expected values of θ_0, $\frac{k}{k-1}$. We examine the case where $k = 1.8$, $k = 2$, and $k = 2.2$ while setting $c_F = 5$, $g_\theta = -0.02$. Regarding Figure 9, there are some points worth noting.[63] First, for any policy variable, the number of differentiated goods is small when k is high. When k is high, the expected value of initial adjusted quality is low, which discourages entry in the transition process. A discouragement to entry leads to a small number of differentiated goods in the steady state. Second, for any variable, E is large when k is high. In the case of section 3.1, if $g_Q = 0$ is satisfied, E is small when k is high. In the case of section 3.1, the absence of the effect that high k leads to large E through small g_Q causes this result. However, in the case of the present section, the number of differentiated goods is greater than zero, which decreases with high k. Then, the effect that high k leads to large E through the effect that small n leads to small resources devoted to disruptive innovation always works.

Finally, we examine the effect of degrees of externality from the average quality as in section 3.2. We

[63] When k is different, θ_0^{c1} takes a different value with given Θ. Then, we draw three broken lines in the second column of Figure 3 to confirm whether the equilibrium is $\theta_0^{c1} \geq \theta_0^{min}$, or $\theta_0^{c1} < \theta_0^{min}$ when $k = 1.8$, $k = 2$, and $k = 2.2$ respectively.

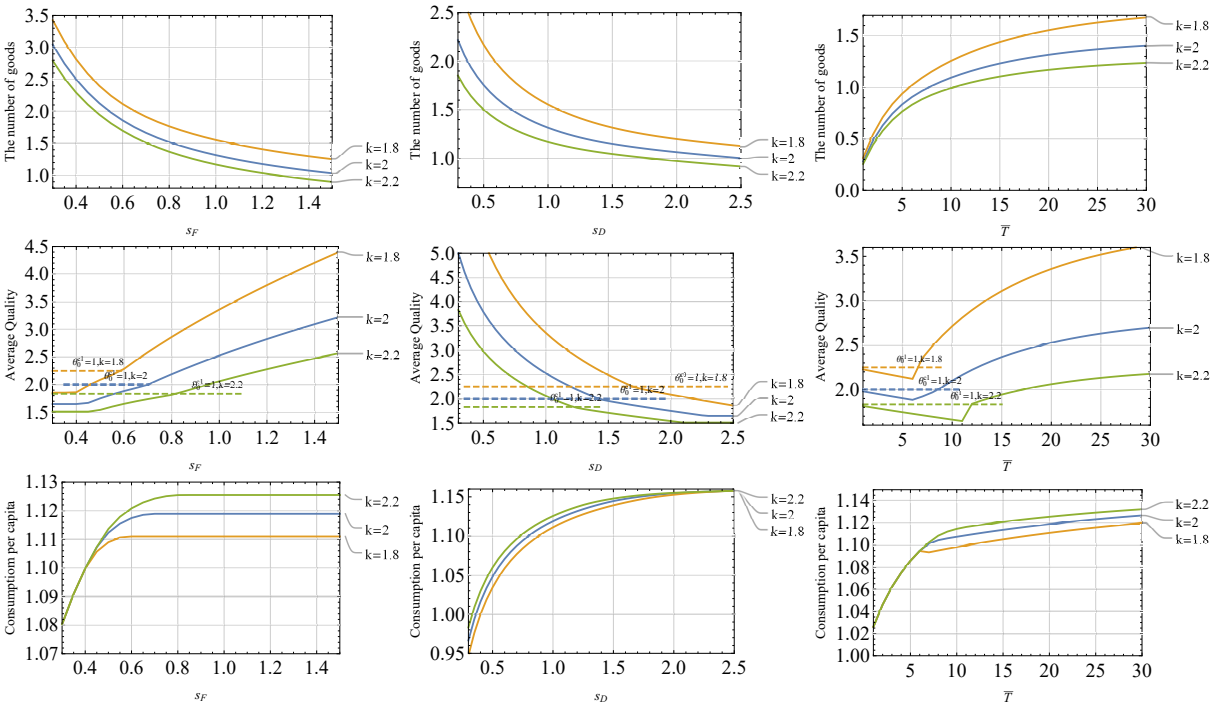

Figure 9: A numerical analysis changing k, where $C_F(t) = \frac{1}{Q(t)} c_F$ and $C_D(t) = \frac{1}{Q(t)} c_D$

specify R&D technology as:[64]

$$C_F(t) = \frac{1}{n(t)[\Theta]^{\gamma}} c_F \quad C_D(t) = \frac{1}{n(t)[\Theta]^{\gamma}} c_L \ . \tag{71}$$

In Figure 10, we confirm the effect of changing policy variables with different degrees of externality from the average quality, γ. We examine the case where $\gamma = 1$, $\gamma = 1.3$, $\gamma = 1.5$, $\gamma = 1.8$, $\gamma = 2$, $\gamma = 2.01$, $\gamma = 2.1$, and $\gamma = 2.15$ while setting $c_F = 5$, $g_\theta = -0.02$, and $k = 2$. The first row of Figure 10 expresses the relationship between the number of differentiated goods, n, and policy variables when $\gamma < k$ is satisfied. The second row of Figure 10 expresses the relationship between the number of differentiated goods, n, and policy variables when $\gamma \geq k$ is satisfied. The third (fourth) row of Figure 10 expresses the relationship between the average quality of differentiated goods, Θ (consumption per capita, E), and policy variables. Regarding figure 10, there are some points worth noting. First, if $\gamma < k$ is satisfied, n is a decreasing function of s_F. Second, if $\gamma = k$ is satisfied, n is independent of s_F. Third, if $\gamma > k$ is satisfied, n is an increasing function of s_F. Interpretation of these results is similar to the reason we obtain proposition 3.4. Fourth, if $\theta_0^{\min} \leq \theta_0^{c1}$ is satisfied, Θ is independent from γ. This result is somewhat counterintuitive because from (77), Θ is small with high γ with given n. However, high γ leads

[64] In Appendix E, we derive the equilibrium and confirm an important analytical result.

