Disruptive Innovation by Heterogeneous Incumbents and Economic Growth: When do incumbents switch to new technology?

Ohki, Kazuyoshi

31 October 2019
Disruptive Innovation by Heterogeneous Incumbents and Economic Growth:

When do incumbents switch to new technology? *

Kazuyoshi Ohki†

October, 30, 2019

Abstract

In this paper, we construct a tractable endogenous growth model to examine heterogeneous incumbents’ current technology-switching behavior. Then, we examine the effects of policies such as a subsidy for innovation by incumbents, a subsidy for innovation by entrants, and the extension of patent length. Our setting suggests interesting and counterintuitive results. High quality incumbents tend to be less likely to conduct innovation, which is inconsistent with Schumpeter’s hypothesis. A subsidy for innovation by entrants decreases the average quality of differentiated goods. Moreover, it may decrease the growth rate of the economy if the positive spillover of innovation from average quality production is adequately large. Aggregate innovation can be small even when the population size is large if the barriers to entry are extremely high.

Keywords: Economic Growth, R&D, Firm-Heterogeneity, Innovation by Incumbents, IPR Policy

JEL classification: O31, O32, O33, O34, O41

*The author is grateful to Tatsuro Iwaisako and the participants of the RoMacS Workshop in Kanazawa for their valuable comments. The author also gratefully acknowledges financial support from a Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Young Scientists (No.19K13646). Any remaining errors are the responsibility of the author.

†Faculty of Economics and Management, Institute of Human and Social Sciences, Kanazawa University, Kakuma, Kanazawa 920-1192 JAPAN. email:kazuyoshi.ohki@gmail.com
1 Introduction

Since the eighteenth century, the world economy has grown through the accumulation of capital stock, population growth, and technological improvements. Technological improvements, which are the result of research and development (R&D) activities, have perhaps played the most important role for economic development in recent years. Many researchers have examined the effects of R&D activities via an endogenous R&D-based growth model. First, Romer (1990), Segerstrom, Anant and Dinopoulos (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992) constructed very simple R&D-based growth models that focused solely on innovation by entrants (or followers).\footnote{Followers are defined as firms once having state-of-the-art technology that have been leapfrogged by another firm.} The weak point of these models is that they cannot examine the innovative activity of incumbents (or leaders).\footnote{Bartelsman and Doms (2000) report that 75 percent of total factor productivity growth results from R&D activities by incumbent firms.} Second, Thompson and Waldo (1994), Aghion, Harris, and Vickers (1997), Peretto (1998), Segerstrom and Zorniercek (1999), Aghion, Harris, Howitt, and Vickers (2001), Etro (2004), Segerstrom (2007), Ledezma (2013), and Kiedaisch (2015) construct advanced R&D-based growth models that can examine the innovative activity of incumbents and leaders that have technological superiority or first move advantage. These models still have weak points in that they cannot capture the heterogeneity of innovative activity by incumbents or they restrict their analysis to a particular industry.\footnote{In the real economy, there are many industries, so firm size and strategic incentives differ, and industries can act heterogeneously with the same economic policy.} Recently, Klette and Kortum (2004), Acemoglu and Akcigit (2012), Denicolo and Zanchettin (2012), Acemoglu and Cao (2015), Akcigit and Kerr (2018), Parello (2019), and Iwaisako and Ohki (2019) attempted to construct more advanced R&D-based growth models that can examine the innovative activity of both heterogeneous incumbents and entrants (or both heterogeneous leaders and followers) in many industries. The present paper examines a series of these studies.\footnote{Some studies examine economic growth or welfare under firm heterogeneity, without, however, examining the endogenous innovative activity of heterogeneous incumbents (or leaders): Melitz (2003), Minniti, Parello, and Segerstrom (2013), Chu, Cozzi, Furukawa, and Liao (2017), and Chu, Cozzi, Fan, Furukawa, and Liao (2019).}

The present paper considers innovation by both heterogeneous incumbents and entrants. Incumbents are heterogeneous in the quality of their invented differentiated goods and efficiency of production, which are drawn from an exogenous distribution when they invent new goods. The contribution of our paper is that we consider the incumbents’ technology-switching behavior. We assume that the quality of their invented differentiated goods declines exogenously with the passage of time, which decreases their instan-
taneous profit. Then, incumbents decide to wash their hands of current technology and switch to new technology. The timing of the switch is determined endogenously. When they switch to new technology, they disrupt their current position to obtain a new sequential profit flow. In this paper, we call this incumbent behavior “disruptive innovation”, which is defined by Christensen (1997). Using this model, we examine how a subsidy (tax) for innovation by entrants, a subsidy (tax) for innovation by incumbents, and the extension (shortening) of patent length affect the growth rate of an economy and the average quality of differentiated goods.

The main results of our analysis are as follows. First, incumbents with high-quality products tend not to conduct disruptive innovation. This result is inconsistent with Schumpeter’s hypothesis; however, it is consistent with the finding of Christensen (1997), who observed this in, for example, the hard-disk and excavating equipment industries. From this point of view, the present paper examines the relationship between economic growth and R&D activity, analytically capturing the exciting property discovered by Christensen (1997). Second, in our basic model, a subsidy for innovation by entrants, a subsidy for innovation by incumbents, and the extension of patent length increase the growth rate of the economy. This result is consistent with the idea that generous treatment of firms stimulates the incentive to entry, which is pointed out in many related studies. Third, a subsidy for innovation by entrants decreases the average quality of differentiated goods, while that for innovation by incumbents and an extension of patent length increase it. In our model, average quality is an increasing function of disruptive innovation by incumbents. It is then a natural result that a subsidy for innovation by incumbents increases the average quality of differentiated goods because it stimulates the incentive to conduct disruptive innovation. Extension of patent length also stimulates disruptive innovation by incumbents because, as noted above, it stimulates innovation by entrants, which decreases the instantaneous profit of incumbents. A decrement of instantaneous profit decreases the incentive to put off disruptive innovation, thereby stimulating the incentive to conduct disruptive innovation. A subsidy for innovation by entrants has two opposite effects on the incentive to conduct disruptive innovation. On one hand, it decreases the instantaneous profit of incumbents, which increases the incentive to engage in disruptive innovation. On the other hand, it decreases the expected benefit from disruptive innovation, which increases the incentive to put

\footnote{In the economic context, except for our paper, Igami (2017) examines the theory of Christensen (1997), focusing on the evolution of market structure using the empirical industrial organization literature.}

\footnote{This result is interesting in that the extension of patent length makes incumbents producing at not so high quality switch their current technology earlier.}
off disruptive innovation. We show analytically that the latter effect always dominates the former, and that a subsidy for innovation by entrants decreases the average quality of differentiated goods. Fourth, if the positive spillover of innovation from average quality is sufficiently large, a subsidy for innovation by entrants does decrease the growth rate of the economy. This counterintuitive result comes from our original framework in which heterogeneous incumbents conduct disruptive innovation. A subsidy for innovation by entrants decreases average quality, which decreases the expected benefit of conducting disruptive innovation. This effect decreases the value of incumbents given the growth rate of the economy, and then the growth rate of the economy must decrease to satisfy the free entry condition. In section 3.2, we show analytically that this phenomenon dominates the effect that a subsidy for innovation by entrants decreases the cost of entry when the positive spillover of innovation from average quality is sufficiently large. Fifth, if there is no negative externality of innovation from market size, and if barriers to entry are extremely high, then aggregate innovation is small when population size is large. This counterintuitive result also comes from our new framework in which heterogeneous incumbents conduct disruptive innovation. When barriers to entry are extremely high, the number of entrants goes to zero. Then, innovation is conducted only by incumbents. Large population size makes instantaneous profit high if there is no negative externality of innovation from market size, which increases the incentive to put off switching from current technology. As shown in section 3.3, this effect decreases aggregate disruptive innovation.

Recently, the definition of innovation has become increasingly diverse, and some studies examine diverse innovation simultaneously. On one hand, innovation to create new products and capture market leadership is defined as “exploration innovation”, “product innovation”, and “external innovation”. On the other hand, innovation to improve product lines that they are currently serving is defined as “exploitation innovation”, “process innovation”, “internal innovation”, and “incremental innovation”. In this context, our model is interpreted as follows. On one hand, entrants conduct innovation to capture market leadership; this innovation is categorized as “exploration innovation”, “product innovation”, and

7 Although in the present paper, disruptive innovation does not affect the growth rate of the economy directly, this result may be one of the explanations of the scale effect puzzle: many related papers construct endogenous growth models showing that the growth rate of the economy is high when population size is large; however, Jones (1995) pointed out that empirical studies do not support this result. By changing our setting with felicity, one can construct a model in which disruptive innovation by incumbents affects the growth rate of the economy directly, and that may show that large population size discourages innovation where barriers to entry are high, which offsets the effect that large population size encourages economic growth through innovation by entrants.


9 Some papers define “radical innovation” or “drastic innovation” as innovation in which the degree of progress is large or invention is epoch-making.
“external innovation”. On the other hand, incumbents conduct innovation to switch from the aged prod-
uct lines they are currently serving to a new one; this activity cannot be categorized under the above
definition. Following Christensen (1997), we define this activity as “disruptive innovation”.

Most related literature on the innovative activity of both heterogeneous incumbents and entrants 
examines “exploitation innovation”, “process innovation”, “internal innovation”, and “incremental inno-
vation” by incumbents, and “exploration innovation”, “product innovation”, and “external innovation” 
by entrants or incumbents. One exception is Iwaisako and Ohki (2019). They unintentionally ex-
amine the “disruptive innovation” by leaders using the extended quality ladder model. In their model,
the quality advantage over followers is caused by the exogenous distribution, and then heterogeneity 
exists between leaders. Their unique setting is that leaders have the opportunity to redraw the quality 
advantage, and then leaders abandon their current position and innovate to earn a larger profit flow. 

They find a negative relationship between the quality of the invented good and leaders’ motivation to 
conduct innovation, which is also found in the present paper. This result is contrary to Schumpeter’s 
hypothesis that large firms tend to conduct R&D activity proportionally more than smaller ones, which 
is supported by Acemoglu and Cao (2015) and Akcigit and Kerr (2018) in the theoretical context by 
analyzing “incremental innovation” by incumbents. However, this difference is natural because the def-
inition of innovation in their model is clearly different, and our result is consistent with Christensen’s idea 
that large firms tend to conduct large “incremental innovation” but much less “disruptive innovation”.

In the near future, progressive technology, such as solid-state batteries and artificial intelligence, will 
play a role in reconstructing existing markets, and the presence of “disruptive innovation” is expected 
to rise. We think both “incremental innovation” and “disruptive innovation” play important roles in the 
development of the economy, so the relationships between these two types of innovation and economic

---

10 The model constructed in the present paper and that constructed by Iwaisako and Ohki (2019) are completely different except on one point: both models consider “disruptive innovation”. The present model is based on the variety expansion model; however, Iwaisako and Ohki (2019) is based on the quality ladder model. On that point, their model cannot examine the effect of extending patent length. Additionally, they cannot capture the relative obsolescence effect, which plays a crucial role in section 3.2 in our paper, unless generalizing the constant elasticity of substitution (CES) utility function. The present model also constructs a general equilibrium; however, Iwaisako and Ohki (2019) use a quasi-linear utility function. Although this assumption makes analysis easy, it restricts the range of analysis, and their model cannot examine the resource effect as examined in section 3.3. The reason they choose such a setting in spite of these weak points is that their model can capture the intensity of innovative activity, and the probability of successful innovation is determined endogenously, which makes distribution of incumbents important. (In the present model, the probability of success is assumed to be one, and key endogenous variables do not depend on the distribution of incumbents.) In their model, growth rate of the economy is affected through the change of the endogenous distribution of incumbents, the “distribution effect”, which is never obtained from the present or related papers.

11 Because they do not consider absolute or relative obsolescence, and their setting permits followers to leapfrog, the leaders’ motivation for innovation is simply earning higher profits. The important point is that leaders abandon their current position, and this setting shows the typical property of “disruptive innovation”.

growth models have to be examined carefully. This paper is the first step in this research plan.

The remainder of our paper is structured as follows. In section 2, we construct a general equilibrium model without specifying R&D technology. As for the homogeneous growth model, the properties of the long-run equilibrium crucially depend on the specification of R&D technology. In section 3.1, we construct the simplest model in which the degree of innovation by incumbents and entrants are independent of labor market conditions. In section 3.2, we generalize the positive externality from the average quality of differentiated goods, which is strongly affected by incumbents’ innovation. In section 3.3, we remove our assumption imposed only for simplicity, and conduct a numerical analysis using a model in which the degree of innovation by incumbents and entrants is dependent on labor market conditions. In section 3.4, we remove the positive externality from the number of differentiated goods, and examine the steady-state equilibrium where the number of differentiated goods is determined endogenously instead of the growth rate.

2 Model

We construct an infinite representative agent model where the productivity of production and the quality of differentiated goods, which change as an exogenous law of motion, differ between incumbents. Using this model, we analyze the relationship between incumbents’ technology-switching behavior, new inventions by entrants, and economic growth. We normalize the wage rate as the numeraire, \( w(t) = 1 \). We focus on the balanced growth path (BGP) equilibrium, where all variables grow at a constant rate.

2.1 Households

There are \( L(t) \) households at time \( t \) that grow at exogenous rate \( g_L \). Household members live forever and are endowed with one unit of labor, which is supplied inelastically. Each household maximizes its discounted utility:

\[
U = \int_0^{\infty} e^{-\rho t} u(t) dt,
\]

where \( \rho \) is a subjective discount rate and \( u(t) \) represents instantaneous utility from consumption at time \( t \). Consumption goods are provided by one industry. Within that industry, there is a continuum of horizontally differentiated goods, and preferences are expressed in the form of a CES in accordance with
Dixit and Stiglitz (1977):

\[
u(t) = \left[ \int_0^{n(t)} q(j, t) x(j, t)^{\frac{1}{\sigma}} \, dj \right]^\frac{\sigma}{\sigma - 1},
\]

where \( q(j, t) \) and \( d(j, t) \) denote the quality and consumption volume of incumbent \( j \) at time \( t \), respectively, and \( \sigma > 1 \) is the elasticity of substitution of differentiated goods.

Solving the static utility maximization problem of the household, we now derive the per capita demand for each differentiated good:

\[
x(j, t) = \left[ \frac{q(j, t)}{p(j, t)} \right]^\sigma P(t)^{\sigma - 1} E(t),
\]

where \( P(t) = \left[ \int_0^{n(t)} q(j, t)^\sigma p(j, t)^{-(\sigma - 1)} \, dj \right]^\frac{1}{\sigma - 1} \) denotes the price index of differentiated goods at time \( t \), and \( E(t) \) is the per capita expenditure at time \( t \).

Given (3), inter-temporal utility maximization yields \( \frac{\dot{E}(t)}{E(t)} = r(t) - \rho \), where \( r(t) \) is the interest rate at time \( t \). In the BGP equilibrium, the growth rate of expenditure \( g_E \) is constant. Thus, the interest rate is also constant, and expressed as:

\[
r = \rho + g_E.
\]

2.2 Production

Each differentiated good is produced by incumbents. They have heterogeneous production technologies and quality for the goods they produce. Incumbents having productivity \( \phi \) must hire \( \frac{1}{\sigma} x \) units of labor to produce \( x \) units of goods, and there is no fixed cost to produce differentiated goods. The instantaneous profit of incumbent \( j \) is expressed as \( \pi(j, t) = x(j, t) L(t) (p(j, t) - 1) \), where \( p(j, t) \) is the price of incumbent \( j \). As incumbents supply goods monopolistically, they choose the optimal price \( p(j, t) = \frac{\sigma}{\sigma - 1} \frac{1}{\phi(j,t)} \). Using (3), we obtain the incumbents’ quantity of production having productivity \( \phi \) and quality \( q \) as:

\[
x(j, t) L(t) = [q(j, t) \phi(j, t)]^\sigma \left( \frac{\sigma - 1}{\sigma} \right)^\sigma P(t)^{\sigma - 1} E(t) L(t).
\]

We define the composite factor of the incumbents’ property (hereafter referred to as adjusted quality) as:

\[
\theta(j, t) \equiv q(j, t)^\sigma \phi(j, t)^{\sigma - 1}.
\]
Because all incumbents with the same adjusted quality behave symmetrically, we index incumbents hereafter by $\theta$ instead of $j$. Then, the labor demand, $(x/\phi)L$, and profit, $\pi$, of incumbents having adjusted quality $\theta$ are expressed as:

$$\frac{x(\theta, t) L(t)}{\phi} = \theta(t) \frac{\sigma - 1}{\sigma} \frac{E(t)L(t)}{Q(t)}, \quad (7)$$

$$\pi(\theta, t) = \theta(t) \frac{1}{\sigma} \frac{E(t)L(t)}{Q(t)}, \quad (8)$$

where $Q(t) \equiv n(t) \Theta(t)$ expresses the improvement level of differentiated goods in this industry, which is composed of the number of differentiated goods, $n(t)$, and average quality, $\Theta(t) = \frac{\theta_{\text{max}}}{\theta_{\text{min}}} \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \mu(\theta, t) d\theta$, at time $t$. Here, $\mu(\theta, t)$ is the distribution of incumbents having adjusted quality $\theta$ at time $t$.\textsuperscript{13}

### 2.3 Entry

Entrants must hire $s_F C_F(t)$ units of labor to invent new differentiated goods at time $t$, and draw an initial adjusted quality $\theta_0$ from a given distribution $G(\theta_0)$. Here, $s_F$ is the subsidy (tax) for innovation by entrants when $s_F < 1$ ($s_F > 1$). We specify the assumed Pareto distribution as:\textsuperscript{14}

$$G(\theta_0) = 1 - \theta_0^{-k}, \quad 1 \leq \theta_0 < \infty, \quad (9)$$

where the expected value of $\theta_0$ is given by $k/(k - 1)$, which decreases in $k > 1$. After the invention, entrants enter the market and become incumbents. Invented differentiated goods are protected by a perfect but finite intellectual property rights policy, and then incumbents can supply goods monopolistically without risk of imitation until their patent protection expires. They can obtain a monopoly profit throughout $\bar{T}$ periods at most, where $\bar{T}$ is patent length. Initial instantaneous profits of an incumbent entering at time $s$ and drawing $\theta_0$ is expressed as:

$$\pi(\theta_0, 0, s) = \frac{\theta_0}{\sigma} \frac{E(s)L(s)}{Q(s)}, \quad (10)$$

\textsuperscript{13}Although $\mu(\theta, t)$ must be determined endogenously, the equilibrium of our economy does not depend on $\mu(\theta, t)$. Thus, we do not pay attention to $\mu(\theta, t)$.

\textsuperscript{14}In general, the cumulative distribution function of the Pareto distribution is expressed as $G(\theta_0) = 1 - a^k \theta_0^{-k}, \quad a \leq \theta_0 < \infty$. In this paper, we simplify $a = 1$. Then, the density function is expressed as $g(\theta_0) = k \theta_0^{-k-1}$.\textsuperscript{14}
We assume that the adjusted quality of each incumbent grows at the exogenous rate $g_{\theta_0}$. The number of households grows at exogenous growth rate $g_L$, and the improvement level of differentiated goods, $Q$, and aggregate expenditure, $E$, grow at endogenous growth rates $g_Q$ and $g_E$, respectively. Then, an instantaneous profit of $\tau$ times passed incumbents entering at time $s$, and drawing $\theta_0$ is expressed as:

$$\pi(\theta_0, \tau, s) = \exp[-g_Q - g_\theta - g_E - g_L] \frac{\theta_0 E(s)L(s)}{Q(s)}. \quad (11)$$

2.4 Disruptive Innovation

From (11), instantaneous profits decrease with decreases in own adjusted quality, $\theta_0$, which captures absolute obsolescence or saturation of demand. It also decreases as the improvement level of differentiated goods, $Q$, increases, which captures the effect of relative obsolescence. In this paper, we consider that incumbents have the opportunity to wash their hands of current and switch to new technology by conducting R&D activity. We call this activity disruptive innovation. Incumbents must hire $s_D C_D(t)$ units of labor to conduct disruptive innovation, and then invent a new differentiated good and reenter the market at time $t$. They draw an initial adjusted quality $\theta_0$ from a given distribution $G(\theta_0)$. Here, $s_D$ is the subsidy (tax) for innovation by incumbents when $s_D < 1 (s_D > 1)$. We assume that $C_D(t)$ is strictly smaller than $C_F(t)$.

The value of incumbents entering at time $s$ consists of aggregate discounted instantaneous profits from time $s$ to the timing of their technology switch and expected discounted net benefit from disruptive innovation, which is the difference between the expected initial value of incumbents and the cost of technology-switching. Then, the initial values of incumbents entering at time $s$ and drawing $\theta_0$ is expressed as:

$$V(\theta_0, 0, s) = \int_0^T \exp[-r\tau] \pi(\theta_0, \tau, s) d\tau + \exp[-rT] \left[EV(s + \tilde{T}) - s_D C_D(s + \tilde{T})\right]. \quad (12)$$

where $EV(t) = \int_{\theta_0}^{\theta_0^{\max}} V(\theta_0, 0, t) dG(\theta_0)$ is the expected initial value of incumbents entering at time $t$, and $\tilde{T}$ is the timing of conducting disruptive innovation, which is determined to satisfy the following

---

15 When incumbents conduct disruptive innovation, the growth rate of adjusted quality of each incumbent, $g_{\theta_0}$, must be negative to satisfy a sufficient condition, which is discussed later. In this paper, we assume that $g_{\theta_0}$ is constant; thus, $g_{\theta_0}$ must be negative.

