Real indeterminacy and dynamics of asset price bubbles in general equilibrium

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Real indeterminacy and dynamics of asset price bubbles in general equilibrium

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Abstract

In a simple infinite-horizon exchange economy with a single consumption good and a financial asset, real indeterminacy and asset price bubble may arise. We show how heterogeneity (in terms of preferences, endowments) and short-sale constraints affect the emergence and the dynamics of asset price bubbles as well as the equilibrium indeterminacy. We also bridge the literature on bubbles in models with infinitely lived agents and that in OLG models.

Keywords: asset price bubble, real indeterminacy, borrowing constraint, intertemporal equilibrium, infinite horizon.
JEL Classifications: D53, E44, G12.

1 Introduction

The existing literature of rational asset price bubbles has focused on two kinds of frameworks: overlapping generations models and infinite-horizon general equilibrium models with many agents. More attentions have been paid for the emergence and implications of pure bubble asset (i.e., fiat money) in OLG models since the influential paper of Tirole (1985).[1] However, as recognized by (Kocherlakota, 2008) and Martin and Ventura (2018), our understanding of asset price bubbles in general equilibrium models with infinitely lived agents is far from complete.[2]

> *"However, despite the widespread belief in the existence of bubbles in the real world, it is difficult to construct model economies in which bubbles exist in equilibrium."* (Kocherlakota, 2008)

This paper aims to address basic and open questions about rational asset price bubbles in general equilibrium: Why do asset price bubbles arise in equilibrium? How to

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[1] See Brunnermeier and Oehmke (2012) and Martin and Ventura (2018) for excellent surveys.

[2] In such models, it is difficult to characterize or compute the equilibrium. It is also not easy to provide non-trivial examples of equilibrium.

compute asset price bubble as a function of borrowing limits and other fundamentals? What are their effects on the economic agents' consumptions and trading?

To do so, we consider a simple infinite-horizon general equilibrium model with a finite number of agents, where there are only one consumption good and one financial asset as Lucas' tree (Lucas, 1978). Our model has two key ingredients: (1) agents are heterogeneous (in terms of endowments and preferences), and (2) the financial friction which takes the form of short-sale constraint (i.e., the asset quantity that each agent can buy does not exceed an exogenous limit). As usual, we say that there is a bubble in equilibrium if the asset price exceeds the fundamental value of the asset (defined as the present value of dividend streams).

The literature of bubbles in infinite-horizon general equilibrium models has shown several conditions ruling out asset price bubbles. A famous no-bubble theorem in Santos and Woodford (1997) states that, under mild conditions, bubbles are ruled out if the present value of aggregate endowments is finite. This condition still holds in a model with debt constraints (Werner, 2014) and in a model with land and collateral constraints (Bosi et al., 2018b). In our model with short-sale constraints, we can also obtain a similar result. Our paper is different from these papers because we do not require the assumption of uniform impatience to prove this result.

Motivated by the fact that most of no-bubble conditions are based on endogenous variables, we contribute to the literature by providing conditions (based on fundamentals) under which bubbles are ruled out. The first one shows the role of the borrowing limits: there is no equilibrium with bubble if borrowing limits are high enough. The second one shows the role of impatience: under the assumption of uniform impatience, there is no bubble if agents prefer strongly the present. The intuition is simple: if agents prefer strongly the present, they do not buy asset in the long run and hence bubbles are ruled out. This is similar to the situation in finite-horizon models in which no one buys asset at the last period and so there is no bubble.

The famous finding in Santos and Woodford (1997) and our above results do not show a clear way to construct models with bubbles because it says nothing about the trading in equilibrium. Our next contribution is to establish that, in an equilibrium with bubbles, there exist two agents whose assets holdings fluctuate over time (i.e., they do not converge). Moreover, we prove that, if bubbles arise in equilibrium, there exist two agents whose borrowing constraints bind (their asset holding equals the borrowing limit) infinitely many dates. This finding is consistent with but stronger than that of Kocherlakota (1992) who shows that, if there is a bubble, the limit infimum of the differences between asset holding and borrowing limit equals zero.

These insightful properties concerning the asset trading imply that a model with bubble must contain at least 2 heterogeneous agents. By consequence, to build a model with bubble, we focus on a model having two agents, and characterize the equilibrium in which borrowing constraints of both agents bind infinitely many dates. Notice that such an equilibrium exists only if (i) the borrowing limits are low and (ii) *the benchmark economy*–the economy without asset– has a so-called *seesaw effect* (i.e. the subjective interest rate of one agent is higher than that of another agent at infinitely many dates while being lower at infinitely many other dates).

Focusing on such equilibrium, we find that bubbles are ruled out if the value of endowments (discounted by using the interest rates of *the benchmark economy*) of the agent who buys asset vanishes in the infinity. By consequence, there does not exist bubbles if the benchmark economy has high interest rates. The basic idea is that the income of asset buyers must be high enough so that these agents are willing to buy the asset, even the asset price exceeds the fundamental value. This result can be viewed as an extension of the no-bubble condition of Tirole (1985) from OLG models (it states that there is no bubble if the steady state interest rates of the economy without bubble asset is higher than the population growth rate) to our general equilibrium model with infinitely lived agents. Tirole (1985) needs the convergence of interest rates of the economy without asset while we do not require such convergence. Our paper is the first one making clear the connection between bubble à la Tirole (1985) and that in infinite-horizon general equilibrium models.

Although the existing literature has given some examples of bubbles (see an overview below), none of them show how the emergence and the dynamics of asset price bubbles depend on economic fundamentals such as endowments, dividends, and borrowing limits. In our model, we manage to do so. Precisely, we show via a number of examples that; when the benchmark economy has low interest rates, bubbles are more likely to arise if (1) asset supply is low, (2) borrowing limits of agents are low, (3) the level of heterogeneity (proxied by the differences between agents' fundamentals such as endowments, initial asset holdings, rates of time preferences) is high, and (4) asset dividends are low with respect to agents' endowments. It should be noticed that the emergence of bubbles in our model does not violate individual transversality conditions (TVC) which ensures the optimality of individuals.

Let us explain the basic mechanism of asset price bubbles in our model. The heterogeneity ensures that in any period there is at least one agent who needs to save as much as possible by buying the asset. When the asset supply and borrowing limits are low, the asset price would be high (even higher than its fundamental value) because using this asset is the only way to smooth consumption.[3] In particular, our model suggests that bubbles may appear if there are (i) an asymmetric growth in terms of endowments of agents (for example, endowments of one agent grow at even dates but those of other agents grow at odd dates) and (ii) a shortage of financial assets (i.e., low asset supply and low dividends).

We also point out that not only bubbles but also real indeterminacy may arise in our simple model (a single consumption good and a single security). The idea behind is that asset prices, in some cases, can be recursively computed, and hence the sequence of prices will be computed as a function of the initial price. Therefore, any value can be an equilibrium price at the initial date if it is low enough so that the price and the bubble component of assets in the future will not be too high so that agents can buy them. Since this real indeterminacy is associated with the emergence of bubbles, the sources of the indeterminacy are agents' heterogeneity and short-sale constraints.

Last but not least, our paper makes clear the relationship between financial asset, bubble and welfare. We prove that the allocation of equilibrium in a model with financial asset strictly Pareto dominates the autarkic allocation. The basic intuition is that the financial asset provides two ways (saving and borrowing) to smooth consumption. Thanks to this, agents can transfer their wealth from dates with high endowment to dates with low endowment. So, the financial asset is welfare improving. In the case of pure bubble asset (without dividends), the economy without bubble coincides with

[3] We can prove that, if we introduce a new asset with which agents can borrow without limit, there will be no bubble.

the one without asset. As a result, we may interpret that a pure bubble asset may be welfare improving. However, we should not interpret that bubbles are always welfare improving because when dividends are positive, the social welfare generated by an equilibrium with bubble may be lower than that generated by another equilibrium without bubble.

Related literature. We survey examples of asset price bubbles in general equilibrium models with infinitely lived agents.[4] First, we focus on the asset having zero dividend and positive supply (i.e., fiat money). Bewley (1980) (Section 13), Townsend (1980), Kocherlakota (1992) (Example 1) and Scheinkman and Weiss (1986) show that, when borrowing is not allowed, fiat money may have positive value in infinite-horizon general equilibrium models. Santos and Woodford (1997) present several examples of this kind of bubbles. Their examples 4.1, 4.2 study fiat money in deterministic models while and their example 4.4 investigates fiat money in a stochastic model. Hirano and Yanagawa (2017) give sufficient conditions for the existence of stochastic bubbles of an asset without dividend and study how the existence of bubbles, economic growth, welfares depend on the degree of pledgeability. Unlike these studies, in our examples there may be a continuum of bubbly equilibria.

Second, we focus on the asset with positive dividends. Santos and Woodford (1997)'s example 4.3 studies bubbles of an asset with positive dividends but zero net supply in a deterministic model. Santos and Woodford (1997)'s example 4.5 investigates bubbles of the Lucas' tree as in our model but in a stochastic model and there is a single representative household. In this example, they introduce a sequence of non-stationary stochastic discount factors and show that bubbles may exist under a state-price process but not under another state-price process. Bosi et al. (2018b) introduce different concepts of land bubbles and provides an example where a land bubble arises but individual land bubbles are ruled out. Le Van and Pham (2016) (Section 6.1) and Bosi et al. (2017a) provide examples of bubbles of the Lucas' tree, where the asset price may be multiple (due to the portfolio effect) but the consumption is not affected by the existence of bubbles. Our added-value with respect to Le Van and Pham (2016), Bosi et al. (2017a) is that the indeterminacy in our model is real and the asset price affects agents' consumptions. Bloise and Citanna (2019) provide a sufficient condition (based on trade and the punishment for default) for the existence of bubble of an asset with vanishing dividends (i.e., dividends converge to zero) of an equilibrium whose sequence of allocations converges. In our paper, we do not impose any convergence, and agents' consumptions and asset prices may fluctuate or converge over time, depending on the economy's fundamentals.[5]

The rest of the paper is organized as follows. Section 2 presents the framework and provides fundamental properties of equilibrium. Sections 3 provides no-bubble conditions in a general framework. Section 4 presents a number of specific models where bubbles arise. Section 5 concludes. Technical proofs are gathered in the appendices.

[4] Brunnermeier and Oehmke (2012) and Martin and Ventura (2018) provide more complete surveys on bubbles in other frameworks (e.g., models with asymmetric information and heterogeneous beliefs or overlapping generations models).

[5] In these examples, the intertemporal utility function is time-separable. Araujo et al. (2011) consider the utility functions $\sum_{t\geq 0}\zeta_{i,t}u(c_{i,t})+\epsilon_i \inf_{t\geq 0} u_i(c_{i,t})$ and show that the parameter ϵ_i plays the key role on the existence of bubbles.

2 An exchange economy with short-sale constraints

Consider an infinite-horizon discrete-time model with short-sale as in Kocherlakota (1992). There are a finite number m of agents, a single consumption good and an asset. The asset structure is similar to Lucas' tree (Lucas, 1978) with exogenous dividend stream $(d_t)_t$. Denote $c_{i,t}, b_{i,t}$ the consumption and asset holding of agent i at date t while q_t is the asset price at date t. Agent i maximizes her intertemporal utility $\sum_{t=0}^{+\infty} \beta_{i,t} u_i(c_{i,t})$ subject to the following constraints:

1. Physical constraints: $c_{i,t} \geq 0 \ \forall t$.
2. Budget constraint: $c_{i,t} + q_t b_{i,t} \leq e_{i,t} + (q_t + d_t) b_{i,t-1} \ \forall t$, where $e_{i,t} > 0$ is the exogenous endowment of agent i at date t and $b_{i,-1}$ is endogenously given.
3. Borrowing constraint (or short-sale constraint): $b_{i,t} \geq -b_i^* \ \forall t$ where $b_i^* \geq 0$ is an exogenous borrowing limit.

An equilibrium is a list of prices and allocations $(q_t, (c_{i,t}, b_{i,t})_t)_{t\geq 0}$ satisfying three conditions: (1) given price, the allocation $(c_{i,t}, b_{i,t})_t$ is a solution of the optimization problem of agent i (i.e., $\sum_{t=0}^{+\infty} \beta_{i,t} u_i(c_{i,t}) \geq \limsup_{T\to\infty} \sum_{t=0}^{T} \beta_{i,t} u_i(c'_{i,t})$ for any sequence (c'_i, b'_i) satisfying physical, budget and borrowing constraints), and (2) market clearing conditions: $\sum_i b_{i,t} = L$ and $\sum_i c_{i,t} = \sum_i e_{i,t} + L d_t \ \forall t \geq 0$, and (3) $q_t > 0 \ \forall t$.

Denote $W_t \equiv \sum_i e_{i,t} + L d_t$ the aggregate resource at date t. We require standard assumptions in the rest of the paper.

Assumption 1. *Assume that u_i is concave, strictly increasing, and continuously differentiable for any i. We also assume that $\beta_{i,t} > 0$, $e_{i,t} > 0$, $d_t \geq 0$, $\sum_t \beta_{i,t} u_i(W_t) < \infty$, $\sum_t \beta_{i,t} < \infty \ \forall i, t$, and $L > 0$.*

Assumption 2. *There exists an increasing function $v(c)$ such that $u'_i(c)c \leq v(c) \ \forall c$ and $\sum_t \beta_{i,t} v(W_t) < \infty$.*

Notice that when $u_i(c) = ln(c)$ or $u_i(0)$ is finite, Assumption 2 is a direct consequence of Assumption 1.

We start by the following result providing necessary and sufficient conditions under which a list of prices and allocation constitutes an equilibrium.

Proposition 1. *Let Assumption 1 be satisfied.*

1. *If $(q, (c_i, b_i)_i)$ is an equilibrium, then we have*

$$\beta_{i,t} u'_i(c_{i,t}) = \lambda_{i,t} \tag{1a}$$
$$\lambda_{i,t} q_t = \lambda_{i,t+1}(q_{t+1} + d_{t+1}) + \eta_{i,t}, \quad \eta_{i,t}(b_{i,t} + b_i^*) = 0, \quad \eta_{i,t} \geq 0. \tag{1b}$$

   In addition, if Assumption 2 holds, then $\lim_{t\to\infty} \lambda_{i,t} q_t (b_{i,t} + b_i^*) = 0$.

2. *If the sequences $(q, (c_i, b_i)_i)$ and (λ_i, η_i) satisfy*
   (a) *$c_{i,t}, b_{i,t}, \lambda_{i,t}, \eta_{i,t}, \geq 0$, $q_t > 0$, $b_{i,t} \geq -b_i^*$, $c_{i,t} + q_t b_{i,t} = e_{i,t} + (q_t + d_t) b_{i,t-1} \ \forall i, t$;*
   (b) *First-order conditions (1a-1b), and market clearing conditions;*
   (c) *Transversality conditions: $\lim_{t\to\infty} \lambda_{i,t} q_t (b_{i,t} + b_i^*) = 0 \ \forall i$;*

(d) *The series* $\sum_{t=0}^{\infty} \beta_{i,t} u_i(c_{i,t})$ *converges*

then $(q, (c_i, b_i)_i)$ *is an equilibrium.*

Proof. See Appendix A. □

It is interesting to notice that when $u_i(0) \geq 0\ \forall i$, the second statement of Proposition 1 still holds if we replace $\lim_{t\to\infty} \lambda_{i,t} q_t (b_{i,t} + b_i^*) = 0\ \forall i$ by $\liminf_{t\to\infty} \lambda_{i,t} q_t (b_{i,t} + b_i^*) = 0\ \forall i$.[6]

Kocherlakota (1992) considers the function $\sum_t \beta_i^t u_i(c_{i,t})$ and states a similar result but he requires that $u_i(c) \leq 0\ \forall c$ or $u_i(c) \geq 0\ \forall x$ (to ensure that the sum $\sum_t \beta_i^t u_i(c_{i,t})$ always converges). Of course, this condition is not satisfied if $u_i(c) = ln(c)$. By contrast, our result applies to unbounded utility functions, including $u_i(c) = ln(c)$. Our result is related to Proposition 1 in Bosi et al. (2018b). The difference is that we impose exogenous borrowing limits while Bosi et al. (2018b) consider collateral constraints and the borrowing limits depends on prices of assets in the future.

Following the standard literature (Kocherlakota, 1992; Santos and Woodford, 1997), we introduce the notion of rational asset price bubbles.[7]

Definition 1. *Consider an equilibrium. The sequence of discount factors* $(R_t)_t$ *is defined by* $R_{t+1} q_t = q_{t+1} + d_{t+1}$. *The fundamental value of the asset is* $FV_0 \equiv \sum_{t=1}^{\infty} Q_t d_t$ *where* $Q_t \equiv \frac{1}{R_1 \cdots R_t}$. *We say that there is a bubble in equilibrium if* $q_0 > FV_0$.

