Leontief Paradox Explored A New Trade Pattern When Countries Have Different Technologies

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December 2015

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Abstract – The trade pattern can be counted as the number one topic for international economics. Based either on the concept of Trefler's effective factor endowments (Trefler, 1993) or on the concept of Fisher and Marshall's virtual factor endowments (Fisher and Marshall, 2008), the trade pattern described by Leontief tests is right. This paper demonstrates that there are three trade patterns: the Heckscher-Ohlin trade, the Leontief trade, and the factor conversion trade (or one-side Leontief trade) when countries have different technologies. Methodologically, the popular sign predictions in empirical studies include both the Leontief trade and the Heckscher-Ohlin trade. Therefore, the sign predictions' results cannot be used to reject the Leontief paradox. The factor conversion trade occurs when a model is with the existence of factor intensity reversals (FIRs). Many studies[2] have demonstrated evidence of the factor intensity reversals (FIRs). They mean the Leontief trade. Another new finding of this study is that the Leontief trade can occur when FIRs do not present.

Keywords

Heckscher-Ohlin-Ricardo model, Leontief Trade, and Factor Conversion Trade, factor content of trade, Leontief Paradox.

JEL code: F10

1. Introduction

The trade pattern can be counted as the number one question in international economics. Ricardo found the first trade pattern of comparative advantage sourced from productivities different across countries. Heckscher and Ohlin illustrated another trade pattern of comparative advantage from different factor endowments under the same technologies across countries. This study demonstrated that the Leontief trade described in the Leontief paradox is a new trade pattern when countries have different productivities. It is also a trade pattern of comparative advantage. The evidence of factor intensity reversals from the empirical studies is direct support for the Leontief trade of this paper.

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[2] Kurokawa (2011), Takahashi (2004), Simpson (2016), and Kozo and Yoshinori (2017) have demonstrated the evidences of the factor intensity reversals (FIRs) in their empirical studies. This means that they presented Leontief trade.

The Leontief test (Leontief, 1953) showed that the US as a capital abundant country exported its labor-intensive commodities. It counters the common sense of international economics. Baldwin (1971) tested 1962 US trade data, the conclusion was the same as Leontief made. Leamer (1980) reformatted the Leontief data and showed that the capital/labor ratio embodied in production exceeded the capital/labor ratio in consumption. Other tests in the last century by using various countries' data showed that the results are half-and-half being consistent with Heckscher-Ohlin theories.

The Leontief Paradox simulated the HOV studies to explore new approaches to explain trade patterns in international economics. Most studies in this field are to incorporate different technologies with the Heckscher-Ohlin framework. Leontief (1953) himself proposed the concept of productivity-equivalent factor (workers) to try to explain his test results. Trefler (1993) extended the concept and implemented it with factor-argument parameters in his model. The implement is also very useful for theoretical analyses. Fisher and Marshall (2008) provided another insight approach to involve different technologies by the virtual endowments and the conversion matrix. Fisher (2011) proposed important terms of the goods price diversification cone and the intersection of goods price diversification cones.

Kwok and Yu (2005) re-investigated the 52 countries data by the approach of differentiated factor intensity techniques and concluded that Leontief paradox "is found to be either disappeared or eased".

Jones (1956) and Robinson (1956) argued that FIR could have been responsible for the Leontief Paradox in the US data. Wong (1995, pp128) thought that one possible reason for the Leontief paradox is the presence of the FIR. When FIRs occur, Leamer(1995) referred it to "factor price insensitivities". Jones (1956) examined the possible trade results for the existence of FIRs, he wrote "If the relatively labor abundant country exports its labor-intensive commodity, it must do so in exchange for the commodity that, in the relatively capital-abundant foreign country, is produced by labor-intensive techniques. Thus if one country satisfies the theorem, the other country cannot". This is the first study describing the trade features of the FIRs. The consequence of trade when countries are with the presence of FIRs still is mysterious, although it is curiosum in the studies of international economics.

Deardorff (1985) presented the diversification cones of the FIRs. He studied the double factor intensity reversals. He suggested a way to turn any model with FIRs into one without it, and vice versa, by simply redefining goods.

Feenstra (2004, p11) described the reality of the FIRs, "While FIRs might seem like a theoretical curiosum, they are actually quite realistic". He thought that the FIR is a typical case of factor technologies different across countries. He implied that the FIR is by the cone reversals. Minhas (1962) first reported finding the evidence of FIRs. He also first provided a production function to form a case of FIRs. He investigated industry data for 19 countries. He found FIRs in 5 countries. Leontief (1964) revisited the Minhas test and showed fewer cases of FIRs.

Kurokawa (2011) showed "clear-cut evidence for the existence of the skill intensity reversal" in his empirical study of the USA-Mexico economy. Sampson (2016) interpreted his assignment reversals of skill workforce between North and South by factor intensity reversal. Takahashi (2004) studied the postwar Japan economy. He interpreted Japan's economic growth by capital-intensity reversal. Reshef (2007) studied the model with factor intensity reversals in skill, which can explain the North-South skill premia increase well. Kozo and Yoshinori (2017) found the existence of factor intensity reversals in their study also. They wrote, "Using newly developed region-level data, however, we argue that the abandonment of factor intensity reversals in the empirical analysis has been premature. Specifically, we find that the degree of the factor intensity reversals is higher than that found in previous studies on average". The FIRs are not just textbook interesting. The theory studies about FIRs are much behind international trade observations. This study displayed that the FIRs always associated with the factor conversion trade.

Guo (2015) proposed a solution to the general trade equilibrium for the $2 \times 2 \times 2$ Heckscher-Ohlin model by Dixit and Norman's Integrated World Equilibrium (see Dixit and Norman 1980). He demonstrated that equalized factor prices and world prices for the commodity are the function of the world factor endowments (the rental-wage ratio equals to the world labor-capital ratio as $r/w = L^W/K^W$). He illustrated that the equalized factor price makes sure that countries gain from trade. Guo (2019) extended the price-trade equilibrium to the Trefler model. It shows that the domestic rental-wage ratio equals the world's effective labor-capital ratio measured by domestic productivities. He illustrated that localized factor prices make sure that countries gain from trade. They are helpful to understand the Leontief trade and the factor conversion trade of this study.

This paper focuses on the discussion of Leontief's test conclusion that an actual capital abundant country export net its labor-intensive commodity[3]. The paper showed that theoretically, there are three trade patterns: the Heckscher-Ohlin trade, the Leontief trade, and the factor conversion trade when countries have different productivities. The Heckscher-Ohlin trade is widely known, which is that each country exports the commodity that uses its (actual) abundant factor intensively. The Leontief trade, we described here, is that each country exports the commodity that uses its actual scarce factor intensively. Most understanding of the cause of the Leontief paradox is by the presence of FIRs in the model structure. This study demonstrates that Leontief trade can occur without the presence of FIRs. The factor conversion trade is that one country does Heckscher-Ohlin trade, another does the Leontief trade, in which both countries export the same factor services and import the same factor services. However, both the Leontief trade and the factor conversion trade are still under the generalized trade pattern that a country exports the commodity that uses its effective abundant factor intensively. When the FIRs present in model structure, international trade converts the worldwide effective abundant factor into the worldwide effective scarce factor.