16 This assumption means that incumbents have cost advantages over entrants, and the related literature provides several plausible interpretations of the assumption; for example, because only incumbents have state-of-the-art technology in their industries, they have better experience, ability, or knowledge of R&D activities in their industries.
condition:

\[ \pi \left( \theta_0, \tilde{T}, s \right) = [r - g_C] \exp \left[ g_C \tilde{T} \right] \left[ EV (s) - sD C_D (s) \right], \]  \tag{13} 

where \( g_C \) is the growth rate of innovation cost.\(^\text{17}\) This is a necessary condition with respect to \( \tilde{T} \) to maximize (12), and expresses the trade-off between marginal benefit and marginal loss from putting off the disruptive innovation. When incumbents put off the disruptive innovation, they can obtain additional instantaneous profits; however, the expected net benefit from disruptive innovation decreases because the cumulative discount rate of expected net benefit from disruptive innovation increases. If the marginal benefit from delaying disruptive innovation is larger (smaller) than that of the marginal loss, incumbents move the timing of disruptive innovation later (forward). Thus, the timing of disruptive innovation is determined to satisfy that the marginal benefit from delaying the disruptive innovation is equal to that of the marginal loss. A sufficient condition with respect to \( \tilde{T} \) to maximize (12) is expressed as:

\[ - [g_Q - g_\theta - g_E] - g_C < 0. \] \tag{14} 

This condition implies that the marginal loss from delaying the disruptive innovation has to increase more than that of marginal benefit when incumbents conduct disruptive innovation. Equation (13) determines the interior solution of \( \tilde{T} \) as a function of \( \theta_0 \), which corresponds to one-to-one. From (8) and (13), we can recognize the positive relationship between \( \theta_0 \) and \( \tilde{T} \), and then incumbents drawing high \( \theta_0 \) tend to engage in disruptive innovation later.

In addition to the case of the interior solution, we have to consider the case of a corner solution. On one hand, for incumbents drawing low \( \theta_0 \) at time \( s \), the marginal benefit from delaying the disruptive innovation can be strictly less than that of the marginal loss even at the moment when they enter the market:

\[ \pi \left( \theta_0, 0, s \right) - [r - g_C] \left[ EV (s) - sD C_D (s) \right] < 0. \]

Incumbents drawing such a low \( \theta_0 \) decide to conduct disruptive innovation immediately, rather than

\(^\text{17}\) We assume, for simplicity, that the growth rate of \( EV \) equals that of \( C_D \), \( g_{EV} = g_C \equiv g_{C_D} \), which implies a growth rate of net benefit from disruptive innovation equal to the growth rate of innovation cost, \( \frac{d (EV (t) - sD C_D (t))}{dt} = g_C \). This assumption is satisfied when cost of disruptive innovation by incumbents and the cost of innovation by entrants grow at same rate, which can be confirmed later.
producing differentiated goods. We define the critical level of initial adjusted quality in terms of whether the incumbent produces as \( \theta_0^{c1} \), which satisfies the following condition:

\[
\pi (\theta_0^{c1}, 0, s) - [r - g_C] [EV (s) - s_D C_D (s)] = 0.
\] (15)

On the other hand, for incumbents drawing a high \( \theta_0 \) at time \( s \), the marginal benefit from delaying the disruptive innovation can be strictly greater than that of the marginal loss even at time \( s + \bar{T} \):

\[
\pi (\theta_0, \bar{T}, s) - [r - g_C] \exp [g_C \bar{T}] [EV (s) - s_D C_D (s)] > 0.
\]

Incumbents drawing such a high \( \theta_0 \) conduct no disruptive innovation before their patent protection expires. We assume, for simplicity, that incumbents drawing such a high \( \theta_0 \) switch their technology immediately after their patent protection expires. We define the critical level of initial adjusted quality in terms of whether the incumbent conducts disruptive innovation before their patent protection expires as \( \theta_0^{c2} \), which satisfies the following condition:

\[
\pi (\theta_0^{c2}, \bar{T}, s) - [r - g_C] \exp [g_C \bar{T}] [EV (s) - s_D C_D (s)] = 0.
\] (16)

Thus, we can write the timing of disruptive innovation as a function of initial adjusted quality:

\[
\hat{T} = 0 \quad \text{for} \quad \theta_0 \leq \theta_0^{c1}
\]

\[
\hat{T} = \hat{T} (\theta_0) \quad \text{for} \quad \theta_0^{c1} < \theta_0 < \theta_0^{c2}
\]

\[
\hat{T} = \bar{T} \quad \text{for} \quad \theta_0 \geq \theta_0^{c2}
\] (17)

where \( \frac{d \hat{T} (\theta_0)}{d \theta_0} > 0 \) is satisfied.

From (11), (13), (15) and (16), we can express \( \theta_0^{c2} \) and \( \hat{T} (\theta_0) \) as a function of \( \theta_0^{c1} \):

\[
\theta_0^{c2} = \exp \left[ -g_0 \bar{T} \right] \theta_0^{c1}
\]

\[
\hat{T} (\theta_0) = \frac{1}{g_0} \ln \left[ \frac{\theta_0}{\theta_0^{c1}} \right].
\] (18)
From (12), we can express the initial values of an incumbent entering at time $s$ and drawing $\theta_0$ as:

$$
V_0 (\theta_0, 0, s) = \left[ EV (s) - s_D C_D (s) \right] \quad \text{for} \quad \theta_0 \leq \theta_0^{c_1}
$$

$$
V_1 (\theta_0, 0, s) = \int_0^{\bar{T} (\theta_0)} \exp [-r \tau] \pi (\theta_0, \tau, s) \, d\tau + \exp \left[-[r - g c] \bar{T} (\theta_0)\right] \left[ EV (s) - s_D C_D (s) \right] \quad \text{for} \quad \theta_0^{c_1} < \theta_0 \leq \theta_0^{c_2},
$$

$$
V_2 (\theta_0, 0, s) = \int_0^{T} \exp [-r \tau] \pi (\theta_0, \tau, s) \, d\tau + \exp \left[-[r - g c] T \right] \left[ EV (s) - s_D C_D (s) \right] \quad \text{for} \quad \theta_0^{c_2} < \theta_0
$$

where $\bar{T}$, $\theta_0^{c_1}$, and $\theta_0^{c_2}$ satisfy (13) and (18), respectively. The first term represents the gain from the monopoly profit and the second represents the gain from disruptive innovation.

### 2.5 Free Entry Condition

When we calculate an expected initial value of incumbents, we have to consider the magnitude of the correlation between the minimum initial adjusted quality, $\theta_0^{\min}$, and the critical level of initial adjusted quality, $\theta_0^{c_1}$, $\theta_0^{c_2}$. First, if $\theta_0^{\min} \leq \theta_0^{c_1}$ is satisfied, we obtain an equilibrium that incumbents drawing $\theta_0 < \theta_0^{c_1}$ conduct disruptive innovation immediately, incumbents drawing $\theta_0^{c_1} < \theta_0 < \theta_0^{c_2}$ conduct disruptive innovation after $\bar{T} (\theta_0)$ periods have passed from entry, and incumbents drawing $\theta_0^{c_2} \leq \theta_0$ conduct disruptive innovation after $T$ periods have passed from the entry. Second, if $\theta_0^{c_1} < \theta_0^{\min} \leq \theta_0^{c_2}$ is satisfied, we obtain an equilibrium that incumbents drawing $\theta_0^{\min} \leq \theta_0 < \theta_0^{c_2}$ conduct disruptive innovation after $\bar{T} (\theta_0)$ periods have passed from entry, and incumbents drawing $\theta_0^{c_2} \leq \theta_0$ conduct disruptive innovation after $T$ periods have passed.

Third, if $\theta_0^{c_2} < \theta_0^{\min}$ is satisfied, we obtain an equilibrium in which all incumbents conduct disruptive innovation after $\bar{T}$ periods have passed from entry. Then, we can express an expected initial value of incumbents as:

$$
EV = \int_{\theta_0^{c_1}}^{\theta_0^{c_2}} V_0 (\theta_0, 0, s) \, dG (\theta_0) + \int_{\theta_0^{c_1}}^{\theta_0^{c_2}} V_1 (\theta_0, 0, s) \, dG (\theta_0) + \int_{\theta_0^{c_1}}^{\theta_0^{c_2}} V_2 (\theta_0, 0, s) \, dG (\theta_0) \quad \text{if} \quad \theta_0^{\min} \leq \theta_0^{c_1}
$$

$$
EV = \int_{\theta_0^{\min}}^{\theta_0^{c_2}} V_1 (\theta_0, 0, s) \, dG (\theta_0) + \int_{\theta_0^{c_2}}^{\theta_0^{\max}} V_2 (\theta_0, 0, s) \, dG (\theta_0) \quad \text{if} \quad \theta_0^{c_1} < \theta_0^{\min} \leq \theta_0^{c_2}.
$$

$$
EV = \int_{\theta_0^{\min}}^{\theta_0^{\max}} V_2 (\theta_0, 0, s) \, dG (\theta_0) \quad \text{if} \quad \theta_0^{c_2} < \theta_0^{\min}
$$

---

18 Incumbents drawing $\theta_0 \leq \theta_0^{c_1}$ conduct disruptive innovation and gain no monopolistic profit.

19 In this case, there are no incumbents conducting disruptive innovation at the moment when they invent.

20 In this case, there are no incumbents conducting disruptive innovation before their patent protection expires.
The free entry condition is satisfied in each period. The expected gain from entry at time $t$ equals to zero, and we obtain the following condition:

$$
\int_{\theta_{0}^{\text{min}}}^{\theta_{0}^{\text{1}}} V_{0} (\theta_{0}, 0, t) \, dG (\theta_{0}) + \int_{\theta_{0}^{\text{1}}}^{\theta_{0}^{\text{2}}} V_{1} (\theta_{0}, 0, t) \, dG (\theta_{0}) + \int_{\theta_{0}^{\text{2}}}^{\theta_{0}^{\text{max}}} V_{2} (\theta_{0}, 0, t) \, dG (\theta_{0}) \leq s_{F} C_{F} (t) \quad \text{if} \quad \theta_{0}^{\text{min}} \leq \theta_{0}^{\text{1}} \leq \theta_{0}^{\text{2}} \leq \theta_{0}^{\text{max}}.
$$

When an equilibrium value of $g_{n}$ is positive, (21) is satisfied with equality. When an equilibrium value of $g_{n}$ is negative, the expected gain from entry is too small for entrants to enter the market. Then, there is no new entry, and the number of differentiated goods does not increase. Thus, we obtain an interior equilibrium in which the number of differentiated goods remains constant, $g_{n} = 0$, if the expected gain from entry is strictly negative, even when $g_{n} = 0$.

### 2.6 Labor Market

From (7), aggregate labor demand for production at time $t$ is:

$$
n(t) \int_{\theta_{0}^{\text{min}}}^{\theta_{0}^{\text{max}}} \frac{x(\theta, t) L(t)}{\phi} \mu(\theta, t) \, d\theta = \frac{\sigma - 1}{\sigma} E(t) L(t). \quad (22)
$$

Because $C_{F} (t)$ units of labor are required to invent one new differentiated good, $n(t) g_{n} C_{F} (t)$ units of labor are required to invent $\dot{n}(t)$ new differentiated goods, where $g_{n} \equiv \dot{n}(t) / n(t)$ is the growth rate of the number of differentiated goods. Aggregate labor demand for innovation by entrants at time $t$ is expressed as:

$$
I_{F} (t) = n(t) g_{n} C_{F} (t). \quad (23)
$$

---

21 Note that this does not mean that incumbents exit the market.
Because $C_D(t)$ units of labor are required to conduct disruptive innovation, aggregate labor demand for innovation by incumbents at time $t$ is expressed as:

$$I_D(t) = \nu n(t) C_D,$$  

(24)

where $\nu$ is the ratio of the number of differentiated goods conducting disruptive innovation to the aggregate number of differentiated goods, which is expressed as:22

$$\nu = \begin{cases} \left[\theta_0^1\right]^k g_n [-g_n]^{[g_n]} & \text{if } \theta_0^\text{min} \leq \theta_0^1 \\
\frac{\left[\theta_0^1\right]^k g_n}{1 - \exp[-g_n + k[-g_n]]} - g_n & \text{if } \theta_0^1 < \theta_0^\text{min} \leq \theta_0^2, \\
\frac{\exp[-g_n T]}{1 - \exp[-g_n T]} g_n & \text{if } \theta_0^2 < \theta_0^\text{min} \end{cases}$$  

(25)

Labor supply is exogenously given, so the labor market clearing condition is:

$$\sigma - 1 E(t) L(t) + \nu n(t) C_D(t) + g_n n(t) C_F(t) = L(t).$$  

(26)

### 2.7 Average Quality of Differentiated Goods

The average quality of differentiated goods, $\Theta$, is calculated as the sum of the quality of differentiated goods divided by the number of differentiated goods produced at time $t$:23

$$\Theta = \begin{cases} \frac{k}{k-1} \left[\theta_0^1\right]^k & \text{if } \theta_0^\text{min} \leq \theta_0^1 \\
\frac{\left[\theta_0^1\right]^k g_n}{1 - \exp[-g_n+k[-g_n]]} + \frac{k}{k-1} \left[1 - \left[\theta_0^1\right]^k \exp[-g_n+k[-g_n]]\right] & \text{if } \theta_0^1 < \theta_0^\text{min} \leq \theta_0^2 \\
\frac{\exp[-g_n T]}{1 - \exp[-g_n T]} g_n & \text{if } \theta_0^2 < \theta_0^\text{min} \end{cases}$$  

(27)

If $\theta_0^\text{min} \leq \theta_0^1$ is satisfied, $\Theta$ is an increasing function of $\theta_0^1$ and independent from $g_n$ and $g_\theta$. First, from (18), high $\theta_0^1$ leads to low $\bar{T}$ for any incumbents, which means that incumbents drawing $\theta_0 \leq \theta_0^2$ make disruptive innovation early. Because of absolute obsolescence, this effect makes average quality high. Second, the average quality of any vintage is identical:

$$\int_{\theta_0^1(T)}^{\theta_0(T)} \frac{\exp[\theta_0 T] k \theta_0^{-k} d\theta_0}{\int_{\theta_0^1(T)}^{\theta_0(T)} k \theta_0^{-k} d\theta_0} = \frac{k}{k-1} \theta_0^1.$$  

The reason for

\[\text{In appendix A, we derive (25).}
\[\text{In appendix B, we derive (27).} \]
this is on one hand, the adjusted quality of any incumbent decreases at the rate of $g_\theta$. Then, the average quality of old differentiated goods tends to be low, and this effect becomes large when the absolute value of $g_\theta$ is high. On the other hand, incumbents producing low-quality goods conduct disruptive innovation earlier, and differentiated goods having low adjusted quality goods go out of production over time. This raises the average quality, and the effect becomes large when the absolute value of $g_\theta$ is high because a high absolute value of $g_\theta$ decreases the incentive to put off disruptive innovation. These two effects offset completely, irrespective of the value of $g_\theta$. Then, the ratio of old to new differentiated goods does not matter when $\theta_0^{\text{min}} \leq \theta_0^{\text{cl}}$ is satisfied.

If $\theta_0^{02} < \theta_0^{\text{min}}$ is satisfied, $\Theta$ is an increasing function of $g_n$, a decreasing function of the absolute value of $g_\theta$, and independent of $\theta_0^{\text{cl}}$. First, no incumbents conduct disruptive innovation before their patent protection expires when $\theta_0^{02} < \theta_0^{\text{min}}$ is satisfied. Then, the adjusted quality of all incumbent decreases at the rate $g_\theta$, and the average quality of old differentiated goods tends to be low. When the absolute value of $g_\theta$ is high, the average quality of old differentiated goods rapidly renders them obsolete. Then, the total average quality is low with a high absolute value of $g_\theta$. When $g_n > 0$, the number of old differentiated goods, which are on average of low quality, is lower than that of new differentiated goods, which are on average of high quality. This effect becomes large when $g_n$ is large, and then, the total average quality is high with high $g_n$. Second, even an incumbent drawing the lowest initial adjusted quality does not conduct disruptive innovation until their patent protection expires when $\theta_0^{02} < \theta_0^{\text{min}}$ is satisfied. Then, $\tilde{T} = \bar{T}$ is satisfied for any incumbent, and $\Theta$ is not affected by $\theta_0^{\text{cl}}$.

If $\theta_0^{\text{cl}} < \theta_0^{\text{min}} < \theta_0^{02}$ is satisfied, $\Theta$ is an increasing function of $\theta_0^{\text{cl}}$ and $g_n$, and a decreasing function of the absolute value of $g_\theta$. The reason that $\Theta$ is an increasing function of $\theta_0^{\text{cl}}$ is the same in the case of $\theta_0^{\text{cl}} \leq \theta_0^{\text{cl}}$. The reason that $\Theta$ is an increasing function of $g_n$ and a decreasing function of the absolute value of $g_\theta$ is as follows: when $\theta_0^{\text{cl}} < \theta_0^{\text{min}} < \theta_0^{02}$ is satisfied, there are some periods during which even an incumbent drawing the lowest initial adjusted quality does not conduct disruptive innovation. Then, the average quality of old differentiated goods is lower than that of new differentiated goods. Thus, the total average quality is high with high $g_n$ or a low absolute value of $g_\theta$, which is similar to the case of $\theta_0^{02} < \theta_0^{\text{min}}$.

Regardless of the difference between $\theta_0^{\text{cl}}$ and $\theta_0^{\text{min}}$, $\Theta$ is a decreasing function of $k$, and is constant as long as $\theta_0^{\text{cl}}$ is constant.\textsuperscript{24} Because the improvement level of differentiated goods is $Q(t) \equiv n(t) \Theta$, the

\textsuperscript{24}Note that the expected value of $\theta_0$ is a decreasing function of $k$. 

14
growth rate of improvement level of differentiated goods is expressed as:

\[ g_Q = g_n. \]  

(28)

2.8 Growth Rate

We call the growth rate of this economy the growth rate of instantaneous utility. Substituting (5) and (6) into (2) yields:

\[ u(t) = Q(t)^\frac{1}{\sigma} \frac{\sigma - 1}{\sigma} E(t). \]  

(29)

Then, the growth rate of this economy is expressed as

\[ g_u = \frac{1}{\sigma - 1} g_Q + g_E. \]  

(30)

3 Equilibrium

In the previous section, we constructed a closed model; however, we have yet to specify the innovation technology. The property of the long-run equilibrium is crucially dependent on how we specify the innovation technology; therefore, we derive several types of equilibriums by specifying several forms of innovation technology.

3.1 The Case Where \( C_F(t) = \frac{E(t) L(t)}{Q(t)} c_F \) and \( C_D(t) = \frac{E(t) L(t)}{Q(t)} c_D \)

In this section, we specify the innovation technology as:

\[ C_F(t) = \frac{E(t) L(t)}{Q(t)} c_F \quad C_D(t) = \frac{E(t) L(t)}{Q(t)} c_D. \]  

(31)

These technology functions contain a positive externality from the improvement level of differentiated goods, and a negative externality from market scale.\(^{25}\) By specifying this formula, we can derive the simplest BGP equilibrium, where the equilibrium value of \( \bar{\theta}_0^1, \bar{\theta}_0^2, \bar{T}(\bar{\theta}_0), g_Q, \) and \( \Theta, \) which are important

\(^{25}\) When the improvement level of differentiated goods is high, much knowledge is accumulated, and then, innovation cost tends to be low. When the market scale is large, the costs of marketing their differentiated goods or research are high. Then, the innovation cost, as an initial investment cost of releasing a new differentiated good, tends to be high.
endogenous variables in this paper, are determined independent of the labor market clearing condition. Another merit for this specification is that $g_Q$ is independent of population; thus, there is no scale effect.

From (26) and (31), the labor market clearing condition is expressed as:

$$E(t) \left[ \frac{\sigma - 1}{\sigma} + \nu \frac{c_D}{\Theta} + g_Q \frac{c_F}{\Theta} \right] = 1. \quad (32)$$

From (32), in the BGP equilibrium, $E(t)$ must be constant, and we obtain:

$$g_E = 0. \quad (33)$$

From (31) and (33), the growth rate of the innovation cost is:

$$g_C = g_L - g_Q. \quad (34)$$

From (14), (33), and (34), the growth rate of the adjusted quality has to be negative:

$$g_\theta < 0. \quad (35)$$

Substituting (4), (11), (21), (34) and (31) into (13) and (18) yields:

$$\theta_0^1(g_Q) = \sigma [\rho - g_L + g_Q] [s_F c_F - s_D c_D]$$

$$\theta_0^2(g_Q) = \exp \left[ \left( -g_\theta \hat{T} \right) \theta_0^3(g_Q) \right].$$

$$\hat{T}(\theta_0, g_Q) = \frac{1}{-g_\theta} \ln \left( \frac{\theta_0}{\theta_0^3(g_Q)} \right). \quad (36)$$

Equation (36) determines the equilibrium value of $\theta_0^1$, $\theta_0^2$, and $\hat{T}$ with given $g_Q$. We can confirm $\frac{\partial \hat{T}(\theta_0, g_Q)}{\partial \theta_0} > 0$ and $\frac{\partial^2 \hat{T}(\theta_0, g_Q)}{\partial \theta_0 \partial \theta_0} < 0$, and we get following proposition.