According to the asset pricing equation $q_t = \frac{q_{t+1} + d_{t+1}}{R_{t+1}}$, we have $q_0 = \sum_{s=1}^{t} Q_s d_s + Q_t q_t\ \forall t \geq 1$. So, there is a bubble iff $\lim_{t\to} Q_t q_t > 0$. In a particular case where $d_t = 0\ \forall t$, the fundamental value equals zero; in this case, there is a bubble iff the asset price is strictly positive (Tirole, 1985).

Our main goal is to understand conditions under which rational asset price bubbles may arise (or be ruled out) in equilibrium as well as the implications of bubbles.

3 No-bubble conditions

Our goal in this section is to find out new conditions (based on fundamentals) under which bubbles cannot appear.

3.1 The role of borrowing constraints

The relationship between the existence of bubble and borrowing constraints is questioned by Kocherlakota (1992). However, he did not investigate whether borrowing constraints are binding or not in equilibrium with bubbles. The following result explores such a relationship and shows our contribution with respect to Kocherlakota (1992) as well as the connection between the existence of bubble and the trading on the asset market.

Proposition 2 (bubble existence and borrowing constraint)**.** *Let Assumption 1, 2 be satisfied. If there is a bubble in equilibrium, then we have:*

[6] See Remark 3 in Appendix A for a proof.

[7] We refer to Bosi et al. (2017a, 2018b) for alternatives concepts of bubbles.

1. *(Kocherlakota, 1992)* $\liminf_{t\to\infty}(b_{i,t} + b_i^*) = 0 \ \forall i$.

2. *There exist* 2 *agents whose borrowing constraints bind infinitely often. Formally, there exist 2 agents, say* i, j, *and 2 infinite sequences* $(i_n)_n, (j_n)$ *such that* $b_{i,i_n} + b_i^* = 0$ *and* $b_{j,j_n} + b_j^* = 0$ *for all* n.

3. *There exist* 2 *agents* i *and* j *such that the sequences* $(b_{i,t})_t$ *and* $(b_{j,t})_t$ *do not converge.*

Proof. See Appendix A. □

Points 2 and 3, which are new with respect to the existing literature, show that the existence of bubbles implies the fluctuations of asset trading of at least 2 agents. They lead to the following result showing the role of borrowing limits (b_i^*).

Corollary 1. *Let Assumption 1, 2 be satisfied. If there is* T *such that* $b_i^* d_t > e_{i,t}$ $\forall i, \forall t \geq T$, *then there is no equilibrium with bubble.*[8]

3.2 Interest rates, impatience and bubble

A famous result in Santos and Woodford (1997) states that, under the assumption of uniform impatience (see infra), bubbles are ruled out if the present value of total future resources is finite (this condition was named "high implied interest rates" by Alvarez and Jermann (2000)).[9] In our model with short-sale constraints, we can also prove a similar result.

Corollary 2 (the role of present value of endowments). *Let Assumption 1, 2 be satisfied. There is no bubble if*

$$\sum_t Q_t(\sum_i e_{i,t}) < \infty. \tag{2}$$

Proof. See Appendix A. □

Unlike Santos and Woodford (1997), we do not require the uniform impatience. Instead, we use transversality conditions in Proposition 1 to prove (2).

A direct consequence of Corollary 2 is that there is no bubble if $\inf_t \frac{d_t}{\sum_i e_{i,t}} > 0$. To the best of our knowledge, there is only this condition (based on exogenous parameters) in the literature, which rules out bubbles. Notice that in the case of zero dividends ($d_t = 0 \ \forall t$), this condition does not help us to understand asset price bubbles.

Our goal in this subsection is to find out other conditions (based on fundamentals) under which bubbles cannot appear. To do so, we borrow the concept "uniform impatience" in the existing literature (Magill and Quinzii, 1996; Levine and Zame, 1996; Magill and Quinzii, 1994). Given a consumption plan $c = (c_t)_{t\geq 0}$, a date t, a vector

[8]To prove this result, suppose that there is an equilibrium with bubble. According to point 2 of Proposition 2, there is an agent i and an infinite sequence $(i_n)_n$ such that $b_{i,i_n} + b_i^* = 0 \ \forall n$. Let n be such that $i_n > T$. We have $c_{i,i_n+1} = e_{i,i_n+1} - d_{i_n+1}b_i^* - q_{i_n+1}(b_i^* + b_{i,i_n}) \leq e_{i,i_n+1} - d_{i_n+1}b_i^* < 0$, a contradiction.

[9]Theorem 6.1 in Huang and Werner (2000) provides a version of Santos and Woodford (1997)'s Theorem 3 in a model with debt constraints. Proposition 12 in Bosi et al. (2018b) shows a related result concerning the bubbles of land.

$(\gamma, \delta) \in (0,1) \times \mathbb{R}_+$, we define another consumption plan, called $z = z(c, t, \gamma, \delta)$, by $z_s = c_s \ \forall s < t$, $z_t = c_t + \delta$, $z_s = \gamma c_s \ \forall s > t$. We also denote $U_i^T(c) = \sum_{t=0}^{T} \beta_{i,t} u_i(c_{i,t})$ and $U_i(c) \equiv \limsup_{t\to\infty} U_i^T(c)$.

Assumption 3 (Uniform impatience)**.** *There exists $\gamma \in (0,1)$ such that for all consumption plan $c = (c_t)$ with $0 \leq c_t \leq W_t \ \forall t$, we have*

$$U_i\Big(z(c, t, \gamma', W_t)\Big) > U_i(c) \ \forall i, \forall t, \forall \gamma' \in [\gamma, 1).$$

Proposition 1 in Pascoa et al. (2011) provides sufficient conditions for the uniform impatience. Notice that they only consider the case where $u_i(c) \geq 0 \ \forall c$. Under well-known utility functions, the following result helps us to understand when the uniform impatience holds.

Lemma 1. *1. If $u_i(c) = ln(c)$ and there exists $\gamma \in (0,1)$ such that $\beta_{i,t} > -\frac{\ln(\gamma)}{ln(2)} \sum_{s=t+1}^{\infty} \beta_{i,s}$ $\forall t$, then the uniform impatience holds.*

2. If $u_i(c) = \frac{c^{1-\sigma}}{1-\sigma}$ where $\sigma > 0$, and there exists $\gamma \in (0,1)$ such that $\beta_{i,t}\frac{2^{1-\sigma}-1}{1-\sigma} W_t^{1-\sigma} + (\gamma^{1-\sigma} - 1) \sum_{s=t+1}^{\infty} \beta_{i,s} \frac{W_s^{1-\sigma}}{1-\sigma} > 0 \ \forall t$, then the uniform impatience holds.

Proof. See Appendix A. □

Our contribution can be stated as follows.

Proposition 3. *Assume that Assumptions 1, 2, 3 hold and $e_{i,t} - d_t b_i^* > 0 \ \forall i, \forall t$. There is no bubble if*

$$\lim_{T\to\infty} W_T \prod_{t=0}^{T-1} \max_i \frac{\beta_{i,t+1} u_i'(e_{i,t+1} - d_{t+1} b_i^*)}{\beta_{i,t} u_i'(\sum_i e_{i,t} + L d_t)} = 0. \tag{3}$$

This leads to two consequences.

1. *When $u_i(c) = ln(c)$, $\beta_{i,t} = \beta^t \ \forall i, \forall t$, and $\frac{1-\beta}{\beta} > -\frac{ln(\gamma)}{ln(2)}$ with $\gamma \in (0,1)$, there is no bubble if*

$$\lim_{T\to\infty} \beta^T W_T \cdots W_1 W_0 \prod_{t=0}^{T-1} \max_i \frac{1}{e_{i,t+1} - d_{t+1} b_i^*} = 0. \tag{4}$$

2. *When $u_i(c) = \frac{c^{1-\sigma}}{1-\sigma}$ where $\sigma > 0$, and there exists $\gamma \in (0,1)$ such that $\frac{2^{1-\sigma}-1}{1-\sigma} W_t^{1-\sigma} + (\gamma^{1-\sigma} - 1) \sum_{s=t+1}^{\infty} \beta^{s-t} \frac{W_s^{1-\sigma}}{1-\sigma} > 0 \ \forall t$, there is no bubble if*

$$\lim_{T\to\infty} \beta^T W_T \prod_{t=0}^{T-1} \max_i \frac{W_t^{\sigma}}{(e_{i,t+1} - d_{t+1} b_i^*)^{\sigma}} = 0. \tag{5}$$

Proof. See Appendix A. □

Proposition 3 and Corollary 1 contribute to the literature by providing conditions (based on fundamentals) under which bubbles are ruled out. When borrowing limits are large, Corollary 1 shows that bubbles do not arise. If borrowing limits are low (in the sense that $e_{i,t} - d_t b_i^* > 0 \ \forall i, \forall t$), Proposition 3 indicates that bubbles do not exist if the agents prefer strongly the present (formally, $\beta_{i,t+1}/\beta_{i,t}$ is low). In a particular case, where $\beta_{i,t} = \beta^t$ with β is low enough, there is no bubble. Notice that, when there is T such that $\beta_{i,t} = 0 \ \forall i, \forall t > T$, we recover a T-horizon model where we have $q_0 = \sum_{s=1}^{T} Q_s d_s$ and $q_s = 0 \ \forall s > T$, and therefore, there is no bubble.

When $d_t = 0 \ \forall t$, conditions (3-5) do not depend on borrowing limits b_i^*. So, bubbles may be ruled out even borrowing limits are too low. This in turn suggests that financial frictions are only necessary conditions for asset price bubbles.

4 Models with bubbles

We are now interested in constructing model economies in which bubbles arise. Proposition 2 shows that such models must contain at least 2 heterogeneous agents. So, we should focus on a model with two types of agents, say A and B. Suggesting by points 2 and 3 of Proposition 2, we look at equilibria in which borrowing constraints of agent A (agent B) binds at any even (odd) date. Formally, we aim to characterize economies where there is an equilibrium such that

$$b_{a,2t} = -b_a^*, \quad b_{b,2t} = L + b_a^*, \quad b_{a,2t+1} = L + b_b^*, \quad b_{b,2t+1} = -b_b^*. \tag{6}$$

With these asset holdings, we have that

$$c_{a,0} = e_{a,0} + (q_0 + d_0) b_{a,-1} + q_0 b_a^*, \quad c_{b,0} = e_{b,0} + (q_0 + d_0) b_{b,-1} - q_0 (L + b_a^*) \tag{7a}$$
$$c_{a,2t-1} = e_{a,2t-1} - b_a^* d_{2t-1} - q_{2t-1} H, \quad c_{b,2t-1} = e_{b,2t-1} + d_{2t-1}(L + b_a^*) + q_{2t-1} H \tag{7b}$$
$$c_{a,2t} = e_{a,2t} + d_{2t}(L + b_b^*) + q_{2t} H, \quad c_{b,2t} = e_{b,2t} - d_{2t} b_b^* - q_{2t} H \tag{7c}$$

where $H \equiv L + b_a^* + b_b^*$ and $b_{a,-1}, b_{b,-1}$ are given. Observe that such equilibrium exists only if the borrowing limits b_a^*, b_b^* are low.

4.1 The role of interest rates of the benchmark economy

We firstly find necessary conditions (based on fundamentals) of the existence of bubble. Our intuition is to look at the benchmark economy, i.e., the economy without asset. In such economy, we have $c_{i,t} = e_{i,t} \ \forall i, t$. We now define the sequences $(R_{a,t}^*), (R_{b,t}^*), (R_t^*)$ by

$$1 = \frac{\gamma_{a,t-1} u_a'(e_{a,t})}{u_a'(e_{a,t-1})} R_{a,t}^*, \quad 1 = \frac{\gamma_{b,t-1} u_b'(e_{b,t})}{u_b'(e_{b,t-1})} R_{b,t}^*, \quad \text{and } R_t^* \equiv \min(R_{a,t}^*, R_{b,t}^*). \tag{8a}$$

where $\gamma_{b,t} \equiv \frac{\beta_{b,t+1}}{\beta_{b,t}}, \gamma_{a,t+1} \equiv \frac{\beta_{a,t}}{\beta_{a,t+1}} \ \forall t \geq 0$.

$R_{a,t}^*$ (resp., $R_{b,t}^*$) represents the subjective real interest rate of agent A (resp., B) while R_t^* is the real interest rate between dates $t-1$ and t in the benchmark economy.

According to FOCs in Proposition 1, we have that, for any $t \geq 1$

$$\begin{cases} 1 = \gamma_{a,2t-1} \dfrac{u'_a\big(e_{a,2t} + d_{2t}(L + b^*_b) + q_{2t}H\big)}{u'_a\big(e_{a,2t-1} - b^*_a d_{2t-1} - q_{2t-1}H\big)} \dfrac{q_{2t} + d_{2t}}{q_{2t-1}} \\ 1 = \gamma_{b,2t} \dfrac{u'_b\big(e_{b,2t+1} + d_{2t+1}(L + b^*_a) + q_{2t+1}H\big)}{u'_b\big(e_{b,2t} - b^*_b d_{2t} - q_{2t}H\big)} \dfrac{q_{2t+1} + d_{2t+1}}{q_{2t}} \end{cases}$$

and

$$\begin{cases} \gamma_{a,2t-1} \dfrac{u'_a(e_{a,2t} + d_{2t}(L + b^*_b) + q_{2t}H)}{u'_a(e_{a,2t-1} - b^*_a d_{2t-1} - q_{2t-1}H)} \geq \gamma_{b,2t-1} \dfrac{u'_b(e_{b,2t} - d_{2t}b^*_b - q_{2t}H)}{u'_b(e_{b,2t-1} + d_{2t-1}(L + b^*_a) + q_{2t-1}H)} \\ \gamma_{b,2t} \dfrac{u'_b(e_{b,2t+1} + d_{2t+1}(L + b^*_a) + q_{2t+1}H)}{u'_b(e_{b,2t} - d_{2t}b^*_b - q_{2t}H)} \geq \gamma_{a,2t} \dfrac{u'_a(e_{a,2t+1} - b^*_a d_{2t+1} - q_{2t+1}H)}{u'_a(e_{a,2t} + d_{2t}(L + b^*_b) + q_{2t}H)} \end{cases}$$

Since the dividends and asset prices are non-negative, these FOCs imply that $R^*_{2t} = R^*_{a,2t} \leq R^*_{b,2t}, R^*_{2t+1} = R^*_{b,2t+1} \leq R^*_{a,2t+1}$ $\forall t \geq 1$. We can interpret that the benchmark economy has a so-called *seesaw effect*.

We also see that $R_{t+1} \equiv \frac{q_{t+1}+d_{t+1}}{q_t} \geq R^*_{t+1}$ $\forall t \geq 2$ which means that the interest rate of the benchmark economy is lower than that of our economy with asset. The value of asset price bubble is $b_0 = q_0 - FV_0 = \lim_{t\to\infty} \frac{q_t}{R_1 \cdots R_t}$. Since the function u'_i is decreasing, we have

$$\frac{q_t}{R_1 \cdots R_t} \leq \frac{q_t}{R^*_1 \cdots R^*_t} \frac{u'_b(c_{b,1})}{u'_b(c_{b,0})} \frac{u'_b(e_{b,0})}{u'_b(e_{b,1})} \ \forall t \geq 2.$$

The positivity of the consumptions implies that $Hq_t \leq e_t$, where we denote $e_{2t} \equiv e_{b,2t}$ and $e_{2t+1} \equiv e_{a,2t+1}$. So, there is no bubble if $\lim_{t\to\infty} \frac{e_t}{R^*_1 \cdots R^*_t} = 0$. Summing up, we obtain the following result showing the role of interest rates of the economy without asset.

Proposition 4 (the role of interest rates of economies without asset)**.** *Consider a model with two agents. Assume that the sequence (q_t), asset holdings are given by (6) and agents' consumptions given by (7a-7c) constitute an equilibrium. We have*

$$R_t \geq R^*_t \ \forall t \geq 2 \tag{9}$$

$$R^*_{b,2t} \geq R^*_{a,2t}, \quad R^*_{b,2t+1} \geq R^*_{a,2t+1} \ \forall t \geq 1 \quad \textit{(seesaw property)}. \tag{10}$$

Moreover, there is no bubble if

$$\lim_{t\to\infty} \frac{e_t}{R^*_1 \cdots R^*_t} = 0. \tag{11}$$

The term $\frac{e_t}{R^*_1 \cdots R^*_t}$ represents the value (discounted by using the interest rates of the benchmark economy) of endowment of the agent who buys asset in the economy with asset. Proposition 4 implies that, if there is bubble, the sequence of these discounted values either diverges or converges to a strictly positive value. In the case of convergence, the existence of bubble requires that $\lim_{t\to\infty} \frac{e_t}{R^*_1 \cdots R^*_t} > 0$. The basic idea behind is that the income of asset buyers must be high enough so that these agents are willing to buy the asset even the asset price exceeds its fundamental value.