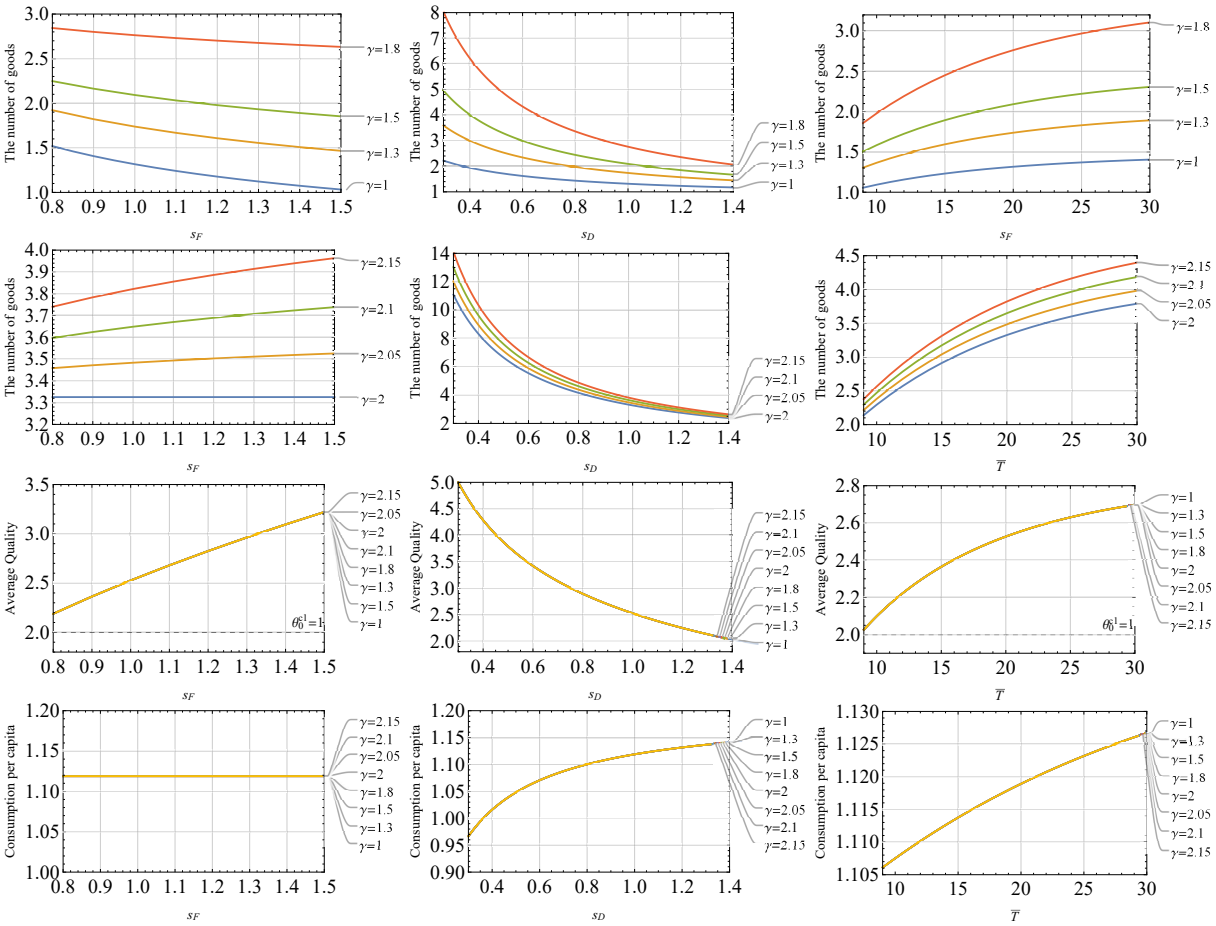

Figure 10: A numerical analysis changing γ, where $C_F(t) = \frac{1}{Q(t)} c_F$ and $C_D(t) = \frac{1}{Q(t)} c_D$

to large n, which increases the marginal loss from putting off disruptive innovation and makes disruptive innovation large. As shown in Appendix E, these two opposite effects offset each other completely. Fifth, if $\theta_0^{\min} \leq \theta_0^{c1}$ is satisfied, E is also independent of γ. Although Θ is independent of γ, high γ enables incumbents to conduct disruptive innovation with few resources because of the high positive externality, which makes E large. High γ leads to large n, which increases the number of firms conducting disruptive innovation; this effect then leads to E being small. As shown in Appendix E, these two opposite effects offset each other completely.

4 Conclusion

In this paper, we constructed a tractable model where heterogeneous incumbents conduct innovation to switch to new technology, and entrants conduct innovation to enter the market. We documented new insights that have never been obtained from growth models not considering heterogeneous incumbents' opportunities to switch their technology. Furthermore, there are some possibilities to develop our model because we intended to construct a tractable model that could be used to examine more complex situations.

The first possibility is theoretical. In our setting, disruptive innovation affects the growth rate only through indirect effects: the degree of disruptive innovation affects the average quality, which affects instantaneous profit and the cost of innovation by entrants, which indirectly affects the growth rate. One can construct a model, for example, in which the degree of disruptive innovation affects the expected initial quality of differentiated goods, which in turn enables the average quality to increase permanently; then, disruptive innovation affects the growth rate directly. We think that this extension offers a new method to resolve the scale effect puzzle.

A second possibility is application. In our setting, the behavior of incumbents is limited to production and disruptive innovation. Of course, there are many potential behaviors of incumbents. Thus, one can consider technology transfer and extend the present model to the North–South model. This can also consider business expansion and marketing activity conducted by heterogeneous incumbents, and then examine how these factors affect disruptive innovation and economic growth. Another extension of our model is in regard to replacement risk: to examine the incumbents' patent protection activity.

A third possibility is policy analysis. In our setting, we examine the effects of a subsidy (tax) for

innovation by incumbents and entrants and patent length. Other plausible policies, such as patent breadth and forward protection, should be examined in future studies.

References

[1] Acemoglu, D., Akcigit, U., 2012, Intellectual property rights policy, competition and innovation, Journal of the European Economic Association 10(1), 1–42.

[2] Acemoglu, D., Cao, D., 2015, Innovation by entrants and incumbents, Journal of Economic Theory 157, 255–294.

[3] Aghion, P., Harris, C., Howitt, P., Vickers, J., 2001, Competition, imitation and growth with step-by-step innovation, Review of Economic Studies 68(3), 467–492.

[4] Aghion, P., Harris, C., Vickers, J., 1997, Competition and growth with step-by-step innovation: an example, European Economic Review 41(3–5), 771–782.

[5] Aghion, P., Howitt, P., 1992, A model of growth through creative destruction, Econometrica 60(2), 323–351.

[6] Akcigit, U., Kerr, W., 2018, Growth through heterogeneous innovations, Journal of Political Economy 126(4), 1374–1443.