**Proposition 3.1.** An incumbent drawing a high initial adjusted quality and earning a large profit tends to invest in disruptive innovation (switch from current technology) later. When patent length is finite, some incumbents drawing a high initial adjusted quality conduct no disruptive innovation before their patent protection expires.

This proposition is consistent with the finding of Iwaisako and Ohki (2019), who also examined
disruptive innovation using a quality ladder model. The result of this proposition is caused by the weak-
form Arrow effect: incumbents with a high adjusted quality lose much of their value when they wash
their hands of current technology; thus, incumbents with a high adjusted quality tend not to conduct
disruptive innovation.\footnote{In the context of empirical studies, as in Igami (2017), this effect is called “cannibalization”.

We also confirm $\frac{\partial \theta_{c_1}}{\partial g_Q} > 0$, $\frac{\partial \theta_{c_2}}{\partial g_Q} > 0$ and $\frac{\partial \tilde{T}(\theta_0, g_Q)}{\partial g_Q} < 0$, and interpret these results as follows. A marginal increment of $g_Q$ decreases the growth rate of the expected net benefit from disruptive innovation, which increases the marginal loss from putting off disruptive innovation. Then, incumbents move the
timing of disruptive innovation forward.\footnote{Strictly speaking, there are two other effects from a marginal increment of $g_Q$. One is a marginal increment of $g_Q$ that decreases instantaneous profit, which decreases the marginal benefit from putting off the disruptive innovation. The other is a marginal increment of $g_Q$ that decreases the level of expected net benefit from disruptive innovation, which in turn decreases the marginal loss from putting off disruptive innovation. We can easily confirm that these two opposite effects offset each other; then, the only effect that remains is a marginal increment of $g_Q$ that decreases the growth rate of the expected net benefit from disruptive innovation.}

We depict the relationship between $\tilde{T}$ and $\theta_0$ in Figure 3.1. The
horizontal axis represents $\theta_0$ and the vertical axis represents $\tilde{T}$. The timing of disruptive innovation, $\tilde{T}$,
slopes upward in $\theta_0$ with given $g_Q$, and shifts downward when $g_Q$ increases.

Moreover, we can confirm
$\frac{\partial \theta_{c_1}}{\partial s_D} < 0$, $\frac{\partial \theta_{c_2}}{\partial s_D} < 0$, $\frac{\partial \tilde{T}(\theta_0, g_Q)}{\partial s_D} > 0$, $\frac{\partial \theta_{c_1}}{\partial s_F} > 0$, $\frac{\partial \theta_{c_2}}{\partial s_F} > 0$, $\frac{\partial \tilde{T}(\theta_0, g_Q)}{\partial s_F} < 0$, $\frac{\partial \tilde{T}(\theta_0, g_Q)}{\partial \tilde{T}} = 0$, $\frac{\partial \theta_{c_2}}{\partial \tilde{T}} > 0$ and $\frac{\partial \tilde{T}(\theta_0, g_Q)}{\partial \tilde{T}} = 0$. We can interpret these results as fol-
lows. First, on one hand, a subsidy (tax) for disruptive innovation by incumbents decreases (increases)
the cost of disruptive innovation, which then increases (decreases) the expected discounted net benefit
from disruptive innovation with given $g_Q$. On the other hand, a subsidy (tax) for innovation by entrants
stimulates (mitigates) competition, which decreases (increases) the expected initial value of incumbents,
and the expected discounted net benefit from disruptive innovation decreases (increases) with given $g_Q$.

Figure 3.1. Relationship between $\tilde{T}$ and $\theta_0$ in Figure 3.1. Here, $g_Q < g_Q'$. 

17
Thus, there is no difference between incumbents drawing \( \theta \) and \( \theta \) disruptive innovation is immediately after their entry, it does not affect the value of incumbents drawing if \( \theta \) has passed after entry. That is, an incumbent drawing does not affect the timing of disruptive innovation as long as \( g_Q \) does not change.\(^{28}\)

Substituting (4), (11) and (31) into (19) yields:

\[
V_0 (\theta_0, 0, s) = \frac{EL(s)}{Q(s)} X_0 \quad for \quad \theta_0 \leq \theta_0^1 (g_Q) \\
V_1 (\theta_0, 0, s) = \frac{EL(s)}{Q(s)} X_1 (\theta_0, g_Q) \quad for \quad \theta_0^1 (g_Q) < \theta_0 < \theta_0^2 (g_Q) , \quad (37) \\
V_2 (\theta_0, 0, s) = \frac{EL(s)}{Q(s)} X_2 (\theta_0, g_Q) \quad for \quad \theta_0 \geq \theta_0^2 (g_Q) 
\]

and

\[
X_0 = \left[ s_{FCF} - s_{DCD} \right] \\
X_1 (\theta_0, g_Q) = \left[ \frac{\theta_0^1 \cdot \exp \left[ -[\rho - gL + gQ - g\theta]T(\theta_0, g_Q) \right]}{p - gL + gQ - g\theta} + \exp \left[ -[\rho - gL + gQ] \tilde{T}(\theta_0, g_Q) \right] \left[ s_{FCF} - s_{DCD} \right] \right] , \\
X_2 (\theta_0, g_Q) = \frac{\theta_0^1 \cdot \exp \left[ -[\rho - gL + gQ - g\theta]T \right]}{p - gL + gQ - g\theta} + \exp \left[ -[\rho - gL + gQ] \tilde{T} \right] \left[ s_{FCF} - s_{DCD} \right] 
\]

where \( \theta_0^1, \theta_0^2 \), and \( \tilde{T}(\theta_0, g_Q) \) satisfy (36). From these equations, we can obtain some properties with respect to the value of incumbents. First, we can confirm \( \frac{\partial X_0}{\partial g_Q} = 0, \frac{\partial X_0}{\theta_0} = 0, \frac{\partial^2 X_0}{\partial g_Q \partial \theta_0} = 0, \frac{\partial X_0}{\partial \theta_0} < 0, \frac{\partial X_0}{\partial s_F} > 0 \), \( \frac{\partial^2 X_0}{\partial g_Q \partial s_F} = 0 \), \( \frac{\partial^3 X_0}{\partial g_Q \partial \theta_0} = 0 \), and \( \frac{\partial X_0}{\partial T} = 0 \), and interpret these results as follows. All incumbents drawing \( \theta_0 \leq \theta_0^1 \) decide not to produce differentiated goods and conduct disruptive innovation immediately. Thus, there is no difference between incumbents drawing \( \theta_0 \leq \theta_0^1 \) (\( \frac{\partial X_0}{\partial \theta_0} = 0, \frac{\partial^2 X_0}{\partial g_Q \partial \theta_0} = 0, \frac{\partial^2 X_0}{\partial g_Q \partial s_F} = 0 \)).

Because \( g_Q \) does not affect the expected net benefit from disruptive innovation as long as the timing of disruptive innovation is immediately after their entry, it does not affect the value of incumbents drawing \( \theta_0 \leq \theta_0^1 \) (\( \frac{\partial X_0}{\partial g_Q} = 0 \)). A subsidy (tax) for disruptive innovation by incumbents and a tax (subsidy) for innovation by entrants increases (decreases) the expected net benefit from disruptive innovation. Thus, a marginal decrement (increment) of \( s_F \) and a marginal increment (decrement) of \( s_F \) increases the value of incumbents drawing \( \theta_0 \leq \theta_0^1 \) (\( \frac{\partial X_0}{\partial s_F} < 0, \frac{\partial X_0}{\partial s_F} > 0 \)). Because incumbents drawing \( \theta_0 \leq \theta_0^1 \) conduct disruptive innovation before patent protection expires, a marginal change of patent length has no direct

\(^{28}\)The reason \( \frac{\partial^2 x^2}{\partial T^2} > 0 \) is trivial as follows. We suppose that the incumbent drawing \( \theta_0 \) conducts disruptive innovation as soon as \( \tilde{T} \) time has passed after entry. In this case, \( \theta_0 = \theta_0^2 \) is satisfied. When patent length extends \( \tilde{T} \) from \( T \), what happens? Because \( \frac{\partial \tilde{T}(\theta_0, g_Q)}{\partial \theta_0} = 0 \), the incumbent drawing \( \theta_0 \) continues to conduct disruptive innovation when \( \tilde{T} \) time has passed after entry. That is, an incumbent drawing \( \theta_0 \) conducts disruptive innovation before patent protection expires, \( \theta_0 < \theta_0^2 \).
effect on their value ($: \frac{\partial X_0}{\partial \theta} = 0$).

Second, we can confirm $\frac{\partial X_1(\theta_0,g)}{\partial \theta} > 0$, $\frac{\partial X_1(\theta_0,g)}{\partial T} = 0$, $\frac{\partial^2 X_1(\theta_0,g)}{\partial \theta \partial g} < 0$, $\frac{\partial^2 X_1(\theta_0,g)}{\partial \theta \partial g} < 0$, $\frac{\partial X_1(\theta_0,g)}{\partial g} < 0$, $\frac{\partial^2 X_1(\theta_0,g)}{\partial \theta \partial g} < 0$, $\frac{\partial X_1(\theta_0,g)}{\partial T} < 0$, $\frac{\partial^2 X_1(\theta_0,g)}{\partial \theta \partial g} < 0$, $\frac{\partial^2 X_1(\theta_0,g)}{\partial \theta \partial g} < 0$, and $\frac{\partial^2 X_1(\theta_0,g)}{\partial \theta \partial g} < 0$, and we interpret these results as follows. Incumbents drawing $\theta_0^1 \leq \theta_0 \leq \theta_0^2$ enjoy monopoly profit until they conduct disruptive innovation, which is optimally chosen to maximize their value. A marginal increment of $\theta_0$ increases discounted instantaneous monopoly profit for any period, which increases the value of incumbents drawing $\theta_0^1 \leq \theta_0 \leq \theta_0^2$ ($: \frac{\partial X_1(\theta_0,g)}{\partial \theta} > 0$). A marginal increment of $\theta_0$, on one hand, decreases the discounted instantaneous monopoly profit of incumbents drawing high $\theta_0$, larger relative to incumbents drawing low $\theta_0$. A marginal increment of $g_Q$, on the other hand, decreases the discounted net benefit from the disruptive innovation of incumbents drawing high $\theta_0$, larger relative to incumbents drawing low $\theta_0$. Thus, a marginal increment of $g_Q$ decreases the value of incumbents drawing high $\theta_0$, larger relative to incumbents drawing low $\theta_0$ ($: \frac{\partial^2 X_1(\theta_0,g)}{\partial \theta \partial g} < 0$). For the same reason as $X_0$, a marginal decrement (increment) of $s_D$ and a marginal increment (decrement) of $s_F$ increases the value of incumbents ($: \frac{\partial X_1(\theta_0,g)}{\partial s_D} < 0$, $\frac{\partial X_1(\theta_0,g)}{\partial s_D} > 0$). A marginal decrement (increment) of $s_D$ and a marginal increment (decrement) of $s_F$ increases (decreases) the expected discounted net benefit from disruptive innovation, which increases (decreases) the value of incumbents drawing high $\theta_0$, smaller relative to incumbents drawing low $\theta_0$ because incumbents drawing high $\theta_0$ tend to set the timing of disruptive innovation later, and then obtain a smaller benefit from disruptive innovation relative to incumbents drawing low $\theta_0$ ($: \frac{\partial^2 X_1(\theta_0,g)}{\partial \theta \partial g} < 0$, $\frac{\partial^2 X_1(\theta_0,g)}{\partial \theta \partial g} < 0$). For the same reason as $X_0$, a marginal change of patent length has no direct effect on their value ($: \frac{\partial X_1(\theta_0,g)}{\partial T} = 0$).
Finally, we can confirm \( \frac{\partial X_2(\theta_0, g_Q)}{\partial \theta_0} > 0, \frac{\partial X_2(\theta_0, g_Q)}{\partial s_Q} < 0, \frac{\partial^2 X_2(\theta_0, g_Q)}{\partial \theta_0 \partial s_Q} < 0, \frac{\partial X_2(\theta_0, g_Q)}{\partial s_D} < 0, \frac{\partial X_2(\theta_0, g_Q)}{\partial x_F} > 0, \)
\[
\frac{\partial^2 X_2(\theta_0, g_Q)}{\partial \theta_0 \partial s_D} = 0, \quad \frac{\partial^2 X_2(\theta_0, g_Q)}{\partial \theta_0 \partial x_F} = 0 \text{, and } \frac{\partial X_2(\theta_0, g_Q)}{\partial t} > 0, \]
and interpret these results as follows. Incumbents drawing \( \theta_0 > \theta_0^2 \) enjoy monopoly profits until their patent protection expires. They conduct innovation immediately after their patent protection expires, irrespective of \( \theta_0 \). A marginal increment of \( \theta_0 \) increases the discounted instantaneous monopoly profit for any period, and then increases the incumbents’ value drawing \( \theta_0 > \theta_0^2 \) (\( : \frac{\partial X_2(\theta_0, g_Q)}{\partial \theta_0} > 0 \)). For the same reason as \( X_1 \), a marginal increment of \( g_Q \) decreases the instantaneous profit and the value of incumbents drawing high \( \theta_0 \), larger relative to incumbents drawing low \( \theta_0 \) (\( : \frac{\partial X_2(\theta_0, g_Q)}{\partial s_Q} < 0, \frac{\partial^2 X_2(\theta_0, g_Q)}{\partial \theta_0 \partial s_Q} < 0 \)). For the same reason as \( X_0 \) and \( X_1 \), a marginal decrement (increment) of \( s_D \) and a marginal increment (decrement) of \( s_F \) increase the value of incumbents (\( : \frac{\partial X_2(\theta_0, g_Q)}{\partial s_D} < 0, \frac{\partial X_2(\theta_0, g_Q)}{\partial s_F} > 0 \)). Because incumbents drawing \( \theta_0 > \theta_0^2 \) conduct disruptive innovation after \( T \) time has passed from the entry, irrespective of \( \theta_0 \), there is no difference in expected discounted net benefit from disruptive innovation between incumbents drawing \( \theta_0 > \theta_0^2 \). Thus, a marginal change of \( s_D \) and \( s_D \) has the same effect on the value of incumbents drawing \( \theta_0 > \theta_0^2 \) (\( : \frac{\partial^2 X_2(\theta_0, g_Q)}{\partial \theta_0 \partial s_D} = 0, \frac{\partial^2 X_2(\theta_0, g_Q)}{\partial \theta_0 \partial s_F} = 0 \)). Because a marginal increment of \( T \) extends periods enjoying monopoly profit, it increases the value of incumbents drawing \( \theta_0 > \theta_0^2 \) (\( : \frac{\partial X_2(\theta_0, g_Q)}{\partial \theta_0} > 0 \)).

Substituting (31) and (37) into (21), we obtain:
\[
\left\{ \begin{array}{ll}
\frac{\theta_0^2}{\theta_0^1} X_0 dG(\theta_0) + \frac{\theta_0^2}{\theta_0^1} X_1 dG(\theta_0) + \frac{\theta_0^2}{\theta_0^1} X_2 dG(\theta_0) = s_F c_F & \text{if } \theta_0^1 \leq \theta_0^1 \\
\frac{\theta_0^2}{\theta_0^1} X_1 dG(\theta_0) + \frac{\theta_0^2}{\theta_0^1} X_2 dG(\theta_0) = s_F c_F & \text{if } \theta_0^1 < \theta_0^1 \leq \theta_0^2 \\
\frac{\theta_0^2}{\theta_0^1} X_2 dG(\theta_0) = s_F c_F & \text{if } \theta_0^2 < \theta_0^2 \end{array} \right.
\]  

Equation (38) determines the equilibrium value of \( g_Q \).

In the following, we investigate the properties of the equilibrium value of \( g_Q \). Because \( \frac{\partial X_a}{\partial s_{g_Q}} = 0 \),
\[ \frac{\partial X_2}{\partial g_Q} < 0, \frac{\partial X_4}{\partial g_Q} < 0, \frac{\partial X_4}{\partial s_D} < 0, \frac{\partial X_4}{\partial x_F} < 0 \text{ and } \frac{\partial X_4}{\partial s_D} < 0 \text{ are satisfied, we can obtain:} \]
\[
\frac{dg_Q}{ds_D} = - \left( \int_{\theta_0^1}^{\theta_0^1} \frac{\partial X_2(\theta, g_Q)}{\partial x_Q} dG(\theta) + \int_{\theta_0^1}^{\theta_0^1} \frac{\partial X_4(\theta, g_Q)}{\partial x_F} dG(\theta) + \int_{\theta_0^1}^{\theta_0^1} \frac{\partial X_4(\theta, g_Q)}{\partial s_D} dG(\theta) \right) < 0.
\]

\(^{29}\text{In appendix C, we confirm these results analytically.}\)
This condition implies that a subsidy (tax) for disruptive innovation by incumbents increases (decreases) the growth rate of the improvement level of differentiated goods. The intuition is as follows. A subsidy (tax) for disruptive innovation by incumbents increases (decreases) the expected discounted net benefit from disruptive innovation for all incumbents, and then the expected initial value of incumbents increases with a given $g_Q$. An increment (decrement) of expected initial value of incumbents accelerates (decelerates) entry, which continues until the free entry condition is satisfied. An acceleration (deceleration) of entry raises (lowers) the growth rate of the number of differentiated goods $g_n$, which raises (lowers) the growth rate of the improvement level of differentiated goods $g_Q$.

Because $\frac{\partial X_0}{\partial g_Q} = 0$, $\frac{\partial X_1}{\partial g_Q} < 0$, $\frac{\partial X_2}{\partial g_Q} < 0$, $c_F = \frac{\partial X_0(\theta_0, g_Q)}{\partial s_F} \geq \frac{\partial X_1(\theta_0, g_Q)}{\partial s_F}$, and $\bar{T} > 0$ are satisfied, we can obtain:

$$\frac{dg_Q}{ds_F} = \frac{c_F - \theta_{c_1}(g_Q) \frac{\partial X_0}{\partial s_F} dG(\theta_0) - \theta_{c_2}(g_Q) \frac{\partial X_1(\theta_0, g_Q)}{\partial s_F} dG(\theta_0) - \theta_{c_3}(g_Q) \frac{\partial X_2(\theta_0, g_Q)}{\partial s_F} dG(\theta_0)}{\theta_{c_2}(g_Q) \frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} dG(\theta_0) + \theta_{c_3}(g_Q) \frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q} dG(\theta_0)} < 0. \quad (40)$$

This condition implies that a subsidy (tax) for innovation by entrants increases (decreases) the growth rate of the improvement level of differentiated goods. The intuition is as follows. A subsidy (tax) for innovation by entrants affects equation (38) through two channels. On one hand, it decreases (increases) the expected discounted net benefit from disruptive innovation for all incumbents, and then the expected initial value of incumbents, $EV$, decreases (increases) with a given $g_Q$. On the other hand, it decreases (increases) the cost of initial investment for entrants, $s_F C_F$. Because the latter effect dominates the former, the net expected benefit from entry, $EV - s_F C_F$, increases (decreases) with a given $g_Q$. This accelerates (decelerates) entry, which continues until the free entry condition is satisfied. An acceleration (deceleration) of entry raises (lowers) the growth rate of the number of differentiated goods $g_n$, which raises (lowers) the growth rate of the improvement level of differentiated goods $g_Q$.

\[\text{When } \bar{T} > 0, \ c_F > \frac{\partial X_1(\theta_0, g_Q)}{\partial s_F} \text{ and } c_F > \frac{\partial X_2(\theta_0, g_Q)}{\partial s_F} \text{ are satisfied. Then, the numerator of this equation is unambiguously positive.}\]
Because $\frac{\partial X_0}{\partial g_Q} = 0$, $\frac{\partial X_1}{\partial g_Q} < 0$, $\frac{\partial X_2}{\partial g_Q} < 0$, $\frac{\partial X_0}{\partial \bar{T}} = 0$, $\frac{\partial X_1}{\partial \bar{T}} = 0$, and $\frac{\partial X_2}{\partial \bar{T}} > 0$ are satisfied, we can obtain:

$$
\frac{dg_Q}{d\bar{T}} = -\frac{\theta_{max}^0}{\theta_{c2}^0(g_Q)} \int \frac{\partial X_2(\theta_0, g_Q)}{\partial \bar{T}} dG(\theta_0) > 0.
$$

(41)

This condition implies that extending (shortening) the patent length increases (decreases) the growth rate of the improvement level of differentiated goods. The intuition is as follows. Extending (shortening) the patent length extends (shortens) the period incumbents drawing $\theta_0 > \theta_{c2}^0$ enjoy monopoly profit, and the value of incumbents drawing $\theta_0 > \theta_{c2}^0$ increases (decreases), which increases (decreases) the expected initial value of incumbents with a given $g_Q$. An increment (decrement) of expected initial value of incumbents accelerates (decelerates) entry, which continues until the free entry condition is satisfied. An acceleration (deceleration) of entry raises (lowers) the growth rate of the number of differentiated goods $g_n$, which raises (lowers) the growth rate of the improvement level of differentiated goods $g_Q$.