The paper provides three ways to display three trade patterns from different views. The first is by the concept of the effective factor endowments in Trefler (1993) model. It also shows that trade patterns are trade consequences. The second way is by the analyses of the sighs of trade flows and the signs of factor content of trades by the virtual endowments. The last one is by using the intersection of output diversification price cones (see Fisher, 2011) to present trade directions based on the generalized HOV

[3] We facus on the discussion of the general trade pattern that Leontief described in the conlusion of his tests. We will not examine whether USA enconomy in 1942 really exports net labor services.

analyses only[4]. This method provides numerical results of three trade patterns very simply, without using any other logic or derived principles in international economics.

Most scholars thought that the issue of Leontief paradox is over and the paradox is explained well now. They believed that the conclusion of the Leontief test is not right even considering technology differences across countries. One reason for the formation of the view is that Leontief trade has no theoretical background. Another reason is that some empirical studies declared that the conclusion of the Leontief test is not right. This study shows that the sign prediction criteria in empirical studies cannot be used to deny Leontief trade. They based on the effective factor endowments or on the virtual factor endowments. This implies that the sign predictions can explain all trade patterns well, including the Leontief trade and conversion trade. They have no function to identify what trade patterns are.

The paper is organized into seven sections. Section 2 reviews some related terms: the model structure (the existence of FIRs and the nonexistence of FIRs), the goods price diversification cone, and the intersection of goods price diversification cones. Section 3 discusses the generalized trade pattern when countries have different technologies. Section 4 displays the trade patterns by Trefler model. Section 5 illustrates the trade patterns by the virtual endowments. Section 7 provides a numerical way to demonstrate the trade patterns by the intersection of goods price diversification cones. Section 7 reviews HOV empirical studies and illustrates that the prediction signs, commonly used, favor all of the trade patterns. The final section is the concluding remarks.

2. Preliminaries: Heckscher-Ohlin-Ricardo Model and Intersection of Output Price Diversification Cones

2.1 Heckscher-Ohlin-Ricardo Model

We refer to the Heckscher-Ohlin framework with general technology differences across countries as the Heckscher-Ohlin-Ricardo model. Davis (1995) used this name first. Morrow (2010) named his study as Ricardian-Heckscher-Ohlin comparative advantage. The two-cone analyses in Deardorff (2002) paved solid foundations for the source of technology differences in the Heckscher-Ohlin framework. Fisher and Marshall (2008) developed the virtual endowments to incorporate different technologies with the Heckscher-Ohlin framework. Trefler (1993) model with the diagonal parameter matrix is a special case of the Heckscher-Ohlin-Ricardo model. It serves as a transaction from the Hicks-Neutral productivity difference across countries to full technology differences across countries. We will use both of them in this study to illustrate the trade patterns.

The Heckscher-Ohlin-Ricardo Model inherits all assumptions of the Heckscher-Ohlin model, except the assumptions of the same technologies and no factor intensity reversals. We take the basic assumptions for the Heckscher-Ohlin-Ricardo model as (1) different technologies across countries, (2) identical

[4] The general trade equilibrium from Trefler can display the three trade patterns directly. However, it may be somewhat sudden for some readers to confirm the derivations immediately from an unpublished manuscript. Other ways are alternative.

homothetic taste, (3) perfect competition in the commodities and factors markets, (4) no cost for international exchanges of commodities, (5) factors are completely immobile across countries but that can move costlessly between sectors within a country, (6) constant return of scale, (7) full employment of factor resources.

We denote the $2 \times 2 \times 2$ Heckscher-Ohlin-Ricardo model as

$$A^h X^h = V^h \qquad (h = H, F) \qquad \text{(2-1)}$$

$$(A^h)' W^{h*} = P^* \qquad (h = H, F) \qquad \text{(2-2)}$$

where A^h is the 2×2 matrix of factor input requirements with elements a_{ki} (r,w), $i = 1,2; k = K, L; and\ h = H, F$; V^h is the 2×1 vector of factor endowments with elements K as capital and L as labor; X^h is the 2×1 vector of output; W^{h*} is the 2×1 vector of factor prices with elements r as rent and w as wage; P^* is the 2×1 vector of commodity prices with elements p_1^* and p_2^*, when trade reached market equilibrium.

2.2 Model Structures

There are two model structures for the $2 \times 2 \times 2$ Heckscher-Ohlin-Ricardo model. One is the nonexistence of factor intensity reversals. Another is the existence of factor intensity reversals [5].

In the model of the nonexistence of FIRS, if the home country is capital (first factor) intensive in commodity 1, the foreign country is capital intensive in commodity 1 also. It can be characterized, for the $2 \times 2 \times 2$ model, by

$$|A^H||A^F| > 0 \qquad \text{(2-3)}$$

where $|A^h|$ is the determinant of technology matrix A^h of country h, $h = H, F$. $|A^h| > 0$ means that it is capital-intensive in sector 1 for country h. $|A^h| < 0$ means labor-intensive in sector 1 for country h.

Another description of the model pattern is by the cost requirement ratio ranks. The following two ranks are typical for the nonexistence of FIRs (2-3),

$$\frac{a_{K1}^H}{a_{K2}^H} > \frac{a_{K1}^F}{a_{K2}^F} > \frac{a_{L1}^H}{a_{L2}^H} > \frac{a_{L1}^F}{a_{L2}^F} \qquad \text{(2-4)}$$

or

$$\frac{a_{K1}^H}{a_{K2}^H} > \frac{a_{K1}^F}{a_{K2}^F} > \frac{a_{L1}^F}{a_{L2}^F} > \frac{a_{L1}^H}{a_{L2}^H} \qquad \text{(2-5)}$$

In the model of the presence of FIRs, if the home country is capital intensive in sector 1, the foreign country is capital intensive in sector 2. It can be characterized by

$$|A^H||A^F| < 0 \qquad \text{(2-6)}$$

The following two cost requirement ratio ranks are typical for (2-6),

$$\frac{a_{K1}^H}{a_{K2}^H} > \frac{a_{L1}^F}{a_{L2}^F} > \frac{a_{L1}^H}{a_{L2}^H} > \frac{a_{K1}^F}{a_{K2}^F} \qquad \text{(2-7)}$$

or

$$\frac{a_{K1}^H}{a_{K2}^H} > \frac{a_{L1}^F}{a_{L2}^F} > \frac{a_{L1}^H}{a_{L2}^H} > \frac{a_{K1}^F}{a_{K2}^F} \qquad \text{(2-8)}$$

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[5] Deardorff (1985) used this classification.