[7] Basu, S., 1996, Procyclical productivity: increasing returns or cyclical utilization? The Quarterly Journal of Economics 111, 719–751.

[8] Bartelsman, E.J., Doms, M., 2000, Understanding productivity: lessons from longitudinal microdata, Journal of Economic Literature 38(3), 569–594.

[9] Christensen, C., 1997, The innovator's dilemma: when new technologies cause great firms to fail, Boston, MA, USA: Harvard Business School Press.

[10] Chu, A., Cozzi, G., Furukawa, Y., Liao, C.H., 2017. Inflation and economic growth in a Schumpeterian model with endogenous entry of heterogeneous firms, European Economic Review, 98(C), 392–409.

[11] Chu, A., Cozzi, G., Fang, H., Furukawa, Y., Liao, C.H., 2019. Innovation and inequality in a monetary Schumpeterian model with heterogeneous households and firms, Review of Economic Dynamics, 34, 141–164.

[12] Cohen, W.M., 1995, Empirical studies of innovative activity, Handbook of the Economics of Innovation and Technological Change, 182–264.

[13] Cohen, W.M., Levin, R.C., 1989, Empirical studies of innovation and market structure, Handbook of Industrial Organization, 1059–1107.

[14] Denicolo, V., Zanchettin, P., 2012, Leadership cycles in a quality-ladder model of endogenous growth, Economic Journal, 122, 618–650.

[15] Dixit, A., Stiglitz, J., 1977, Monopolistic competition and optimum product diversity, The American Economic Review, 67(3), 297–308.

[16] Dinopoulos, E., Segerstrom, P., 2010, Intellectual property rights, multinational firms and economic growth, Journal of Development Economics, 92(1), 13–27.

[17] Etro, F., 2004, Innovation by leaders, Economic Journal 114(495), 281–303.

[18] Grossman, G., Helpman, E., 1991, Innovation and growth in the global economy, MIT Press, Cambridge, MA.

[19] Igami, M., 2017, Estimating the innovator's dilemma: structural analysis of creative destruction in the hard disk drive industry, 1981–1998, Journal of Political Economy 125(3), 798–847.

[20] Iwaisako, T., Ohki, K., 2019, Innovation by heterogeneous leaders, The Scandinavian Journal of Economics 121(4), 1673–1704.

[21] Jones, C.I., 1995, Time series tests of endogenous growth models, Quarterly Journal of Economics 110, 495–525.

[22] Jones, C.I., Williams, J.C., 2000, Too much of a good thing? The economics of investment in R&D. Journal of Economic Growth 5, 65–85.

[23] Kiedaisch, C., 2015, Intellectual property rights in a quality-ladder model with persistent leadership, European Economic Review 80(C), 194–213.

[24] Klette, T.J., Kortum, S., 2004, Innovating firms and aggregate innovation, Journal of Political Economy, 112(5), 986–1018.

[25] Ledezma, I., 2013, Defensive strategies in quality ladders, Journal of Economic Dynamics and Control 37, 176–194.

[26] Melitz, M., 2003, The impact of trade on intra-industry reallocations and aggregate industry productivity, Econometrica, 71, 1695–1725.

[27] Minniti, A., Parello, C., Segerstrom, P.S., 2013, A Schumpeterian growth model with random quality improvements, Economic Theory 52, 755–791.

[28] Norrbin, S.C., 1993, The relation between price and marginal cost in US industry: a contradiction. Journal of Political Economy 101, 1149–1164.

[29] Ohki, K., 2015, Analysis of transition dynamics caused by technological breakthroughs -cause of productivity slowdown and drop in existing firms' stock prices-, Structural Change and Economic Dynamics 35, 12–25.

[30] Parello, C., 2019, R&D policy and competition in a Schumpeterian growth model with heterogeneous firms, Oxford Economic Papers, 71(1), 187–202.

[31] Peretto, P, 1998, Technological change and population growth, Journal of Economic Growth 3(4), 283–311.

[32] Segerstrom, P., Anant, T.C.A., Dinopoulos, E., 1990, A Schumpeterian model of the product life cycle, American Economic Review, 80(5), 1077–1091.

[33] Segerstrom, P., 1998. Endogenous growth without scale effects. American Economic Review 88, 1290–1310.

[34] Segerstrom, P.S., 2007, Intel economics, International Economic Review 48, 247–280.

[35] Segerstrom, P.S., Zornierek, J.M., 1999, The R&D incentives of industry leaders, International Economic Review 40, 745–766.

[36] Thompson, P., Waldo, D., 1994, Growth and trustified capitalism, Journal of Monetary Economics 34, 455–462.

[37] Romer, P., 1990, Endogenous technological change, Journal of Political Economy 98, 71–102.

[38] World Bank, 2007, World Bank indicators. Report, World bank, Washington

A Appendix

We define $\nu(t)$ as the ratio of the number of differentiated goods embodying disruptive innovation to the aggregate number of differentiated goods at time t. The number of differentiated goods invented at time t is the number of differentiated goods embodying disruptive innovation at time t plus the number of differentiated goods invented by entrants at time t, which is expressed as $n(t)[\nu(t)+g_n]$.

Differentiated goods invented at time t having initial adjusted quality $\theta_0 \leq \theta_0^{c1}$ embody disruptive innovation immediately, and then the number of differentiated goods invented at time t and disruptive innovation conducted at time t are expressed as $n(t)[\nu(t)+g_n]\int_{\theta_0^{\min}}^{\theta_0^{c1}} dG(\theta_0)$.

Differentiated goods invented at time $t-\tilde{T}$ having initial adjusted quality $\theta_0\left(\tilde{T}\right)$ conduct disruptive innovation at time t, where $\theta_0\left(\tilde{T}\right)$ is the initial adjusted quality corresponding to be embodied disruptive innovation after $\tilde{T}$ periods have passed from the entry, which is derived from (13). Because the number of differentiated goods invented at time $t-\tilde{T}$ is $\exp\left[-g_n\tilde{T}\right]n(t)\left[\nu\left(t-\tilde{T}\right)+g_n\right]$, the number of differentiated goods invented at time $t-\tilde{T}$ that conduct disruptive innovation at time t is expressed as $\exp\left[-g_n\tilde{T}\right]n(t)\left[\nu\left(t-\tilde{T}\right)+g_n\right]g\left(\theta_0\left(\tilde{T}\right)\right)$. Because there are incumbents who have chosen the interior solution of the timing of disruptive innovation, having initial adjusted quality from $\theta_0(0)$ to $\theta_0\left(\bar{T}\right)$, the number of differentiated goods having initial adjusted quality $\theta_0^{c1}<\theta_0<\theta_0^{c2}$ that conduct disruptive innovation at time t is expressed as $\int_{\theta_0^{c1}}^{\theta_0^{c2}} \exp\left[-g_n\tilde{T}(\theta_0)\right]n(t)\left[\nu\left(t-\tilde{T}(\theta_0)\right)+g_n\right]dG(\theta_0)$.