Although condition (11) is obtained in a two-agent model, it is new with respect to the literature of rational bubbles in infinite-horizon general equilibrium models. Notice that it is not implied by the well-known no-bubble condition $\sum_t Q_t(\sum_i e_{i,t}) < \infty$ (Santos and Woodford, 1997; Werner, 2014; Bosi et al., 2018b) because $R_t \geq R_t^*$. The novelty of condition (11) is to show the importance of interest rates of the economy without asset (these interest rates are exogenous) on the emergence of bubbles in the economy with assets.

Condition (11) allows us to establish the connection between the literature of bubbles in OLG models and that in infinite-horizon models. Indeed, let us compare it with the main result in the influential paper Tirole (1985) who studies a pure bubble asset (i.e., asset pays no dividend) in an overlapping generations model. He proves that there is no bubble if the steady state interest rates of the economy without bubble asset is higher than the population growth rate. Condition (11) can be interpreted as a high interest rates condition (in the stationary case, i.e., $e_t = e$, $R_t^* = R^*\ \forall t$, it becomes $R^* > 1$). So, our result is consistent with that in Tirole (1985). The difference is that we do not require the convergence of interest rates R_t^* as in Tirole (1985).

Remark 1 (interest rates in the economy with adjusted endowments)**.** *Assume that borrowing limits are low enough so that* $e_{a,2t}-d_{2t}b_a^*, e_{a,2t-1}-b_a^*d_{2t-1}, e_{b,2t+1}-d_{2t+1}b_b^*, e_{b,2t}-b_b^*d_{2t}$ *are strictly positive. By using the same argument in Proposition 4, we can prove that there is no bubble if*

$$\lim_{t\to\infty} \frac{e_t}{R_1^d \cdots R_t^d} = 0. \tag{12}$$

where R_t^d *is defined by*

$$1 = \frac{\gamma_{a,2t-1}u_a'\big(e_{a,2t} - d_{2t}b_a^*\big)}{u_a'\big(e_{a,2t-1} - b_a^*d_{2t-1}\big)}R_{2t}^d, \quad 1 = \frac{\gamma_{b,2t}u_b'\big(e_{b,2t+1} - d_{2t+1}b_b^*\big)}{u_b'\big(e_{b,2t} - b_b^*d_{2t}\big)}R_{2t+1}^d \tag{13}$$

which can be interpreted as the interest rate of the economy with adjusted endowments.

4.2 Examples of bubbles with logarithmic utility functions

In this section, we will provide several examples of bubbles. We will work under logarithmic utility functions. We start by giving a condition under which a sequence is a system of prices.

Lemma 2. *Assume that* $u_i(c) = ln(c)\ \forall i = a, b$.
1. *If* $(q_t)_t$*, asset holdings given by (6) and agents' consumptions given by (7a-7c) constitute an equilibrium, then*

$$q_0 = (q_1 + d_1)\frac{\gamma_{b,0}(e_{b,0} + (q_0 + d_0)b_{b,-1} - q_0b_{b,0})}{e_{b,1} + d_1(L + b_a^*) + q_1H} \tag{14a}$$

$$q_{2t-1} = (q_{2t} + d_{2t})\frac{\gamma_{a,2t-1}(e_{a,2t-1} - b_a^*d_{2t-1} - q_{2t-1}H)}{e_{a,2t} + d_{2t}(L + b_b^*) + q_{2t}H} \tag{14b}$$

$$q_{2t} = (q_{2t+1} + d_{2t+1})\frac{\gamma_{b,2t}(e_{b,2t} - b_b^*d_{2t} - q_{2t}H)}{e_{b,2t+1} + d_{2t+1}(L + b_a^*) + q_{2t+1}H} \tag{14c}$$

2. *Conversely,* $(q_t)_t$*, asset holdings given by (6) and agents' consumptions given by (7a-7c) constitute an equilibrium if (14a-14c) hold,* $\gamma_{a,2t-1} \geq \gamma_{b,2t-1}$*,* $\gamma_{b,2t} \geq \gamma_{a,2t}\ \forall t$*,* *and*

$$e_{a,2t-1} \geq e_{b,2t-1} + (L + 2b_a^*)d_{2t-1} + 2Hq_{2t-1} \ \forall t \geq 1 \tag{15a}$$
$$e_{b,2t} \geq e_{a,2t} + (L + 2b_b^*)d_{2t} + 2Hq_{2t} \ \forall t \geq 1 \tag{15b}$$
$$e_{b,0} \geq e_{a,0} + d_0(b_{a,-1} - b_{b,-1}) + q_0(L + 2b_a^* + b_{a,-1} - b_{b,-1}) \tag{15c}$$

Proof. See Appendix B. It should be noticed that conditions (14a-14c) are part of FOCs which are necessary. Conditions (15a-15c) imply the TVCs. □

The FOCs (14a-14c) can be rewritten as

$$\begin{cases} \dfrac{e_{b,1} - b_b^* d_1}{q_1 + d_1} & = \gamma_{b,0}\dfrac{e_{b,0} + d_0 b_{b,-1}}{q_0} - \gamma_{b,0}(L + b_a^* - b_{b,-1}) - H \\ \dfrac{e_{a,2t} - d_{2t} b_a^*}{q_{2t} + d_{2t}} & = \dfrac{\gamma_{a,2t-1}(e_{a,2t-1} - b_a^* d_{2t-1})}{q_{2t-1}} - H(\gamma_{a,2t-1} + 1) \\ \dfrac{e_{b,2t+1} - d_{2t+1} b_b^*}{q_{2t+1} + d_{2t+1}} & = \dfrac{\gamma_{b,2t}(e_{b,2t} - b_b^* d_{2t})}{q_{2t}} - H(\gamma_{b,2t} + 1). \end{cases} \tag{16}$$

From the system (16), we observe that q_t is strictly increasing in q_{t-1} but $\frac{q_{t-1}}{q_t+d_t}$ is strictly decreasing in q_{t-1}. By consequence, q_t is strictly increasing in q_0 and $\frac{1}{R_t} = \frac{q_{t-1}}{q_t+d_t}$ is strictly decreasing in q_0. Thus, the fundamental value $FV_0 = \sum_{t\geq 1} Q_t d_t$ is strictly decreasing in q_0. This implies that the asset price bubble $B_0 \equiv q_0 - FV_0$ is strictly increasing in q_0.

Notation. For $x > 0$, the sequence $(q_t)_{t\geq 0}$ defined by $q_0 = x$ and the system (16), is unique. So, we denote this sequence by $(q_t(x))_t$.

Notice that $(q_t(x))$ may violate conditions (15a), (15b). According to Lemma 2, the sequence $(q_t(x))_t$ is a price sequence of an equilibrium if

$$q_t(x) > 0 \ \forall t \tag{17a}$$
$$e_{a,2t-1} \geq e_{b,2t-1} + (L + 2b_a^*)d_{2t-1} + 2Hq_{2t-1}(x) \tag{17b}$$
$$e_{b,2t} \geq e_{a,2t} + (L + 2b_b^*)d_{2t} + 2Hq_{2t}(x) \tag{17c}$$
$$e_{b,0} \geq e_{a,0} + d_0(b_{a,-1} - b_{b,-1}) + x(L + 2b_a^* + b_{a,-1} - b_{b,-1}) \tag{17d}$$

We see that: x is an asset price with bubble iff $B_0(x) \equiv x - FV_0(x) > 0$. The following result states useful properties of equilibrium with bubbles

Denote $\mathcal{B}_0$ of all the values $x > 0$ such that the sequence $(q_t(x))_{t\geq 0}$ satisfies the system (17a-17d). The following result presents some useful properties of the set $\mathcal{B}_0$.

Lemma 3. *The set $\mathcal{B}_0$ is bounded and connected (in the sense that, if $x, y \in \mathcal{B}_0$ and $x < y$, then $(x, y) \subset \mathcal{B}_0$). So, if the set $\mathcal{B}_0$ is non-empty, either it contains a unique element or it is an interval. By consequence, we have that:*

1. *There is at most one bubble-less equilibrium.*

2. *If $\mathcal{B}_0$ contains at least 2 elements, there are a continuum of bubbly equilibria.*

To prove these properties, let $x, y \in \mathcal{B}_0$ with $x < y$, and let $z \in (x, y)$. Since $q_t(\cdot)$ is an increasing function, we have $q_t(y) > q_t(z) > q_t(x) > 0$. By verifying all conditions in the definition of $\mathcal{B}_0$, we get that $z \in \mathcal{B}_0$. The two last points of Lemma 3 are from the property that $B_0(x) \equiv x - FV_0(x)$ is strictly increasing in x.

In the next subsections, we will present several examples where bubbles arise.

4.2.1 Asset without dividends

We focus on the case of fiat money or pure bubble asset (i.e., $d_t = 0\ \forall t$). To simplify our exposition, we introduce some notations.

$$\gamma_{2t} \equiv \gamma_{b,2t} = \frac{\beta_{b,2t+1}}{\beta_{b,2t}}, \quad \gamma_{2t-1} \equiv \gamma_{a,2t-1} = \frac{\beta_{a,2t}}{\beta_{a,2t-1}} \tag{18a}$$

$$e_{2t} \equiv e_{b,2t}, \quad e_{2t-1} \equiv e_{a,2t-1}, \quad w_{2t} \equiv e_{a,2t}, \quad w_{2t-1} \equiv e_{b,2t-1} \tag{18b}$$

We can verify that $\frac{\gamma_{t-1}e_{t-1}}{w_t} = \frac{1}{R^*_t}$.

$$\Gamma_t \equiv \frac{\gamma_{t-1}e_{t-1}}{w_t} \cdots \frac{\gamma_0 e_0}{w_1} = \frac{1}{R^*_1 \cdots R^*_t} \tag{19a}$$

$$D_t \equiv \frac{1+\gamma_{t-1}}{w_t} + \frac{1}{R^*_t}\frac{1+\gamma_{t-2}}{w_{t-1}} + \cdots + \frac{1}{R^*_t \cdots R^*_2}\frac{1+\gamma_0\frac{L+b^*_a-b_{b,-1}}{L+b^*_a+b^*_b}}{w_1} \tag{19b}$$

We now provide necessary and sufficient conditions under which bubbles arise in equilibrium.

Proposition 5 (continuum equilibria with bubble)**.** *Assume that $d_t = 0\ \forall t$ and $u_i(c) = ln(c)\ \forall i = a, b$.*

1. *If $(q_t)_{t\geq 0}$, asset holdings given by (6) and agents' consumptions given by (7a-7c) constitute an equilibrium with bubble, then we have*

$$\frac{1}{Hq_t} = \frac{1}{Hq_0}\Gamma_t - D_t\ \forall t \tag{20}$$

By consequence, we have $q_0 \leq \frac{\Gamma_t}{HD_t}\ \forall t$ and therefore

$$\sup_t \left(\frac{HD_t}{\Gamma_t}\right) < \infty \text{ and } \sum_{t=1}^{\infty} \frac{R^*_1 \cdots R^*_t}{e_t} < \infty \tag{21}$$

2. *Assume, in addition, that $\gamma_{a,2t+1} \geq \gamma_{b,2t+1}$, $\gamma_{b,2t} \geq \gamma_{a,2t}$, $e_t - w_t > 0$ (i.e., $e_{b,2t} > e_{a,2t}$, $e_{a,2t+1} > e_{b,2t+1}$) $\forall t$.*

If

$$\sup_t \frac{H\left(D_t + \frac{2}{e_t-w_t}\right)}{\Gamma_t} < \infty \tag{22}$$

then any sequence $(q_t)_{t\geq 0}$ determined by

$$q_0 \in (0, \bar{q}), \quad \frac{1}{Hq_t} = \frac{1}{Hq_0}\Gamma_t - D_t\ \forall t \geq 1 \tag{23a}$$

$$\text{where } \bar{q} \equiv \min\left\{\inf_{t\geq 1}\left(\frac{\Gamma_t}{H\left(D_t + \frac{2}{e_t-w_t}\right)}\right), \frac{e_{b,0}-e_{a,0}}{L+2b^*_a+b_{a,-1}-b_{b,-1}}\right\} > 0 \tag{23b}$$

is a system of prices of an equilibrium at which asset holdings are given by (6) and agents' consumptions are given by (7a-7c). Moreover, all such equilibria are bubbly.

Proof. See Appendix B.1. □

Condition (21) indicates that interest rates of the economy without asset must be low. Notice that condition (21) also implies that $lim_{t\to\infty}\frac{e_t}{R_1^*\cdots R_t^*} = \infty$, i.e., the present value of endowment of the autarkic economy is infinite. It means that the no-bubble condition (11) in Proposition 4 is violated.

Condition (22) is a key in Proposition 5. We can compute that

$$\frac{H\big(D_t + \frac{2}{e_t - w_t}\big)}{\Gamma_t} = \frac{Hw_1\cdots w_{t-1}}{e_0\cdots e_{t-1}}\frac{1+\gamma_{t-1}}{\gamma_0\cdots\gamma_{t-1}} + \frac{Hw_1\cdots w_{t-2}}{e_0\cdots e_{t-2}}\frac{1+\gamma_{t-2}}{\gamma_0\cdots\gamma_{t-2}} + \\ + \cdots + \frac{H}{e_0}\Big(\frac{1}{\gamma_0} + \frac{L+b_a^* - b_{b,-1}}{L+b_a^*+b_b^*}\Big) + \frac{2Hw_1\cdots w_t}{e_0\cdots e_{t-1}(e_t - w_t)}\frac{1}{\gamma_0\cdots\gamma_{t-1}}. \tag{24}$$

So, (22) can be satisfied for a large class of parameters (for example, $\gamma_t = \gamma \in (0,1)$ and $w_t = e_t x$ where $x \in (0,\gamma)$.

Proposition 5 suggests that when the economy without asset has low interest rates, an equilibrium with binding borrowing constraints has bubbles if the initial price q_0 is low enough in the sense that $q_0 \leq \bar{q}$. It is useful to understand how the upper bound $\bar{q}$ depends on fundamentals. According to (24), we observe that $\bar{q}$ is decreasing in the asset supply L, borrowing limits b_a^*, b_b^*, the endowment ratio $\frac{w_t}{e_t}$, the initial asset holding $b_{a,-1}$ of agent A, and $\bar{q}$ is increasing in the rate of time preference γ_t, the initial asset holding $b_{b,-1}$ of agent B.

In a specific case as in the following example, we can fully understand why bubbles may arise in a seesaw economy.

Example 1. *Assume that $\beta_{i,t} = \beta^t$ where $\beta \in (0,1)$ and $d_t = 0\ \forall t$. Assume also that $b_{a,-1} = L + b_b^*$, $b_{b,-1} = -b_b^*$, and endowments are*

$$e_{a,2t-1} = e, \quad e_{a,2t} = w, \quad e_{b,2t+1} = w, \quad e_{b,2t} = e$$

where $e, w > 0$ (so $e_t = e > 0, w_t = w > 0\ \forall t$).

1. *If $\frac{\beta e}{w} \leq 1$ (i.e., $R^* \geq 1$), there is no bubble.*
2. *If $\frac{\beta e}{w} > 1$ (i.e., $R^* < 1$: low interest rate condition), then the initial price of any equilibrium with bubble must satisfy condition $q_0 \leq \frac{1}{H}\frac{\beta e - w}{1+\gamma}$. Conversely, we have:*
   (a) *There is a unique equilibrium with initial price $q_0 = \frac{1}{H}\frac{\beta e - w}{1+\beta}$. Moreover, this equilibrium is stationary in the sense that $q_t = \frac{1}{H}\frac{\beta e - w}{1+\beta} > 0\ \forall t$.*
   (b) *For any value x in the interval $(0, \frac{1}{H}\frac{\beta e - w}{1+\beta})$, the sequence (q_t) determined by $q_0 = x$ and $\frac{1}{Hq_{t+1}} = \frac{\beta e}{w}\frac{1}{Hq_t} - \frac{1+\beta}{w}\ \forall t \geq 0$, is a system of price of an equilibrium with bubble. Moreover, (1) q_t is decreasing in t and converges to zero, (2) the interest rate $R_t \equiv q_t/q_{t-1}$ is decreasing in t and converges to $R^* = \frac{w}{\beta e} < 1$.*

Proof. See Appendix B.1. □

This example can be viewed as a version of the main result in Tirole (1985) (Proposition 1) for an exchange general equilibrium model with infinitely lived agents and short-sale constraints. Moreover, we can explicitly compute the maximum level of initial price bubble (which equals $\frac{1}{H}\frac{\beta e - w}{1+\beta}$) while it is explicit in Tirole (1985).