2.3 The output price diversification cone and Intersection of two output price diversification cones

Fisher (2011) proposed the terms of "the goods price diversification cone" and "the intersection of two price diversification cones". The goods price diversification cone is the counterpart of the diversification cone of factor endowments. It is a very important concept for price-trade equilibriums. The intersection of two price diversification cones illustrates what makes sure that the rewards of two sets of localized factor prices are positive when countries have different technologies.

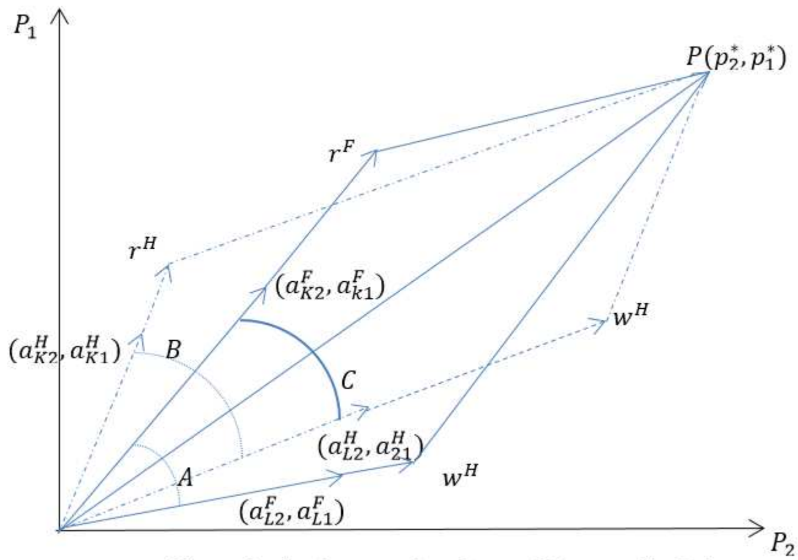

Figure 1 the Intersection Cone of Commodity Price

We draw the cost requirement vectors (a_{K2}^H, a_{K1}^H), (a_{K2}^H, a_{K1}^H), (a_{K2}^F, a_{K1}^F), and (a_{L2}^F, a_{L1}^F) in the ranks (2-5) in Figure 1. After multiplying each of them by their payments of respective factors, these vectors create two cones of commodity prices, labeled as cone A and cone B. There is an overlapped part of two cones of commodity prices. The overlap of two cones is the intersection cone of two output price cones, labeled as cone C, which is the space spanned by vectors (a_{K2}^F, a_{K1}^F) and (a_{L2}^H, a_{L1}^H). It is clear from Figure 1 that the rewards for four factors of the two countries will be positive, if and only if the world common commodity price vector (p_1^* , p_2^*) lies in the intersection cone of two output price diversification cones.

The intersection of two commodity price cones for the case of inequity (2-4) can be expressed in algebra as

$$\frac{a_{K1}^F}{a_{K2}^F} > \frac{p_1^*}{p_2^*} > \frac{a_{L1}^H}{a_{L2}^H} \qquad (2\text{-}9)$$

It identifies a full set of commodity prices for all possible equilibriums. It also shows how the factor price localized when world commodity prices formed.

3. Generalized Trade Pattern

When countries have different productivities, the effective factor abundance determines the trade direction of factor services. When countries have general technology differences, the virtual factor abundance determines the trade direction of factor services. Fisher and Marshall (2008) used their prediction sign as

$$(V_k^{vi} - s^i \sum_j V_k^{vj}) F_k^{vi} > 0 \tag{3-1}$$

where V_k^{vj} is the element of vector V^{vj} which is defined as $V^{vj} = A^0\, y^j$. V_k^{vj} is the factor service needed to product country j's commodity y^j using a reference to technology matrix in country i as A^0 . F_k^{vi} is the factor service exported by country i. For the $2 \times 2 \times 2$ economy, it can be written by the notation of this paper as

$$(K^h - s^h(K^h + K^{fbh}))F_K^h > 0 \qquad (h = H, F) \tag{3-2}$$

$$(L^h - s^h(L^h + L^{fbh}))F_L^h > 0 \qquad (h = H, F) \tag{3-3}$$

where K^{fbh} and L^{fbh} are the virtual factor endowments of country *f* measured by referring to the technology of county *h*, it is from

$$\begin{bmatrix} K^{fbh} \\ L^{fbh} \end{bmatrix} = (A^h A^{f^{-1}}) \begin{bmatrix} K^f \\ L^f \end{bmatrix} \tag{3-4}$$

The world virtual endowments measured by the technology of country h are

$$V^{Wbh} = \begin{bmatrix} K^{Wbh} \\ L^{Wbh} \end{bmatrix} = \begin{bmatrix} K^h \\ L^h \end{bmatrix} + \begin{bmatrix} K^{fbh} \\ L^{fbh} \end{bmatrix} \tag{3-5}$$

Bernhofen (2011, p104) mentioned, "A country's factor content is defined using the country's domestic technology matrix". This is a very important point. We need to use domestic (or local) technology matrix to express a country's factor endowments and the country's virtual factor abundances. Our three characters' superscripts look rare. However, it is easy to identify effective (or equivalent) factor endowments by referring to each country's productivities.

Equations (3-1) through (3-3) implies the following the generalized HOV theorem directly as,

The general trade rule of factor contents – Each country exports the service of its effective (virtual) abundant factor and imports the services of its effective (virtual) scarce factor.

Trefler (1993) illustrated that both the factor price equalization hypothesis and the HOV theorem hold for his model. Definitely, the Heckscher theorem holds for the Trefler factor equivalent model also. Fisher (2011) also mentioned that under the virtual endowments assumptions, the classical Heckscher-Ohlin theory holds true when technologies and factor prices are identical to those of the reference country. Both Trefler(1993) and Fisher (2011) implied that a generalized Heckscher-Ohlin theorem held. It can be stated as:

The general trade rule of commodities - Each country exports the commodity that uses its effective (virtual) abundant factor intensively and imports the commodity that uses its effective (virtual) scarce factor intensively.

Both the generalized Heckscher-Ohlin theorem and them generalized HOV theorem explain the three trade patterns of this paper well.