From (39), (40), and (41), we can write the equilibrium value of $g_Q$, which is determined by (38), as a function of policy variables:

$$
g_Q^* = g_Q(s_D, s_F, \bar{T}),
$$

(42)

where $\frac{\partial g_Q(s_D, s_F, \bar{T})}{\partial s_D} < 0$, $\frac{\partial g_Q(s_D, s_F, \bar{T})}{\partial s_F} < 0$, and $\frac{\partial g_Q(s_D, s_F, \bar{T})}{\partial \bar{T}} > 0$.

Substituting (33) and (42) into (30), we can confirm that the growth rate of this economy increases proportionally with $g_Q$, and is expressed as:

$$
g_u^* = \frac{1}{\sigma - 1} g_Q^*.
$$

(43)

Substituting (42) into (36), we obtain the equilibrium value of $\theta_{c1}^0$, $\theta_{c2}^0$, and $\bar{T}(\theta_0)$. In the following, we investigate the property of the equilibrium value of $\theta_{c1}^0$, $\theta_{c2}^0$, and $\bar{T}(\theta_0)$. 

22
Differentiating (36) with $s_D$, we can obtain:\(^\text{31}\)

\[
\frac{d\theta^1_c(q_g)}{ds_D} = \sigma \frac{\partial g(f(s_D,s_F,T))}{\partial s_D} \left[ s_{FCF} - s_{DCD} \right] - \sigma [r + g_c] c_F < 0
\]

\[
\frac{d\theta^2_c(q_g)}{ds_D} = \exp \left[ \left[-g_c \right] \hat{T} \right] \frac{\partial g_c(q_g)}{\partial s_D} < 0 \cdot \quad (44)
\]

\[
\frac{d\hat{T}(\theta_0, q_g)}{ds_D} = -\frac{1}{\left[-g_c \right] \hat{T}} \frac{\partial g_c(q_g)}{\partial s_D} > 0
\]

This condition implies that a subsidy (tax) for disruptive innovation by incumbents moves the timing of disruptive innovation by each incumbent earlier (later). The intuition is as follows. On one hand, a subsidy (tax) for disruptive innovation by incumbents increases (decreases) the expected discounted net benefit from disruptive innovation, and then the marginal loss from putting off disruptive innovation increases (decreases). On the other hand, a subsidy (tax) for disruptive innovation by entrants stimulates (mitigates) innovation by entrants, which decreases (increases) the growth rate of the expected net benefit from disruptive innovation, $g_C$, and then the marginal loss from putting off disruptive innovation increases (decreases). Because these two effects are in same direction, we can obtain the above unambiguous result.

As shown in Appendix D, we obtain:\(^\text{32}\)

\[
\frac{d\theta^1_c(q_g)}{ds_F} = \sigma \frac{\partial g_c(s_D,s_F,T)}{\partial s_F} \left[ s_{FCF} - s_{DCD} \right] + \sigma [r + g_c] c_F > 0
\]

\[
\frac{d\theta^2_c(q_g)}{ds_F} = \exp \left[ \left[-g_c \right] \hat{T} \right] \frac{\partial g_c(q_g)}{\partial s_F} > 0 \cdot \quad (45)
\]

\[
\frac{d\hat{T}(\theta_0, q_g)}{ds_F} = -\frac{1}{\left[-g_c \right] \hat{T}} \frac{\partial g_c(q_g)}{\partial s_F} < 0
\]

This condition implies that a subsidy (tax) for innovation by entrants moves the timing of disruptive innovation by each incumbent later (earlier). The intuition is as follows. On one hand, a subsidy (tax) for innovation by entrants decreases (increases) the expected discounted net benefit from disruptive innovation, and thus, the marginal loss from putting off disruptive innovation decreases (increases). On the other hand, a subsidy (tax) for innovation by entrants stimulates (mitigates) innovation by entrants, which decreases (increases) the growth rate of the expected net benefit from disruptive innovation, $g_C$, and thus, the marginal loss from putting off disruptive innovation increases (decreases). Although these two effects are in opposite directions, the former effect dominates the latter; thus, we can obtain an

\(^{31}\)Since $\frac{\partial g_c(s_D,s_F,T)}{\partial s_D} < 0$, $\frac{\partial g_c(q_g)}{\partial s_D}$ is unambiguously negative. Because $\theta^1_c(q_g)$ is an increasing function of $\theta^1_c(q_g)$ and $\hat{T}(\theta_0, q_g)$ is a decreasing function of $\theta^1_c(q_g)$, $\frac{\partial g_c(q_g)}{\partial s_D}$ is negative and $\frac{d\hat{T}(\theta_0, q_g)}{ds_D}$ is positive.

\(^{32}\)Since $\frac{\partial g_c(q_g)}{\partial s_F} < 0$, the sign of $\frac{\partial g_c(q_g)}{\partial s_F}$ seems to be ambiguous. However, we can obtain an unambiguous result by calculating in detail.
unambiguous result. 33

Because $\frac{\partial g_C(s_F, s_D, T)}{\partial T} > 0$, we can obtain:

$$
\frac{d\bar{\theta}^{-1}(g_C)}{dT} = \sigma \frac{\partial g_C(s_D, s_F, T)}{\partial T} \left[ s_F C_F - s_D C_D \right] > 0
$$

$$
\frac{d\bar{\theta}^{-2}(g_C)}{dT} = \exp \left[ \left[ -g_S \bar{T} \right] \frac{d\bar{\theta}^{1}(g_C)}{dT} \right] > 0.
$$

$$
\frac{d\bar{T}(\theta, g_C)}{dT} = -\frac{1}{\left[ -g_S \bar{\theta} \right]} \frac{d\bar{\theta}^{1}(g_C)}{dT} < 0
$$

(46)

This condition implies that extending (shortening) the patent length moves the timing of disruptive innovation by each incumbent forward (later). This result is somewhat counterintuitive because a longer patent length makes incumbents decide front-loading switch of their current technology. However, the interpretation is simple: extending the patent length stimulates innovation by entrants, which decreases the growth rate of the expected net benefit from disruptive innovation, $g_C$, and then the marginal loss from putting off disruptive innovation increases. 34

From (44), (45), and (46), we can write the equilibrium value of $\theta_0^{1*}$ as a function of the policy variable:

$$
\theta_0^{1*} = \theta_0^{1*} (s_D, s_F, \bar{T}),
$$

(47)

where $\frac{\partial \theta_0^{2*}(s_D, s_F, T)}{\partial s_D} < 0$, $\frac{\partial \theta_0^{1*}(s_D, s_F, T)}{\partial s_F} > 0$ and $\frac{\partial \theta_0^{1*}(s_D, s_F, T)}{\partial T} > 0$. 35

We summarize the above discussion in the following proposition:

**Proposition 3.2.** When we specify the innovation technology as $C_F(t) = E(t)L(t)c_F$, and $C_D(t) = \frac{E(t)L(t)}{Q(t)}c_D$, we obtain the following results.

- **A subsidy (tax) for disruptive innovation by incumbents stimulates (mitigates) innovation by entrants, and then increases (decreases) the growth rate of the economy.**

- **A subsidy (tax) for disruptive innovation by incumbents moves the timing of disruptive innovation by each incumbent earlier (later), and the number of incumbents conducting disruptive innovation**
before their patent protection expires increases (decreases).

- A subsidy (tax) for innovation by entrants stimulates (mitigates) innovation by entrants, and then increases (decreases) the growth rate of the economy.

- A subsidy (tax) for innovation by entrants moves the timing of disruptive innovation by each incumbent later (earlier), and the number of incumbents conducting disruptive innovation before their patent protection expires decreases (increases).

- Extension (shortening) of patent length stimulates (mitigates) innovation by entrants, and then increases (decreases) the growth rate of the economy.

- Extending (shortening) the patent length moves the timing of disruptive innovation by each incumbent earlier (later), and increases (decreases) the number of incumbents conducting disruptive innovation before their patent protection expires.

Next, we derive the equilibrium values of the average quality of differentiated goods, Θ. From the first row of (27) and (47), if \(θ^c_0 ≥ \theta^m_0\) is satisfied, we obtain the equilibrium value of the average quality of differentiated goods as:

\[
Θ^* = \frac{k}{k-1} \theta^c_0^1.
\]

(48)

which is an increasing function of the expectation of initial adjusted quality, \(k/(k-1)\), and is an increasing function of the equilibrium value of the critical level of initial adjusted quality in terms of whether the incumbent conducts production, \(θ^c_0^1\). \(^{36}\) The intuition is as follows. A high expectation of initial adjusted quality makes adjusted quality with given \(g_0\) high at any time. High \(θ^c_0^1\), which means low \(\tilde{T}\), denotes that incumbents conduct disruptive innovation earlier, and then incumbents having low adjusted quality tend to exit the market earlier. Thus, high adjusted quality differentiated goods tend to remain, which raises the average quality of remaining differentiated goods.

Because we have already confirmed \(\frac{\partial \theta^c_0^1}{\partial s_D} < 0\), \(\frac{\partial \theta^c_0^1}{\partial s_F} > 0\), and \(\frac{\partial \theta^c_0^1}{\partial \tilde{T}} > 0\), we can confirm \(\frac{\partial \Theta^*}{\partial s_D} < 0\), \(\frac{\partial \Theta^*}{\partial s_F} > 0\), and \(\frac{\partial \Theta^*}{\partial \tilde{T}} > 0\). Then, we obtain following proposition:

\(^{36}\) If \(θ^c_0^1 > \theta^m_0\) is not satisfied, the equilibrium values of the average quality of differentiated goods is determined from the second or third row of (27). We refer to this case when we conduct numerical analysis.
Proposition 3.3. When we specify the innovation technology as $C_F(t) = \frac{F(t) L}{Q(t)} c_F$, and $C_D(t) = \frac{E(t) L}{Q(t)} c_D$, and when the average quality of differentiated goods is sufficiently large, we get the following results.

- A subsidy (tax) for disruptive innovation by incumbents stimulates (mitigates) disruptive innovation by incumbents, and then increases (decreases) the average quality of differentiated goods.
- A subsidy (tax) for innovation by entrants mitigates (stimulates) disruptive innovation by incumbents, and then decreases (increases) the average quality of differentiated goods.
- Extending (shortening) patent length stimulates (mitigates) disruptive innovation by incumbents, and then increases (decreases) the average quality of differentiated goods.

Substituting the equilibrium value of $g_Q$, $\nu$, and $\Theta$ into (32), we obtain the equilibrium value of consumption per capita, $E$. From (32), consumption per capita is affected by policy variables through three channels. First, consumption per capita is a decreasing function of $g_Q$. An increment of $g_Q$ requires resources devoted to innovation activity by entrants, which decreases resources devoted to production activity. Second, consumption per capita is a decreasing function of $\nu$. An increment of $\nu$ requires resources devoted to innovation activity by incumbents, which decreases resources devoted to production activity. Third, consumption per capita is an increasing function of $\Theta$. An increment of $\Theta$ strengthens the positive externality of innovation activity, which requires fewer resources with a given level of innovation activity by entrants and incumbents. Then, resources devoted to production activity increase with an increment of $\Theta$. From these three effects, it is difficult to confirm analytically the effects of policy variables on consumption per capita. In the following, we conduct a numerical analysis to confirm the effects of policy variables visually.

Following precedent studies, we set parameter values $\rho = 0.04$, $g_L = 0.01$, and $\sigma = 6.37$ In addition, we set $c_D = 1$ as the standard relative to $c_F$.

In Figure 1, we confirm the effect of changing policy variables with different efficiency of innovation by entrants, $c_F$. We examine the case where $c_F = 3$, $c_F = 5$, and $c_F = 7$ while setting $g_{\theta} = -0.02$, $k = 2$. The first row of Figure 1 expresses the relationship between the growth rate of $Q$, $g_Q$, and

---

37 From Jones and Williams (2000), we interpret the interest rate as the expected internal rate of return to R&D projects, and we set $\rho = 0.04$. From the World Development Indicators (World Bank 2007), the average annual rate of population growth in the US between 1975 and 2007 was around 1%, so we set $g_L = 0.01$. From Basu (1996) and Norrbin (1993), the average price markup over marginal cost, $\sigma = 6$, has been estimated as ranging between 1.05 and 1.4, so we set $\sigma = 6$, which
reduced when \( \bar{\theta} \) decreases. This benefit is enjoyed only by incumbents drawing a high initial adjusted quality. Incumbents must increase more to satisfy the free entry condition. Because \( \frac{\partial g}{\partial q_l} < 0 \), \( \frac{\partial X}{\partial q_l} > 0 \), and \( \frac{\partial \bar{X}}{\partial q_l} > 0 \) are satisfied, the expected value of incumbents decreases less when \( q_l \) is large and the value of \( q_l \) is small. When \( s_F \) and \( s_D \) are very low, \( g \) is large when \( \bar{g} \) is small. When \( s_F \) and \( s_D \) are very low, \( g \) is large when \( \bar{g} \) is large, and then \( g \) must increase more to satisfy the free entry condition. High \( \bar{T} \) increases the expected value of incumbents with given \( g \) through extending the duration of instantaneous profit. This benefit is enjoyed only by incumbents drawing a high initial adjusted quality. Incumbents enjoying this effect decrease when \( T \) is very high, and the effect that high \( T \) enlarges the expected value of incumbents is reduced when \( T \) is very high.

\[ \frac{\partial X}{\partial q} > 0 \]

\[ \frac{\partial \bar{X}}{\partial q} > 0 \]

A subsidy for innovation for incumbents and entrants increases the expected profit from entry, and thus, the expected value of incumbents must increase through an increment of \( g \) to satisfy the free entry condition. Because \( \frac{\partial g}{\partial q} < 0 \), \( \frac{\partial X}{\partial q} > 0 \), and \( \frac{\partial \bar{X}}{\partial q} > 0 \) are satisfied, the expected value of incumbents increases less when \( q \) is large and the value of \( q \) is small. When \( s_F \) and \( s_D \) are very low, \( g \) is large, and then \( g \) must increase more to satisfy the free entry condition. High \( T \) increases the expected value of incumbents with given \( g \) through extending the duration of instantaneous profit. This benefit is enjoyed only by incumbents drawing a high initial adjusted quality. Incumbents enjoying this effect decrease when \( T \) is very high, and the effect that high \( T \) enlarges the expected value of incumbents is reduced when \( T \) is very high.

The second row of Figure 1 expresses the relationship between the average quality of differentiated goods, \( \Theta \), and policy variables. The broken line of \( \theta_0^1 = 1 \) helps us confirm whether the equilibrium is \( \theta_0^1 \geq \theta_{0}^{\min} \) or \( \theta_0^1 < \theta_{0}^{\min} \). If the average quality is greater (smaller) than the broken line, \( \theta_0^1 \geq \theta_{0}^{\min} \) (\( \theta_0^1 < \theta_{0}^{\min} \)) is satisfied. First, if \( \theta_0^1 \geq \theta_{0}^{\min} \) is satisfied, we can confirm that the average quality, \( \Theta \), is a decreasing function of \( s_D \) and an increasing function of \( s_F \). This result is consistent with proposition

38The first column examines the effect of a subsidy (tax) on innovation for entrants, the second examines the effect of a subsidy (tax) on disruptive innovation for incumbents, and the third examines the effect of patent length.

39A subsidy for innovation for incumbents and entrants increases the expected profit from entry, and thus, the expected value of incumbents must decrease through an increment of \( g \) to satisfy the free entry condition. Because \( \frac{\partial \bar{X}}{\partial q} < 0 \), \( \frac{\partial X}{\partial q} > 0 \), and \( \frac{\partial \bar{X}}{\partial q} > 0 \) are satisfied, the expected value of incumbents decreases less when \( q \) is large and the value of \( q \) is small. When \( s_F \) and \( s_D \) are very low, \( g \) is large, and then \( g \) must increase more to satisfy the free entry condition. High \( T \) increases the expected value of incumbents with given \( g \) through extending the duration of instantaneous profit. This benefit is enjoyed only by incumbents drawing a high initial adjusted quality. Incumbents enjoying this effect decrease when \( T \) is very high, and the effect that high \( T \) enlarges the expected value of incumbents is reduced when \( T \) is very high.

40Since we set \( k = 2 \), \( \theta_0^1 = 1 \) is satisfied when \( \Theta = 2 \), from (48).
3.3. Second, if $\theta_0^c \geq \theta_0^{min}$ is satisfied, $\Theta$, is an increasing function of $\bar{T}$ when $g_Q > 0$ is satisfied, however, is independent of $\bar{T}$ when $g_Q = 0$. An increment of $\bar{T}$ makes disruptive innovation active through a increment of $g_Q$, and the this effect does not work as long as $g_Q = 0$. Third, if $\theta_0^c < \theta_0^{min}$ is satisfied, $\Theta$ can increase despite a small $s_F$. This is because the average quality of any vintage decreases over time if $\theta_0^c < \theta_0^{min}$. Small $s_F$ leads to high $g_Q$, which increases the ratio of new differentiated goods, which have a high quality on average. If this effect dominates the effect that small $s_F$ leads to small disruptive innovation, the average quality can be raised with a small $s_F$. We can confirm this case when $s_F$ is low in the second column of the left panel. Fourth, if $\theta_0^c < \theta_0^{min}$ is satisfied, $\Theta$ can increase with a small $\bar{T}$. When $\bar{T}$ is small, $\theta_0^c$ becomes small, and tends to be less than $\theta_0^{min}$. In this case, all incumbents conduct no disruptive innovation until their patent protection expires, and then a small $\bar{T}$ makes the timing of disruptive innovation for all incumbents earlier. An earlier technological switch for all incumbents raises the average quality of differentiated goods. We can confirm this case when $\bar{T}$ is low in the second column of the right panel. Fifth, if $\theta_0^c \geq \theta_0^{min}$ is satisfied, $\Theta$ is large when $c_F$ is high for any policy variables. On one hand, a high $c_F$ makes the expected return to disruptive innovation high and stimulates disruptive innovation. On the other hand, a high $c_F$ leads to a small $g_Q$, which discourages disruptive innovation. These two effects are in the opposite direction; however, our numerical result shows that the former effect dominates the latter. Sixth, if $\theta_0^c < \theta_0^{min}$ is satisfied, $\Theta$ can be greater when $c_F$ is small. When $\theta_0^c < \theta_0^{min}$, the average quality is an increasing function of $g_Q$. Then, the effect that a high $c_F$ leads to a small $g_Q$, which decreases $\Theta$ with a given level of disruptive innovation, is added. If this additive effect dominates the effect that a high $c_F$ leads to active disruptive innovation with a given $g_Q$, $\Theta$ is large with a low $c_F$. We can confirm this case when $s_F$ is low in the second column of the left panel. Seventh, if $\theta_0^c < \theta_0^{min}$ and $g_Q = 0$ are satisfied, $\Theta$ is independent of $c_F$. If $\theta_0^c < \theta_0^{min}$ is satisfied, the effect that a high $c_F$ leads to active disruptive innovation does not work. Moreover, if $g_Q = 0$ is satisfied, the effect that a high $c_F$ leads to a small $g_Q$, which decreases $\Theta$, does not work. We can confirm this case when $s_D$ is high in the second column of the center panel. 41

The third row of Figure 1 expresses the relationship between consumption per capita, $E$, and policy variables. First, if $g_Q > 0$ is satisfied, $E$ is an increasing function of $s_F$. An increment of $s_F$ decreases $g_Q$ and increases $\nu$ and $\Theta$. 42 On one hand, a decrement of $g_Q$ and an increment of $\Theta$ leads to consumption

41 Using L'Hôpital’s rule in the third row of (27) when $g_n = 0$, $\Theta = \frac{k}{k-1} \frac{1}{\bar{T}} \frac{1 - \exp\left[-g_Q\bar{T}\right]}{-g_Q}$. We can easily confirm that $\Theta$ is independent of $c_F$. 42 If $\theta_0^c < \theta_0^{min}$ is satisfied, an increment of $s_F$ can decrease $\Theta$, however, this effect is negligibly small.
per capita being large, while on the other hand, an increment of \( \nu \) reduces consumption per capita. Our numerical result shows that the former effect dominates the latter. Second, if \( g_Q = 0 \) is satisfied, \( E \) is a decreasing function of \( s_F \). In this case, the effect that an increment of \( s_F \) decreases \( g_Q \) does not work. Then, the effect that an increment of \( s_F \) increases \( \Theta \) cannot dominate the effect that an increment of \( s_F \) increases \( \nu \), as shown in our numerical result.\(^{43}\) Third, if \( g_Q > 0 \) is satisfied, \( E \) is an increasing function of \( s_D \). An increment of \( s_D \) decreases \( g_Q \), \( \nu \) and \( \Theta \). On the one hand, a decrement of \( g_Q \) and \( \nu \) leads to large consumption per capita, on the other hand, a decrement of \( \Theta \) leads to small consumption per capita. Our numerical result shows that the former effect dominates the latter. Fourth, even if \( g_Q = 0 \) is satisfied, \( E \) is an increasing function of \( s_D \). In this case, the effect that an increment of \( s_D \) decreases \( g_Q \) does not work; however, only the effect that an increment of \( s_D \) decreases \( \nu \) dominates the effect that an increment of \( s_F \) decreases \( \Theta \), as shown in our numerical result. Fifth, if \( g_Q > 0 \) is satisfied, \( E \) temporarily decreases with high \( \bar{T} \) when \( g_Q \) increases rapidly; however, \( E \) shifts from decreasing to increasing when the increment of \( g_Q \) becomes small, and \( E \) converges to a finite value. On one hand, high \( \bar{T} \) leads to large \( g_Q \) and large disruptive innovation by incumbents drawing \( \theta_0 < \theta_0^{c2} \), which makes consumption per capita small. On the other hand, high \( \bar{T} \) leads to large \( \Theta \), and high \( \bar{T} \) makes incumbents drawing \( \theta_0 > \theta_0^{c2} \) put off disruptive innovation, which leads to large consumption per capita. Our numerical result shows that the former effects dominate the latter when \( g_Q \) increases rapidly; however, the former effects become weak when the increment of \( g_Q \) becomes small, and then the latter effects dominate the former.