To sum up, the existence of bubble requires low interest rates of the economy without asset. Moreover, when such interest rates are low, bubbles are more likely to arise if

1. Asset supply L is low. (Asset shortage)
2. Borrowing limits b_a^* and b_b^* are low. (Financial frictions matter.)
3. The initial asset $b_{b,-1}$ is high and/or the initial asset $b_{a,-1}$ is low. (Heterogeneity matters.)
4. The endowment ratios $\frac{e_{b,2t}}{e_{a,2t}}$ and $\frac{e_{a,2t+1}}{e_{b,2t+1}}$ are high. (Heterogeneity matters.)
5. The rates of time preference $\frac{\beta_{b,2t+1}}{\beta_{b,2t}}$ and $\frac{\beta_{a,2t}}{\beta_{a,2t-1}}$ are high.

Equilibrium indeterminacy and bubbles

Under above conditions, not only asset price bubbles but also real indeterminacy arise. It is interesting to notice that our model contains only 1 consumption good and 1 asset. Our framework indicates that financial frictions and heterogeneity may generate real indeterminacy.

We now investigate the properties of consumptions in the bubbly equilibria. In equilibrium, the agent B buys asset at date $2t$ ($b_{b,2t} = L + b_a^*$) and the agent A buys asset at date $2t+1$ ($b_{a,2t+1} = L + b_b^*$). Consumptions are given by

$$
\begin{aligned}
c_{a,0} &= e_{a,0} + q_0(b_{a,-1} + b_a^*), & c_{b,0} &= e_{b,0} + q_0(b_{b,-1} - L - b_a^*)\\
c_{a,2t} &= e_{a,2t} + q_{2t}H, & c_{b,2t} &= e_{b,2t} - q_{2t}H\\
c_{a,2t+1} &= e_{a,2t+1} - q_{2t+1}H, & c_{b,2t+1} &= e_{b,2t+1} + q_{2t+1}H
\end{aligned}
$$

Recall that q_t is increasing in q_{t-1} and hence in q_0. So, for any $t \geq 1$, we observe that (1) the consumptions $c_{a,2t}$ is increasing in q_0 but $c_{a,2t-1}$ decreasing in q_0 and (2) $c_{b,2t}$ is decreasing in q_0 but $c_{b,2t-1}$ is increasing in q_0.

(Inequality). We have $\frac{c_{a,2t}}{c_{b,2t}}$ is increasing in q_{2t} and so is in q_0. $\frac{c_{a,2t+1}}{c_{b,2t+1}}$ is decreasing in q_{2t+1} and so is in q_0.

Notice that the bubbleless equilibrium has the consumption allocation $(e_i)_{i=1}^m$ which coincides with that of the autarkic equilibrium. Since the utility function is strictly concave, we can easily prove that $U_i(c_i) > U_i(e_i)$. So, its allocation is strictly Pareto dominated by that of bubbly equilibrium. This point is consistent with Proposition 4 in Townsend (1980).

The number of agents matters

Assume that there are n_a agents of type A and n_b agents of type B. For the sake of simplicity, we assume that $n_a = n_b = n$. In this case, the asset holding of agents is

$$b_{a,2t} = -b_a^*, \quad b_{b,2t} = \frac{L + nb_a^*}{n} = \frac{L}{n} + b_a^*, \quad b_{a,2t+1} = \frac{L}{n} + b_b^*, \quad b_{b,2t+1} = -b_b^*$$
$$L_n \equiv \frac{L}{n}, \quad H_n \equiv L_n + b_a^* + b_b^*$$

With these asset holdings, we have that

$$\begin{aligned} c_{a,0} &= e_{a,0} + (q_0 + d_0)b_{a,-1} + q_0 b_a^*, \quad c_{b,0} = e_{b,0} + (q_0 + d_0)b_{b,-1} - q_0(L_n + b_a^*) \\ c_{a,2t-1} &= e_{a,2t-1} - b_a^* d_{2t-1} - q_{2t-1}H_n, \quad c_{b,2t-1} = e_{b,2t-1} + d_{2t-1}(L_n + b_a^*) + q_{2t-1}H_n \\ c_{a,2t} &= e_{a,2t} + d_{2t}(L_n + b_b^*) + q_{2t}H_n, \quad c_{b,2t} = e_{b,2t} - d_{2t}b_b^* - q_{2t}H_n \end{aligned}$$

By applying our above results, bubbles are more likely to arise when L_n is low (the number of agents n is high).

4.2.2 Assets with positive dividends

In this subsection, we will study the emergence and dynamics of bubbles of assets having positive dividends. According to the asset pricing equation $q_t = \frac{q_{t+1}+d_{t+1}}{R_{t+1}}$, we have $q_t Q_t = q_{t+1}Q_{t+1}(1 + \frac{d_{t+1}}{q_{t+1}})$. By iterating, we get that $q_0 = q_T Q_T \prod_{t=1}^{T}(1 + \frac{d_t}{q_t})$. Bubbles exist if and only if $\lim_{t\to} Q_t q_t > 0$, i.e., the discounted value of 1 unit of the asset does not vanish in the infinity. Therefore, this happens if and only if $\lim_{t\to\infty} \prod_{t=1}^{T}(1 + \frac{d_t}{q_t}) < \infty$, or equivalently

$$\sum_t \frac{d_t}{q_t} < \infty \tag{27}$$

This means that there is a bubble if the price q_t goes faster than the dividend d_t. With equilibrium allocations given by (6) and (7a-7c), since consumptions are positive, we have $e_{b,2t} - b_b^* d_{2t} > Hq_{2t}$ and $e_{a,2t-1} - b_a^* d_{2t-1} > q_{2t-1}H$ for any t. By consequence, we obtain the following result.

Lemma 4. *The existence of bubble implies that*

$$\sum_t \frac{d_{2t}}{e_{b,2t} - b_b^* d_{2t}} < \infty \text{ and } \sum_t \frac{d_{2t-1}}{e_{a,2t-1} - b_a^* d_{2t-1}} < \infty. \tag{28a}$$

This means that the existence of bubbles in equilibrium requires a low level of dividends with respect to the agents' endowment in the future. The intuition behind is that, the emergence of bubble requires that the asset price goes faster than the dividend. Since there is always trading, the income of asset buyers, and therefore their endowments must go faster than the dividend. Bloise and Citanna (2019) wrote that "It might seem paradoxical that a Lucas tree is priced at its fundamental value as long as it provides dividends". According to (27) and (28a), there would exist no paradox. Indeed, the existence of bubble depends on the relationship between assets dividends and prices. In a simple case where $d_t = d > 0\ \forall t$, there is a bubble iff $\sum_t (1/q_t) < \infty$

which we can interpret that asset prices goes to infinity fast enough. As we will prove in Example 4 below, this can happen.

In order to provide conditions under which there are a continuum of bubbly equilibria, we need to introduce some notations.

$$\begin{cases} a_1 \equiv \frac{\gamma_{b,0}(e_{b,0}+d_0 b_{b,-1})}{e_{b,1}-b_b^* d_1} \\ a_{2t} \equiv \frac{\gamma_{a,2t-1}(e_{a,2t-1}-b_a^* d_{2t-1})}{e_{a,2t}-b_a^* d_{2t}} \\ a_{2t+1} \equiv \frac{\gamma_{b,2t}(e_{b,2t}-b_b^* d_{2t})}{e_{b,2t+1}-b_b^* d_{2t+1}} \end{cases} \quad \begin{cases} H_1 \equiv \frac{\gamma_{b,0}(L+b_a^*-b_{b,-1})+H}{e_{b,1}-b_b^* d_1} \\ H_{2t} \equiv \frac{H(1+\gamma_{a,2t-1})}{e_{a,2t}-b_a^* d_{2t}} \\ H_{2t+1} \equiv \frac{H(1+\gamma_{b,2t})}{e_{b,2t+1}-b_b^* d_{2t+1}} \end{cases} \quad \begin{cases} \bar{q}_0 \equiv \frac{e_{b,0}-e_{a,0}-d_0(b_{a,-1}-b_{b,-1})}{L+2b_a^*+b_{a,-1}-b_{b,-1}} \\ \bar{q}_{2t-1} \equiv \frac{e_{a,2t-1}-e_{b,2t-1}-(L+2b_a^*)d_{2t-1}}{2H} \\ \bar{q}_{2t} \equiv \frac{e_{b,2t}-e_{a,2t}-(L+2b_b^*)d_{2t}}{2H}. \end{cases}$$

Define (R_t^d) by $1 = a_t R_t^d$. Then, we can interpret (R_t^d) as the interest rates of the economy with adjusted endowments. It should be noticed that, if there is no dividend ($d_t = 0\ \forall t$) or agents are prevented from borrowing ($b_a^* = b_b^* = 0$), then $R_t^d = R_t^*$.

Observe that the inequalities (15a-15c) can be rewritten as $q_t \leq \bar{q}_t\ \forall t$. The FOCs (14a-14c) can be rewritten as

$$\frac{1}{q_t + d_t} = \frac{a_t}{q_{t-1}} - H_t\ \forall t \geq 1, \text{ or equivalently } q_t = \frac{q_{t-1}}{a_t - H_t q_{t-1}} - d_t\ \forall t \geq 1 \tag{29}$$

If (q_t) is a sequence of price, we must have

$$\frac{a_t d_t}{1 + d_t H_t} < q_{t-1} < \frac{a_t}{H_t}\ \forall t \geq 1. \tag{30}$$

So, the equilibrium price at each date must be bounded.

We now state the main result in this section, which shows that bubbles may arise under strong heterogeneity and low dividends.

Proposition 6 (multiple equilibria with bubbles)**.** *Let $u_i(c) = ln(c)\ \forall i = a, b$. Assume that $H_t > 0$, $a_{t+1}/H_{t+1} < \bar{q}_t\ \forall t$ and there are sequences $(\alpha_t), (\sigma_t)$ such that*

$$0 < \alpha_t < 1 < \sigma_t \tag{31a}$$

$$\textit{Strong heterogeneity or low interest rates: } a_{t+1} > \frac{H_{t+1}}{H_t} \frac{\alpha_t}{\alpha_{t+1}(1-\alpha_t)} \tag{31b}$$

$$\textit{Low dividends: } \frac{d_t}{d_{t+1}} > \frac{\sigma_{t+1}}{\sigma_t - 1} a_{t+1} \tag{31c}$$

$$1 - (\sigma_t - 1) d_t H_t > 0 \tag{31d}$$

$$\frac{\sigma_1 a_1 d_1}{1 + d_1 H_1} < \frac{\alpha_1 a_1}{H_1} \tag{31e}$$

Then, any sequence $(q_t)_{t\geq 0}$ determined by the system (14a-14c) and $q_0 \in (\frac{\sigma_1 a_1 d_1}{1+d_1 H_1}, \frac{\alpha_1 a_1}{H_1})$, is a system of prices of an equilibrium in which asset holdings are given by (6) and agents' consumptions are given by (7a-7c); and for such equilibrium, we have

$$\frac{\sigma_t a_t d_t}{1 + d_t H_t} < q_{t-1} < \frac{\alpha_t a_t}{H_t}\ \forall t \geq 1. \tag{32}$$

Moreover, Lemma 3 implies that there are a continuum bubbly equilibria.

Proof. See Appendix B.2. □

Remark 2 (strong heterogeneity and low interest rate condition)**.** *We interpret condition (31b) as strong heterogeneity because we observe that*

$$\frac{a_{2t+1}H_{2t}}{H_{2t+1}} = \frac{\gamma_{b,2t}(1+\gamma_{a,2t-1})}{1+\gamma_{b,2t}}\frac{e_{b,2t}-b_b^* d_{2t}}{e_{a,2t}-b_a^* d_{2t}}$$
$$\frac{a_{2t}H_{2t-1}}{H_{2t}} = \frac{\gamma_{a,2t-1}(1+\gamma_{b,2t-2})}{1+\gamma_{a,2t-1}}\frac{e_{a,2t-1}-b_a^* d_{2t-1}}{e_{b,2t-1}-b_b^* d_{2t-1}}.$$

Since the interest rates of the economy with adjusted endowments are $R_t^d = 1/a_t$, condition (31b) can also be interpreted as a "low interest rate condition".

To the best of our knowledge, Proposition 6 is the first result showing the existence of multiple equilibria with bubbles of assets with positive dividends in deterministic general equilibrium models. It is important to notice that there are exogenous parameters satisfying all conditions in Proposition 6. Indeed, we can choose parameters as follows.

1. Choose $\alpha_t = \alpha$, $\sigma_t = \sigma$ $\forall t$.
2. Choose $\gamma_{i,t} = \beta \in (0,1)$. In this case, we have

   $$\frac{a_1}{H_1} = \frac{\beta(e_{b,0}+d_0 b_{b,-1})}{\beta(L+b_a^*-b_{b,-1})+H}, \frac{a_{2t}}{H_{2t}} = \frac{\beta(e_{a,2t-1}-b_a^* d_{2t-1})}{(1+\beta)H}, \frac{a_{2t+1}}{H_{2t+1}} = \frac{\beta(e_{b,2t}-b_b^* d_{2t})}{(1+\beta)H}$$

   So, condition $\frac{a_{t+1}}{H_{t+1}} < \bar{q}_t$ is equivalent to

   $$\frac{\beta(e_{b,0}+d_0 b_{b,-1})}{\beta(L+b_a^*-b_{b,-1})+H} < \frac{e_{b,0}-e_{a,0}-d_0(b_{a,-1}-b_{b,-1})}{L+2b_a^*+b_{a,-1}-b_{b,-1}} \quad (33a)$$
   $$\frac{\beta(e_{a,2t-1}-b_a^* d_{2t-1})}{(1+\beta)H} < \frac{e_{a,2t-1}-e_{b,2t-1}-(L+2b_a^*)d_{2t-1}}{2H} \quad (33b)$$
   $$\frac{\beta(e_{b,2t}-b_b^* d_{2t})}{(1+\beta)H} < \frac{e_{b,2t}-e_{a,2t}-(L+2b_b^*)d_{2t}}{2H}. \quad (33c)$$

3. Choose $e_{b,2t+1}, e_{a,2t}$ such that $H_t = h > 0$ $\forall t$. Hence, $\frac{H_{t+1}}{H_t} = 1$.
4. Given that (d_t) is low, we can choose $e_{b,2t}, e_{a,2t+1}$ sufficiently high so that $(1-\alpha)a_{t+1} > 1$ and (33a-33c) hold. (This is a low interest rates condition.)
5. Choose (d_t) and $\frac{d_{t+1}}{d_t}$ low enough such that (31b), (31d) are satisfied and $\frac{\sigma a_1 d_1}{1+d_1 H_1} < \frac{\alpha a_1}{H_1}$. (This is a low dividends condition.)

Although Proposition 6 provides a sufficient condition under which there are a continuum of equilibria with bubbles, it would be useful to give examples with explicit parameters. We firstly focus on parameters satisfying the following assumption.

Assumption 4. *Assume that $\gamma_{i,t} = \beta \in (0,1)$ (i.e., $\beta_{i,t} = \beta^t$) and endowments are*

$$e_{a,2t-1} = b_a^* d_{2t-1} + e, \quad e_{a,2t} = b_a^* d_{2t} + w, \quad e_{b,2t-1} = b_b^* d_{2t-1} + w, \quad e_{b,2t} = b_b^* d_{2t} + e$$

where $e, w > 0$.

Under this specification, we have $a_t = a = \frac{\beta e}{w}$ and $H_t = h \equiv \frac{H(\beta+1)}{w}$ $\forall t$ and the system of price satisfies

$$\frac{1}{q_t + d_t} = \frac{a}{q_{t-1}} - h \ \forall t \geq 1, \text{ or equivalently } q_t = \frac{q_{t-1}}{a - hq_{t-1}} - d_t \ \forall t \geq 1 \tag{35}$$

In this case, we have the following result which is useful when finding examples of bubbles.

Proposition 7. *Let $u_i(c) = ln(c)$ $\forall i = a, b$ and Assumption 4 be satisfied. Assume that (q_t) is the price of an equilibrium in which asset holdings are given by (6) and agents' consumptions are given by (7a-7c).*

1. *If $a < 1$, there is no bubble.*
2. *If $a > 1$, then there are only three cases*
   (a) *There is no bubble.*
   (b) *The equilibrium is bubbly and q_t converges to zero.*
   (c) *The equilibrium is bubbly, $q_t > \frac{a-1}{h}$ $\forall t$, and q_t converges to $\frac{a-1}{h}$.*

   Moreover, when $a > 1$, there is almost one equilibrium satisfying $q_t > \frac{a-1}{h}$ $\forall t$, conditions (6) and (7a-7c).

Proof. See Appendix B.2. □

According to Proposition 7, in equilibrium with bubbles, the asset price q_t converges either to zero or to $(a-1)/h$.[10] We start with an example where q_t converges to zero.