Differentiated goods invented at time $t-\bar{T}$ having initial adjusted quality $\theta_0 \geq \theta_0^{c2}$ conduct disruptive innovation at time t, and then the number of differentiated goods invented at time $t-\bar{T}$ that conduct disruptive innovation at time t is expressed as $\exp\left[-g_n\bar{T}\right]n(t)\left[\nu\left(t-\bar{T}\right)+g_n\right]\int_{\theta_0^{c2}}^{\theta_0^{\max}} dG(\theta_0)$.

If $\theta_0^{\min} \leq \theta_0^{c1}$ is satisfied, we can express $\nu(t)$ as:

$$\nu(t) = \nu(t) Z_{\nu 1} + g_n Z_{g_n 1},$$

where

$$\begin{aligned} Z_{\nu,1} &= \int_{\theta_0^{\min}}^{\theta_0^{c1}} dG(\theta_0) + \int_{\theta_0^{1}}^{\theta_0^{2}} \exp\left[-(g_n+g_\nu)\tilde{T}(\theta_0)\right] dG(\theta_0) + \exp\left[-(g_n+g_\nu)\bar{T}\right] \int_{\theta_0^{c2}}^{\theta_0^{\max}} dG(\theta_0) \\ Z_{g_n,1} &= \int_{\theta_0^{\min}}^{\theta_0^{c1}} dG(\theta_0) + \int_{\theta_0^{1}}^{\theta_0^{2}} \exp\left[-g_n\tilde{T}(\theta_0)\right] dG(\theta_0) + \exp\left[-g_n\bar{T}\right] \int_{\theta_0^{c2}}^{\theta_0^{\max}} dG(\theta_0) \end{aligned}.^{65}$$

As long as $\tilde{T}(\theta_0)$, θ_0^{c1}, and θ_0^{c2} are constant, $Z_{\nu,1}$ and $Z_{g_n,1}$ are constant. When $Z_{\nu,1}$ and $Z_{g_n,1}$ are constant, ν is constant. Thus, $g_\nu = 0$ is satisfied, and we obtain:

$$\nu = \frac{Z_1}{1-Z_1} g_n,$$

where $Z_1 = \int_{\theta_0^{\min}}^{\theta_0^{c1}} dG(\theta_0) + \int_{\theta_0^{1}}^{\theta_0^{2}} \exp\left[-g_n\tilde{T}(\theta_0)\right] dG(\theta_0) + \exp\left[-g_n\bar{T}\right] \int_{\theta_0^{c2}}^{\theta_0^{\max}} dG(\theta_0)$. Substituting (9) and (18) into this function, we obtain the first row of (25).

If $\theta_0^{c1} < \theta_0^{\min} < \theta_0^{c2}$ is satisfied, there are no incumbents conducting disruptive innovation immediately. Then, we can express $\nu(t)$ as:

$$\nu = \frac{Z_2}{1-Z_2} g_n,$$

where $Z_2 = \int_{\theta_0^{\min}}^{\theta_0^{2}} \exp\left[-g_n\tilde{T}(\theta_0)\right] dG(\theta_0) + \exp\left[-g_n\bar{T}\right] \int_{\theta_0^{c2}}^{\theta_0^{\max}} dG(\theta_0)$. Substituting (9) and (18) into this function, we obtain the second row of (25).

If $\theta_0^{c2} < \theta_0^{\min}$ is satisfied, there are no incumbents conducting disruptive innovation before their patent protection expires. Then, we can express $\nu(t)$ as:

$$\nu = \frac{Z_3}{1-Z_3} g_n,$$

where $Z_3 = \exp\left[-g_n\bar{T}\right] \int_{\theta_0^{\min}}^{\theta_0^{\max}} dG(\theta_0)$. Substituting (9) and (18) into this function, we obtain the third row of (25).

From the above discussion, we can derive ν when $g_n > 0$; however, this relational expression is valueless when $g_n = 0$.[66] Then, we have to find another way to derive ν.

When $g_n = 0$, the number of new invented differentiated goods invented only by incumbents is identical for all periods, and the expected value of their duration is also identical. Thus, we can express

[65] Here, g_ν is the growth rate of ν.
[66] When $g_n = 0$, $Z_1 = Z_2 = Z_3 = 1$ is satisfied, and then $\nu = \nu Z + g_n Z$ becomes $\nu = \nu$.

the ratio of the number of differentiated goods conducting disruptive innovation to the aggregate number of differentiated goods as an inverse number of the expected value of duration. We can derive the expected value of duration, $E\left(\tilde{T}\right)$, as

$$\begin{aligned}
E\left(\tilde{T}\right) &= 0\int_{\theta_0^{\min}}^{\theta_0^{c1}} dG\left(\theta_0\right) + \int_{\theta_0^{c1}}^{\theta_0^{c2}} \tilde{T}\left(\theta_0\right) dG\left(\theta_0\right) + \bar{T}\int_{\theta_0^{c2}}^{\theta_0^{\max}} dG\left(\theta_0\right) \quad if \quad \theta_0^{\min} \leq \theta_0^{c1} \\
E\left(\tilde{T}\right) &= \int_{\theta_0^{\min}}^{\theta_0^{c2}} \tilde{T}\left(\theta_0\right) dG\left(\theta_0\right) + \bar{T}\int_{\theta_0^{c2}}^{\theta_0^{\max}} dG\left(\theta_0\right) \quad if \quad \theta_0^{c1} < \theta_0^{\min} \leq \theta_0^{c2} \\
E\left(\tilde{T}\right) &= \bar{T}\int_{\theta_0^{\min}}^{\theta_0^{\max}} dG\left(\theta_0\right) \quad if \quad \theta_0^{c2} < \theta_0^{\min}
\end{aligned} .$$

Substituting (9) and (18) into this function, we obtain

$$\begin{aligned}
\nu &= \frac{\left[\theta_0^{c1}\right]^k k[-g_\theta]}{1-\exp\left[-k[-g_\theta]\bar{T}\right]} \quad if \quad \theta_0^{\min} \leq \theta_0^{c1} \\
\nu &= \frac{k[-g_\theta]}{1-\exp\left[-k[-g_\theta]\bar{T}\right]\left[\theta_0^{c1}\right]^{-k} - k\ln\left[\theta_0^{c1}\right]} \quad if \quad \theta_0^{c1} < \theta_0^{\min} \leq \theta_0^{c2} \\
\nu &= \frac{1}{\bar{T}} \quad if \quad \theta_0^{c2} < \theta_0^{\min}
\end{aligned} .$$

Using L'Hopital's rule, we can confirm that (25) coincides with the above equation. Thus, ν can be expressed as (25), even when $g_n = 0$.