We also confirm that these two effects are balanced, and that \( E \) does not change when \( \bar{T} \) is very large.

Sixth, if \( g_Q = 0 \) is satisfied, \( E \) is an increasing function of \( \bar{T} \). If \( g_Q = 0 \) is satisfied, the effect that high \( \bar{T} \) leads to large innovation by both incumbents and entrants does not work. Then, \( E \) increases with high \( \bar{T} \) through the effect that high \( \bar{T} \) moves the timing of disruptive innovation for incumbents drawing \( \theta_0 > \theta_0^{c2} \) later. Seventh, if \( g_Q > 0 \) is satisfied, \( E \) is large when \( c_F \) is high for any policy variable. High \( c_F \) requires large resources with a given \( g_Q \), which directly makes consumption per capita small. When \( c_F \) is high, for any policy variable, \( g_Q \) is small, and \( \nu \) and \( \Theta \) are large. On one hand, small \( g_Q \) and large \( \Theta \) make consumption per capita large. On the other hand, large \( \nu \) and high \( c_F \) itself make consumption per capita small. Our numerical result shows that the former effect dominates the latter. Eighth, if \( g_Q = 0 \) is satisfied, \( E \) is small when \( c_F \) is high. In this case, the effect that high \( c_F \) leads to small \( g_Q \) and the

\(^{43}\)When \( g_Q = 0 \), the labor market clearing condition is expressed as

\[
E = \left[ \frac{r - 1}{\sigma} + \frac{k - 1}{H} \frac{[\theta_0^{c1}]^{k - 1} - [\theta_0^{c2}]^{k - 1}}{1 - \exp(-[\theta_0^{c2}]^{k - 1})} c_D \right]^{-1}. 
\]

From this function, \( E \) is a decreasing function of \( \theta_0^{c2} \).

29
direct resource effect of $c_F$ do not work. Our numerical result shows that the effect that high $c_F$ leads to large $\Theta$ cannot dominate the effect that high $c_F$ makes large $\nu$.

In Figure 2, we confirm the effect of changing policy variables with different growth rates of adjusted quality, $g_Q$. We examine the case that $g_Q = -0.01$, $g_Q = -0.02$, and $g_Q = -0.03$ while setting $c_F = 5$, $k = 2$. Although a tendency of the effect of changing policy variables is similar to the above numerical analysis, there are some points worth noting. First, $g_Q$ is small when the absolute value of $g_Q$ is high for any policy variable. When the obsolescence rate is high, the profit of incumbents decreases rapidly, and then the expected value of incumbents with given $g_Q$ is low, which discourage entry. Second, if $g_Q > 0$ is satisfied, $\Theta$ is small when the absolute value of $g_Q$ is high. A high obsolescence rate makes $g_Q$ small ($g_C$ large), which decreases the marginal loss from putting off disruptive innovation and makes disruptive innovation small. Third, if $g_Q = 0$ and $\theta_0^{c1} \geq \theta_0^{m1}$ are satisfied, $\Theta$ is not affected by $g_Q$. As discussed in section 2.7, the average quality of any vintage is not affected by $g_Q$ as long as $\theta_0^{c1} \geq \theta_0^{mmin}$ is satisfied. We can confirm this case when $s_F$ is high in the second column of the left panel. Fourth, if $g_Q = 0$ and $\theta_0^{c1} < \theta_0^{mmin}$ are satisfied, $\Theta$ is small when the absolute value of $g_Q$ is high, which is also discussed in section 2.7. We can confirm this case when $s_D$ is high in the second column of the center panel. Fifth, if $g_Q > 0$ is satisfied, $E$ is large when the absolute value of $g_Q$ is high. A high obsolescence rate makes $g_Q$ and $\Theta$ small. From (25), a high obsolescence rate increases $\nu$ through the direct effect; however, it decreases $\nu$.

Figure 2: A numerical analysis of changing $g_Q$
through a decrement of $g_Q$. On one hand, small $g_Q$ and the indirect effect of $\nu$ lead to large consumption per capita. On the other hand, small $\Theta$ and the direct effect of $\nu$ lead to small consumption per capita. Our numerical result shows that the former effect dominates the latter. Sixth, if $g_Q = 0$ is satisfied, $E$ is small when the absolute value of $g_\theta$ is high. In this case, the effect that a high absolute value of $g_\theta$ leads to small $g_Q$ and the indirect effect of $\nu$ does not work. Then, only the effect that a high absolute value of $g_\theta$ leads to small $\Theta$ and the direct effect of $\nu$ remains, which leads to consumption per capita being small.

In Figure 3, we confirm the effect of changing policy variables with a different expected value of $\theta_0$, $\frac{k}{k-1}$. We examine the case where $k = 1.8$, $k = 2$, and $k = 2.2$ while setting $c_F = 5$, $g_\theta = -0.02$.\footnote{When $k$ is different, $\theta_0^{k_1}$ takes a different value with given $\Theta$. Then, we draw three broken lines in the second column of figure 3 to confirm whether the equilibrium is $\theta_0^{k_1} \geq \theta_0^{min}$, or $\theta_0^{k_1} < \theta_0^{min}$ when $k = 1.8$, $k = 2$, and $k = 2.2$ respectively.} In regard to Figure 3, there are some points worth noting.\footnote{\cite{44} \cite{45}} First, for any policy variable, $g_Q$ is small when $k$ is high. When $k$ is high, the expected value of the initial adjusted quality is low, which lowers expected profit. Then, the expected value of incumbents with given $g_Q$ is small, which discourages entry. Second, for any policy variable, $\Theta$ is small when $k$ is high. High $k$ makes $g_Q$ small, which raises the incentive to put off disruptive innovation. Moreover, high $k$ itself leads to average quality with a given level of disruptive innovation, and then $\Theta$ is small with high $k$ even when $g_Q$ is satisfied. Third, if $g_Q > 0$ is
satisfied, $E$ is large when $k$ is high. High $k$ makes $g_Q$, $\nu$, and $\Theta$ small. On one hand, small $g_Q$ and $\nu$ lead to large consumption per capita. On the other hand, small $\Theta$ leads to small consumption per capita.

Our numerical result shows that the former effect dominates the latter. Fourth, if $g_Q = 0$ is satisfied, $E$ is small when $k$ is high. When $g_Q = 0$ is satisfied, the effect that high $k$ leads to small $g_Q$ does not work.

Our numerical result shows that the effect that high $k$ leads to $\nu$ being small dominates the effect that high $k$ makes $\Theta$ small.

### 3.2 The Case Where $C_F(t) = E(t)E(t)\gamma c_F$ and $C_D(t) = E(t)E(t)\gamma c_L$

In this section, we generalize the externality from the improvement level of differentiated goods as:

\[
C_F(t) = \frac{E(t)E(t)}{n(t)\Theta} c_F \quad C_D(t) = \frac{E(t)E(t)}{n(t)\Theta} c_L.
\]  

These technology functions contain a positive externality from average quality and the number of differentiated goods, and a negative externality from market scale. We restrict the parameter $\gamma \geq 1$. For simplicity, we also restrict our analysis to where $\theta^0_{10} = 1$ and $g_Q > 0$ are satisfied.

The labor market clearing condition is expressed as:

\[
E(t) \left[ \frac{\sigma - 1}{\sigma} + \frac{\nu}{\Theta} c_D + \frac{g_Q}{\Theta} c_F \right] = 1.
\]  

(50)

From (50), in the BGP equilibrium, $g_E = 0$ must be satisfied, and then the growth rate of innovation cost is $gC = gL - gQ$.\(^{47}\)

Necessary conditions for the maximization problem $\tilde{T}$ are expressed as:\(^{48}\)

\[
\theta^0_{10} (g_Q) = \left[ \sigma \left[ \rho - g_L + g_Q \right] (s_F c_F - s_D c_D) \right] + \left[ k - 1 \right] \gamma \right]^{1 - 1 - \gamma}
\]

\[
\theta^0_{20} (g_Q) = \exp \left[ -g_0 \tilde{T} \right] \theta^0_{10} (g_Q)
\]

\[
\tilde{T} (\theta_0, g_Q) = \frac{1}{g_0} \ln \left[ \frac{\theta_0}{\theta^0_{10}} \right]
\]

Because we generalize the externality from the improvement level of differentiated goods, the incentive for disruptive innovation is affected by the average quality of differentiated goods. On one hand, a high av-

---

\(^{46}\)When $\gamma = 1$, this specification coincides with the previous section.

\(^{47}\)If $g_0 < 0$ must be satisfied not to break the sufficient condition of the maximization problem of $\tilde{T}$.

\(^{48}\)(15) is expressed as $\theta^0_{10} (g_Q) = \sigma \left[ \rho - g_L + g_Q \right] (s_F c_F - s_D c_D) \left[ \Theta \right]^{1 - \gamma}$. If $\theta^0_{10} \geq \theta^0_{10} = 1$ is satisfied, we obtain (51).
verage quality of differentiated goods decreases the expected net benefit from disruptive innovation, which
decreases the marginal loss from putting off disruptive innovation. On the other hand, a high average
quality of differentiated goods also decreases the instantaneous profit of incumbents, which decreases the
marginal benefit from putting off disruptive innovation. When $\gamma = 1$, these two opposing effects offset
each other completely. For $\gamma > 1$, the former effect dominates the latter, and then the high average
quality of differentiated goods decreases the marginal loss from putting off disruptive innovation relative
to the marginal benefit. Thus, incumbents move the timing of disruptive innovation later with given $g_Q$, and
the average quality of differentiated goods, $\Theta = \frac{L}{k-1} \theta_0^1$, is smaller than when $\gamma = 1$ with given $g_Q$.

The initial values of an incumbent entering at time $s$ and drawing $\theta_0$ is expressed as:

$$
V(\theta_0, 0, s) = \begin{cases} 
\frac{E[L(\bar{s})]}{n(s)} \left[ \frac{k}{k-1} \theta_0^1 (g_Q) \right]^{-\gamma} X_0 & \text{for } \theta_0 \leq \theta_0^1 \\
\frac{E[L(\bar{s})]}{n(s)} \left[ \frac{k}{k-1} \theta_0^1 (g_Q) \right]^{-\gamma} X_1 (\theta_0, g_Q, \theta_0^1 (g_Q)) & \text{for } \theta_0^1 \leq \theta_0 \leq \theta_0^2, \\
\frac{E[L(\bar{s})]}{n(s)} \left[ \frac{k}{k-1} \theta_0^1 (g_Q) \right]^{-\gamma} X_2 (\theta_0, g_Q, \theta_0^1 (g_Q)) & \text{for } \theta_0 > \theta_0^2
\end{cases}
$$

where

$$
X_0 = \left[ s_{FCF} - s_{DCD} \right]
$$

$$
X_1 (\theta_0, g_Q, \theta_0^1 (g_Q)) = \left[ \frac{\theta_0^1}{\sigma} \exp \left[ -[\rho - g_L + g_Q - g_T(\theta_0, g_Q)] \left[ \frac{k}{k-1} \theta_0^1 (g_Q) \right]^{-\gamma} \right] \right],
$$

$$
X_2 (\theta_0, g_Q, \theta_0^1 (g_Q)) = \left[ \frac{\theta_0^1}{\sigma} \exp \left[ -[\rho - g_L + g_Q - g_T(\theta_0, g_Q)] \left[ \frac{k}{k-1} \theta_0^1 (g_Q) \right]^{-\gamma} \right] \right],
$$

where $\theta_0^1$, $\theta_0^2$, and $T(\theta_0, g_Q)$ satisfy (51). The value of the incumbent is composed of two parts: aggregate
discounted instantaneous profits and expected discounted net profit from disruptive innovation. On one
hand, aggregate discounted instantaneous profits are proportional to $EL/Q$, while on the other hand, the
expected discounted net benefit from disruptive innovation is proportional to the innovation technology,
as in (31) and (49). In the model of section 3.1, for simplicity, we specify innovation technology to
be proportional to $EL/Q$, which makes the value of the incumbent proportional to $EL/Q$. In the
present section, we generalize the externality from the average quality of differentiated goods, and then
the expected discounted net benefit from disruptive innovation is proportional to $EL \left( Q[\Theta]^{-\gamma} \right)$. An
increment of $\Theta$ decreases aggregate discounted instantaneous profits to be smaller than the expected
discounted net benefit from disruptive innovation when \( \gamma > 1 \). Because we multiply \( \Theta^{-\gamma} \) before \( X_0, X_1, \) and \( X_2 \) to simplify the free entry condition, \( X_1 \) and \( X_2 \) are increasing functions of \( \Theta \).

The free entry condition is expressed as:

\[
\int_{1}^{\infty} X_0 dG (\theta_0) + \int_{\theta_0^c (g_Q)}^{\infty} X_1 (\theta_0, g_Q, \theta_0^c (g_Q)) dG (\theta_0) + \int_{\theta_0^c (g_Q)}^{\infty} X_2 (\theta_0, g_Q, \theta_0^c (g_Q)) dG (\theta_0) = s_F c_F. \tag{54}
\]

(51) and (54) determine the equilibrium value of \( g_Q \). Substituting the equilibrium value of \( g_Q \) into (51), we obtain the equilibrium values of \( \theta_0^c, \theta_0^e \), and \( \tilde{T} \). Substituting the equilibrium value of \( \theta_0^c \) into (27), we obtain the equilibrium value of \( \Theta \). Substituting the equilibrium value of \( g_Q \) into (30), we obtain the equilibrium value of the growth rate of this economy, \( g_u \). Substituting the equilibrium values of \( g_Q \) and \( \theta_0^c \) into (50), we obtain the equilibrium value of \( E \).

Regarding the effect of a subsidy (tax) for innovation by entrants, we obtain the interesting result:

\[
\frac{dg_Q}{ds_F} < 0 \quad \text{if} \quad \gamma < k
\]

\[
\frac{dg_Q}{ds_F} = 0 \quad \text{if} \quad \gamma = k
\]

\[
\frac{dg_Q}{ds_F} > 0 \quad \text{if} \quad \gamma > k \tag{55}
\]

We can interpret this result as follows. As discussed in the previous section, a decrement of \( s_F \) (the subsidy for entrants) simultaneously makes \( EV \) and the cost of innovation by entrants small; however, \( EV \) decreases less than the cost of innovation by entrants when \( \gamma = 1 \). Thus, \( g_Q \) must increase, which decreases \( EV \), to satisfy the free entry condition. When \( \gamma > 1 \) is satisfied, \( X_1 \) and \( X_2 \) are increasing functions of \( \Theta \). A decrement of \( s_F \) decreases the expected profit from disruptive innovation with given \( g_Q \), which mitigates disruptive innovation and makes \( \Theta \) small. Then, the subsidy for entrants makes \( X_1 \) and \( X_2 \) small through a decrement of \( \Theta \) with given \( g_Q \), and this effect is strengthened with high \( \gamma \). Thus, when \( \gamma \) is sufficiently high (\( \gamma > k \)), a decrement of \( s_F \) decreases \( EV \) more than the cost of innovation by entrants. Thus, \( g_Q \) must decrease to satisfy the free entry condition.

Regarding the effect of a subsidy (tax) for disruptive innovation by incumbents and the effect of extending (shortening) patent length, we obtain a result similar to that in the previous section, as well

\[\text{In appendix (D), we can confirm (55) analytically. The signs of } \frac{dg_Q}{d\bar{T}_F}, \frac{dg_Q}{d\bar{T}_F}, \frac{d\theta_0}{ds_F}, \frac{d\theta_0}{d\bar{T}_F}, \text{ and } \frac{d\theta_0}{d\bar{T}_F} \text{ are the same as those in the previous section.}\]
as the following proposition:

**Proposition 3.4.** When we specify the innovation technology as $C_F(t) = E(t)L(t)\gamma c_F$ and $C_D(t) = \frac{E(t)L(t)}{n(t)|\Theta|}c_L$, we get the following results.

If $\gamma < k$ is satisfied,

- A subsidy (tax) for disruptive innovation by entrants stimulates (mitigates) innovation by entrants, and then increases (decreases) the growth rate of the economy.

If $\gamma = k$ is satisfied,

- A subsidy (tax) for disruptive innovation by entrants does not affect innovation by entrants or the growth rate of the economy.

If $\gamma > k$ is satisfied,

- A subsidy (tax) for disruptive innovation by entrants mitigates (stimulates) innovation by entrants, and then decreases (increases) the growth rate of the economy.

Regardless of the magnitude of the correlation between $\gamma$ and $k$, we obtain the following results.

- A subsidy (tax) for disruptive innovation by incumbents and an extension (shortening) of patent length stimulate (mitigate) innovation by entrants, and then increase (decrease) the growth rate of the economy.

- A subsidy (tax) for disruptive innovation by incumbents, tax (subsidy) for innovation by entrants, and extension (shortening) of patent length move the timing of disruptive innovation by each incumbent earlier (later), and the number of incumbents conducting disruptive innovation before their patent protection expires increases (decreases).

This proposition shows that a subsidy for entrants decreases not only disruptive innovation by incumbents, but also innovation by entrants if the positive externality from the average quality is sufficiently strong, which decreases both the growth rate of the economy and the average quality of differentiated goods. Many related studies have concluded that support for entrants and the deregulation of barriers to entry stimulate economic growth.50 The present paper considers firm heterogeneity and the timing of

---

50 Regarding a welfare analysis, however, related studies have noted that there can be some cases where support for entrants and deregulation of barriers to entry harm social welfare.
incumbents switching their current technology, and these settings make the average quality endogenous. Because support for entrants and deregulation of barriers decrease the incentive to switch current technology, the average quality of differentiated goods decreases endogenously. If the positive externality of innovation from average quality is strong, support for entrants decreases the value of incumbents more relative to entry cost. Then, the incentive to entry decreases, which decreases the value of incumbents more. This interesting result implies that deregulation does not always stimulate economic growth, and that the role of disruptive innovation is important.

Next, we conduct a numerical analysis focused on $\gamma < k$. In Figure 4, we confirm the effect of changing policy variables with different $\gamma < k$. We examine the case where $\gamma = 1$, $\gamma = 1.3$, $\gamma = 1.5$, and $\gamma = 1.8$ while setting $c_F = 5$, $g_0 = -0.02$, $k = 2$, and $\sigma = 16$. Regarding Figure 4, there are some points worth noting. First, for any policy variables, $g_Q$ is large when $\gamma$ is high. When $\gamma$ is high, the cost of innovation decreases more than the expected value of incumbents with given $g_Q$, which encourages entry. Then, $g_Q$ must be large with high $\gamma$ to satisfy the free entry condition. Second, for any policy variables, $\Theta$ is large when $\gamma$ is high. This result is somewhat counterintuitive because from (51), $\Theta$ is small with high $\gamma$ with given $g_Q$. However, high $\gamma$ leads to large $g_Q$, which increases the marginal loss from putting off disruptive

---

51. Note that a decrement of $s_F$ itself decreases the expected benefit from disruptive innovation with given $g_Q$.

52. Note that we restrict our analysis to where $g_0^1 > g_0^{min} = 1$ and $g_Q > 0$ are satisfied.

53. We set a high $\sigma = 16$ to obtain a plausible growth rate. When $\sigma = 16$, the markup is 1.06.
innovation and makes disruptive innovation large. Our numerical result shows that this effect dominates the former. Third, for any policy variables, $E$ is small when $\gamma$ is high. High $\gamma$ leads to large $g_Q, \nu$, and $\Theta$. On one hand, large $\Theta$ leads to consumption per capita being large; on the other hand, large $g_Q$ and $\nu$ lead to consumption per capita being small. Our numerical result shows that the latter effect dominates the former.

In Figure 5, we confirm the effect of changing policy variables with different $\gamma \geq k$. We examine the case where $\gamma = 2, \gamma = 2.05, \gamma = 2.1, \gamma = 2.15$ while setting $c_F = 5, q = -0.02, k = 2, \sigma = 16$. Regarding Figure 5, there are some points worth noting. First, $g_Q$ is independent of $s_F$ when $\gamma = k$. Second, $g_Q$ is an increasing function of $s_F$ when $\gamma > k$ is satisfied. These features are consistent with proposition 3.4. Third, we can confirm that this other feature is similar to Figure 4.