Example 2 (multiple equilibria with bubble and $q_t \to 0$)**.** Let $u_i(c) = ln(c)$ $\forall i = a, b$ and Assumption 4 be satisfied. Assume that there exists σ such that $1 < \sigma$ and

$$\text{Low interest rate condition: } \frac{\beta e}{w} > 1 \tag{36a}$$

$$\text{Low dividends condition: } \begin{cases} \frac{\sigma-1}{\sigma}\frac{d_t}{d_{t+1}} > \frac{\beta e}{w} \\ d_t < \frac{w}{(\sigma-1)(\beta+1)H} \\ d_t < \frac{\frac{1-\beta}{1+\beta}e - w}{H} \\ \frac{\sigma a d_1}{1+d_1\frac{H(\beta+1)}{w}} < \frac{\beta e - w}{H(\beta+1)} \end{cases} \tag{36b}$$

$$\text{and } \frac{\beta(e_{b,0} + d_0 b_{b,-1})}{\beta(L + b_a^* - b_{b,-1}) + H} < \frac{e_{b,0} - e_{a,0} - d_0(b_{a,-1} - b_{b,-1})}{L + 2b_a^* + b_{a,-1} - b_{b,-1}} \tag{36c}$$

Then, any sequence $(q_t)_{t\geq 0}$ determined by the system (14a-14c) and

$$q_0 \in \Big(\frac{\sigma a d_1}{1 + d_1 h}, \frac{a-1}{h}\Big]$$

[10]This result is related to Propositions 2 and 3 in Bosi et al. (2018a). The difference is that Bosi et al. (2018a) consider an OLG model with descending altruism while we study a general equilibrium model with infinitely lived agents.

is a system of prices of an equilibrium at which asset holdings are given by (6) and agents' consumptions are given by (7a-7c). Moreover, Lemma 3 implies that there are a continuum bubbly equilibria.

For any equilibrium with $q_0 < \frac{a-1}{h}$ (including bubbly equilibrium), the asset priceq_t decreasingly converges to zero.

Proof. See Appendix B.2. □

Under conditions in Proposition 7 and $a > 1$, there is almost one bubbly equilibrium such that q_t converges to a strictly positive value. We provide an example of this case.

Example 3 (an equilibrium with bubble and $q_t \to q > 0$)**.** Let $u_i(c) = ln(c)$ $\forall i = a, b$ and Assumption 4 be satisfied. Assume also that $a > 1$. Let $x > 0$ such that $\frac{x+1}{x} > a > 1$ and define the sequence (d_t) by

$$\frac{1}{d_t} = \Big(\frac{x+1}{xa}\Big)^t\Big(\frac{1}{d_0} - \frac{hx(x+1)}{1-(a-1)x}\Big) + \frac{hx(x+1)}{1-(a-1)x} \tag{37a}$$

$$0 < d_0 < \frac{1-(a-1)x}{hx(x+1)}, \quad d_0 < \frac{\frac{1-\beta}{1+\beta}e - w}{H} \tag{37b}$$

Observe that $0 < hxd_t < 1$ $\forall t$ and $xd_t + d_t = \frac{axd_{t-1}}{1-hxd_{t-1}}$. Moreover, $\sum_t d_t < \infty$.

Define the sequence (q_t) by $q_t = \frac{a-1}{h} + xd_t$ $\forall t$. Then (q_t) is a system of prices of an equilibrium at which asset holdings are given by (6) and agents' consumptions are given by (7a-7c). Moreover, q_t decreasingly converges to $\frac{a-1}{h}$.

In this equilibrium, we have $\sum_t(d_t/q_t) = \sum_t(\frac{d_t}{\frac{a-1}{h}+xd_t}) < \sum_t d_t\frac{h}{a-1} < \infty$. So, this equilibrium experiences a bubble.

Proof. See Appendix B.2. □

In Examples 2 and 3, the economy is uniformly bounded and the dividend goes to zero. The following result shows in an economy with unbounded and asymmetric growth, bubbles may arise and the asset price goes to infinity.

Example 4 (growth economy and multiple equilibria with $q_t \to \infty$)**.** Let $u_i(c) = ln(c)$ $\forall i = a, b$, $\gamma_{i,t} = \beta \in (0, 1)$ (i.e., $\beta_{i,t} = \beta^t$). Assume that $d_t = d > 0$ $\forall t$ and endowments are

$$e_{a,2t-1} = b^*_a d_{2t-1} + e_{2t-1}, \quad e_{a,2t} = b^*_a d_{2t} + w_{2t}$$
$$e_{b,2t-1} = b^*_b d_{2t-1} + w_{2t-1}, \quad e_{b,2t} = b^*_b d_{2t} + e_{2t}$$

Let α and σ be such that $0 < \alpha < 1 < \sigma$. Assume that, for any t,

$$\frac{e_{b,0} - e_{a,0} - d_0(b_{a,-1} - b_{b,-1})}{L + 2b^*_a + b_{a,-1} - b_{b,-1}} > \frac{\beta(e_{b,0} + d_0 b_{b,-1})}{\beta(L + b^*_a - b_{b,-1}) + H}$$
$$\frac{1-\beta}{1+\beta}e_t - w_t > Hd$$
$$w_{t+1} > \frac{\sigma}{\sigma-1}\beta e_t, \quad e_t > \frac{1}{\beta(1-\alpha)}w_t,$$
$$w_t > (\sigma-1)H(\beta+1)d.$$

Notice that the two first conditions ensure that $\frac{a_{t+1}}{H_{t+1}} < \bar{q}_t \ \forall t$.

According to Proposition 6, any sequence $(q_t)_{t\geq 0}$ determined by the system (14a-14c) and $q_0 \in (\frac{\sigma a_1 d_1}{1+d_1 H_1}, \frac{\alpha a_1}{H_1})$, is a system of prices of an equilibrium in which asset holdings are given by (6) and agents' consumptions are given by (7a-7c). By consequence, Lemma 3 implies that there are a continuum bubbly equilibria.

In this example, endowments of both agents go to infinity. However, there is an asymmetric growth: $\frac{e_t}{w_t} > \frac{1}{\beta(1-\alpha)} > 1$ (or equivalently $\frac{e_{a,2t-1}-b_a^* d_{2t-1}}{e_{b,2t-1}-b_b^* d_{2t-1}} > \frac{1}{\beta(1-\alpha)}$ and $\frac{e_{b,2t}-b_b^* d_{2t}}{e_{a,2t}-b_a^* d_{2t}} > \frac{1}{\beta(1-\alpha)}$).

Example 5 (an equilibrium with bubbles and q_t may fluctuate over time)**.** Consider a particular case where $\beta_{i,t} = \beta^t \ \forall i, \forall t$ where $\beta \in (0,1)$, $b_a^* = b_b^* = 0$ (no short-sales) and $e_{b,2t+1} = e_{a,2t} = 0$. In this case, $\gamma_{b,2t} = \gamma_{a,2t-1} = \beta < 1$, $H = L$ and there is a unique equilibrium satisfying condition (6)

$$q_{2t} = \frac{\beta}{(1+\beta)L} e_{b,2t} \text{ and } q_{2t-1} = \frac{\beta}{(1+\beta)L} e_{a,2t-1}. \tag{40}$$

In other words, the set $\mathcal{B}_0$ contains a unique element. This equilibrium experiences a bubble iff $\sum_t d_t/q_t < \infty$ which now becomes $\sum_t \frac{d_{2t}}{e_{b,2t}} + \sum_t \frac{d_{2t-1}}{e_{a,2t-1}} < \infty$. So, we recover (28a) and this corresponds to the key condition in examples of bubbles in Section 5.1.1 in Bosi et al. (2018b).

We now look at the consumption

$$c_{a,0} = e_{a,0} + (q_0 + d_0) b_{a,-1}, \quad c_{b,0} = e_{b,0} + (q_0 + d_0) b_{b,-1} - q_0 L \tag{41a}$$
$$c_{a,2t-1} = e_{a,2t-1} - q_{2t-1} L, \quad c_{b,2t-1} = e_{b,2t-1} + d_{2t-1} L + q_{2t-1} L \tag{41b}$$
$$c_{a,2t} = e_{a,2t} + d_{2t} L + q_{2t} L, \quad c_{b,2t} = e_{b,2t} - q_{2t} L \tag{41c}$$

Since $Lq_{2t} = \frac{\beta}{1+\beta} e_{b,2t}$ and $Lq_{2t-1} = \frac{\beta}{1+\beta} e_{a,2t-1}$, we see that $c_{a,2t-1}$ and $c_{b,2t}$ do not depend on $(d_t)_t$ but $c_{a,2t}$ (resp., $c_{b,2t-1}$) is strictly increasing in d_{2t} (resp., d_{2t-1}). So, when dividends decrease, bubbles will be more likely to arise but the individual welfares will be lower.

5 Conclusion

In general equilibrium models with infinitely lived agents, we have provided new conditions (based on fundamentals) under which assets (with or without dividend) do not generate price bubbles. In general, the formation of bubble is associated to the fluctuations of asset trading. However, the emergence of bubble is not a matter of a single factor but the result of an interaction between heterogeneous agents in an imperfect market. Since bubbles and equilibrium outcomes are determined simultaneously in equilibrium, we should not say that bubbles affect equilibrium outcomes or vice-versa. Instead, they are caused by economic fundamentals.

We have provided several examples where bubbles and real indeterminacy arise in a model economy with two kinds of agents. Our basic idea is that when the economy without asset has low interest rates and cannot allow agents to efficiently smooth their consumption, agents may buy an asset even its price is higher than its fundamental value. Our analyses suggest that bubbles are more likely to arise if (1) heterogeneity of agents takes at any period, (2) borrowing limits are tight, (3) the interest rates of the benchmark economy are low so that agents are willing to buy assets at a high price, (4) there is an asset shortage (the asset supply is low or asset dividends are low with respect to agents' endowment).

A Appendix: Proofs for Section 2

Proof of Proposition 1. Part 1. It is easy to see that $q_t > 0\ \forall t$. Indeed, if $q_t = 0$ for some t, we can increase $b_{i,t}$ and obtain a higher income in $t+1$ and increase $c_{i,t+1}$: a contradiction.

To prove the FOCs, it suffices to prove that $q_t\beta_{i,t}u_i'(c_{i,t}) \geq \beta_{i,t+1}u_i'(c_{i,t+1})(q_{t+1}+d_{t+1})$ and we have equality if $b_{i,t}+b_i^* > 0$. Fix $t \geq 0$ and consider another allocation $(c_{i,s}', b_{i,s}')_s$ given by $(c_{i,s}', b_{i,s}') = (c_{i,s}, b_{i,s})\ \forall s \notin \{t, t+1\}$ and $(c_{i,s}', b_{i,s}')_{s=t,t+1}$ determined by

$$\underbrace{c_{i,t}-\epsilon}_{c_{i,t}'} + q_t(b_{i,t}+\frac{\epsilon}{q_t}) = e_{i,t} + (q_t+d_t)b_{i,t-1}$$

$$\underbrace{c_{i,t+1}+(q_{t+1}+d_{t+1})\frac{\epsilon}{q_t}}_{c_{i,t+1}'} + q_{t+1}b_{i,t+1} = e_{i,t+1} + (q_{t+1}+d_{t+1})(b_{i,t}+\frac{\epsilon}{q_t}).$$

where $\epsilon > 0$ is low enough so that $c_{i,t}-\epsilon > 0$.

By the optimality $(c_{i,t}, b_{i,t})_t$, we have

$$\beta_{i,t}u_i(c_{i,t}) + \beta_{i,t+1}u_i(c_{i,t+1}) \geq \beta_{i,t}u_i(c_{i,t}') + \beta_{i,t+1}u_i(c_{i,t+1}'), \text{ and hence}$$
$$\beta_{i,t}\frac{u_i(c_{i,t})-u_i(c_{i,t}-\epsilon)}{\epsilon} \geq \beta_{i,t+1}\frac{u_i\Big(c_{i,t+1}+(q_{t+1}+d_{t+1})\frac{\epsilon}{q_t}\Big)-u_i(c_{i,t+1})}{\epsilon\frac{q_{t+1}+d_{t+1}}{q_t}}\frac{q_{t+1}+d_{t+1}}{q_t}.$$

Let ϵ tend to zero, we get that $q_t\beta_{i,t}u_i'(c_{i,t}) \geq \beta_{i,t+1}u_i'(c_{i,t+1})(q_{t+1}+d_{t+1})$.

If $b_{i,t}+b_i^* > 0$, we can do as above but with $\epsilon < 0$ and get that $q_t\beta_{i,t}u_i'(c_{i,t}) \leq \beta_{i,t+1}u_i'(c_{i,t+1})(q_{t+1}+d_{t+1})$. Therefore, we have the equality.

We finally define $\lambda_{i,t} \equiv \beta_{i,t}u_i'(c_{i,t})$ and $\eta_{i,t} \equiv \lambda_{i,t}q_t - \lambda_{i,t+1}(q_{t+1}+d_{t+1})$.

We now prove the TVCs. The FOCs imply that the sequence $(\lambda_{i,t}q_t)_t$ is decreasing. Moreover, we have

$$\lambda_{i,t}q_tb_{i,t} = \Big(\lambda_{i,t+1}(q_{t+1}+d_{t+1})+\eta_{i,t}\Big)b_{i,t} = \lambda_{i,t+1}(q_{t+1}+d_{t+1})b_{i,t} - \eta_{i,t}b_i^*$$

We rewrite the budget constraint of agent i at date t as follows

$$\lambda_{i,t}(c_{i,t}-e_{i,t}) = \lambda_{i,t}(q_t+d_t)b_{i,t-1} - \lambda_{i,t}q_tb_{i,t}$$

By taking the sum of budget constraints from $t=0$ until T and using (1b), we get that

$$\sum_{t=0}^{T}\lambda_{i,t}(c_{i,t}-e_{i,t}) = \sum_{t=0}^{T}\Big(\lambda_{i,t}(q_t+d_t)b_{i,t-1} - \lambda_{i,t}q_tb_{i,t}\Big) \quad \text{(A.1)}$$

$$= \lambda_{i,0}(q_0+d_0)b_{i,-1} - \lambda_{i,T}q_Tb_{i,T} + \sum_{t=1}^{T}\eta_{i,t}b_i^* \quad \text{(A.2)}$$

and hence $\lambda_{i,0}(q_0+d_0)b_{i,-1} + \sum_{t=0}^{T}\lambda_{i,t}e_{i,t} + \sum_{t=1}^{T}\eta_{i,t}b_i^* = \lambda_{i,T}q_Tb_{i,T} + \sum_{t=0}^{T}\lambda_{i,t}c_{i,t}$.

We will prove that $\lim_{T\to+\infty} \lambda_{i,T} q_T(b_{i,T} + b^*)$ exists in $\mathbb{R}^+$. Recall that the sequence $(\lambda_{i,t} q_t)_t$ is positive and decreasing. So, $\lim_{t\to+\infty} \lambda_{i,t} q_t$ exists and is in $\mathbb{R}^+$. We have $-b_i^* \leq b_{i,t} = L - \sum_{j\neq i} b_{j,t} \leq L + \sum_i b_i^* \forall t$, and hence

$$-\infty < \liminf_{T\to+\infty} \lambda_{i,T} q_T b_{i,T} \leq \limsup_{T\to+\infty} \lambda_{i,T} q_T b_{i,T} < \infty.$$

Under our assumptions, we have $\sum_t \lambda_{i,t} c_{i,t} < \infty$. Indeed, we have $\sum_t \lambda_{i,t} c_{i,t} = \sum_t \beta_{i,t} u_i'(c_{i,t}) c_{i,t} \leq \sum_t \beta_{i,t} v(c_{i,t}) \leq \sum_t \beta_{i,t} v(\sum_i e_{i,t} + L d_t) < \infty$.