4. **Analyses of Trade Patterns By Trefler Model**

4.1 Leontief Trade

The implementation of the equivalent-productivity unit in the Trefler (1993) model is by

$$A^H = \Pi A^F = \begin{bmatrix} \pi_K & 0 \\ 0 & \pi_L \end{bmatrix} A^F \tag{4-1}$$

where Π is a 2×2 diagonal matrix, its element π_k is factor productivity-argument parameter, $k = K, L$. This composes a typical Trefler model as

$$A^H X^H = V^H, \qquad (A^H)' W^H = P^H \tag{4-2}$$

$$\Pi^{-1} A^H X^F = V^F, \qquad (\Pi^{-1} A^H)' W^F = P^F \tag{4-3}$$

The world effective factor endowments by referring to the home productivities are

$$K^{WbH} = K^H + \pi_K K^F \quad , \quad L^{WbH} = L^H + \pi_L L^H \tag{4-4}$$

The world effective factor endowments by referring to foreign productivities are

$$K^{WbF} = K^F + K^H/\pi_K , \qquad L^{WbF} = L^F + L^H/\pi_L \tag{4-5}$$

A unique feature for the Trefler model is that it is with a single commodity price diversification cone, although it is with two cones of factor diversifications. Its cost ratio ranks, which show the rays of commodity price cones in algebra, are with the following relationship

$$\frac{a_{K1}^H}{a_{K2}^H} = \frac{a_{K1}^F}{a_{K2}^F} = \frac{a_{K1}^H \pi_k}{a_{K2}^H \pi_k} > \frac{P_1^*}{P_2^*} > \frac{a_{L1}^H}{a_{L2}^H} = \frac{a_{L1}^F}{a_{L2}^F} = \frac{a_{L1}^H \pi_L}{a_{L2}^H \pi_L} \tag{4-6}$$

Feenstra and Taylor (2012, pp103) defined the effective factor endowment as the actual amount of a factor found in a country times its productivity (Effective factor endowment = Actual factor endowment • Factor productivity). Supposing home country to be the reference of productivity and assuming the home country's productivity to be 1, the equivalent factor endowment is just the effective factor endowment.

When actual factor abundance is consistent with effective factor abundance, it leads to the Heckscher-Ohlin trade. Otherwise, it leads to the Leontief trade. We assume that the home country is actual capital abundance as[6]

$$\frac{K^H}{L^H} > \frac{K^F}{L^F} \tag{4-7}$$

We also assume that the home country is effective labor abundant as[7]

$$\frac{K^H}{L^H} < \frac{K^{FbH}}{L^{FbH}} = \frac{\pi_K K^F}{\pi_L L^F} \tag{4-8}$$

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[6] Actual capital abundance of home country can be expressed by $\frac{K^H}{L^H} > \frac{K^F}{L^F}$ or $\frac{K^H}{L^H} > \frac{K^W}{L^W}$. Actually, there are $\frac{K^H}{L^H} > \frac{K^W}{L^W} > \frac{K^F}{L^F}$.

[7] The equivalent capital abundance of home country can be expressed by $\frac{K^H}{L^H} > \frac{K^{FbH}}{L^{FbH}}$ or $\frac{K^H}{L^H} > \frac{K^{WbH}}{L^{WbH}}$. Actually, there are $\frac{K^H}{L^H} > \frac{K^{WbH}}{L^{WbH}} > \frac{K^{FbH}}{L^{FbH}}$.

Inequality (4-8) is measured by home productivities.

Rewrite it as

$$\frac{K^H L^F}{L^H K^F} > \frac{\pi_K}{\pi_L} \tag{4-9}$$

Rewriting (4-9) again, we have,

$$\frac{K^F}{L^F} > \frac{K^H/\pi_K}{L^H/\pi_L} = \frac{K^{HbF}}{L^{HbF}} \tag{4-10}$$

Inequality (4-10) is measured by foreign productivities. Inequality (4-9) means that the home country is effective labor abundant; inequality (4-10) means that the foreign country is effective capital abundant. Both countries' actual factor abundances are different from their effective factor abundances. Therefore, the home country will export labor service and will import capital service and the foreign country will export capital service and will import labor service. The home country as actual capital abundance will export the commodity that uses labor intensively. In addition, the foreign country as actual labor abundance will export the commodity that uses capital intensively. Those are just the Leontief trade. Both countries do Leontief trade. We provide proof of Leontief trade for the Trefler model by price-trade equilibrium in Appendix A.

Inequation (4-9) is the condition when the Leontief trade occurs.

This is a new understanding of the Leontief trade, which occurs even with the nonexistence of FIRs. The existing studies only described that the FIRs are a possible model structure for the Leontief trade. The scope of the existence of Leontief trade is much larger than before.

4.2 The Factor Conversion Trade

4.2.1 The model with the presence of FIRs

Trefler's (1993) model is also useful to implement a system with the presence of FIRs. We now specify a "Cross-Factor Hicks-Neutral FIRs" model (CF-HN-FIRs model) by assuming that technological matrices of two countries be as

$$A^H = \psi A^F = \begin{bmatrix} 0 & \theta_K \\ \theta_L & 0 \end{bmatrix} A^F \tag{4-11}$$

where ψ is a 2×2 anti-diagonal matrix, its element θ_k is the productivity-cross-factor-argument parameter, $k = K, L$. This composes a model presenting FIRs as

$$A^H X^H = V^H, \qquad (A^H)' W^H = P^H \tag{4-12}$$

$$\psi^{-1} A^H X^F = V^F, \qquad (\psi^{-1} A^H)' W^F = P^F \tag{4-13}$$

The world effective factor endowments by referring to home productivity are

$$K^{WbH} = K^H + \theta_L L^F\ , \qquad L^{WbH} = L^H + \theta_K K^F \tag{4-14}$$

The world effective factor endowments by referring to foreign productivity are

$$K^{WbF} = K^F + L^H/\theta_L\ , \qquad L^{WbF} = L^F + K^H/\theta_K \tag{4-15}$$

When $\theta_K = 1$ and $\theta_L = 1$, we have

$$A^F = \begin{bmatrix} a_{L1}^H & a_{L2}^H \\ a_{K1}^H & a_{K2}^H \end{bmatrix} \qquad (4\text{-}16)$$

The foreign country's technology requirement coefficients for labor are as same as the home country's coefficients for capital. The requirements for factors in the foreign country are switched. It did turn the sector technologies across countries as the way Deardroff (1985) mentioned.

The cost requirement ratio ranks, which indicate the rays of the cones of commodity prices by (2-8), are

$$\frac{a_{K1}^H}{a_{K2}^H} = \frac{a_{L1}^F}{a_{L2}^F} = \frac{a_{L1}^H/\theta_L}{a_{L2}^H/\theta_L} \quad > \quad \frac{a_{L1}^H}{a_{L2}^H} = \frac{a_{K1}^F}{a_{K2}^F} = \frac{a_{K1}^H/\theta_K}{a_{K2}^H/\theta_K} \qquad (4\text{-}17)$$

This is also the case of the single cone of commodity prices.

Expression (4-17) implies that $|A^H||A^F| < 0$. Therefore, the model by (4-11) through (4-13) is with the existence of the FIRs. It results in the factor conversion trade.