3.3 The Case Where $C_F (t) = \frac{1}{Q(t)} c_F$ and $C_D (t) = \frac{1}{Q(t)} c_D$

In the previous two sections, we consider the negative externality of innovation caused by market size. This setting enables us to analyze $g_Q$ and $\Theta$ without considering the labor market clearing condition. However, in this setting, we cannot examine how population size affects key endogenous variables like $g_Q, \Theta$, and $E$. In this section, we eliminate this useful assumption and specify the innovation technology.
as:

\[
C_F(t) = \frac{1}{Q(t)} C_F \quad C_D(t) = \frac{1}{Q(t)} C_D \quad . \tag{56}
\]

These technology functions contain a positive spillover from the improvement level of differentiated goods; however, there is no negative externality from market scale. With this specification, the labor market clearing condition affects the equilibrium value of \( \theta_0^1, \theta_0^2, \bar{T}(\theta_0), g_Q, \) and \( \Theta \); thus, these values are dependent on population size. In this section, we restrict our analysis to where \( \theta_0^1 \geq \theta_0^{\text{min}} = 1 \), and \( g_L = 0 \) are satisfied.\(^{54}\)

The labor market clearing condition is expressed as:

\[
E(t) L \sigma - \frac{1}{\sigma} + \frac{\nu}{\Theta} c_D + \frac{g_Q}{\Theta} c_F = L. \tag{57}
\]

From (57), in the BGP equilibrium, \( g_E = 0 \) must be satisfied, and then the growth rate of innovation cost is \( g_C = -g_Q. \)^{55}

The necessary conditions of the maximization problem of \( \bar{T} \) are expressed as:

\[
\theta_0^1 (g_Q, E) = \frac{1}{T_F} [\rho + g_Q] [s_F c_F - s_D c_D]
\]

\[
\theta_0^2 (g_Q, E) = \exp \left[ -g_0 \bar{T} \right] \theta_0^1 (g_Q) \quad . \tag{58}
\]

\[
\bar{T} (\theta_0, g_Q, E) = \frac{1}{-g_0} \ln \left[ \frac{\theta_0}{\theta_0^1} \right]
\]

Market scale increases the instantaneous profit of incumbents, which increases the marginal benefit from putting off disruptive innovation. In the case of section 3.1, market scale also increases the expected net profit from disruptive innovation, which increases the marginal loss from putting off disruptive innovation; these two opposing effects offset each other completely. In this section, the absence of a negative externality from market scale makes the timing of disruptive innovation depend on market scale, \( EL \). A large \( EL \) only increases the marginal benefit from putting off disruptive innovation, and then incumbents move the timing of disruptive innovation later. When \( EL > 1 \) (\( EL < 1 \)), this effect is greater (smaller) than in the case of section 3.1. Thus, incumbents move the timing of disruptive innovation later (earlier) with given \( g_Q \), and the average quality of differentiated goods, \( \Theta = \frac{1}{k} \theta_0^1 \), is smaller (greater) than the

---

\(^{54}\) If these restrictions are not satisfied, analysis becomes far too complex and beyond the scope of the present paper.

\(^{55}\) \( g_0 < 0 \) must be satisfied to avoid breaking the sufficient condition of the maximization problem of \( \bar{T} \).
The initial values of an incumbent entering at time $s$ and drawing $\theta_0$ is expressed as:

$$ V(\theta_0, 0, s) = \frac{1}{Q(s)} X_0 \quad \text{for} \quad \theta_0 \leq \theta_0^1(g_Q, E) $$

$$ V(\theta_0, 0, s) = \frac{1}{Q(s)} X_1(\theta_0, g_Q, E) \quad \text{for} \quad \theta_0^1(g_Q, E) \leq \theta_0 \leq \theta_0^2(g_Q, E) $$

$$ V(\theta_0, 0, s) = \frac{1}{Q(s)} X_2(\theta_0, g_Q, E) \quad \text{for} \quad \theta_0 > \theta_0^2(g_Q, E) $$

where

$$ X_0 = \frac{\theta_0^1 EL}{\sigma} \left[ s_{FCF} - s_{DCD} \right] $$

$$ X_1(\theta_0, g_Q, E) = \frac{\theta_0^1 EL}{\sigma} \frac{1 - \exp\left[-|\rho + g_Q - g_0| T(\theta_0, g_Q)\right]}{\rho + g_Q - g_0} + \exp\left[-|\rho + g_Q| T(\theta_0, g_Q)\right] \left[ s_{FCF} - s_{DCD} \right] $$

$$ X_2(\theta_0, g_Q, E) = \frac{\theta_0^2 EL}{\sigma} \frac{1 - \exp\left[-|\rho + g_Q - g_0| T\right]}{\rho + g_Q - g_0} + \exp\left[-|\rho + g_Q| T\right] \left[ s_{FCF} - s_{DCD} \right] $$

where $\theta_0^1$, $\theta_0^2$, and $T(\theta_0, g_Q)$ satisfy (58). The value of an incumbent is composed of two parts: aggregate discounted instantaneous profits and expected discounted net benefits from disruptive innovation. On one hand, aggregate discounted instantaneous profits are proportional to $EL/Q$. On the other hand, expected discounted net benefits from disruptive innovation are proportional to $1/Q$. An increment of $EL$ increases aggregate discounted instantaneous profits; however, it does not increase the expected discounted net benefits from disruptive innovation. Because we multiply $1/Q(s)$ before $X_0$, $X_1$ and $X_2$ to simplify the free entry condition, $X_1$ and $X_2$ are increasing functions of $EL$.

The free entry condition is expressed as:

$$ \int_1^{\theta_0^1(g_Q, E)} \int_{\theta_0^1(g_Q, E)}^{\theta_0^2(g_Q, E)} X_0 dG(\theta_0) + \int_{\theta_0^1(g_Q, E)}^{\theta_0^2(g_Q, E)} X_1(\theta_0, g_Q, E) dG(\theta_0) + \int_{\theta_0^1(g_Q, E)}^{\theta_0^2(g_Q, E)} X_2(\theta_0, g_Q, E) dG(\theta_0) = s_{FCF}. \tag{61} $$

From (57), (58), and (61), the equilibrium values of $g_Q$ and $E$ are determined. Unlike the case of section 3.1, this model captures the resource effect; however, it generates a scale effect. Substituting the equilibrium value of $g_Q$ and $E$ into (58), we obtain the equilibrium values of $\theta_0^1$, $\theta_0^2$, and $T$. Substituting the equilibrium value of $\theta_0^1$ into (27), we obtain the equilibrium value of $\Theta$. Substituting the equilibrium value of $g_Q$ into (30), we obtain the equilibrium value of the growth rate of this economy.

Next, we conduct a numerical analysis with this equilibrium. In Figure 6, we confirm the effect of changing policy variables with different population size, $L$. We examine the case $L = 0.8$, $L = 0.9$, $L = 1$, $L = 1.2$. The equilibrium values of $g_Q$ and $E$ are found by numerically solving for $\theta_0^1$, $\theta_0^2$, and $T$. The results show that as $L$ increases, the equilibrium values of $g_Q$ and $E$ decrease. This indicates that an increase in population size leads to a decrease in the expected discounted net benefits from disruptive innovation, which in turn decreases the equilibrium values of $g_Q$ and $E$. The free entry condition is then satisfied for these equilibrium values of $g_Q$ and $E$.
The parameter is set as $c_F = 5$, $g_D = -0.02$, $k = 2$. The curve tagged “basic model” expresses the relationship between key endogenous and policy variables using the model constructed in section 3.1, where the parameter is set as $c_F = 5$, $g_D = -0.02$, $k = 2$, and $g_L = 0$.

Regarding Figure 6, there are some points worth noting. First, for any policy variables, $g_Q$ is large when $L$ is high. When $L$ is high, the instantaneous profit of incumbents is high, which increases the expected value of incumbents with given $g_Q$. Then, $g_Q$ must be large with high $L$ to satisfy the free entry condition.\(^{56}\) Second, the elasticity of $g_Q$ with respect to $s_F$ and $s_D$ is smaller when there is no negative externality from market scale compared with the case of section 3.1. Low $s_F$ and $s_D$ stimulate innovation by entrants, which lowers $E$. In the case of section 3.1, a decrement of $E$ affects the incentive for innovation by entrants through two channels. On one hand, a decrement of $E$ decreases the instantaneous profit of incumbents, which lowers the incentive for innovation by entrants. On the other hand, a decrement of $E$ decreases the cost of innovation by entrants, which raises the incentive for innovation by entrants.

We constructed a model in section 3.1, in which these two effects offset each other completely. However,\(^{56}\) in the model of the previous two sections, high $L$ also increases the cost of innovation by entrants, which offsets completely the effect that high $L$ increases the expected value of incumbents.

---

\(^{56}\)In the model of the previous two sections, high $L$ also increases the cost of innovation by entrants, which offsets completely the effect that high $L$ increases the expected value of incumbents.
in the present section, we remove this useful assumption, and the latter effect does not work. Thus, because of this resource effect, the subsidy for innovation by incumbents and entrants stimulates growth weaker than in the case of section 3.1.  

Third, as suggested in our analytical section, we can confirm that $g_Q$ is greater (smaller) than in the case of section 3.1 when aggregate consumption, $EL$, is greater (smaller) than 1.  

Fourth, if $g_Q > 0$ is satisfied, $\Theta$ is large when $L$ is high. This result is somewhat counterintuitive because from (58), $\Theta$ is small with high $L$ with given $g_Q$. However, high $L$ leads to large $g_Q$, which increases the marginal loss from putting off disruptive innovation and makes disruptive innovation large. Our numerical result shows that this effect dominates the former. Fifth, if $g_Q = 0$ is satisfied, $\Theta$ is small when $L$ is high. In this case, the effect that high $L$ leads to large $g_Q$ does not work, and then only the effect that high $L$ leads to small $\Theta$ with given $g_Q$ remains. Sixth, if $g_Q > 0$ is satisfied, $E$ is small when $L$ is high. High $L$ leads to large $g_Q$, $\nu$, and $\Theta$ if $g_Q > 0$ is satisfied. On one hand, large $\Theta$ makes consumption per capita large, while on the other hand, large $g_Q$ and $\nu$ make consumption per capita small. Our numerical result shows that the latter effect dominates the former. Seventh, if $g_Q = 0$ is satisfied, $E$ is large when $L$ is high. In this case, the effect that high $L$ leads to small $g_Q$ does not work. Our numerical result shows that the effect that high $L$ leads to small $\nu$ dominates the effect that $L$ leads to small $\Theta$.

Many growth models derive a positive relationship between population size and the degree of innovation (economic growth rate). However, as pointed out by Jones (1995), a positive relationship between population size and economic growth rate is not observed in the real economy. Many papers attempt to remove this irrelevant property in various ways. The result obtained in this section has a potential to offer one new solution to the scale effect. We extend the instantaneous utility function as used in Ohki (2015):

$$u(t) = \left[ \int_0^{n_1} q_1(j,t) (x_1(j,t))^{x-1} \, dj \right]^\alpha \left[ \int_0^{n_2} q_2(j,t) (x_1(j,t))^{x-1} \, dj \right]^{\frac{\sigma}{\sigma-1}} \left( \int_0^{\alpha} \right)^{\frac{\sigma}{\sigma-1}} (62)$$

where $q_i(j,t)$ and $d_i(j,t)$ denote the quality and consumption volume of incumbent $j$ in industry $i$ at time $t$, respectively, and $\alpha$ denotes the market share of industry 1. We assume that barriers to entry in

---

57 On one hand, the benefit of the setting of the present section is that we can capture the resource effect and examine the effect of population size. On the other hand, the disadvantage of the setting of the present section is that we cannot conduct analytical analysis, and that numerical calculation takes a long time. Because the difference between the two models is qualitatively small, we think that the assumption used in section 3.1 has sufficient validity.

58 See the first and fourth rows of Figure 6.
industry 1, $c_{F1}$, are too high for any entrants, $g_{n1} = 0$. In industry 2, barriers to entry are not too high, and thus, new entrants enter the market, $g_{n2} > 0$.

From the above discussion, high $L$ allows active innovation by incumbents and entrants if new entrants enter the market ($g_n = g_Q > 0$) in the equilibrium. However, high $L$ stifles innovation by incumbents if new entrants do not enter the market ($g_n = g_Q = 0$) in the equilibrium. If $\alpha$ is large, the share of industry where new entrants do not enter the market is large. Thus, if $\alpha$ is large, the latter effect can dominate the former, and we can show that high $L$ leads to a small degree of aggregate innovation. Because our model is designed so that innovation by incumbents does not increase the growth rate of the economy directly, we cannot observe a negative relationship between population size and the growth rate of the economy. However, an appropriate extension of our model enables us to design a model in which innovation by incumbents raises the growth rate of the economy directly. In doing so, we will construct a new endogenous growth model that overcomes the scale effect puzzle.

### 3.4 The Case Where $C_F(t) = \frac{E(t)L(t)}{e} c_F$ and $C_D(t) = \frac{E(t)L(t)}{e} c_L$

So far, we have analyzed the economy where the number of differentiated goods grows endogenously. In the context of the endogenous growth model, especially in the homogeneous model, the economy where the number of differentiated goods is constant over the long run is also examined. In this section, we examine a model where the number of differentiated goods is constant over the long run in consideration of firm heterogeneity and disruptive innovation. We specify the innovation technology as:

$$C_F(t) = \frac{E(t)L(t)}{e} c_F \quad \text{and} \quad C_D(t) = \frac{E(t)L(t)}{e} c_L.$$  \hspace{1cm} (63)

These technology functions contain a positive externality from the average quality of differentiated goods and a negative externality from market scale; however, there is no positive externality from the number of differentiated goods.

From (26) and (63), the labor market clearing condition is expressed as:

$$E(t) \left[ \frac{\sigma - 1}{\sigma} + \frac{\nu}{e} n(t) c_L + \frac{g_Q}{e} n(t) c_F \right] = 1.$$  \hspace{1cm} (64)

From (64), in the steady-state equilibrium, $g_E = 0$ and $g_n = 0$ must be satisfied; then, $g_Q = 0$, $g_C = 0$, and
\( g_u = 0 \) are satisfied. In this economy, the steady-state equilibrium value of \( n \) is determined endogenously instead of \( g_Q \).

The necessary conditions of the maximization problem of \( \bar{T} \) are expressed as:

\[
\begin{align*}
\theta_0^1 (n) &= n \sigma \left[ \rho - g_L \right] \left[ s_{FCF} - s_{DCD} \right] \\
\theta_0^2 (n) &= \exp \left[ \left[ -g_0 \bar{T} \right] \theta_0^3 (n) \right] \\
\bar{T} (\theta_0, n) &= \frac{1}{g_0} \ln \left[ \frac{\theta_0}{\theta_0^3 (n)} \right]
\end{align*}
\]

(65)

Large \( n \) decreases the instantaneous profit of incumbents, which in turn decreases the marginal benefit from putting off disruptive innovation. In the case of section 3.1, large \( n \) also decreases the expected benefit from disruptive innovation, which decreases the marginal loss from putting off disruptive innovation; these two opposite effects offset each other completely. Then, the absence of a positive externality from the number of differentiated goods makes the timing of disruptive innovation depend on \( n \), which decreases only the marginal benefit from putting off disruptive innovation, and incumbents move the timing of disruptive innovation earlier with large \( n \).

The initial values of an incumbent entering at time \( s \) and drawing \( \theta_0 \) is expressed as:

\[
\begin{align*}
V (\theta_0, 0, s) &= \frac{EL}{\theta_0} X_0 \quad \text{for} \quad \theta_0 \leq \theta_0^1 \\
V (\theta_0, 0, s) &= \frac{EL}{\theta_0} X_1 (\theta_0, n) \quad \text{for} \quad \theta_0^1 \leq \theta_0 \leq \theta_0^2 \\
V (\theta_0, 0, s) &= \frac{EL}{\theta_0} X_2 (\theta_0, n) \quad \text{for} \quad \theta_0 > \theta_0^2
\end{align*}
\]

(66)

where

\[
\begin{align*}
X_0 &= \left[ s_{FCF} - s_{DCD} \right] \\
X_1 (\theta_0, n) &= \frac{1 - \exp \left[ \frac{-\rho - g_0}{\rho - g_0} \bar{T} (\theta_0, n) \right] \theta_0}{\theta_0^3} + \left[ s_{FCF} - s_{DCD} \right] \exp \left[ -\rho \bar{T} (\theta_0, n) \right] \\
X_2 (\theta_0, n) &= \frac{1 - \exp \left[ \frac{-\rho - g_0}{\rho - g_0} \bar{T} (\theta_0, n) \right] \theta_0}{\theta_0^3} + \left[ s_{FCF} - s_{DCD} \right] \exp \left[ -\rho \bar{T} \right]
\end{align*}
\]

(67)

where \( \theta_0^1, \theta_0^2, \) and \( \bar{T} (\theta_0, n) \) satisfy (65). The value of incumbents is composed of two parts: aggregate discounted instantaneous profits and expected discounted net benefits from disruptive innovation. On one hand, aggregate discounted instantaneous profits are proportional to \( EL/Q \). On the other hand,

\( g_0 < 0 \) must be satisfied to avoid breaking the sufficient condition of the maximization problem of \( \bar{T} \).

\( \bar{T} \) in the model of section 3.1, \( g_n = 0 \) is satisfied if the expected profit from the entry is too small. In this case, the number of differentiated goods is constant; however, this is not determined endogenously, rather, it is exogenously given or derived from the span when \( g_n > 0 \) is satisfied and the exogenous initial value of \( n \). In the model of section 3.1, \( n \) does not affect other endogenous variables, which is clearly different from the model constructed in the present section.
expected discounted net benefits from disruptive innovation are proportional to $EL/\Theta$. An increment to $n$ decreases aggregate discounted instantaneous profits; however, it does not decrease expected discounted net benefits from disruptive innovation. Because we multiply $EL/\Theta$ before $X_0$, $X_1$, and $X_2$ to simplify the free entry condition, $X_1$ and $X_2$ are decreasing functions of $n$.

The free entry condition is expressed as:

$$
\int_1^\infty \frac{\theta_1^2(n)}{\varphi_1^2(n)} X_0 dG(\theta_0) + \int_1^{\infty} X_1(\theta_0, n) dG(\theta_0) + \int_1^{\infty} X_2(\theta_0, n) dG(\theta_0) = s_F c_F, \quad (68)
$$

where $\theta_0^1$, $\theta_0^2$, and $\bar{T}$ satisfy (65), and $X_0$, $X_1$, and $X_2$ satisfy (67), which determines the equilibrium value of $n$. Then, we can write the equilibrium value of $n$ as a function of policy variable:

$$
n^* = n(s_D, s_F, \bar{T}), \quad (69)
$$

where $\frac{\partial n_s}{\partial s_D} < 0$, $\frac{\partial n_s}{\partial s_F} < 0$, and $\frac{\partial n_s}{\partial \bar{T}} > 0$. Substituting (69) into (65), we obtain the equilibrium value of $\theta_0^1$.

Then, we can write the equilibrium value of $n$ as a function of the policy variable:

$$
\theta_0^{1*} = \theta_0^1(s_D, s_F, \bar{T}), \quad (70)
$$

where $\frac{\partial \theta_0^{1*}}{\partial s_D} < 0$, $\frac{\partial \theta_0^{1*}}{\partial s_F} > 0$, and $\frac{\partial \theta_0^{1*}}{\partial \bar{T}} > 0$. Substituting (70) into (27), we obtain the equilibrium value of the average quality. Substituting (69) and (70) into (64), we obtain the equilibrium value of $E$.

We summarize the above discussion in the following proposition:

**Proposition 3.5.** When we specify the innovation technology as $C_F(t) = \frac{E(t)L(t)}{e^t} c_F$, and $C_D(t) = \frac{E(t)L(t)}{e^t} c_L$, we obtain the following results.

- A subsidy (tax) for disruptive innovation by incumbents stimulates (mitigates) innovation by entrants in the transition process, and then increases (decreases) the number of differentiated goods in the steady state; however, it does not affect the growth rate in the steady state.

- A subsidy (tax) for innovation by entrants stimulates (mitigates) innovation by entrants in the transition process, and then increases (decreases) the number of differentiated goods in the steady state.

Since $\theta_0^2$ and $\bar{T}(\theta_0)$ are expressed as functions of $\theta_0^1$, we can also obtain the equilibrium value of these variables.
transition process, and then increases (decreases) the number of differentiated goods in the steady state; however, it does not affect the growth rate in the steady state.

- Extending (shortening) patent length stimulates innovation by entrants in the transition process, and then increases (decreases) the number of differentiated goods in the steady state; however, it does not affect the growth rate in the steady state.

- Regarding disruptive innovation by incumbents and the average quality of differentiated goods, the results obtained in this section are qualitatively the same as those in section 3.1.