Summing up, we obtain that $\sum_t \lambda_{i,t} c_{i,t} < \infty$. Since $\sum_t \lambda_{i,t} c_{i,t} < \infty$, both series $\sum_t \lambda_{i,t} e_{i,t}$ and $\sum_t \eta_{i,t} b_i^*$ converge. By consequence, $\lim_{T\to+\infty} \lambda_{i,T} q_T b_{i,T}$ exists in $\mathbb{R}$. Therefore $\lambda_{i,T} q_T(b_{i,T} + b^*)$ converges and

$$\lim_{T\to+\infty} \lambda_{i,T} q_T(b_{i,T} + b^*) = \lim_{T\to+\infty} \lambda_{i,T} q_T b_{i,T} + \lim_{T\to+\infty} \lambda_{i,T} q_T b_i^* \in \mathbb{R}$$

There are two cases:

- Case (a): If $\liminf_{t\to+\infty}(b_{i,t} + b_i^*) = 0$, then $\lim_{t\to+\infty} \lambda_{i,t} q_t(b_{i,t} + b^*) = 0$ because $\lambda_{i,t} q_t \leq \lambda_{i,0} q_0 \ \forall t$.
- Case (b): If $\liminf_{t\to+\infty}(b_{i,t} + b_i^*) > 0$, then there exist $\alpha > 0$ and T such that $b_{i,t} + b_i^* > \alpha, \forall t \geq T$. In this case $\eta_{i,t} = 0, \forall t \geq T$. For simplicity of the proof, assume $T = 0$.
  We know that $\lim_{t\to+\infty} \lambda_{i,t} q_t$ exists. Let $\zeta = \lim_{t\to+\infty} \lambda_{i,t} q_t$. We claim that $\zeta = 0$. Assume the contrary: $\zeta > 0$. In this case $\zeta = \lim_{\tau\to+\infty} \lambda_{i,T+\tau+1} q_{T+\tau+1} \leq \lambda_{i,T} q_T \ \forall T$. Construct a sequence $(c'_{i,t}, b'_{i,t})$ as follows:

  $$c'_{i,0} = c_{i,0} + \frac{\zeta\alpha}{\lambda_{i,0}}, \ c'_{i,t} = c_{i,t}, \forall t \geq 1, \quad b'_{i,t} = b_{i,t} - \frac{\zeta\alpha}{q_t \lambda_{i,t}}, \forall t \geq 0$$

  Since $b'_{i,t} \geq -b_i^* + \alpha - \frac{\zeta\alpha}{q_t\lambda_{i,t}} = -b_i^* + \alpha(1 - \frac{\zeta}{q_t\lambda_{i,t}}) \geq -b_i^*, \forall t$, the sequence $(c'_{i,t}, b'_{i,t})$ satisfies physical, budget and borrowing constraints. However $\sum_{t=0}^{+\infty} \beta_{i,t} u_i(c'_{i,t}) > \sum_{t=0}^{+\infty} \beta_{i,t} u_i(c_{i,t})$ which is a contradiction. Hence $\zeta = 0$, i.e. $\lim_{t\to\infty}\{q_t \lambda_{i,t}\} = 0$. Since $b_{i,t} + b_i^* = L - \sum_{j\neq i} b_{j,t} + b_i^* \leq L + \sum_i b_i^* \forall t$, we get

  $$\lambda_{i,t} q_t(L + \sum_i b_i^*) \geq \lambda_{i,t} q_t(b_{i,t} + b_t^*) \geq \lambda_{i,t} q_t \alpha.$$

  This implies $\lim_{t\to\infty} \lambda_{i,t} q_t(b_{i,t} + b_t^*) = 0$.

Considering the two cases (a) and (b), we get $\lim_{t\to\infty} \lambda_{i,t} q_t(b_{i,t} + b_t^*) = 0$. The proof is complete.

Part 2 (sufficient condition). It suffices to prove the optimality of the allocation (c_i, b_i). Consider another sequence (c'_i, b'_i) satisfying physical, budget and borrowing constraint. We have, for any T,

$$
\begin{aligned}
\sum_{t=0}^{T}\lambda_{i,t}(c_{i,t}-c'_{i,t}) &\geq \sum_{t=0}^{T}\lambda_{i,t}\Big(e_{i,t}+(q_t+d_t)b_{i,t-1}-q_tb_{i,t}-e_{i,t}-(q_t+d_t)b'_{i,t-1}+q_tb'_{i,t}\Big)\\
&=\sum_{t=0}^{T-1}\lambda_{i,t+1}(q_{t+1}+d_{t+1})(b_{i,t}-b'_{i,t})-\sum_{t=0}^{T-1}\lambda_{i,t}q_t(b_{i,t}-b'_{i,t})-q_T\lambda_{i,T}(b_{i,T}-b'_{i,T})\\
&=-q_T\lambda_{i,T}(b_{i,T}-b'_{i,T})+\sum_{t=0}^{T-1}\Big(\lambda_{i,t+1}(q_{t+1}+d_{t+1})-\lambda_{i,t}q_t\Big)(b_{i,t}-b'_{i,t})\\
&=-q_T\lambda_{i,T}(b_{i,T}-b'_{i,T})+\sum_{t=0}^{T-1}\eta_{i,t}(b'_{i,t}-b_{i,t})\\
&=-q_T\lambda_{i,T}(b_{i,T}+b^*_i-(b'_{i,T}+b^*_i))+\sum_{t=0}^{T-1}\eta_{i,t}(b'_{i,t}+b^*_i-(b_{i,t}+b^*_i))\\
&\geq -q_T\lambda_{i,T}(b_{i,T}+b^*_i)+\sum_{t=0}^{T-1}\eta_{i,t}(b'_{i,t}+b^*_i)\geq -q_T\lambda_{i,T}(b_{i,T}+b^*_i).
\end{aligned}
$$

Therefore, we have

$$\sum_{t=0}^{T}\Big(\beta_{i,t}u(c_{i,t})-\beta_{i,t}u(c'_{i,t})\Big)\geq\sum_{t=0}^{T}\lambda_{i,t}(c_{i,t}-c'_{i,t})\geq -q_T\lambda_{i,T}(b_{i,T}+b^*_T).$$

Denote $U_T\equiv\sum_{t=0}^{T}\beta_{i,t}u(c_{i,t})$ and $U'_T\equiv\sum_{t=0}^{T}\beta_{i,t}u(c'_{i,t})$. Observe that $\lim_{T\to\infty}U_T$ exists.

If $\lim_{T\to\infty}q_T\lambda_{i,T}(b_{i,T}+b^*_T)=0$, then $\limsup_{T\to\infty}U'_T\leq\lim_{T\to\infty}U_T$; we have finished our proof.

Remark 3. *If $u_i(0)\geq 0$, then the series $\sum_{t=0}^{\infty}\lambda_{i,t}u_i(c_{i,t})$ always converges. By consequence, conditions $U_T\geq U'_T-q_T\lambda_{i,T}(b_{i,T}+b^*_T)$ $\forall T$ and $\liminf_{T\to\infty}q_T\lambda_{i,T}(b_{i,T}+b^*_T)=0$ implies that $\lim_{T\to\infty}U_T\geq\lim_{T\to\infty}U'_T=\limsup_{T\to\infty}U'_T$.*

□

Proof of Corollary 2. Suppose that $\sum_t Q_te_{i,t}<\infty$ $\forall i$. Budget constraint of agent i implies that $Q_tc_{i,t}+Q_tq_tb_{i,t}=Q_te_{i,t}+Q_t(q_t+d_t)b_{i,t-1}$. By summing over t and noticing that $Q_tq_t=Q_{t+1}(q_{t+1}+d_{t+1})$, we have $\sum_{t=0}^{T}Q_tc_{i,t}+Q_Tq_Tb_{i,T}=\sum_{t=0}^{T}Q_te_{i,t}+(q_0+d_0)b_{i,-1}$ $\forall t$. Since $\sum_t Q_te_{i,t}<\infty$ and $(Q_Tq_Tb_{i,T})$ is bounded (because $b_{i,T}$ and Q_Tq_T are bounded), the series $\sum_t Q_tc_{i,t}$ converges, and so does the sequence $(Q_Tq_Tb_{i,T})_T$. If there is a bubble, we have $q_t>0$ $\forall t$ and $\lim_{t\to\infty}Q_tq_t>0$. By consequence, $(b_{i,t})$ converges for any i. Market clearing conditions imply that there is an agent i such that $b_i\equiv\lim_{t\to\infty}b_{i,t}>0$. So, borrowing constraints of agent i do not bind from some date on, say T. Hence, $\frac{\lambda_{i,t+1}}{\lambda_{i,t}}=\frac{q_t}{q_{t+1}+d_{t+1}}=\frac{1}{R_{t+1}}$ $\forall t\geq T$. This implies that $Q_t=Q_T\frac{\lambda_{i,t}}{\lambda_{i,T}}$ $\forall t\geq T$. By combining with the TVC, we get that $\lim_{t\to\infty}Q_tq_t(b_{i,t}+b^*_i)=0$ $\forall i$. This is impossible because $\lim_{t\to\infty}(b_{i,t}+b^*_i)>0$ and $\lim_t Q_tq_t>0$. □

Proof of Lemma 1. We have, for $T>t$,

$$U_i^T\Big(z(c_i,t,\gamma',W_t)\Big)=\sum_{s=0}^{t-1}\beta_{i,s}u_i(c_{i,s})+\beta_{i,t}u_i\big(c_{i,t}+W_t\big)+\sum_{s=t+1}^{T}\beta_{i,s}u_i(\gamma'c_{i,s}).$$

1. If $u_i(c) = ln(c)$, we have

$$U_i^T\Big(z(c_i,t,\gamma',W_t)\Big) - U_i^T(c_i) = \beta_{i,t}\Big(u_i(c_{i,t}+W_t) - u_i(c_{i,t})\Big) + \sum_{s=t+1}^{T}\beta_{i,s}\Big(u_i(\gamma' c_{i,s}) - u_i(c_{i,s})\Big)$$

$$= \beta_{i,t}ln\Big(1+\frac{W_t}{c_{i,t}}\Big) + \ln(\gamma')\sum_{s=t+1}^{T}\beta_{i,s} \geq \beta_{i,t}ln(2) + \ln(\gamma)\sum_{s=t+1}^{T}\beta_{i,s} \quad \forall \gamma' \geq \gamma.$$

So, we have the uniform impatience if $\beta_{i,t} > -\frac{\ln(\gamma)}{ln(2)}\sum_{s=t+1}^{\infty}\beta_{i,s}\ \forall t$.
2. If $u_i(c) = \frac{c^{1-\sigma}}{1-\sigma}$, we have

$$U_i^T\big(z(c_i,t,\gamma',W_t)\big) - U_i^T(c_i) = \beta_{i,t}\Big(\frac{\big(c_{i,t}+W_t\big)^{1-\sigma}}{1-\sigma} - \frac{\big(c_{i,t}\big)^{1-\sigma}}{1-\sigma}\Big) + (\gamma^{1-\sigma}-1)\sum_{s=t+1}^{T}\beta_{i,s}\frac{\big(c_{i,s}\big)^{1-\sigma}}{1-\sigma}$$

$$\geq \beta_{i,t}\frac{2^{1-\sigma}-1}{1-\sigma}W_t^{1-\sigma} + (\gamma^{1-\sigma}-1)\sum_{s=t+1}^{T}\beta_{i,s}\frac{W_s^{1-\sigma}}{1-\sigma}$$

where the last inequality is come from $c_{i,t} \leq W_t\ \forall i, \forall t$, the function $u_i(c+W_t) - u_i(c)$ is decreasing in c, and $\gamma < 1$. So, the uniform impatience holds if $\beta_{i,t}\frac{2^{1-\sigma}-1}{1-\sigma}W_t^{1-\sigma} + (\gamma^{1-\sigma} - 1)\sum_{s=t+1}^{\infty}\beta_{i,s}\frac{W_s^{1-\sigma}}{1-\sigma} > 0\ \forall t$. □

Proof of Proposition 2. We mainly use Proposition 1.

1. Suppose that there exists i such that $\liminf_{t\to\infty}(b_{i,t}+b_i^*) > 0$. In this case, there exists T such that $b_{i,t} + b_i^* > 0\ \forall t \geq T$. So, $\frac{\lambda_{i,t+1}}{\lambda_{i,t}} = \frac{q_t}{q_{t+1}+d_{t+1}} = \frac{1}{R_{t+1}}\ \forall t \geq T$. This implies that $Q_t = Q_T\frac{\lambda_{i,t}}{\lambda_{i,T}}\ \forall t \geq T$. By combining with the TVC, we get that $\lim_{t\to\infty} Q_t q_t(b_{i,t}+b_i^*) = 0$ $\forall i$. This is impossible because $\liminf_{t\to\infty}(b_{i,t}+b_i^*) > 0$ and $\lim_t Q_t q_t > 0$.

2. We firstly prove that: there exist an agent, say agent i, and an increasing sequence $(i_n)_n$ such that $b_{i,i_n} + b_i^* = 0$ for all $n = 0, 1, \ldots$. Indeed, assume, by contradiction, that for any agent i, there exists T such that $b_{i,t} > b_i^*\ \forall t \geq T$. As discussed above, we obtain that $\lim_{t\to\infty} Q_t q_t(b_{i,t}+b_i^*) = 0\ \forall i$. Taking the sum over i and using market clearing conditions, we get that $\lim_{t\to\infty} Q_t q_t = 0$, i.e., there is no bubble, a contradiction.

We now consider other agents $j \in \{2, \cdots, m\}$. Suppose that for any $j \geq 2$, there is T_j such that $b_{j,t} + b_j^* > 0\ \forall t \geq T_j$. So, $\frac{\lambda_{j,t+1}}{\lambda_{j,t}} = \frac{q_t}{q_{t+1}+d_{t+1}} = \frac{1}{R_{t+1}}\ \forall j \geq 2, \forall t \geq T \equiv \max_{j\geq 2} T_j$, which implies that $Q_t = Q_T\frac{\lambda_{j,t}}{\lambda_{j,T}}\ \forall j \geq 2, t \geq T$. By combining with the TVC, we get that $\lim_{t\to\infty} Q_t q_t(b_{j,t}+b_j^*) = 0$. Since bubbles exist, we have $\lim_{t\to\infty} Q_t q_t > 0$. We then get that $\lim_{t\to\infty}(b_{j,t}+b_j^*) = 0$. Market clearing conditions imply that $\lim_{t\to\infty} b_{1,t} + b_1^* = L - \lim_{t\to\infty}\sum_{j\geq 2} b_{j,t} + b_1^* = L + \sum_{i=1}^{m} b_i^* > 0$. So, there exists T_1 such that $b_{1,t} + b_1^* > 0$ $\forall t \geq T_1$, a contradiction. By consequence, there exist an agent, say agent 2, and an increasing sequence $(j_n)_n$ such that $b_{j,j_n} + b_j^* = 0$ for all $n = 0, 1, \ldots$.

3. Suppose that there are $m-1$ agents such that their asset holding converges. By market clearing conditions, the asset holding of all agents converges. So, there is an agent i such that $\lim_{t\to\infty} b_{i,t} > 0$. According to point 1, this is impossible. □

Proof of Proposition 3. We need the following intermediate results (Lemmas 5, 6, 7).

Lemma 5. *At each date t, there exists i such that $b_{i,t} \geq b_{i,t+1}$ and borrowing constraint is not binding ($b_{i,t} + b_i^* > 0$).*

Proof. Define i_0 such that $b_{i_0,t} - b_{i_0,t+1} = \max_i\{b_{i,t} - b_{i,t+1}\}$. Then, we have $b_{i_0,t} - b_{i_0,t+1} \geq 0$. We consider two cases.

Case 1: If $b_{i_0,t} - b_{i_0,t+1} > 0$, then $b_{i_0,t} + b_i^* > b_{i_0,t+1} + b_i^* \geq 0$.