The Cross-Factor Hicks-Neutral FIRs model essentially is a Trefler (1993) model. By assuming $V^{FbH} = \psi V^F$ and $W^{FbH} = \psi^{-1} W^F$, we have the mapped equivalent-productivity version of the CF-NH-FIR model as

$$A^H X^H = V^H, \qquad (A^H)' W^H = P^H \qquad (4\text{-}18)$$

$$A^H X^F = V^{FbH}, \qquad (A^H)' W^{FbH} = P^F \qquad (4\text{-}19)$$

For this version of the model, both factor price equalization hypothesis and the HOV theorem hold. Statistically, for empirical studies, the CS-HN-FIRs model is not crazed any more than the Hicks-Neutral Trefler model.

4.2.2 Both countries are effective factor abundance at the same factor

One property of the CS-HN-FIRs model is that both countries are effective factor abundant at the same factor. Assume that the home country is effective capital abundance, which is measured by home productivities,

$$\frac{K^H}{L^H} > \frac{K^{FbH}}{L^{FbH}} = \frac{\theta_L L^F}{\theta_K K^F} \qquad (4\text{-}20)$$

It can be rewritten as

$$\frac{K^F}{L^F} > \frac{L^H/\theta_K}{K^H/\theta_L} = \frac{K^{FbH}}{L^{FbH}} \qquad (4\text{-}21)$$

It means that the foreign country is effective-capital abundance also. It is measured by foreign productivities. The factor abundance at the same factor in two countries causes both countries to export the service of the same factor and to imports the service of the same factor as

$$\text{Sign}(F_K^H) = \text{Sign}(F_K^F), \quad \text{Sign}(F_L^H) = \text{Sign}(F_L^F) \qquad (4\text{-}22)$$

We call it the factor conversion trade. Appendix B is the equilibrium solution for the $2 \times 2 \times 2$ Cross-Factor Hicks-Neutral FIRs model, which demonstrates (4-22) analytically.

However, the trade volume balance definitely holds as

$$T^H = -T^F \qquad (4\text{-}23)$$

The home country is capital intensive at commodity 1, it will export commodity 1. The foreign county is capital intensive at commodity 2, it will export commodity 2.

Timing two sides of (4-23) by the home technology matrix A^H yields

$$F^H = -A^H T^F = -A^H A^{F^{-1}} F^F = -F^{FbH} \tag{4-24}$$

where F^{FbH} is the vector of the foreign factor content of trade measured by the home country's technology. This equation implies that the home country's factor content of trade equals negatively to the foreign country's factor content of trade measured by referring to the home country's technology. The conversion trade is always symmetrical and balanced under this meaning. It is actually under the generalized Heckscher-Ohlin theory. The conversion trade sounds odd for a "normal" understanding of international economics. Actually, it is normal and is not with any paradox theoretically.

Leontief (1953) initialed the idea of the productivity-equivalent unit. Trefler (1993) implemented it in an artful HOV model. Leontief proposed his idea mostly to try to demonstrate that his test conclusion is right. Trefler (1993) only emphasized that Leontief's idea of the productivity-equivalent unit is right. Most scholars cited Trefler work to illustrate that the trade pattern described by the Leontief test is not right. This paper used the concept of equivalent productivity unit Leontief initialed and Trefler implemented to demonstrate that both Leontief's method of productivities-equivalent unit and Leontief trade are right.

With factor content reversal, both countries will consume more on their effective scarce factor. International trade adjusts the consumption of factor content not only quantitatively but also in quality when countries have different productivities.

Example 1 in Appendix C provides a numerical illustration for the conversion trade.

Appendix D is the Conversion Trade for Many Factors, Many Commodities.

5. Analyses of Trade Patterns by Fisher and Marshall's Virtual Endowments

Fisher and Marshall (2008) provided an approach to incorporate the general different technologies fully within the Heckscher-Ohlin framework. The $2 \times 2 \times 2$ Heckscher-Ohlin-Ricardo model in section (2) reflects the full technology differences that Fisher and Marshall Analyzed. It is a true model of two cones of commodity prices. We present only the most tricky trade pattern, the factor conversion trade, in this section. We also present geometric expressions of the Leontief trade and the factor conversion trade.

5.1 Factor Conversion Trade

Leamer (1984, pp 8-9) provides a unique way to demonstrate the Heckscher-Ohlin theorem analytically. He showed that for the $2 \times 2 \times 2$ Heckscher-Ohlin model if the home country is capital abundant as

$$(K^H/K^W) > s > (L^H/L^W) \tag{5-1}$$

the excess factor supplies have signs

$$F^H = \begin{bmatrix} F_K^H \\ F_L^H \end{bmatrix} = \begin{bmatrix} + \\ - \end{bmatrix} \quad (5\text{-}2)$$

In addition, if the home country is capital intensive in commodity 1, the signs of trade flow will be[8]

$$T^H = (A^H)^{-1}F^H = \begin{bmatrix} + & - \\ - & + \end{bmatrix}\begin{bmatrix} + \\ - \end{bmatrix}=\begin{bmatrix} + \\ - \end{bmatrix} \quad (5\text{-}3)$$

Therefore, the home country will export commodity 1 and import commodity 2.

We now generalize Leamer's analysis with the Heckscher-Ohlin-Ricardo model by using virtual factor endowments.

With the Heckscher-Ohlin-Ricardo model (2-1) and (2-2), the vector of the commodity exports in the home country is the difference between production output X^H and consumption C^H:

$$T^H = X^H - C^H = A^{H-1}(V^H - s^H V^{WbH}) \quad (5\text{-}4)$$

which is A^{H-1} times the vector of excess factor supplies,

$$F^H = V^H - s^H V^{WbH} = \begin{bmatrix} K^H - s^H K^{WbH} \\ L^H - s^H L^{WbH} \end{bmatrix} = \begin{bmatrix} K^{WbH}(K^H/K^{WbH} - s^H) \\ L^{WbH}(L^H/L^{WbH} - s^H) \end{bmatrix} \quad (5\text{-}5)$$

The vector of the commodity exports in the foreign country is

$$T^F = X^F - C^F = A^{F-1}(V^F - s^F V^{WbF}) \quad (5\text{-}6)$$

which is A^{F-1} times the vector of excess factor supplies:

$$F^F = V^F - s^F V^{WbF} = \begin{bmatrix} K^F - s^F K^{WbF} \\ L^F - s^F L^{WbF} \end{bmatrix} = \begin{bmatrix} K^{WbF}(K^H/K^{WbF} - s^F) \\ L^{WbF}(L^H/L^{WbF} - s^F) \end{bmatrix} \quad (5\text{-}7)$$