B Appendix

In this appendix, we derive (27). At time $t-\tilde{T}$, $[\nu+g]\exp\left[-g_n\tilde{T}\right]n\left(t\right)$ differentiated goods are invented. At time t, differentiated goods invented $\tilde{T}$ periods before remain if inventors draw initial adjusted quality $\theta_0 \geq \theta_0\left(\tilde{T}\right)$. Because the adjusted quality of differentiated goods invented $\tilde{T}$ periods before having initial adjusted quality θ_0 is expressed as $\exp\left[g_\theta\tilde{T}\right]\theta_0$, the sum of adjusted quality of remaining differentiated goods, which are invented at time $t-\tilde{T}$, at time t is expressed as:

$$[\nu+g]\exp\left[-g_n\tilde{T}\right]n\left(t\right)\int_{\theta_0\left(\tilde{T}\right)}^{\theta_0^{\max}} \exp\left[g_\theta\tilde{T}\right]\theta_0 dG\left(\theta_0\right)$$

At time t, because there are differentiated goods invented between time $t-\bar{T}$ and time t, the sum of the adjusted quality of remaining differentiated goods, which were invented between time $t-\bar{T}$ and time t,

at time t is expressed as:

$$\int_{0}^{\bar{T}} [\nu + g] \exp\left[-g_n \tilde{T}\right] n(t) \int_{\theta_0(\tilde{T})}^{\theta_0^{\max}} \exp\left[g_\theta \tilde{T}\right] \theta_0 dG(\theta_0) d\tilde{T}.$$

In a similar manner, we can derive the number of differentiated goods at time t as:

$$\int_{0}^{\bar{T}} [\nu + g] \exp\left[-g_n \tilde{T}\right] n(t) \int_{\theta_0(\tilde{T})}^{\theta_0^{\max}} dG(\theta_0) d\tilde{T}.$$

The average quality of differentiated goods, Θ, is calculated as the sum of the quality of differentiated goods produced at time t divided by the number of differentiated goods at time t, and then we obtain:

$$\Theta = \frac{\int_{0}^{\bar{T}} \exp\left[-g_n \tilde{T}\right] \int_{\theta_0(\tilde{T})}^{\theta_0^{\max}} \exp\left[g_\theta \tilde{T}\right] \theta_0 dG(\theta_0) d\tilde{T}}{\int_{0}^{\bar{T}} \exp\left[-g_n \tilde{T}\right] \int_{\theta_0(\tilde{T})}^{\theta_0^{\max}} dG(\theta_0) d\tilde{T}}.$$

Substituting (9) and (18) into the above equation, we obtain the first row of (27).

If $\theta_0^{c1} < \theta_0^{\min}$ is satisfied, all incumbents produce differentiated goods and put off disruptive innovation for some time. We define T^c as the time when the incumbent drawing the lowest initial adjusted quality, $\theta_0^{\min}$, conducts disruptive innovation. Then, we obtain the average quality of the differentiated goods as:

$$\Theta = \frac{\int_{0}^{\tilde{T}^c} \exp\left[-g_n \tilde{T}\right] \left[\int_{\theta_0^{\min}}^{\theta_0^{\max}} \exp\left[g_\theta \tilde{T}\right] \theta_0 dG(\theta_0)\right] d\tilde{T} + \int_{\tilde{T}^c}^{\bar{T}} \exp\left[-g_n \tilde{T}\right] \left[\int_{\theta_0(\tilde{T})}^{\theta_0^{\max}} \exp\left[g_\theta \tilde{T}\right] \theta_0 dG(\theta_0)\right] d\tilde{T}}{\int_{0}^{\tilde{T}^c} \exp\left[-g_n \tilde{T}\right] \left[\int_{\theta_0^{\min}}^{\theta_0^{\max}} dG(\theta_0)\right] d\tilde{T} + \int_{\tilde{T}^c}^{\bar{T}} \exp\left[-g_n \tilde{T}\right] \left[\int_{\theta_0(\tilde{T})}^{\theta_0^{\max}} dG(\theta_0)\right] d\tilde{T}}.$$

Substituting (9) and (18) into the above equation, we obtain the second row of (27).

If $\theta_0^{c2} < \theta_0^{\min}$ is satisfied, all incumbents produce differentiated goods until their patent protection expires, and then we obtain the average quality of the differentiated goods as:

$$\Theta = \frac{\int_{0}^{\bar{T}} \exp\left[-g_n \tilde{T}\right] \left[\int_{\theta_0^{\min}}^{\theta_0^{\max}} \exp\left[g_\theta \tilde{T}\right] \theta_0 dG(\theta_0)\right] d\tilde{T}}{\int_{0}^{\bar{T}} \exp\left[-g_n \tilde{T}\right] \left[\int_{\theta_0^{\min}}^{\theta_0^{\max}} dG(\theta_0)\right] d\tilde{T}}.$$

Substituting (9) and (18) into the above equation, we obtain the second row of (27).