As follows, we conduct a numerical analysis. Although a numerical analysis where $C_F(t) = \frac{E(t)L(t)}{\theta}c_F$ and $C_D(t) = \frac{1}{\theta}c_D$ is similar to the case discussed in section 3.1, there are some points to be considered. In Figure 7, we confirm the effect of changing variables with different efficiency of innovation by entrants, $c_F$. We examine the case where $c_F = 3$, $c_F = 5$, and $c_F = 7$ while setting $g_0 = -0.02$, $k = 2$. Regarding Figure 7, there are some points worth noting. First, $n$ is a decreasing function of $s_F$ and $s_D$, and an increasing function of $T$, which is consistent with proposition 3.5. Second, for any policy variable, the number of differentiated goods is large when $c_F$ is low. This result is interpreted as follows: a high $c_F$ imposes a high cost to enter the market, which discourages entry in the transition process. Discouragement of entry keeps the number of differentiated goods small in the steady state.
Third, if $\theta_0^\text{min} \leq \theta_0^c$ is satisfied, $E$ is independent of $s_F$. In the steady state, only disruptive innovation by incumbents is conducted. The resources devoted to disruptive innovation increases with large $n$ and $\nu$ and small $\Theta$. High $s_F$ leads to small $n$ and large $\nu$ and $\Theta$; however, as shown in appendix E, these three effects offset each other completely. We can confirm this in the third row of the left panel when $s_F$ is high. Fourth, if $\theta_0^\text{min} > \theta_0^c$ is satisfied, $E$ increases with high $s_F$. The effect that large $\nu$ leads to small $E$ is stronger than the effect that large $\Theta$ leads to large $E$. Then, high $s_F$ leads to small $E$ through large $\nu$ and $\Theta$. If $\theta_0^\text{min} > \theta_0^c$ is satisfied, this effect is weaker than the case where $\theta_0^\text{min} \leq \theta_0^c$ is satisfied because the incentive to conduct disruptive innovation is too small. Thus, this effect cannot dominate the effect that high $s_F$ leads to large $E$ through small $n$. Fifth, for any $\bar{T}$, $E$ does not decrease with high $\bar{T}$. In the case of section 3.1, $E$ temporarily decreases with high $\bar{T}$ when $g_Q$ increases rapidly. However, in the case of the present section, our numerical result suggests that this does not happen. Sixth, if $\theta_0^\text{min} \leq \theta_0^c$ is satisfied, $E$ is independent of $c_F$. High $c_F$ leads to small $n$, large $\nu$, and large $\Theta$. As shown in appendix E, these three effects offset each other completely. Seventh, if $\theta_0^\text{min} > \theta_0^c$ is satisfied, $E$ increases with high $c_F$. This interpretation is similar to the reason $E$ increases with high $s_F$ if $\theta_0^\text{min} \leq \theta_0^c$ is satisfied.

In Figure 8, we confirm the effect of changing policy variables with a different growth rate of adjusted quality, $g_\theta$. We examine the case where $g_\theta = -0.01$, $g_\theta = -0.02$, and $g_\theta = -0.03$ while setting $c_F = 5$, $k = 2$. Regarding Figure 8, there are some points worth noting. First, for any policy variable, the number of differentiated goods is large when the absolute value of $g_\theta$ is low. We interpret this result as follows: a high absolute value of $g_\theta$ decreases instantaneous profit rapidly, which discourages entry in the transition process. Discouragement of entry means there will be a small number of differentiated goods in the steady state. Second, for any $s_F$, $\Theta$ is strictly high when the absolute value of $g_\theta$ is low. In the case of section 3.1, $\Theta$ takes same value when $g_Q = 0$ and $\theta_0^\text{min} \leq \theta_0^c$ are satisfied. However, in the case of the present section, the number of differentiated goods is greater than zero, which decreases with a high absolute value of $g_\theta$. Then, the effect that a low absolute value of $g_\theta$ leads to a large $\Theta$ through large $n$ still works, even when $s_F$ is very high. Third, for any variable, $E$ is large when the absolute value of $g_\theta$ is low. In the case of section 3.1, if $g_Q > 0$ is satisfied, $E$ is small when the absolute value of $g_\theta$ is high. In the case of section 3.1, a high absolute value of $g_\theta$ leads to large $E$ through less innovation.

\[62\] In the case of section 3.1, an increment of $\bar{T}$ increases $g_Q$ rapidly, which increases the resources devoted to innovation by entrants, and then decreases the resources devoted to production. However, in the present case, innovation is conducted only by incumbents in steady-state equilibrium. Without this effect, a difference between the case of section 3.1 and the present case is created.
by incumbents. However, in the case of the present section, disruptive innovation is conducted only by incumbents in steady-state equilibrium, and then, this effect does not work.

In Figure 9, we confirm the effect of changing policy variables with different expected values of $\theta_0$, $\frac{k}{k+1}$. We examine the case where $k = 1.8$, $k = 2$, and $k = 2.2$ while setting $c_F = 5$, $g_\theta = -0.02$. Regarding Figure 9, there are some points worth noting. First, for any policy variable, the number of differentiated goods is small when $k$ is high. When $k$ is high, the expected value of initial adjusted quality is low, which discourages entry in the transition process. A discouragement to entry leads to a small number of differentiated goods in the steady state. Second, for any variable, $E$ is large when $k$ is high. In the case of section 3.1, if $g_Q = 0$ is satisfied, $E$ is small when $k$ is high. In the case of section 3.1, the absence of the effect that high $k$ leads to large $E$ through small $g_Q$ causes this result. However, in the case of the present section, the number of differentiated goods is greater than zero, which decreases with high $k$. Then, the effect that high $k$ leads to large $E$ through the effect that small $n$ leads to small resources devoted to disruptive innovation always works.

Finally, we examine the effect of degrees of externality from the average quality as in section 3.2. We

---

63 When $k$ is different, $\theta_0^1$ takes a different value with given $\Theta$. Then, we draw three broken lines in the second column of Figure 3 to confirm whether the equilibrium is $\theta_0^1 \geq \theta_0^{1^{\text{max}}}$, or $\theta_0^1 < \theta_0^{1^{\text{max}}}$ when $k = 1.8$, $k = 2$, and $k = 2.2$ respectively.
from the average quality, expresses the relationship between the number of differentiated goods, $\gamma = 2$

Regarding figure 10, there are some points worth noting. First, if $\gamma > k$ is satisfied, $\Theta$ is independent of $s_F$. Second, if $\gamma = k$ is satisfied, $n$ is independent of $s_F$. Third, if $\gamma > k$ is satisfied, $n$ is an increasing function of $s_F$. Interpretation of these results is similar to the reason we obtain proposition 3.4. Fourth, if $\theta_{0,0}^{\text{min}} \leq \theta_{0,0}^{k}$ is satisfied, $\Theta$ is independent from $\gamma$. This result is somewhat counterintuitive because from (77), $\Theta$ is small with high $\gamma$ with given $n$. However, high $\gamma$ leads

$\theta_{0,0}^{\text{min}} \leq \theta_{0,0}^{k}$

specify R&D technology as:

$$C_F(t) = \frac{1}{Q(t)} c_F$$
$$C_D(t) = \frac{1}{Q(t)} c_D$$

Figure 9: A numerical analysis changing $k$, where $C_F(t) = \frac{1}{Q(t)} c_F$ and $C_D(t) = \frac{1}{Q(t)} c_D$

In Figure 10, we confirm the effect of changing policy variables with different degrees of externality from the average quality, $\gamma$. We examine the case where $\gamma = 1$, $\gamma = 1.3$, $\gamma = 1.5$, $\gamma = 1.8$, $\gamma = 2$, $\gamma = 2.01$, $\gamma = 2.1$, and $\gamma = 2.15$ while setting $c_F = 5$, $g_0 = -0.02$, and $k = 2$. The first row of Figure 10 expresses the relationship between the number of differentiated goods, $n$, and policy variables when $\gamma < k$ is satisfied. The second row of Figure 10 expresses the relationship between the number of differentiated goods, $n$, and policy variables when $\gamma \geq k$ is satisfied. The third (fourth) row of Figure 10 expresses the relationship between the average quality of differentiated goods, $\Theta$ (consumption per capita, $E$), and policy variables. Regarding figure 10, there are some points worth noting. First, if $\gamma < k$ is satisfied, $n$ is a decreasing function of $s_F$. Second, if $\gamma = k$ is satisfied, $n$ is independent of $s_F$. Third, if $\gamma > k$ is satisfied, $n$ is an increasing function of $s_F$. Interpretation of these results is similar to the reason we obtain proposition 3.4. Fourth, if $\theta_{0,0}^{\text{min}} \leq \theta_{0,0}^{k}$ is satisfied, $\Theta$ is independent from $\gamma$. This result is somewhat counterintuitive because from (77), $\Theta$ is small with high $\gamma$ with given $n$. However, high $\gamma$ leads

$\theta_{0,0}^{\text{min}} \leq \theta_{0,0}^{k}$

In Appendix E, we derive the equilibrium and confirm an important analytical result.
Figure 10: A numerical analysis changing $\gamma$, where $C_F(t) = \frac{1}{q(t)c_F}$ and $C_D(t) = \frac{1}{q(t)c_D}$
to large $n$, which increases the marginal loss from putting off disruptive innovation and makes disruptive innovation large. As shown in Appendix E, these two opposite effects offset each other completely. Fifth, if $\theta_0^{\min} \leq \theta_0^{\varpi}$ is satisfied, $E$ is also independent of $\gamma$. Although $\Theta$ is independent of $\gamma$, high $\gamma$ enables incumbents to conduct disruptive innovation with few resources because of the high positive externality, which makes $E$ large. High $\gamma$ leads to large $n$, which increases the number of firms conducting disruptive innovation; this effect then leads to $E$ being small. As shown in Appendix E, these two opposite effects offset each other completely.

4 Conclusion

In this paper, we constructed a tractable model where heterogeneous incumbents conduct innovation to switch to new technology, and entrants conduct innovation to enter the market. We documented new insights that have never been obtained from growth models not considering heterogeneous incumbents’ opportunities to switch their technology. Furthermore, there are some possibilities to develop our model because we intended to construct a tractable model that could be used to examine more complex situations.

The first possibility is theoretical. In our setting, disruptive innovation affects the growth rate only through indirect effects: the degree of disruptive innovation affects the average quality, which affects instantaneous profit and the cost of innovation by entrants, which indirectly affects the growth rate. One can construct a model, for example, in which the degree of disruptive innovation affects the expected initial quality of differentiated goods, which in turn enables the average quality to increase permanently; then, disruptive innovation affects the growth rate directly. We think that this extension offers a new method to resolve the scale effect puzzle.

A second possibility is application. In our setting, the behavior of incumbents is limited to production and disruptive innovation. Of course, there are many potential behaviors of incumbents. Thus, one can consider technology transfer and extend the present model to the North–South model. This can also consider business expansion and marketing activity conducted by heterogeneous incumbents, and then examine how these factors affect disruptive innovation and economic growth. Another extension of our model is in regard to replacement risk: to examine the incumbents’ patent protection activity.

A third possibility is policy analysis. In our setting, we examine the effects of a subsidy (tax) for
innovation by incumbents and entrants and patent length. Other plausible policies, such as patent breadth and forward protection, should be examined in future studies.

References


We define $\nu(t)$ as the ratio of the number of differentiated goods embodying disruptive innovation to the aggregate number of differentiated goods at time $t$. The number of differentiated goods invented at time $t$ is the number of differentiated goods embodying disruptive innovation at time $t$ plus the number of differentiated goods invented by entrants at time $t$, which is expressed as $n(t) [\nu(t) + g_n]$. Differentiated goods invented at time $t$ having initial adjusted quality $\theta_0 \leq \theta_0^1$ embody disruptive innovation immediately, and then the number of differentiated goods invented at time $t$ and disruptive innovation conducted at time $t$ are expressed as $n(t) [\nu(t) + g_n] \int_{\theta_0^{\min}}^{\theta_0^1} dG(\theta_0)$.

Differentiated goods invented at time $t - \tilde{T}$ having initial adjusted quality $\theta_0 \left(\tilde{T}\right)$ conduct disruptive innovation at time $t$, where $\theta_0 \left(\tilde{T}\right)$ is the initial adjusted quality corresponding to be embodied disruptive innovation after $\tilde{T}$ periods have passed from the entry, which is derived from (13). Because the number of differentiated goods invented at time $t - \tilde{T}$ is expressed as $\exp \left[ -g_n \tilde{T} \right] n(t) \left[ \nu \left( t - \tilde{T} \right) + g_n \right]$, the number of differentiated goods invented at time $t - \tilde{T}$ that conduct disruptive innovation at time $t$ is expressed as $\exp \left[ -g_n \tilde{T} \right] n(t) \left[ \nu \left( t - \tilde{T} \right) + g_n \right] g \left( \theta_0 \left( \tilde{T} \right) \right)$. Because there are incumbents who have chosen the interior solution of the timing of disruptive innovation, having initial adjusted quality from $\theta_0 (0)$ to $\theta_0 (\tilde{T})$, the number of differentiated goods having initial adjusted quality $\theta_0^1 < \theta_0^{\min} < \theta_0^2$ that conduct disruptive innovation at time $t$ is expressed as $\int_{\theta_0^1}^{\theta_0^{\min}} \exp \left[ -g_n \tilde{T} (\theta_0) \right] n(t) \left[ \nu \left( t - \tilde{T} (\theta_0) \right) + g_n \right] dG(\theta_0)$.

Differentiated goods invented at time $t - \tilde{T}$ having initial adjusted quality $\theta_0 \geq \theta_0^2$ conduct disruptive innovation at time $t$, and then the number of differentiated goods invented at time $t - \tilde{T}$ that conduct disruptive innovation at time $t$ is expressed as $\exp \left[ -g_n \tilde{T} \right] n(t) \left[ \nu \left( t - \tilde{T} \right) + g_n \right] \int_{\theta_0^2}^{\theta_0^{\max}} dG(\theta_0)$.

If $\theta_0^{\min} \leq \theta_0^1$ is satisfied, we can express $\nu(t)$ as:

$$\nu(t) = \nu(t) Z_{\nu 1} + g_n Z_{g 1},$$
where

\[ Z_{\nu,1} = \int_{\theta_0^{\min}}^{\theta_0^{c_1}} dG(\theta_0) + \int_{\theta_0^{c_1}}^{\theta_0^{c_2}} dG(\theta_0) + \int_{\theta_0^{c_2}}^{\theta_0^{\max}} dG(\theta_0) \]

\[ Z_{g_n,1} = \int_{\theta_0^{\min}}^{\theta_0^{c_1}} dG(\theta_0) + \int_{\theta_0^{c_1}}^{\theta_0^{c_2}} dG(\theta_0) + \int_{\theta_0^{c_2}}^{\theta_0^{\max}} dG(\theta_0) \]

As long as \( \bar{T}(\theta_0) \), \( \theta_0^{c_1} \), and \( \theta_0^{c_2} \) are constant, \( Z_{\nu,1} \) and \( Z_{g_n,1} \) are constant. When \( Z_{\nu,1} \) and \( Z_{g_n,1} \) are constant, \( \nu \) is constant. Thus, \( g_n = 0 \) is satisfied, and we obtain:

\[ \nu = \frac{Z_1}{1 - Z_2} g_n, \]

where \( Z_1 = \int_{\theta_0^{\min}}^{\theta_0^{c_1}} dG(\theta_0) + \int_{\theta_0^{c_1}}^{\theta_0^{c_2}} dG(\theta_0) + \int_{\theta_0^{c_2}}^{\theta_0^{\max}} dG(\theta_0) \). Substituting (9) and (18) into this function, we obtain the first row of (25).

If \( \theta_0^{c_1} < \theta_0^{\min} < \theta_0^{c_2} \) is satisfied, there are no incumbents conducting disruptive innovation immediately. Then, we can express \( \nu(t) \) as:

\[ \nu = \frac{Z_2}{1 - Z_2} g_n, \]

where \( Z_2 = \int_{\theta_0^{\min}}^{\theta_0^{c_2}} dG(\theta_0) + \int_{\theta_0^{c_2}}^{\theta_0^{\max}} dG(\theta_0) \). Substituting (9) and (18) into this function, we obtain the second row of (25).

If \( \theta_0^{c_2} < \theta_0^{\min} \) is satisfied, there are no incumbents conducting disruptive innovation before their patent protection expires. Then, we can express \( \nu(t) \) as:

\[ \nu = \frac{Z_3}{1 - Z_3} g_n, \]

where \( Z_3 = \exp \left[ -g_n \bar{T}(\theta_0) \right] \). Substituting (9) and (18) into this function, we obtain the third row of (25).

From the above discussion, we can derive \( \nu \) when \( g_n > 0 \); however, this relational expression is valueless when \( g_n = 0 \).\(^{66}\) Then, we have to find another way to derive \( \nu \).

When \( g_n = 0 \), the number of new invented differentiated goods invented only by incumbents is identical for all periods, and the expected value of their duration is also identical. Thus, we can express

\[ \nu = \nu Z + g_n Z \]

---

\(^{65}\) Here, \( g_n \) is the growth rate of \( \nu \).

\(^{66}\) When \( g_n = 0 \), \( Z_1 = Z_2 = Z_3 = 1 \) is satisfied, and then \( \nu = \nu Z + g_n Z \) becomes \( \nu = \nu \).
the ratio of the number of differentiated goods conducting disruptive innovation to the aggregate number of differentiated goods as an inverse number of the expected value of duration. We can derive the expected value of duration, \( E(\tilde{T}) \), as

\[
E(\tilde{T}) = \begin{cases} \\
0 \int_{\theta_0^\text{min}}^{\theta_0^\text{c1}} dG(\theta_0) + \int_{\theta_0^\text{min}}^{\tilde{T}} (\theta_0) dG(\theta_0) + \bar{T} \int_{\theta_0^\text{c2}}^{\theta_0^\text{max}} dG(\theta_0) & \text{if } \theta_0^\text{min} \leq \theta_0^\text{c1} \\
\int_{\theta_0^\text{min}}^{\tilde{T}} (\theta_0) dG(\theta_0) + \bar{T} \int_{\theta_0^\text{c1}}^{\theta_0^\text{max}} dG(\theta_0) & \text{if } \theta_0^\text{c1} < \theta_0^\text{min} \leq \theta_0^\text{c2} \\
\bar{T} \int_{\theta_0^\text{min}}^{\theta_0^\text{c2}} dG(\theta_0) & \text{if } \theta_0^\text{c2} < \theta_0^\text{min}
\end{cases}
\]

Substituting (9) and (18) into this function, we obtain

\[
\nu = \begin{cases} \\
\frac{[\theta_0^\text{c1}]^{-1} k[-g_n]}{1-\exp[-k[-g_n]\bar{T}]} & \text{if } \theta_0^\text{min} \leq \theta_0^\text{c1} \\
\frac{k[-g_n]}{1-\exp[-k[-g_n]\bar{T}] \theta_0^\text{c1} -k \ln[\theta_0^\text{c1}]} & \text{if } \theta_0^\text{c1} < \theta_0^\text{min} \leq \theta_0^\text{c2} \\
\frac{1}{\bar{T}} & \text{if } \theta_0^\text{c2} < \theta_0^\text{min}
\end{cases}
\]

Using L’Hopital’s rule, we can confirm that (25) coincides with the above equation. Thus, \( \nu \) can be expressed as (25), even when \( g_n = 0 \).

B Appendix

In this appendix, we derive (27). At time \( t - \tilde{T} \), \([\nu + g] \exp[-g_n \tilde{T}] \) differentiated goods are invented.

At time \( t \), differentiated goods invented \( \tilde{T} \) periods before remain if inventors draw initial adjusted quality \( \theta_0 \geq \theta_0(\tilde{T}) \). Because the adjusted quality of differentiated goods invented \( \tilde{T} \) periods before having initial adjusted quality \( \theta_0 \) is expressed as \( \exp[g_0 \tilde{T}] \theta_0 \), the sum of adjusted quality of remaining differentiated goods, which are invented at time \( t - \tilde{T} \), at time \( t \) is expressed as:

\[
[\nu + g] \exp[-g_n \tilde{T}] \int_{\theta_0(\tilde{T})}^{\theta_0^\text{max}} \theta_0 dG(\theta_0)
\]

At time \( t \), because there are differentiated goods invented between time \( t - \tilde{T} \) and time \( t \), the sum of the adjusted quality of remaining differentiated goods, which were invented between time \( t - \tilde{T} \) and time \( t \),
at time $t$ is expressed as:

$$
\int_0^T [\nu + g] \exp\left[-g_nT\right] n(t) \int_{\theta_0(T)}^{\theta_{0}^{\text{max}}} \exp\left[g_0T\right] \theta_0 dG(\theta_0) d\tilde{T}.
$$

In a similar manner, we can derive the number of differentiated goods at time $t$ as:

$$
\int_0^\tilde{T} [\nu + g] \exp\left[-g_nT\right] n(t) \int_{\theta_0(T)}^{\theta_{0}^{\text{max}}} dG(\theta_0) d\tilde{T}.
$$

The average quality of differentiated goods, $\Theta$, is calculated as the sum of the quality of differentiated goods produced at time $t$ divided by the number of differentiated goods at time $t$, and then we obtain:

$$
\Theta = \frac{\int_0^T \exp\left[-g_nT\right] \int_{\theta_0(T)}^{\theta_{0}^{\text{max}}} \exp\left[g_0T\right] \theta_0 dG(\theta_0) d\tilde{T}}{\int_0^{\tilde{T}} \exp\left[-g_nT\right] \int_{\theta_0(T)}^{\theta_{0}^{\text{max}}} dG(\theta_0) d\tilde{T}}.
$$

Substituting (9) and (18) into the above equation, we obtain the first row of (27).