The share of GNP is the function of commodity prices. Corresponding to the four rays of the two cones of the commodity prices of two countries, there are four boundaries of shares of GNP. The boundaries of the share of GNP by the rays of the commodity price diversification cone of the home country are

$$s_b^h\left(p\left(\frac{a_{K1}^H}{a_{K2}^H}, 1\right)\right) = \frac{a_{K1}^H x_1^H + a_{K2}^H x_2^H}{a_{K1}^H x_1^w + a_{K2}^H x_2^w} = \frac{K^H}{K^{WbH}} \quad (5\text{-}8)$$

$$s_a^h\left(p\left(\frac{a_{L1}^H}{a_{L2}^H}, 1\right)\right) = \frac{a_{L1}^H x_1^H + a_{L2}^H x_2^H}{a_{L1}^H x_1^w + a_{L2}^H x_2^w} = \frac{L^H}{L^{WbH}} \quad (5\text{-}9)$$

The boundaries of the share of GNP by the rays of the commodity price diversification cone of the foreign country are

$$s_b^F\left(p\left(\frac{a_{K1}^F}{a_{K2}^F}, 1\right)\right) = \frac{a_{K1}^F x_1^F + a_{K2}^F x_2^F}{a_{K1}^H x_1^w + a_{K2}^H x_2^w} = \frac{K^F}{K^{WbF}} \quad (5\text{-}10)$$

$$s_a^F\left(p\left(\frac{a_{L1}^F}{a_{L2}^F}, 1\right)\right) = \frac{a_{L1}^F x_1^F + a_{L2}^F x_2^F}{a_{L1}^F x_1^w + a_{L2}^F x_2^w} = \frac{L^F}{L^{WbF}} \quad (5\text{-}11)$$

We now discuss the case that the mode is with the existence of FIRs, in which $|A^H| > 0$ and $|A^F| < 0$. If the home country is virtual capital abundant as (5-8), there are

$$K^H/K^{WbH} - s^H > 0$$
$$L^H/L^{WbH} - s^H < 0$$

[8] Leamer used the inversion matrix of technology matrix as

$$A^{-1} = \begin{bmatrix} a_{K1} & a_{K2} \\ a_{L1} & a_{L2} \end{bmatrix}^{-1} = \begin{bmatrix} a_{L2} & -a_{K2} \\ -a_{L1} & a_{K1} \end{bmatrix} / |A|$$

where $|A| = a_{L1}a_{L2}\left({a_{K1}}/{a_{L1}} - {a_{K2}}/{a_{L2}}\right) > 0$

The home country's share of GNP must lie in the following range, from (5-8)

$$\frac{K^H}{K^{WbH}} > s^H > \frac{L^H}{L^{WbH}} \tag{5-12}$$

The home country will export the services of capital and import the service of labor. Therefore, the vector of factor content of trade in the home country is with signs

$$F^H = \begin{bmatrix} + \\ - \end{bmatrix} \tag{5-13}$$

The signs of trade flow of the home country from equation (5-13) will be

$$T^H = (A^H)^{-1}F^H = \begin{bmatrix} + & - \\ - & + \end{bmatrix}\begin{bmatrix} + \\ - \end{bmatrix}=\begin{bmatrix} + \\ - \end{bmatrix} \tag{5-14}$$

This is due to the home country is capital intensive in commodity 1 by $|A^H| > 0$.
By the international trade balance, the sign of trade flow in the foreign country is

$$T^F = -T^H = \begin{bmatrix} - \\ + \end{bmatrix} \tag{5-15}$$

The vector of factor content of trade in the foreign country then is with signs

$$F^F = A^F T^F = \begin{bmatrix} - & + \\ + & - \end{bmatrix}\begin{bmatrix} - \\ + \end{bmatrix} = \begin{bmatrix} + \\ - \end{bmatrix} \tag{5-16}$$

This is due to the foreign country is capital intensive in commodity 2 by $|A^F| < 0$.
From (5-7) and (5-16), we know that the foreign country is virtual capital abundant. This leads just the result that each country exports the commodity that uses its virtual abundant factor intensively and imports the commodity that uses its virtual scarce factor intensively.

Equation (5-13) and (5-16) shows that signs of export of the two countries are the same as

$$\text{Sign } (F_K^H) = \text{Sign } (F_K^F) \quad , \quad \text{Sign } (F_L^H) = \text{Sign } (F_L^F) \tag{5-17}$$

It demonstrates that when presenting FIRs, both countries are virtual abundant at the same factor and both export the service of the same factor.

The world commodity price must lie in the intersection of commodity price cones as

$$\frac{a_{L1}^F}{a_{L2}^F} > \frac{a_{L1}^H}{a_{L2}^H} \tag{5-18}$$

The home country's share of GNP, corresponding the intersection of commodity price cone above, should lie in the following range by

$$\frac{L^{HbF}}{L^{WbF}} > s^H > \frac{L^H}{L^{WbH}} \tag{5-19}$$

The range by (5-19) is a part of the range by (5-12). The relationships (5-13) through (5-16) that hold under (5-12) should hold also under (5-19).

5.2 Geometric Presentation of the Leontief Trade and the Factor Conversion Trade

Figure 2 draws a generalized IWE diagram with the Leontief trade. It is a multiscale diagram that