C Appendix

In this appendix, we examine the property of $X_1(\theta_0, g_Q)$ and $X_2(\theta_0, g_Q)$. Differentiating $X_1(\theta_0, g_Q)$ and $X_2(\theta_0, g_Q)$ with g_Q, we obtain:

$$\begin{aligned}
\frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} &= -\frac{X_{10}}{[\rho - g_L + g_Q][\rho - g_L + g_Q - g_\theta]^2}\frac{\theta_0}{\sigma} \\
\frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q} &= -\frac{X_{20}}{[\rho - g_L + g_Q][\rho - g_L + g_Q - g_\theta]^2}\frac{\theta_0}{\sigma}
\end{aligned}$$

where

$$\begin{aligned}
X_{10} &= \left[\begin{array}{l} [\rho - g_L + g_Q]\left[1 - \exp\left[-[\rho - g_L + g_Q - g_\theta]\tilde{T}\right]\right] \\ +[-g_\theta][\rho - g_L + g_Q - g_\theta]\tilde{T}\exp\left[-[\rho - g_L + g_Q - g_\theta]\tilde{T}\right] \end{array}\right] \\
X_{20} &= \left[\begin{array}{l} [\rho - g_L + g_Q]\left[1 - \exp\left[-[\rho - g_L + g_Q - g_\theta]\bar{T}\right]\right] \\ +\left[[\rho - g_L + g_Q - g_\theta]\frac{\theta_0^{c2}}{\theta_0} - [\rho - g_L + g_Q]\right][\rho - g_L + g_Q - g_\theta]\bar{T}\exp\left[-[\rho - g_L + g_Q - g_\theta]\bar{T}\right] \end{array}\right].
\end{aligned}$$

Because X_{10} is unambiguously positive, we can easily confirm $\frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} < 0$.

When θ_0 goes to positive infinity, X_{20} is expressed as:

$$\lim_{\theta_0 \to +\infty} X_{20} = \left[\begin{array}{l} [\rho - g_L + g_Q]\left[1 - \exp\left[-[\rho - g_L + g_Q - g_\theta]\bar{T}\right]\right] \\ -[\rho - g_L + g_Q][\rho - g_L + g_Q - g_\theta]\bar{T}\exp\left[-[\rho - g_L + g_Q - g_\theta]\bar{T}\right] \end{array}\right].$$

Because $\lim_{\theta_0 \to +\infty} X_{20}$ is an increasing function of $\bar{T}$ and $\bar{T} > 0$, $\lim_{\theta_0 \to +\infty} X_{20} > 0$ is satisfied.[67] Because X_{20} is a decreasing function of θ_0, we can confirm that X_{20} is also positive. Thus, we can confirm $\frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q} < 0$.

Differentiating $\frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q}$ and $\frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q}$ with θ_0, we obtain:

$$\begin{aligned}
\frac{\partial}{\partial \theta_0}\frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} &= -\frac{1}{\sigma[\rho - g_L + g_Q]}\left[\frac{X_{10}}{[\rho - g_L + g_Q - g_\theta]^2} + \frac{1 - [-g_\theta]\tilde{T}}{[-g_\theta]}\exp\left[-[\rho - g_L + g_Q - g_\theta]\tilde{T}\right]\right] \\
\frac{\partial}{\partial \theta_0}\frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q} &= -\frac{\lim_{\theta_0 \to +\infty} X_{20}}{[\rho - g_L + g_Q][\rho - g_L + g_Q - g_\theta]^2}\frac{\theta_0}{\sigma}
\end{aligned}.$$

[67] When $\bar{T} = 0$, $\lim_{\theta_0 \to +\infty} X_{20} = 0$.

Then, we can confirm $\frac{\partial}{\partial\theta_0}\frac{\partial X_1(\theta_0,g_Q)}{\partial g_Q} < 0$ and $\frac{\partial}{\partial\theta_0}\frac{\partial X_2(\theta_0,g_Q)}{\partial g_Q} < 0$.[68]

Differentiating $X_1(\theta_0, g_Q)$ and $X_2(\theta_0, g_Q)$ with $[s_F c_F - s_D c_D]$, we obtain:

$$\begin{array}{rcl} \frac{\partial X_1(\theta_0,g_Q)}{\partial[s_F c_F - s_D c_D]} & = & \exp\left[-[\rho - g_L + g_Q]\tilde{T}\right] > 0 \\ \frac{\partial X_2(\theta_0,g_Q)}{\partial[s_F c_F - s_D c_D]} & = & \exp\left[-[\rho - g_L + g_Q]\bar{T}\right] > 0 \end{array}.$$

Differentiating the above function with respect to θ_0, we obtain:

$$\begin{array}{rcl} \frac{\partial}{\partial\theta_0}\frac{\partial X_1(\theta_0,g_Q)}{\partial[s_F c_F - s_D c_D]} & = & -[\rho - g_L + g_Q]\exp\left[-[\rho - g_L + g_Q]\tilde{T}\right]\frac{\partial\tilde{T}}{\partial\theta_0} < 0 \\ \frac{\partial}{\partial\theta_0}\frac{\partial X_2(\theta_0,g_Q)}{\partial[s_F c_F - s_D c_D]} & = & 0 \end{array}.$$

D Appendix

In this appendix, we examine the sign of $\frac{dg_Q}{ds_F}$ and $\frac{d\theta_0^{c1}(g_Q)}{ds_F}$, where R&D technology is expressed as $C_F(t) = \frac{E(t)L(t)}{n(t)[\Theta]^\gamma}c_F$ and $C_D(t) = \frac{E(t)L(t)}{n(t)[\Theta]^\gamma}c_D$.

Differentiating $\theta_0^{c1} = [\sigma[\rho - g_L + g_Q][s_F c_F - s_D c_D]]^{\frac{1}{\gamma}}\left[\frac{k}{k-1}\right]^{\frac{1-\gamma}{\gamma}}$ with s_F yields:

$$\frac{d\theta_0^{c1}}{ds_F} = \frac{\sigma}{\gamma}\left[\sigma[\rho - g_L + g_Q][s_F c_F - s_D c_D]\frac{k}{k-1}\right]^{\frac{1-\gamma}{\gamma}}\left[[\rho - g_L + g_Q]c_F - [s_F c_F - s_D c_D]\left[-\frac{dg_Q}{ds_F}\right]\right]. \tag{72}$$

Substituting (9) and (53) into (54), we obtain:[69]

$$[s_F c_F - s_D c_D][\rho - g_L + g_Q]\left[\frac{\left[1 - \exp\left[-[\rho - g_L + g_Q + k[-g_\theta]]\bar{T}\right]\right]}{[\rho - g_L + g_Q + k[-g_\theta]]}\frac{\left[\theta_0^{c1}\right]^{-k}}{[k-1]}\right] = s_D c_D. \tag{73}$$

Substituting (51) into (73), we obtain:

$$\left[\theta_0^{c1}\right]^{\gamma-k}\frac{\left[1 - \exp\left[-[\rho - g_L + g_Q + k[-g_\theta]]\bar{T}\right]\right]}{[\rho - g_L + g_Q + k[-g_\theta]]} = \left[\frac{k-1}{k}\right]^\gamma k\sigma s_D c_D. \tag{74}$$

[68] $\left[1 - [-g_\theta]\tilde{T}\right]\left[\exp\left[-[\rho + g_Q - g_\theta]\tilde{T}\right]\right]$ is a decreasing function of $\tilde{T}$, and is zero when $\tilde{T} = 0$. Since $\tilde{T}$ is positive, $\left[1 - [-g_\theta]\tilde{T}\right]\left[\exp\left[-[\rho + g_Q - g_\theta]\tilde{T}\right]\right]$ is also positive.