If $\theta_0^1 < \theta_0^{\text{min}}$ is satisfied, all incumbents produce differentiated goods and put off disruptive innovation for some time. We define $T^c$ as the time when the incumbent drawing the lowest initial adjusted quality, $\theta_0^{\text{min}}$, conducts disruptive innovation. Then, we obtain the average quality of the differentiated goods as:

$$
\Theta = \frac{\int_0^{T^c} \exp\left[-g_nT\right] \int_{\theta_0(T)}^{\theta_{0}^{\text{max}}} \exp\left[g_0T\right] \theta_0 dG(\theta_0) d\tilde{T} + \int_{T^c}^\tilde{T} \exp\left[-g_nT\right] \int_{\theta_0(T)}^{\theta_{0}^{\text{max}}} \exp\left[g_0T\right] \theta_0 dG(\theta_0) d\tilde{T}}{\int_0^{T^c} \exp\left[-g_nT\right] \int_{\theta_0(T)}^{\theta_{0}^{\text{max}}} dG(\theta_0) d\tilde{T} + \int_{T^c}^\tilde{T} \exp\left[-g_nT\right] \int_{\theta_0(T)}^{\theta_{0}^{\text{max}}} dG(\theta_0) d\tilde{T}}.
$$

Substituting (9) and (18) into the above equation, we obtain the second row of (27).

If $\theta_0^2 < \theta_0^{\text{min}}$ is satisfied, all incumbents produce differentiated goods until their patent protection expires, and then we obtain the average quality of the differentiated goods as:

$$
\Theta = \frac{\int_0^T \exp\left[-g_nT\right] \int_{\theta_0(T)}^{\theta_{0}^{\text{max}}} \exp\left[g_0T\right] \theta_0 dG(\theta_0) d\tilde{T}}{\int_0^T \exp\left[-g_nT\right] \int_{\theta_0(T)}^{\theta_{0}^{\text{max}}} dG(\theta_0) d\tilde{T}}.
$$

57
Substituting (9) and (18) into the above equation, we obtain the second row of (27).

C Appendix

In this appendix, we examine the property of $X_1(\theta_0, g_Q)$ and $X_2(\theta_0, g_Q)$. Differentiating $X_1(\theta_0, g_Q)$ and $X_2(\theta_0, g_Q)$ with $g_Q$, we obtain:

\[
\frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} = -\frac{X_{10}}{[\rho - gL + gQ][\rho - gL + gQ - g\bar{T}]^2} \frac{\theta_0}{\sigma} \\
\frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q} = -\frac{X_{20}}{[\rho - gL + gQ][\rho - gL + gQ - g\bar{T}]^2} \frac{\theta_0}{\sigma}
\]

where

\[
X_{10} = \left[ [\rho - gL + gQ] \left[ 1 - \exp \left[ - \left[ \rho - gL + gQ - g\bar{T} \right] T \right] \right] \right. \\
+ [-g\bar{T}] \left[ \rho - gL + gQ - g\bar{T} \right] \exp \left[ - \left[ \rho - gL + gQ - g\bar{T} \right] T \right] \\
X_{20} = \left[ [\rho - gL + gQ] \left[ 1 - \exp \left[ - \left[ \rho - gL + gQ - g\bar{T} \right] T \right] \right] \right. \\
+ \left[ \rho - gL + gQ - g\bar{T} \right] \left[ \rho - gL + gQ - g\bar{T} \right] \exp \left[ - \left[ \rho - gL + gQ - g\bar{T} \right] T \right]
\]

Because $X_{10}$ is unambiguously positive, we can easily confirm $\frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} < 0$.

When $\theta_0$ goes to positive infinity, $X_{20}$ is expressed as:

\[
\lim_{\theta_0 \to +\infty} X_{20} = \left[ [\rho - gL + gQ] \left[ 1 - \exp \left[ - \left[ \rho - gL + gQ - g\bar{T} \right] T \right] \right] \right. \\
- \left[ \rho - gL + gQ \right] \left[ \rho - gL + gQ - g\bar{T} \right] \exp \left[ - \left[ \rho - gL + gQ - g\bar{T} \right] T \right]
\]

Because $\lim_{\theta_0 \to +\infty} X_{20}$ is an increasing function of $\bar{T}$ and $\bar{T} > 0$, $\lim_{\theta_0 \to +\infty} X_{20} > 0$ is satisfied.\(^67\) Because $X_{20}$ is a decreasing function of $\theta_0$, we can confirm that $X_{20}$ is also positive. Thus, we can confirm $\frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q} < 0$.

Differentiating $\frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q}$ and $\frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q}$ with $\theta_0$, we obtain:

\[
\frac{\partial}{\partial \theta_0} \left( \frac{\partial X_1(\theta_0, g_Q)}{\partial g_Q} \right) = -\frac{1}{\sigma [\rho - gL + gQ]} \left[ \frac{X_{10}}{[\rho - gL + gQ - g\bar{T}]^2} + \frac{1}{1 - g\bar{T}} \exp \left[ - \left[ \rho - gL + gQ - g\bar{T} \right] T \right] \right] \\
\lim_{\theta_0 \to +\infty} X_{20} \\
\frac{\partial}{\partial \theta_0} \left( \frac{\partial X_2(\theta_0, g_Q)}{\partial g_Q} \right) = \left. \frac{-\frac{X_{20}}{[\rho - gL + gQ][\rho - gL + gQ - g\bar{T}]^2} \frac{\theta_0}{\sigma}} \right|_{\theta_0 \to +\infty}
\]

\(^{67}\)When $T = 0$, $\lim_{\theta_0 \to +\infty} X_{20} = 0$. 

58
Then, we can confirm \( \frac{\partial \theta_0}{\partial \theta_0} \frac{\partial \theta_0}{\partial g_Q} < 0 \) and \( \frac{\partial \theta_0}{\partial \theta_0} \frac{\partial \theta_0}{\partial g_Q} < 0 \).\(^{68}\)

Differentiating \( X_1(\theta_0, g_Q) \) and \( X_2(\theta_0, g_Q) \) with \([s_{FCF} - s_{DCD}]\), we obtain:

\[
\frac{\partial X_1(\theta_0, g_Q)}{\partial [s_{FCF} - s_{DCD}]} = \exp \left[ -[\rho - g_l + g_Q] \hat{T} \right] > 0
\]

\[
\frac{\partial X_2(\theta_0, g_Q)}{\partial [s_{FCF} - s_{DCD}]} = \exp \left[ -[\rho - g_l + g_Q] \hat{T} \right] > 0
\]

Differentiating the above function with respect to \( \theta_0 \), we obtain:

\[
\frac{\partial}{\partial \theta_0} \frac{\partial X_1(\theta_0, g_Q)}{\partial [s_{FCF} - s_{DCD}]} = -[\rho - g_l + g_Q] \exp \left[ -[\rho - g_l + g_Q] \hat{T} \right] \frac{\partial \hat{T}}{\partial \theta_0} < 0
\]

\[
\frac{\partial}{\partial \theta_0} \frac{\partial X_2(\theta_0, g_Q)}{\partial [s_{FCF} - s_{DCD}]} = 0
\]

**D Appendix**

In this appendix, we examine the sign of \( \frac{dg_Q}{ds_F} \) and \( \frac{dg_Q}{ds_F} \), where R&D technology is expressed as \( C_F(t) = \frac{E(t)}{n(t)} c_F \) and \( C_D(t) = \frac{E(t)}{n(t)} c_D \).\(^{69}\)

Differentiating \( \theta_0^1 = [\sigma [\rho - g_l + g_Q] [s_{FCF} - s_{DCD}]^\gamma] \left[ \frac{k}{k-1} \right]^{1-\gamma} \) with \( s_F \) yields:

\[
\frac{d\theta_0^1}{ds_F} = \frac{\sigma}{\gamma} \left[ \sigma [\rho - g_l + g_Q] [s_{FCF} - s_{DCD}] \left[ \frac{k}{k-1} \right]^{1-\gamma} \right] \frac{[\rho - g_l + g_Q] c_F - [s_{FCF} - s_{DCD}] \left[ \frac{dg_Q}{ds_F} \right]}{[\rho - g_l + g_Q] \hat{T} \left[ \frac{[\rho - g_l + g_Q] c_F - [s_{FCF} - s_{DCD}] \left[ \frac{dg_Q}{ds_F} \right]}{[\rho - g_l + g_Q] \hat{T}} \right]}.
\]

(72)

Substituting (9) and (53) into (54), we obtain:\(^{69}\)

\[
[s_{FCF} - s_{DCD}] [\rho - g_l + g_Q] \left[ 1 - \exp \left[ -[\rho - g_l + g_Q + k [-g_\theta]] \hat{T} \right] \right] \left[ \theta_0^1 \right]^{-k} \left[ \frac{k}{k-1} \right]^-\gamma \frac{\left[ \frac{k}{k-1} \right]^{1-\gamma}}{k} = s_{DCD}.
\]

(73)

Substituting (51) into (73), we obtain:

\[
\left[ \theta_0^1 \right]^{-k} \left[ 1 - \exp \left[ -[\rho - g_l + g_Q + k [-g_\theta]] \hat{T} \right] \right] = \left[ \frac{k}{k-1} \right]^{\gamma} k s_{DCD}.
\]

(74)

\(^{68}\) \( [1 - [-g_\theta] \hat{T}] \left[ \exp \left[ -[\rho + g_Q - g_\theta] \hat{T} \right] \right] \) is a decreasing function of \( \hat{T} \), and is zero when \( \hat{T} = 0 \). Since \( \hat{T} \) is positive, \( [1 - [-g_\theta] \hat{T}] \left[ \exp \left[ -[\rho + g_Q - g_\theta] \hat{T} \right] \right] \) is also positive.\(^{69}\)

Regardless of the specification of \( C_F \) and \( C_D \), (73) is satisfied as long as \( \theta_0^1 \geq \theta_0^{1\min} \) is satisfied.
Differentiating (74) with $s_F$, and substituting (72) yields:

$$\frac{[\gamma - k] c_F}{\gamma [s_F c_F - s_D c_D]} = \frac{dgQ}{ds_F} \left[ \frac{1 - [1 + [\rho - g_L + g_Q + k [-g_\theta]] \bar{T}] \exp [-[\rho - g_L + g_Q + k [-g_\theta]] \bar{T}]}{1 - \exp [-[\rho - g_L + g_Q + k [-g_\theta]] \bar{T}] [\rho - g_L + g_Q + k [-g_\theta]]} \right] - \frac{[\gamma - k]}{\gamma [\rho - g_L + g_Q]} \right].$$

(75)

First, left-hand-side of (75) increases with $\gamma$, and brackets of right-hand-side decreases with $\gamma$. Then, $\frac{d}{\gamma} \frac{dgQ}{ds_F} > 0$ must be satisfied; that is, $\frac{dgQ}{ds_F}$ must increase monotonically with high $\gamma$. Second, from (75), $\frac{dgQ}{ds_F} = 0$ must be satisfied when $\gamma = k$. Thus, $\frac{dgQ}{ds_F}$ is negative (positive) when $\gamma < k$ ($\gamma > k$) is satisfied.

Then, we can confirm the sign of $\frac{dgQ}{ds_F}$ expressed as (55).

We can rewrite (75) as:

$$\frac{d\theta^1}{ds_F} = \left[ \frac{1}{\gamma - k} \frac{dgQ}{ds_F} \right]_{\gamma} \frac{g^n}{1} \left[ 1 - [1 + [\rho - g_L + g_Q + k [-g_\theta]] \bar{T}] \exp [-[\rho - g_L + g_Q + k [-g_\theta]] \bar{T}] \right] \left[ 1 - \exp [-[\rho - g_L + g_Q + k [-g_\theta]] \bar{T}] [\rho - g_L + g_Q + k [-g_\theta]] \right].$$

From (55), $\left[ \frac{1}{\gamma - k} \frac{dgQ}{ds_F} \right] > 0$ is satisfied.\(^70\) We can easily confirm that $[1 - [1 + [\rho - g_L + g_Q + k [-g_\theta]] \bar{T}] \exp [-[\rho - g_L + g_Q + k [-g_\theta]] \bar{T}] < 0$ is satisfied when $\bar{T} > 0$. Thus, $\frac{d\theta^1}{ds_F} > 0$ must be satisfied.

In the case of section 3.1, which is the special case of section 3.2, $\gamma = 1$ is satisfied. Because we assume a Pareto distribution where $k > 1$, we can confirm that $\frac{dgQ}{ds_F} < 0$ and $\frac{d\theta^1}{ds_F} > 0$ are satisfied, and we obtain the result of propositions 3.2 and 3.3.

E  Appendix

In this appendix, we conduct an analytical analysis where R&D technology is expressed as $C_F(t) = E(t) L c_F$ and $C_D(t) = E(t) L c_D$.

The labor market clearing condition is expressed as:

$$E \left[ \frac{\sigma - 1}{\sigma} + \frac{\nu}{[\theta]} n c_D + \frac{g^n}{[\theta]} n c_F \right] = 1.$$  \hfill (76)

From this equation, in steady-state equilibrium, $g_E = 0$ and $g_n = 0$ must be satisfied, and then $g_Q = 0$, $g_u = 0$, and $g_C = g_L$ are satisfied.

\(^70\)Since $\frac{d}{\gamma} \frac{dgQ}{ds_F} > 0$, we can confirm $\left[ \frac{1}{\gamma - k} \frac{dgQ}{ds_F} \right] > 0$ is satisfied, even when $\gamma = k$ by using L’Hôpital’s rule.
Necessary conditions for the maximization problem of $\bar{T}$ are expressed as:

\[ \theta_0^1 (n) = \left[ \frac{k-1}{k} \right]^{\frac{n-1}{3}} [n \sigma [\rho - g_L] [s_F c_F - s_D c_D]]^{\frac{1}{3}} \]

\[ \theta_0^2 (n) = \exp \left[ \left[ -g_0 \bar{T} \right] \theta_0^3 (n) \right] \right. \]

\[ \bar{T} (\theta_0, n) = \frac{1}{-\gamma g_0} \ln \left[ \frac{\theta_0}{\theta_0^3 (n)} \right] \]

The initial values of an incumbent entering at time $s$ and drawing $\theta_0$ are expressed as:

\[ V (\theta_0, 0, s) = \int_{\theta_0}^{\theta_0^3 (n)} X_0 \text{ for } \theta_0 \leq \theta_0^3 \]

\[ V (\theta_0, 0, s) = \int_{\theta_0}^{\theta_0^3 (n)} X_1 (\theta_0, n, \theta_0^3 (n)) \text{ for } \theta_0^1 \leq \theta_0 \leq \theta_0^2 \]

\[ V (\theta_0, 0, s) = \int_{\theta_0}^{\theta_0^3 (n)} X_2 (\theta_0, n, \theta_0^3 (n)) \text{ for } \theta_0 > \theta_0^2 \]

where

\[ X_0 = \left[ s_F c_F - s_D c_D \right] \]

\[ X_1 (\theta_0, n, \theta_0^3 (n)) = \left[ 1 - \exp \left[ -g_0 \bar{T} (\theta_0, n) \right] \right] \theta_0 \left[ \frac{k}{k-1} \theta_0^3 (n) \right]^{\gamma-1} \]

\[ + [s_F c_F - s_D c_D] \exp \left[ -[\rho - g_L] \bar{T} (\theta_0, n) \right] \]

\[ X_2 (\theta_0, n, \theta_0^3 (n)) = \left[ 1 - \exp \left[ -g_0 \bar{T} (\theta_0, n) \right] \right] \theta_0 \left[ \frac{k}{k-1} \theta_0^3 (n) \right]^{\gamma-1} \]

\[ + [s_F c_F - s_D c_D] \exp \left[ -[\rho - g_L] \bar{T} \right] \]

where $\theta_0^1$, $\theta_0^2$, and $\bar{T} (\theta_0, g_Q)$ satisfy (77).

We restrict our analysis to where $\theta_0^{\min} \leq \theta_0^3$ is satisfied, and then the free entry condition is expressed as:

\[ \int_{\theta_0^{\min}}^{\theta_0^1} X_0 dG (\theta_0) + \int_{\theta_0^{\min}}^{\theta_0^2} X_1 (\theta_0, n, \theta_0^3 (n)) dG (\theta_0) + \int_{\theta_0^2}^{\theta_0^{\max}} X_2 (\theta_0, n, \theta_0^3 (n)) dG (\theta_0) = c_F \]

where $\theta_0^1$, $\theta_0^2$, and $\bar{T}$ satisfy (77), and $X_0$, $X_1$, and $X_2$ satisfy (79), which determines the equilibrium value of $n$. Substituting the equilibrium value of $n$ into (77), we obtain the equilibrium value of $\theta_0^1$.

Substituting the equilibrium value of $\theta_0^1$ into (27), we obtain the equilibrium value of average quality.

Next, we confirm the sign of $\frac{\partial \theta_0}{\partial s_F}$ and $\frac{\partial \theta_0^1 (n)}{\partial s_F}$. Differentiating $\theta_0^1 = \left[ \frac{k-1}{k} \right]^{\frac{n-1}{3}} [n \sigma [\rho - g_L] [s_F c_F - s_D c_D]]^{\frac{1}{3}}$.
with \( s_F \), we obtain:

\[
\frac{d\theta_0^{-1} s_F}{ds_F \theta_0^{-1}} = \frac{1}{\gamma - k} \frac{dn}{ds_F \gamma - k \frac{dn}{n}}.
\]  

(81)

Because (73) is satisfied regardless of the specification of R&D technology, substituting (77) into (73), we obtain:

\[
[\theta_0^{-1} (n)]^{\gamma-k} \frac{1}{n} = \left[ \frac{k - 1}{k} \right]^{\gamma} \frac{\theta - gL + k [-g_0]}{1 - \exp \left[ -\left[ \theta - gL + k [-g_0] \right] \bar{T} \right]} k s_D c_D.
\]  

(82)

Differentiating (82) with \( s_F \) and substituting (81) yields:

\[
\frac{1}{n} \frac{dn}{ds_F} = \gamma - k \frac{c_F}{s_F c_F - s_D c_D}.
\]  

(83)

From (83), we can confirm the sign of \( \frac{dn}{ds_F} \), expressed as:

\[
\begin{align*}
\frac{dn}{ds_F} &< 0 \quad \text{if} \quad \gamma < k \\
\frac{dn}{ds_F} &= 0 \quad \text{if} \quad \gamma = k \\
\frac{dn}{ds_F} &> 0 \quad \text{if} \quad \gamma > k
\end{align*}
\]  

(84)

From (84) and (81), we can confirm \( \frac{d\theta_0^{-1} (g_Q)}{ds_F} > 0 \).

Next, we confirm that the equilibrium value of average quality \( \Theta \) is independent of \( \gamma \). The free entry condition is rewritten as:

\[
\frac{[\theta_0^{-1} (n)]^k [k - 1] c_D}{s_F c_F - s_D c_D [\rho - gL]} = 1 - \exp \left[ -\left[ \rho - gL + k [-g_0] \right] \bar{T} \right] \bigg[ \rho - gL + k [-g_0] \bigg] s_D.
\]  

(85)

From this equation, we can confirm that \( \theta_0^{-1} \) is independent of \( \gamma \). Thus, \( \Theta = \frac{k}{k-1} \theta_0^{-1} \) is also independent of \( \gamma \).

Finally, we confirm that the equilibrium value of consumption per capita is independent of \( s_F, c_F \), and \( \gamma \) if \( \theta^{-1} \geq \theta_0^{-1} \text{min} \) is satisfied. Because \( g_Q = 0 \) is satisfied, the labor market clearing condition is expressed as:

\[
E \left[ \sigma - 1 \sigma + \frac{\nu}{[\Theta]^{n}} c_D \right] = 1.
\]
Thus, we can confirm the result about consumption per capita by analyzing \( \nu^{\nu}_{\Theta} nc_D \). If \( \theta^c_1 \geq \theta^c_0 \) is satisfied, from the first rows of (25) and (27), we obtain:

\[
\nu^{\nu}_{\Theta} nc_D = \frac{[\theta^c_1]^k (k-1)c_D}{s_{FCF} - s_{DCD} |\rho - g_L| \sigma [1 - \exp \left[ -k[-g_0] T \right]]}.
\] (86)

Substituting (85) into this equation, we obtain:

\[
\nu^{\nu}_{\Theta} nc_D = \frac{1 - \exp \left[ - \left( \rho - g_L + k [-g_0] \right) \bar{T} \right]}{1 - \exp \left[ -k[-g_0] T \right]} \frac{[-g_0]}{[\rho - g_L + k [-g_0]] \sigma s_D}.
\] (87)

From (87), we can confirm that \( \nu^{\nu}_{\Theta} nc_D \) is independent from \( s_{CF}, c_F, \) and \( \gamma \). Thus, \( E \) is also independent of \( s_{CF}, c_F, \) and \( \gamma \).