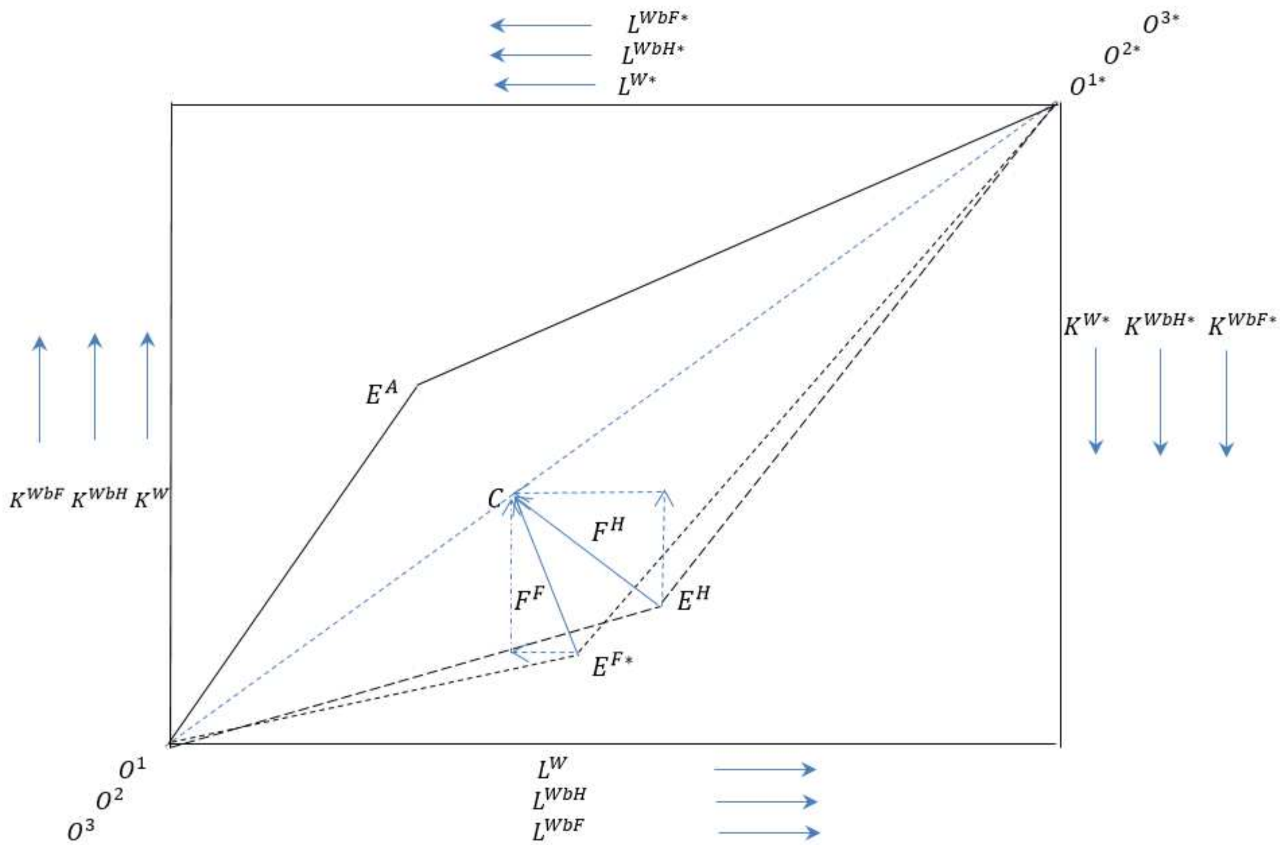

Figure 2 Multiscale IWE Diagram for Leontief Trade

merges the three diagrams together. The densities of each diagram's scales are different. The lower-left corner is three origins for the home country. The right-upper corner is three origins for the foreign country. Dimension O^1O^{1*} is for two countries' actual factor endowments. Dimension O^2O^{2*} is for two countries' virtual factor endowments measured by referring to the home country's technology. Dimension O^3O^{3*} is for two countries' virtual factor endowments measured by referring to the foreign country's technology. The diagram dimension just fits V^{WbH}, V^{WbF}, and V^W, although $V^{WbH} \neq V^{WbF} \neq V^W$. The goal is to make subtle changes to the feature density of each scale to avoid distortion of the factor content of trade and overall message.

We assume here that the ratios of capital to labor employed in the two countries lie in their diversification cone of factor endowments like

$$\frac{a_{K1}^H}{a_{L1}^H} > \frac{K^H}{L^H} > \frac{a_{K2}^H}{a_{L2}^H} \tag{5-20}$$

$$\frac{a_{K1}^F}{a_{L1}^F} > \frac{K^F}{L^F} > \frac{a_{K2}^F}{a_{L2}^F} \tag{5-21}$$

For a giving allocation of actual factor endowments of two countries at E^A, there are two respective allocations of virtual factor endowments E^H and E^{F*}. E^A is the vector from the home origin O^1. It is above the diagonal line. It indicates that the home country is actual capital abundance as

$$\frac{K^H}{L^H} > \frac{K^W}{L^W} \tag{5-22}$$

Point E^H indicates the allocation of virtual factor endowments of two countries, which are measured by referring to the home country's technology. It is below the diagonal line. It signifies that the home country is virtual labor abundant as

$$\frac{K^H}{L^H} < \frac{K^{WbH}}{L^{WbH}} \tag{5-23}$$

Point E^{F*} indicates the allocation of virtual factor endowments of two countries, which are measured by referring to the foreign country's technology. It is below the diagonal line from the view of foreign origin. It signifies that the foreign country is capital abundant as

$$\frac{K^F}{L^F} > \frac{K^{WbF}}{L^{WbF}} \tag{5-24}$$

Inequalities (5-22) and (5-23) implies that the home country is with the Leontief trade. In addition, Inequalities (5-22) and (5-24) implies that the foreign country is with Leontief trade too.

There are two vectors of factor content of trade, $F^H$and F^F, in Figure 2. They both point at C and reach the same share of GNP. Point C represents the trade equilibrium point. It indicates the sizes of the consumption of the two countries. Vector F^H indicates that the home country as an actual capital abundant country exports labor services and imports capital services. Similarly, vector F^F indicates that the foreign country as an actual labor abundant country exports capital services and imports labor services.

When the factor endowments of E^A allocated below the diagonal line, it will be the Heckscher-Ohlin trade.

For the Trefler model, by its single price cone property, E^H and E^{F*} will overlap together in the multiscale diagram.

Figure 3 shows a generalized multiscale IWE diagram with the factor conversion trade. E^A is from home origin. It indicates the allocation of actual factor endowments of two countries. It shows that the home country is actual capital abundance as

$$\frac{K^H}{L^H} > \frac{K^W}{L^W} \tag{5-25}$$

E^H is the vector from home origin. It indicates the allocation of virtual factor endowments of two countries, which are measured by referring to the home country's technology. It is below the diagonal line. It signifies that the home country is virtual labor abundance as