[69] Regardless of the specification of C_F and C_D, (73) is satisfied as long as $\theta_0^{c1} \geq \theta_0^{min}$ is satisfied.

Differentiating (74) with s_F, and substituting (72) yields:

$$\frac{[\gamma - k]\, c_F}{\gamma\,[s_F c_F - s_D c_D]} = \frac{dg_Q}{ds_F}\left[\frac{\left[1-\left[1+[\rho - g_L + g_Q + k\,[-g_\theta]]\,\bar{T}\right]\exp\left[-[\rho - g_L + g_Q + k\,[-g_\theta]]\,\bar{T}\right]\right]}{\left[1-\exp\left[-[\rho - g_L + g_Q + k\,[-g_\theta]]\,\bar{T}\right]\right][\rho - g_L + g_Q + k\,[-g_\theta]]} - \frac{[\gamma - k]}{\gamma\,[\rho - g_L + g_Q]}\right]. \tag{75}$$

First, left-hand-side of (75) increases with γ, and brackets of right-hand-side decreases with γ. Then, $\frac{d}{d\gamma}\frac{dg_Q}{ds_F} > 0$ must be satisfied; that is, $\frac{dg_Q}{ds_F}$ must increase monotonically with high γ. Second, from (75), $\frac{dg_Q}{ds_F} = 0$ must be satisfied when $\gamma = k$. Thus, $\frac{dg_Q}{ds_F}$ is negative (positive) when $\gamma < k$ ($\gamma > k$) is satisfied. Then, we can confirm the sign of $\frac{dg_Q}{ds_F}$ expressed as (55).

We can rewrite (75) as:

$$\frac{d\theta_0^{c1}}{ds_F} = \left[\frac{1}{\gamma - k}\frac{dg_Q}{ds_F}\right]\theta_0^{c1}\frac{\left[1-\left[1+[\rho - g_L + g_Q + k\,[-g_\theta]]\,\bar{T}\right]\exp\left[-[\rho - g_L + g_Q + k\,[-g_\theta]]\,\bar{T}\right]\right]}{\left[1-\exp\left[-[\rho - g_L + g_Q + k\,[-g_\theta]]\,\bar{T}\right]\right][\rho - g_L + g_Q + k\,[-g_\theta]]}.$$

From (55), $\left[\frac{1}{\gamma-k}\frac{dg_Q}{ds_F}\right] > 0$ is satisfied.[70] We can easily confirm that $\left[1-\left[1+[\rho - g_L + g_Q + k\,[-g_\theta]]\,\bar{T}\right]\exp\left[-[\rho - g_L + g_Q\right.\right.$ 0 is satisfied when $\bar{T} > 0$. Thus, $\frac{d\theta_0^{c1}}{ds_F} > 0$ must be satisfied.

In the case of section 3.1, which is the special case of section 3.2, $\gamma = 1$ is satisfied. Because we assume a Pareto distribution where $k > 1$, we can confirm that $\frac{dg_Q}{ds_F} < 0$ and $\frac{d\theta_0^{c1}}{ds_F} > 0$ are satisfied, and we obtain the result of propositions 3.2 and 3.3.

E Appendix

In this appendix, we conduct an analytical analysis where R&D technology is expressed as $C_F(t) = \frac{E(t)L}{[\Theta]^\gamma}c_F$ and $C_D(t) = \frac{E(t)L}{[\Theta]^\gamma}c_D$.

The labor market clearing condition is expressed as:

$$E\left[\frac{\sigma - 1}{\sigma} + \frac{\nu}{[\Theta]^\gamma}nc_D + \frac{g_n}{[\Theta]^\gamma}nc_F\right] = 1. \tag{76}$$

From this equation, in steady-state equilibrium, $g_E = 0$ and $g_n = 0$ must be satisfied, and then $g_Q = 0$, $g_u = 0$, and $g_C = g_L$ are satisfied.

[70] Since $\frac{d}{d\gamma}\frac{dg_Q}{ds_F} > 0$, we can confirm $\left[\frac{1}{\gamma-k}\frac{dg_Q}{ds_F}\right] > 0$ is satisfied, even when $\gamma = k$ by using L'Hopital's rule.

Necessary conditions for the maximization problem of $\tilde{T}$ are expressed as:[71]

$$\begin{aligned} \theta_0^{c1}(n) &= \left[\tfrac{k-1}{k}\right]^{\frac{\gamma-1}{\gamma}} \left[n\sigma\left[\rho - g_L\right]\left[s_F c_F - s_D c_D\right]\right]^{\frac{1}{\gamma}} \\ \theta_0^{c2}(n) &= \exp\left[\left[-g_\theta\right]\bar{T}\right]\theta_0^{c1}(n) \\ \tilde{T}(\theta_0, n) &= \tfrac{1}{-g_\theta}\ln\left[\tfrac{\theta_0}{\theta_0^{c1}(n)}\right] \end{aligned} . \tag{77}$$

The initial values of an incumbent entering at time s and drawing θ_0 are expressed as:

$$\begin{aligned} V(\theta_0, 0, s) &= \tfrac{EL(s)}{[\Theta]^\gamma} X_0 && for \quad \theta_0 \le \theta_0^{c1} \\ V(\theta_0, 0, s) &= \tfrac{EL(s)}{[\Theta]^\gamma} X_1\left(\theta_0, n, \theta_0^{c1}(n)\right) && for \quad \theta_0^{c1} \le \theta_0 \le \theta_0^{c2} \\ V(\theta_0, 0, s) &= \tfrac{EL(s)}{[\Theta]^\gamma} X_2\left(\theta_0, n, \theta_0^{c1}(n)\right) && for \quad \theta_0 > \theta_0^{c2} \end{aligned} , \tag{78}$$