Case 2: If $b_{i_0,t} - b_{i_0,t+1} = 0$, then $b_{i,t} - b_{i,t+1} \leq 0 \ \forall i$. Since $\sum_i (b_{i,t} - b_{i,t+1}) = 0$, we get that $b_{i,t} - b_{i,t+1} = 0$ for every i. Since $\sum_i b_{i,t} > 0$, we can choose i_1 such that $b_{i_1,t} > 0$, we have $b_{i_1,t} = b_{i_1,t+1}$ and $b_{i_1,t} + b_i^* > 0$. □

Lemma 6. *In equilibrium, we have*

$$1 = R_{t+1} \max_i \frac{\beta_{i,t+1} u_i'(c_{i,t+1})}{\beta_{i,t} u_i'(c_{i,t})} \ \forall t \geq 0. \tag{A.3}$$

In addition, if we assume that $e_{i,t} - d_t b_i^* > 0 \ \forall i, \forall t$, then we have

$$\frac{1}{R_{t+1}} \leq \max_i \frac{\beta_{i,t+1} u_i'(e_{i,t+1} - d_{t+1} b_i^*)}{\beta_{i,t} u_i'(\sum_i e_{i,t} + L d_t)} \ \forall t \geq 0. \tag{A.4}$$

Proof. According to FOCs, we have $q_t \geq (q_{t+1} + d_{t+1}) \max_i \frac{\beta_{i,t+1} u_i'(c_{i,t+1})}{\beta_{i,t} u_i'(c_{i,t})}$. Since $\sum_i b_{i,t} > 0$, there is an agent i_t such that $b_{i_t,t} > 0$. Hence, $\eta_{i,t} = 0$. By consequence, $q_t = (q_{t+1} + d_{t+1}) \frac{\beta_{i_t,t+1} u_{i_t}'(c_{i_t,t+1})}{\beta_{i_t,t} u_{i_t}'(c_{i_t,t})}$. Therefore, we get that

$$q_t = (q_{t+1} + d_{t+1}) \max_i \frac{\beta_{i,t+1} u_i'(c_{i,t+1})}{\beta_{i,t} u_i'(c_{i,t})} = R_{t+1} \max_i \frac{\beta_{i,t+1} u_i'(c_{i,t+1})}{\beta_{i,t} u_i'(c_{i,t})}.$$

Let $t \geq 0$, Lemma 5 implies that there exists an agent $i = i(t)$ (depending on t) such that $b_{i(t),t} \geq b_{i(t),t+1}$ and $b_{i(t),t} + b_{i(t)}^* > 0$. Then, we have $\eta_{i(t),t} = 0$ and hence

$$1 = R_{t+1} \frac{\beta_{i(t),t+1} u_{i(t)}'(c_{i(t),t+1})}{\beta_{i(t),t} u_{i(t)}'(c_{i(t),t})}.$$

We observe that $c_{i(t),t+1} = e_{i(t),t+1} + (q_{t+1} + d_{t+1}) b_{i(t),t} - q_{t+1} b_{i(t),t+1} \geq e_{i(t),t+1} - d_{t+1} b_{i(t)}^*$ and $c_{i(t),t} \leq W_t \equiv \sum_i e_{i,t} + L d_t$. By consequence, we get that

$$\frac{1}{R_{t+1}} = \frac{\beta_{i(t),t+1} u_{i(t)}'(c_{i(t),t+1})}{\beta_{i(t),t} u_{i(t)}'(c_{i(t),t})} \leq \frac{\beta_{i(t),t+1} u_{i(t)}'(e_{i(t),t+1} - d_{t+1} b_{i(t)}^*)}{\beta_{i(t),t} u_{i(t)}'(\sum_i e_{i,t} + L d_t)} \leq \max_i \frac{\beta_{i,t+1} u_i'(e_{i,t+1} - d_{t+1} b_i^*)}{\beta_{i,t} u_i'(\sum_i e_{i,t} + L d_t)}$$

□

Lemma 7. *Consider an equilibrium. Take γ in Assumption 3, we have that $(1-\gamma) q_t b_{i,t} \leq W_t$ $\forall i, \forall t$.*

Proof. Suppose, by contradiction, that there exist i and t such that $(1-\gamma) q_t b_{i,t} > W_t$. Let us consider a new allocation of agent i: $z_i := z\big(c_i, t, \gamma, (1-\gamma) q_t b_{i,t}\big)$. We check that this allocation is in the budget set of agent i because

$$\begin{aligned} \big(c_{i,t} + (1-\gamma) q_t b_{i,t}\big) + q_t\big(\gamma b_{i,t}\big) &\leq e_{i,t} + (q_t + d_t) b_{i,t-1} \\ \gamma c_{i,s} + q_s\big(\gamma b_{i,s}\big) = \gamma e_{i,s} + (q_t + d_t)\big(\gamma b_{i,s-1}\big) &\leq e_{i,s} + (q_t + d_t)\big(\gamma b_{i,s-1}\big) \ \forall s \geq t+1 \end{aligned}$$

By Assumption 3, we have

$$U_i(c_i) < U_i\Big(z(c_i, t, \gamma, W_t\Big) < U_i\Big(z\big(c_i, t, \gamma, (1-\gamma) q_t b_{i,t}\big)\Big). \tag{A.5}$$

This is in contradiction to the optimality of (c_i, b_i). □

We now prove Proposition 3. Since points 1 and 2 are direct consequences of (3) and Lemma 1, let us prove (3). According to Lemma 7, we have $(1-\gamma)q_t b_{i,t} \leq W_t \ \forall i, \forall t$. Taking the sum over i, we get $(1-\gamma)q_t L \leq mW_t \ \forall t$. Since $L(1-\gamma) > 0$, we get that

$$q_t \leq \frac{mW_t}{L(1-\gamma)} \ \forall t. \tag{A.6}$$

According to Lemma 6, we have

$$\frac{1}{R_{t+1}} \leq \max_i \frac{\beta_{i,t+1} u_i'(e_{i,t+1} - d_{t+1} b_i^*)}{\beta_{i,t} u_i'(\sum_i e_{i,t} + Ld_t)} \ \forall t \geq 0. \tag{A.7}$$

Recall that there is no bubble iff $\lim_{t\to} Q_t q_t = 0$. By combining these above arguments, there is no bubble if condition (3) is satisfied.

□

B Appendix: Proofs for Section 4.2

Proof of Lemma 2. With our asset allocations, the FOCs become

$$1 \geq \frac{\gamma_{a,2t} u_a'(c_{a,2t+1})}{u_a'(c_{a,2t})} \frac{q_{2t+1} + d_{2t+1}}{q_{2t}}, \quad 1 = \frac{\gamma_{a,2t-1} u_a'(c_{a,2t})}{u_a'(c_{a,2t-1})} \frac{q_{2t} + d_{2t}}{q_{2t-1}} \tag{A.8a}$$

$$1 = \frac{\gamma_{b,2t} u_b'(c_{b,2t+1})}{u_b'(c_{b,2t})} \frac{q_{2t+1} + d_{2t+1}}{q_{2t}}, \quad 1 \geq \frac{\gamma_{b,2t-1} u_b'(c_{b,2t})}{u_a'(c_{b,2t-1})} \frac{q_{2t} + d_{2t}}{q_{2t-1}}. \tag{A.8b}$$

We know that $u'(c) = 1/c$. Denote $H \equiv L + b_a^* + b_b^*$. FOCs now become

$$1 = \frac{\gamma_{a,2t-1} u_a'(c_{a,2t})}{u_a'(c_{a,2t-1})} \frac{q_{2t} + d_{2t}}{q_{2t-1}} = \frac{\gamma_{a,2t-1}(e_{a,2t-1} - b_a^* d_{2t-1} - q_{2t-1} H)}{e_{a,2t} + d_{2t}(L + b_b^*) + q_{2t} H} \frac{q_{2t} + d_{2t}}{q_{2t-1}} \tag{A.9a}$$

$$1 = \frac{\gamma_{b,2t} u_b'(c_{b,2t+1})}{u_b'(c_{b,2t})} \frac{q_{2t+1} + d_{2t+1}}{q_{2t}} = \frac{\gamma_{b,2t}(e_{b,2t} - d_{2t} b_b^* - q_{2t} H)}{e_{b,2t+1} + d_{2t+1}(L + b_a^*) + q_{2t+1} H} \frac{q_{2t+1} + d_{2t+1}}{q_{2t}} \tag{A.9b}$$

$$\frac{\gamma_{a,2t-1} u_a'(c_{a,2t})}{u_a'(c_{a,2t-1})} \geq \frac{\gamma_{b,2t-1} u_b'(c_{b,2t})}{u_b'(c_{b,2t-1})} \tag{A.9c}$$

$$\frac{\gamma_{b,2t} u_b'(c_{b,2t+1})}{u_b'(c_{b,2t})} \geq \frac{\gamma_{a,2t} u_a'(c_{a,2t+1})}{u_a'(c_{a,2t})} \tag{A.9d}$$

The two last inequalities become

$$\gamma_{a,2t-1} \frac{e_{a,2t-1} - b_a^* d_{2t-1} - q_{2t-1} H}{e_{a,2t} + d_{2t}(L + b_b^*) + q_{2t} H} \geq \gamma_{b,2t-1} \frac{e_{b,2t-1} + d_{2t-1}(L + b_a^*) + q_{2t-1} H}{e_{b,2t} - d_{2t} b_b^* - q_{2t} H} \tag{A.10a}$$

$$\gamma_{b,2t} \frac{e_{b,2t} - d_{2t} b_b^* - q_{2t} H}{e_{b,2t+1} + d_{2t+1}(L + b_a^*) + q_{2t+1} H} \geq \gamma_{a,2t} \frac{e_{a,2t} + d_{2t}(L + b_b^*) + q_{2t} H}{e_{a,2t+1} - b_a^* d_{2t+1} - q_{2t+1} H} \tag{A.10b}$$

At the first period, FOCs are

$$\frac{\gamma_{b,0} u_b'(c_{b,1})}{u_b'(c_{b,0})} \geq \frac{\gamma_{a,0} u_a'(c_{a,1})}{u_a'(c_{a,0})} \quad \Leftrightarrow \gamma_{b,0} \frac{c_{b,0}}{c_{b,1}} \geq \gamma_{a,0} \frac{c_{a,0}}{c_{a,1}} \tag{A.11a}$$

$$\Leftrightarrow \gamma_{b,0} \frac{e_{b,0} + (q_0 + d_0) b_{b,-1} - q_0 b_{b,0}}{e_{b,1} + d_1(L + b_a^*) + q_1 H} \geq \gamma_{a,0} \frac{e_{a,0} + (q_0 + d_0) b_{a,-1} - q_0 b_{a,0}}{e_{a,1} - b_a^* d_1 - q_1 H} \tag{A.11b}$$

$$\text{and } 1 = \frac{\gamma_{b,0} u_b'(c_{b,1})}{u_b'(c_{b,0})} \frac{q_1 + d_1}{q_0} = \gamma_{b,0} \frac{e_{b,0} + (q_0 + d_0) b_{b,-1} - q_0 b_{b,0}}{e_{b,1} + d_1(L + b_a^*) + q_1 H} \frac{q_1 + d_1}{q_0} \tag{A.11c}$$

So, we get necessary conditions (14a-14c).

According to Proposition 1, it remains to prove the transversality conditions

$$\lim_{t\to\infty} q_{2t}(b_{a,2t}+b_a^*)\lambda_{a,2t}=0, \quad \lim_{t\to\infty} q_{2t+1}(b_{a,2t+1}+b_a^*)\lambda_{a,2t+1}=0 \tag{A.12a}$$

$$\lim_{t\to\infty} q_{2t}(b_{b,2t}+b_b^*)\lambda_{b,2t}=0, \quad \lim_{t\to\infty} q_{2t+1}(b_{b,2t+1}+b_b^*)\lambda_{b,2t+1}=0. \tag{A.12b}$$

Since $b_{a,2t}=-b_a^*$ and $b_{b,2t-1}=-b_b^*$, they becomes

$$\lim_{t\to\infty} q_{2t+1}(b_{a,2t+1}+b_a^*)\lambda_{a,2t+1}=0 \text{ and } \lim_{t\to\infty} q_{2t}(b_{b,2t}+b_b^*)\lambda_{b,2t}=0 \tag{A.13}$$

$$\text{or equivalently, } \lim_{t\to\infty} q_{2t+1}H\beta_{a,2t+1}\frac{1}{c_{a,2t+1}}=0 \text{ and } \lim_{t\to\infty} q_{2t}H\beta_{b,2t}\frac{1}{c_{b,2t}}=0 \tag{A.14}$$

These conditions are satisfied thank to (15a-15c).

□

B.1 Proofs for Section 4.2.1

Proof of Proposition 5. Part 1. Bubble exists iff $q_t>0\ \forall t$. FOCs (14a-14c) now become

$$\begin{cases}1 &= \gamma_{b,0}\frac{e_{b,0}+q_0 b_{b,-1}-q_0(L+b_a^*)}{e_{b,1}+q_1 H}\frac{q_1}{q_0}\\ 1 &= \frac{\gamma_t(e_t-Hq_t)}{w_{t+1}+Hq_{t+1}}\frac{q_{t+1}}{q_t}\ \forall t\geq 1\end{cases} \Leftrightarrow \begin{cases}\frac{e_{b,1}}{q_1}+H &= \gamma_{b,0}\frac{e_{b,0}}{q_0}-\gamma_{b,0}(L+b_a^*-b_{b,-1})\\ \frac{w_{t+1}}{q_{t+1}}+H &= \gamma_t e_t\frac{1}{q_t}-\gamma_t H\ \forall t\geq 1\end{cases}$$

or equivalently

$$\begin{aligned}\frac{1}{Hq_1} &= \frac{\gamma_{b,0}e_{b,0}}{e_{b,1}}\frac{1}{Hq_0}-\frac{1}{e_{b,1}}\Big(1+\gamma_{b,0}\frac{L+b_a^*-b_{b,-1}}{L+b_a^*+b_b^*}\Big)\\ \frac{1}{Hq_{t+1}} &= \frac{\gamma_t e_t}{w_{t+1}}\frac{1}{Hq_t}-\frac{1+\gamma_t}{w_{t+1}} \quad \forall t\geq 1\end{aligned}$$

From this, we can compute that, for any $t\geq 1$,

$$\begin{aligned}\frac{1}{Hq_t} &= \frac{\gamma_{t-1}e_{t-1}}{w_t}\Big(\frac{\gamma_{t-2}e_{t-2}}{w_{t-1}}\frac{1}{Hq_{t-2}}-\frac{1+\gamma_{t-2}}{w_{t-1}}\Big)-\frac{1+\gamma_{t-1}}{w_t}\\ &= \frac{\gamma_{t-1}e_{t-1}}{w_t}\frac{\gamma_{t-2}e_{t-2}}{w_{t-1}}\frac{1}{Hq_{t-2}}-\frac{\gamma_{t-1}e_{t-1}}{w_t}\frac{1+\gamma_{t-2}}{w_{t-1}}-\frac{1+\gamma_{t-1}}{w_t}\\ &= \cdots = \frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_1 e_1}{w_2}\frac{1}{Hq_1}\\ &\quad -\Big(\frac{1+\gamma_{t-1}}{w_t}+\frac{\gamma_{t-1}e_{t-1}}{w_t}\frac{1+\gamma_{t-2}}{w_{t-1}}+\cdots+\frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_2 e_2}{w_3}\frac{(1+\gamma_1)}{w_2}\Big)\\ &= \frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_0 e_0}{w_1}\frac{1}{Hq_0}\\ &-\Big(\frac{1+\gamma_{t-1}}{w_t}+\frac{\gamma_{t-1}e_{t-1}}{w_t}\frac{1+\gamma_{t-2}}{w_{t-1}}+\cdots+\frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_1 e_1}{w_2}\frac{1+\gamma_{b,0}\frac{L+b_a^*-b_{b,-1}}{L+b_a^*+b_b^*}}{w_1}\Big)\\ &= \frac{1}{Hq_0}\Gamma_t-D_t.\end{aligned}$$

By consequence, we obtain (21).

Condition $q_t > 0$ is equivalent to $D_t/\Gamma_t < 1/(Hq_0)$. We see that

$$\frac{D_t}{\Gamma_t} = \frac{\frac{1+\gamma_{t-1}}{w_t} + \frac{\gamma_{t-1}e_{t-1}}{w_t}\frac{1+\gamma_{t-2}}{w_{t-1}} + \cdots + \frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_1 e_1}{w_2}\frac{1+\gamma_{b,0}\frac{L+b_a^*-b_{b,-1}}{L+b_a^*+b_b^*}}{w_1}}{\frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_0 e_0}{w_1}}$$
$$= \frac{R_1^*\cdots R_{t-1}^*}{e_{t-1}}(1+\frac{1}{\gamma_{t-1}}) + \cdots + \frac{1}{e_0}\Big(\frac{1}{\gamma_0} + \frac{L+b_a^*-b_{b,-1}}{L+b_a^*+b_b^*}\Big)$$

where recall that $\gamma_0 = \gamma_{b,0}$, $w_1 \equiv e_{b,1}$, and $\frac{\gamma_{t-1}e_{t-1}}{w_t} = \frac{1}{R_t^*}$. By combining this with $D_t/\Gamma_t < 1/(Hq_0)$, we get that $\sum_{t=1}^{\infty}\frac{R_1^*\cdots R_t^*}{e_t} < \infty$.

Part 2. We have to check that (1) prices and consumptions are strictly positive, and (2) all conditions in Lemma 2 are satisfied.

Since $e_t - w_t > 0$, condition $q_0 < \frac{\Gamma_t}{H\left(D_t + \frac{2}{e_t - w_t}\right)}$ is equivalent to $\frac{1}{Hq_0}\Gamma_t - D_t > \frac{2}{e_t - w_t}$ which implies that $\Gamma_t > Hq_0 D_t$ and $e_t - w_t > 2Hq_t$. We have $q_t > 0$ because $\Gamma_t > Hq_0 D_t$. Condition $e_t - w_t \geq 2Hq_t \ \forall t$ ensures that consumptions given by (7b-7c) are strictly positive.

Our construction $\frac{1}{Hq_t} = \frac{1}{Hq_0}\Gamma_t - D_t \ \forall t \geq 1$ ensures FOCs (14a-14c) because (14a-14c) become

$$\begin{cases} 1 &= \gamma_{b,0}\frac{e_{b,0}+q_0 b_{b,-1} - q_0(L+b_a^*)}{e_{b,1}+q_1 H}\frac{q_1}{q_0} \\ 1 &= \frac{\gamma_t(e_t - Hq_t)}{w_{t+1}+Hq_{t+1}}\frac{q_{t+1}}{q_t} \ \forall t \geq 1 \end{cases} \Leftrightarrow \begin{cases} \frac{e_{b,1}}{q_1} + H &= \gamma_{b,0}\frac{e_{b,0}}{q_0} - \gamma_{b,0}(L+b_a^*-b_{b,-1}) \\ \frac{w_{t+1}}{q_{t+1}} + H &= \gamma_t e_t \frac{1}{q_t} - \gamma_t H \ \forall t \geq 1 \end{cases}$$

By using condition $e_t - w_t \geq 2Hq_t$, we obtain (15a) and (15b). Moreover, condition $q_0 < \frac{e_{b,0} - e_{a,0}}{L + 2b_a^* + b_{a,-1} - b_{b,-1}}$ implies condition (15c).