where

$$\begin{aligned} X_0 &= \left[s_F c_F - s_D c_D\right] \\ X_1\left(\theta_0, n, \theta_0^{c1}(n)\right) &= \left[\begin{array}{l} \frac{1-\exp\left[-[\rho-g_L-g_\theta]\tilde{T}(\theta_0,n)\right]}{\rho-g_L-g_\theta}\frac{\theta_0}{n\sigma}\left[\frac{k}{k-1}\theta_0^{c1}(n)\right]^{\gamma-1} \\ +\left[s_F c_F - s_D c_D\right]\exp\left[-\left[\rho - g_L\right]\tilde{T}(\theta_0, n)\right] \end{array}\right] \\ X_2\left(\theta_0, n, \theta_0^{c1}(n)\right) &= \left[\begin{array}{l} \frac{1-\exp\left[-[\rho-g_L-g_\theta]\bar{T}\right]}{\rho-g_L-g_\theta}\frac{\theta_0}{n\sigma}\left[\frac{k}{k-1}\theta_0^{c1}(n)\right]^{\gamma-1} \\ +\left[s_F c_F - s_D c_D\right]\exp\left[-\left[\rho - g_L\right]\bar{T}\right] \end{array}\right] \end{aligned} . \tag{79}$$

where θ_0^{c1}, θ_0^{c2}, and $\tilde{T}(\theta_0, g_Q)$ satisfy (77).

We restrict our analysis to where $\theta_0^{\min} \le \theta_0^{c1}$ is satisfied, and then the free entry condition is expressed as:

$$\int_{\theta_0^{\min}}^{\theta_0^{c1}} X_0 dG(\theta_0) + \int_{\theta_0^{c1}}^{\theta_0^{c2}} X_1\left(\theta_0, n, \theta_0^{c1}(n)\right) dG(\theta_0) + \int_{\theta_0^{c2}}^{\theta_0^{\max}} X_2\left(\theta_0, n, \theta_0^{c1}(n)\right) dG(\theta_0) = c_F \tag{80}$$

where θ_0^{c1}, θ_0^{c2}, and $\tilde{T}$ satisfy (77), and X_0, X_1, and X_2 satisfy (79), which determines the equilibrium value of n. Substituting the equilibrium value of n into (77), we obtain the equilibrium value of θ_0^{c1}. Substituting the equilibrium value of θ_0^{c1} into (27), we obtain the equilibrium value of average quality.

Next, we confirm the sign of $\frac{dn}{ds_F}$ and $\frac{d\theta_0^{c1}(g_Q)}{ds_F}$. Differentiating $\theta_0^{c1} = \left[\frac{k-1}{k}\right]^{\frac{\gamma-1}{\gamma}}\left[n\sigma\left[\rho - g_L\right]\left[s_F c_F - s_D c_D\right]\right]^{\frac{1}{\gamma}}$

[71] $g_\theta < 0$ must be satisfied not to violate the sufficient condition of the maximization problem of $\tilde{T}$.

with s_F, we obtain:

$$\frac{d\theta_0^{c1}}{ds_F}\frac{s_F}{\theta_0^{c1}} = \frac{1}{\gamma - k}\frac{dn}{ds_F}\frac{s_F}{n}. \tag{81}$$

Because (73) is satisfied regardless of the specification of R&D technology, substituting (77) into (73), we obtain:

$$\left[\theta_0^{c1}(n)\right]^{\gamma-k}\frac{1}{n} = \left[\frac{k-1}{k}\right]^{\gamma}\frac{[\rho - g_L + k[-g_\theta]]}{1-\exp\left[-[\rho - g_L + k[-g_\theta]]\bar{T}\right]}k\sigma s_D c_D. \tag{82}$$

Differentiating (82) with s_F and substituting (81) yields:

$$\frac{1}{n}\frac{dn}{ds_F} = \frac{\gamma - k}{k}\frac{c_F}{s_F c_F - s_D c_D}. \tag{83}$$

From (83), we can confirm the sign of $\frac{dn}{ds_F}$, expressed as:

$$\begin{array}{lcll} \frac{dn}{ds_F} & < & 0 & if \quad \gamma < k \\ \frac{dn}{ds_F} & = & 0 & if \quad \gamma = k \\ \frac{dn}{ds_F} & > & 0 & if \quad \gamma > k \end{array}, \tag{84}$$

From (84) and (81), we can confirm $\frac{d\theta_0^{c1}(g_Q)}{ds_F} > 0$.

Next, we confirm that the equilibrium value of average quality Θ is independent of γ. The free entry condition is rewritten as:

$$\frac{\left[\theta_0^{c1}\right]^k[k-1]c_D}{[s_F c_F - s_D c_D][\rho - g_L]} = \frac{1-\exp\left[-[\rho - g_L + k[-g_\theta]]\bar{T}\right]}{[\rho - g_L + k[-g_\theta]]s_D}. \tag{85}$$

From this equation, we can confirm that θ_0^{c1} is independent of γ. Thus, $\Theta = \frac{k}{k-1}\theta_0^{c1}$ is also independent of γ.

Finally, we confirm that the equilibrium value of consumption per capita is independent of s_F, c_F, and γ if $\theta^{c1} \geq \theta_0^{\min}$ is satisfied. Because $g_Q = 0$ is satisfied, the labor market clearing condition is expressed as:

$$E\left[\frac{\sigma - 1}{\sigma} + \frac{\nu}{[\Theta]^\gamma}nc_D\right] = 1.$$

Thus, we can confirm the result about consumption per capita by analyzing $\frac{\nu}{[\Theta]^{\gamma}} nc_D$. If $\theta^{c1} \geq \theta_0^{\min}$ is satisfied, from the first rows of (25) and (27), we obtain:

$$\frac{\nu}{[\Theta]^{\gamma}} nc_D = \frac{\left[\theta_0^{c1}\right]^k [k-1]\, c_D}{[s_F c_F - s_D c_D]\,[\rho - g_L]} \frac{[-g_\theta]}{\sigma \left[1 - \exp\left[-k\,[-g_\theta]\,\bar{T}\right]\right]}. \tag{86}$$

Substituting (85) into this equation, we obtain:

$$\frac{\nu}{[\Theta]^{\gamma}} nc_D = \frac{1 - \exp\left[-\left[\rho - g_L + k\,[-g_\theta]\right]\bar{T}\right]}{1 - \exp\left[-k\,[-g_\theta]\,\bar{T}\right]} \frac{[-g_\theta]}{[\rho - g_L + k\,[-g_\theta]]\,\sigma s_D}. \tag{87}$$

From (87), we can confirm that $\frac{\nu}{[\Theta]^{\gamma}} nc_D$ is independent from s_F, c_F, and γ. Thus, E is also independent of s_F, c_F, and γ.