Let us look at $\bar{q}$. We can compute that

$$\frac{HD_t}{\Gamma_t} + \frac{2H}{(e_t - w_t)\Gamma_t} = H\Big(\frac{R_1^*\cdots R_{t-1}^*}{e_{t-1}}(\frac{1}{\gamma_{t-1}}+1) + \cdots + \frac{1}{e_0}(\frac{1}{\gamma_0} + \frac{L+b_a^*-b_{b,-1}}{L+b_a^*+b_b^*})\Big)$$
$$+ \frac{2H}{(e_t - w_t)\frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_0 e_0}{w_1}}$$
$$= \frac{Hw_1\cdots w_{t-1}}{e_0\cdots e_{t-1}}\frac{1+\gamma_{t-1}}{\gamma_0\cdots\gamma_{t-1}} + \frac{Hw_1\cdots w_{t-2}}{e_0\cdots e_{t-2}}\frac{1+\gamma_{t-2}}{\gamma_0\cdots\gamma_{t-2}} +$$
$$+ \cdots + \frac{H}{e_0}(\frac{1}{\gamma_0} + \frac{L+b_a^*-b_{b,-1}}{L+b_a^*+b_b^*}) + \frac{2Hw_1\cdots w_t}{e_0\cdots e_{t-1}(e_t - w_t)}\frac{1}{\gamma_0\cdots\gamma_{t-1}}$$

Recall that $H \equiv L + b_a^* + b_b^*$. As a result, $\frac{H}{\Gamma_t}(D_t + \frac{2}{e_t - w_t})$ is increasing in L, b_a^*, b_b^*, w_t and decreasing in e_t, $\gamma_t, b_{b,-1}$. By consequence, $\bar{q}$ is decreasing in L, b_a^*, b_b^*, w_t, $b_{a,-1}$ and increasing in e_t, $\gamma_t, b_{b,-1}$.

□

Proof of Example 1. Assume that there is a bubble, then we have $q_t > 0 \ \forall t$, and according to FOCs (14a-14c) we obtain that $\frac{1}{Hq_{t+1}} = \frac{\beta e}{w}\frac{1}{Hq_t} - \frac{1+\beta}{w} \ \forall t \geq 0$. We have

$$\frac{1}{Hq_t} = \frac{1}{Hq_0}\Gamma_t - D_t = \frac{1}{Hq_0}(\frac{\beta e}{w})^t - \frac{1+\beta}{w}\Big(1 + \frac{\beta e}{w} + \cdots + (\frac{\beta e}{w})^{t-1}\Big) \tag{A.19}$$

1. If $\frac{\beta e}{w} \leq 1$ (i.e., $R^* \geq 1$), then the right hand side of (A.19) is negative if t is high enough while the left hand side is strictly positive, a contradiction. Therefore, there is no bubble in this case.

2. If $\frac{\beta e}{w} > 1$ (i.e., $R^* < 1$). In this case, we have

$$\frac{1}{Hq_t} = \frac{\left(\frac{\beta e}{w}\right)^t}{Hq_0} - \frac{1+\beta}{w}\frac{\left(\frac{\beta e}{w}\right)^t - 1}{\frac{\beta e}{w} - 1} = \frac{\left(\frac{\beta e}{w}\right)^t}{Hq_0}\Big(1 - Hq_0\frac{1+\beta}{\beta e - w}\big(1 - (\frac{w}{\beta e})^t\big)\Big) \tag{A.20}$$

(a) If $q_0 > \frac{1}{H}\frac{\beta e-w}{1+\beta}$, then $1 - Hq_0\frac{1+\beta}{\beta e-w} < 0$. By consequence, the right hand side is strictly negative when t is high enough, a contradiction. In this case, there is no bubble.

(b) If $q_0 = \frac{1}{H}\frac{\beta e-w}{1+\beta}$, then $1 - Hq_0\frac{1+\beta}{\beta e-w} = 0$. By consequence, we have $q_t = q > 0$ $\forall t \geq 1$. To verify that this is an equilibrium price, we must check conditions (15a-15c) which now become $e - w > 2H\frac{1}{H}\frac{\beta e-w}{1+\beta}$. This is satisfied because $\beta \in (0, 1)$.

(c) If $0 < q_0 < \frac{1}{H}\frac{\beta e-w}{1+\beta}$, then $1-Hq_0\frac{1+\beta}{\gamma e-w} > 0$. In this case, we see that q_t determined by (*A*.20) is positive and it is decreasing in t and $\lim_{t\to\infty} q_t = 0$. Conditions (15a-15c) which now become $e - w > 2Hq_t$ $\forall t$. Since $q_0 < \frac{1}{H}\frac{\beta e-w}{1+\beta}$, we have $2Hq_t \leq 2Hq_0 < e - w$ $\forall t$. So, conditions (15a-15c) are satisfied. Therefore, the sequence $(q_t)_t$ determined by $q_0 < \frac{1}{H}\frac{\beta e-w}{1+\beta}$ and (A.19) constitutes a system of equilibrium price with bubble.

□

B.2 Proofs for Section 4.2.2

Proof of Proposition 6. We will prove, by induction, that

$$\frac{\sigma_s a_s d_s}{1 + d_s H_s} < q_{s-1} < \frac{\alpha_s a_s}{H_s} \; \forall t \geq 1. \tag{A.21}$$

This is satisfied for $t = 1$ because we choose $q_0 \in (\frac{\sigma_1 a_1 d_1}{1+d_1 H_1}, \frac{\alpha_1 a_1}{H_1})$. Assume that it holds for $s = t$. Let us prove it for $s = t + 1$. According to (29) and $q_{t-1} < \frac{\alpha_t a_t}{H_t}$, we have

$$q_t = \frac{(1 + d_t H_t)q_{t-1} - a_t d_t}{a_t - H_t q_{t-1}} < \frac{(1 + d_t H_t)\frac{\alpha_t a_t}{H_t} - a_t d_t}{a_t - H_t\frac{\alpha_t a_t}{H_t}} = \frac{\frac{\alpha_t}{H_t} - (1 - \alpha_t)d_t}{1 - \alpha_t} \tag{A.22}$$

$$< \frac{\alpha_t}{(1 - \alpha_t)H_t} < \frac{\alpha_{t+1}a_{t+1}}{H_{t+1}} \tag{A.23}$$

where the last inequality is from (31b).

The system (29) and condition $\frac{\sigma_t a_t d_t}{1+d_t H_t} < q_{t-1}$ imply that

$$q_t = \frac{(1 + d_t H_t)q_{t-1} - a_t d_t}{a_t - H_t q_{t-1}} > \frac{(1 + d_t H_t)\frac{\sigma_t a_t d_t}{1+d_t H_t} - a_t d_t}{a_t - H_t\frac{\sigma_t a_t d_t}{1+d_t H_t}} = \frac{(\sigma_t - 1)d_t}{1 - \frac{\sigma_t d_t H_t}{1+d_t H_t}}. \tag{A.24}$$

According to (31d), we have $1 - \frac{\sigma_t d_t H_t}{1+d_t H_t} > 0$ which in turn implies that

$$q_t > (\sigma_t - 1)d_t > \sigma_{t+1}a_{t+1}d_{t+1} \tag{A.25}$$

where the last inequality is from (31c). Finally, we get that $q_t > \frac{\sigma_{t+1}a_{t+1}d_{t+1}}{1+H_{t+1}d_{t+1}}$. Therefore, we have just proved (32).

To prove that (q_t) is a price sequence of an equilibrium, we verify that all conditions in Lemma 2 are satisfied. First, since $0 < \alpha_t < 1 < \sigma_t$, condition (32) ensures that $q_t > 0$ $\forall t \geq 0$.

Second, observe that (32) implies that $q_t < \frac{a_{t+1}}{H_{t+1}} < \bar{q}_t$. This shows that conditions (15a-15c) are satisfied. It also ensures that consumptions are strictly positive.

Last, FOCs (14a-14c) are ensured by the system (29).

□

Proof of Proposition 7. Part 1. According to Remark 1, there is no bubble if $a < 1$. Now, consider the case $a > 1$. Suppose that there is a bubble. We must have $\sum_t d_t < \infty$. There are only two cases.

Case 1. If there is t_0 such that $q_{t_0} \leq \frac{a-1}{h}$, then we have

$$q_{t_0+1} - q_{t_0} = \frac{q_{t_0}(hq_{t_0} - (a-1))}{a - hq_{t_0}} - d_t < 0 \tag{A.26}$$

By induction, we have that $q_t < q_{t-1} < (a-1)/h \ \forall t \geq t_0$. By consequence, the sequence q_t decreasingly converges to a value $q \geq 0$. Observe that

$$(q_t + d_t)(a - hq_{t-1}) = q_{t-1}, \text{ and hence } q(a - hq) = q, \tag{A.27}$$

So, either $q = 0$ or $q = (a-1)/h$. Since $q_t < q_{t-1} < (a-1)/h \ \forall t \geq t_0$, the value q must be strictly lower than $(a-1)/h$. As a result, q_t converges to zero.

Case 2. $q_t > \frac{a-1}{h} \ \forall t \geq 0$. Observe that

$$(q_t + d_0 + \cdots + d_t) - (q_{t-1} + d_0 + \cdots + d_{t-1}) = q_t + d_t - q_{t-1} = \frac{q_{t-1}(hq_{t-1} - (a-1))}{a - hq_{t-1}} > 0.$$

So the sequence $(q_t + d_0 + \cdots + d_t)$ is strictly increasing. Since $\sum_t d_t < \infty$ and $q_t < \frac{a}{h}$, this sequence is bounded, and hence converges. As a result, the sequence (q_t) converges. So, it must converge to $\frac{a-1}{h}$.

Part 2. We now prove that there is almost one equilibrium satisfying $q_t > \frac{a-1}{h} \ \forall t$, in which asset holdings are given by (6) and agents' consumptions are given by (7a-7c). Let (q_t) and (q'_t) be two systems of equilibrium prices. We must have $q_t < a/h$ and $q'_t < a/h$.

Define $x_t = q_t - \frac{a-1}{h}, x'_t = q'_t - \frac{a-1}{h}$, then we have $0 < x_t, x'_t < 1/h$ and

$$q_t + d_t = \frac{q_{t-1}}{a - hq_{t-1}} \Leftrightarrow x_t + \frac{a-1}{h} + d_t = \frac{x_{t-1} + \frac{a-1}{h}}{a - h(x_{t-1} + \frac{a-1}{h})} \Leftrightarrow x_t + d_t = \frac{ax_{t-1}}{1 - hx_{t-1}}$$

Similarly, we have $x'_t + d_t = \frac{ax'_{t-1}}{1-hx'_{t-1}}$. Therefore, we get that

$$x_t - x'_t = \frac{a(x_{t-1} - x'_{t-1})}{(1 - hx_{t-1})(1 - hx'_{t-1})} \ \forall t \geq 1. \tag{A.28}$$

We will prove that $x_0 = x'_0$ (which implies that $q_t = q'_t \ \forall t$). Without loss of generality, suppose that $x_0 > x'_0$. According to (A.28), we have $x_t - x'_t > a(x_{t-1} - x'_{t-1}) \ \forall t \geq 1$. Therefore, we have

$$x_t - x'_t > a^t(x_0 - x'_0) \ \forall t \geq 1. \tag{A.29}$$

Since $a > 1$, $a^t(x_0 - x'_0)$ converges to infinity. So, $x_t - x'_t$ also converges to infinity. However, this cannot happen because both x_t and x'_t belong the interval $(0, 1/h)$.

□

Proof of Example 2. First, we have

$$a_{2t} = a_{2t+1} = \frac{\beta e}{w}, \quad H_{2t} = H_{2t+1} = h \equiv \frac{H(\beta + 1)}{w}$$
$$2H\bar{q}_{2t-1} \equiv e - w - Hd_{2t-1}, \quad 2H\bar{q}_{2t} \equiv e - w - Hd_{2t}$$

So, condition $\frac{a_{t+1}}{H_{t+1}} < \bar{q}_t \ \forall t$ becomes

$$\frac{\beta(e_{b,0} + d_0 b_{b,-1})}{\beta(L + b_a^* - b_{b,-1}) + H} < \frac{e_{b,0} - e_{a,0} - d_0(b_{a,-1} - b_{b,-1})}{L + 2b_a^* + b_{a,-1} - b_{b,-1}} \tag{A.30}$$

$$\frac{2\beta e}{1+\beta} < e - w - Hd_{2t-1}, \quad \frac{2\beta e}{1+\beta} < e - w - Hd_{2t}. \tag{A.31}$$

These and condition $\bar{q}_t > 0$ are satisfied because we assume that $d_t < \frac{\frac{1-\beta}{1+\beta}e - w}{H}$.

Second, observe that condition $\frac{\sigma a d_1}{1 + d_1 \frac{H(\beta+1)}{w}} < \frac{\beta e - w}{H(\beta+1)}$ ensures that $\frac{\sigma a d_1}{1+d_1 h} < \frac{a-1}{h}$. So, the interval $(\frac{\sigma a d_1}{1+d_1 h}, \frac{a-1}{h}]$ is well defined.

We next prove that $q_s \in (\frac{\sigma a d_s}{1+d_s h}, \frac{a-1}{h}] \ \forall s \geq 0$. This holds for $s = 0$ because $q_0 \in (\frac{\sigma a d_1}{1+d_1 h}, \frac{a-1}{h}]$. Assume that it holds for $t-1$, we will prove this for t. Indeed, condition $d_t < \frac{w}{(\sigma-1)(\beta+1)H}$ is equivalent to $1 - \frac{\sigma d_t h}{1+d_t h} > 0$. By combining with $\frac{\sigma-1}{\sigma}\frac{d_t}{d_{t+1}} > \frac{\beta e}{w}$, we ge that

$$q_t = \frac{(1+d_t h)q_{t-1} - a d_t}{a - h q_{t-1}} > \frac{(1+d_t h)\frac{\sigma a d_t}{1+d_t h} - a d_t}{a_t - h\frac{\sigma a d_t}{1+d_t h}} = \frac{(\sigma-1)d_t}{1 - \frac{\sigma d_t h}{1+d_t h}} \tag{A.32}$$

$$> (\sigma - 1)d_t > \sigma a_{t+1} d_{t+1} > \frac{\sigma a_{t+1} d_{t+1}}{1 + H_{t+1} d_{t+1}} \tag{A.33}$$

We also have

$$q_t - q_{t-1} = \frac{q_{t-1}(h q_{t-1} - (a-1))}{a - h q_{t-1}} - d_t < 0 \tag{A.34}$$

because $hq_{t-1} < a - 1$. So, we have $q_t < q_{t-1} < (a-1)/h \ \forall t$. This in turn implies that $q_t < (a-1)/h \ \forall t$. By consequence, the sequence q_t decreasingly converges, and hence it cannot converge to $(a-1)/h$. As a result, it converges to zero.

It remains to prove that (q_t) is a price sequence of an equilibrium. To do so, we verify that all conditions in Lemma 2 are satisfied. First, it is easy to see that $q_t > 0 \ \forall t \geq 0$. Second, according to $\frac{a_{t+1}}{H_{t+1}} < \bar{q}_t$, we have $q_t < \frac{a_{t+1}}{H_{t+1}} < \bar{q}_t$. This shows that conditions (15a-15c) are satisfied. It also ensures that consumptions are strictly positive. Last, FOCs (14a-14c) are ensured by the system (29).

□

Proof of Example 3. We see that $1 - (a-1)x > 0$ and $hxd_0 < 1$. So, we can check that $0 < hxd_t < 1$ and $xd_t + d_t = \frac{axd_{t-1}}{1-hxd_{t-1}}$. According to the proof of Proposition 7, the sequence (q_t) defined by $q_t = \frac{a-1}{h} + xd_t \ \forall t$ satisfies: $q_t \in (\frac{a-1}{h}, \frac{a}{h})$ and $q_t + d_t = \frac{q_{t-1}}{a - hq_{t-1}} \ \forall t$. In order to prove that (q_t) is a system of prices of an equilibrium at which asset holdings are given by (6) and agents' consumptions are given by (7a-7c), we verify all conditions in Lemma 2.

As in the proof of Example 2, condition $d_0 < \frac{\frac{1-\beta}{1+\beta}e - w}{H}$, ensures that $a/h < \bar{q}_t \ \forall t$. Thus, $q_t < a/h < \bar{q}_t \ \forall t$. This shows that conditions (15a-15c) are satisfied. It also ensures that consumptions are strictly positive. Last, FOCs (14a-14c) are ensured by the system $q_t + d_t = \frac{q_{t-1}}{a - hq_{t-1}} \ \forall t$.

□

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