$$\frac{K^H}{L^H} < \frac{K^{WbH}}{L^{WbH}} \tag{5-26}$$

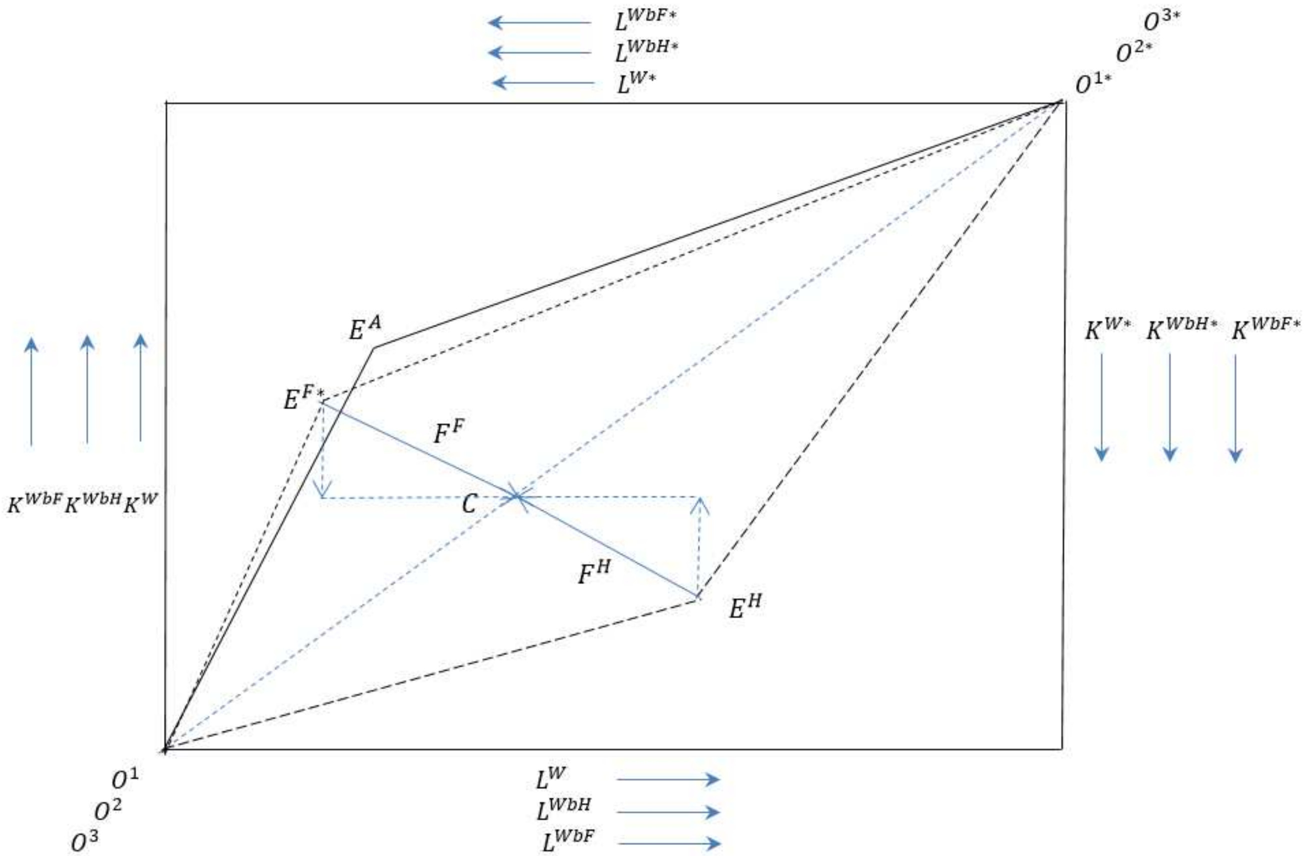

Figure 3 Multiscales IWE Diagram for Conversion Trade

E^{F*} is from foreign origin, It indicates the allocation of the virtual factor endowments of two countries, which are measured by referring to the foreign country's technology. It is below the diagonal line from the view of foreign origin. It signifies that the foreign country is virtual labor abundance as

$$\frac{K^F}{L^F} < \frac{K^{WbF}}{L^{WbF}} \tag{5-27}$$

Inequalities (5-25) and (5-26) indicates that this is a conversion trade since both countries are virtual factor abundance at labor. Inequalities (5-25) and (5-26) implies that the home country is with the Leontief trade since the home country is actual capital abundance and virtual labor abundance. In addition, inequalities (5-25) and (5-27) implies that the foreign country is with the Heckscher-Ohlin trade since the foreign country is actual labor abundance and virtual labor abundance.

Vectors F^F and F^H indicates that both countries export labor services and import capital services. It illustrates how the factor conversion trade formatted.

Under the FIRs model structure, one country does the Heckscher-Ohlin trade; another does the Leontief trade.

Dixit-Norman IWE diagram shows a property that the factor price and commodity price remain the same when the location of factor endowments changes with the box by factor diversification cone. The two diagrams by virtual factor endowments above have a similar property that the prices remain the same when the allocation of outputs changes. This can be very simply demonstrated by the argument, similar to the argument Woodland (2013, p70) made for the property of the Dixit-Norman principle, that change of the allocation of world virtual factor endowments between countries or the change of the allocation of output production leaves the world supply of goods and income unchanged and so supplies will still match the unchanged world demands and the prices will remain the same.

6. Trade Patterns by the Intersection of Output Price Diversification Cones.

We introduce the simplest way to demonstrate three trade patterns by using the intersection of output price diversification cones (see Fisher, 2011). This is a numerical way.

The major logic used here is that the signs of trade volume in (5-14) and (5-16) and the signs of factor content of trade in (5-13) and (5-15) remain the same for all commodity prices that lie in the intersection cone of output price cones. We present the proof of the logic in Appendix E. It implies that any commodity price lies in the intersection cone of commodity price can present trade directions by (5-13) through (5-16) since the GNP share of a country is the function of commodity price.

Corresponding to the intersection cone of commodity price cones (2-5), there is a range of the share of GNP of the home country as

$$s_b\left(p_b\left(\frac{a_{K1}^F}{a_{K2}^F},1\right)\right) > \quad s^h \ > \ s_a\left(p_a\left(\frac{a_{L1}^H}{a_{L2}^H},1\right)\right) \tag{6-1}$$

where

$$s_a\left(p_a\left(\frac{a_{K1}^F}{a_{K2}^F},1\right)\right) = \frac{a_{K1}^F x_1^H + a_{K2}^F x_2^H}{a_{K1}^F x_1^W + a_{K2}^F x_2^W} = \frac{K^{HbF}}{K^{WbF}} \tag{6-2}$$

$$s_b\left(p_b\left(\frac{a_{L1}^H}{a_{L2}^H},1\right)\right) = \frac{a_{L1}^H x_1^H + a_{L2}^H x_2^H}{a_{L1}^H x_1^W + a_{L2}^H x_2^W} = \frac{L^H}{L^{WbH}} \tag{6-3}$$

Giving a share of GNP in the range (5-1), it can predict trade direction and direction of factor content of trade by (5-13) through (5-16). The middle point of the range of share of GNP (5-1) is a good candidate to use to display the trade direction numerically. For the Trefler model, it is just the equilibrium share of GNP of the home country. Appendix E is helpful to understand this.

This is the most direct way to show the three trade patterns. It based on very simple and no arguable logic.

7. Related Discussions

We demonstrated that the Leontief trade and the factor conversion trade are normal trade patterns theoretically based on either the concept of the effective endowments or on the concept of the virtual endowments. The trade pattern described by Leontief's tests is more likely to be true in international trade observation than at any time before. The factor intensity reversals always associated with the factor conversion trade. The factor conversion trade is one kind of Leontief trade. Kurokawa (2011), Takahashi (2004), Simpson (2016), Kozo and Yoshinori (2017) and some other scholars have provided clear evidence of factor intensity reversals. Their studies imply that there exist both the Leontief trade and the factor conversion trade in international trade practice. More studies need to be done for further confirmations. This paper believes that all of the three trade patterns are true in the real life of international trade. Theories and empirical studies both pointed at it yet.

Many empirical HOV studies predicted the trade direction successfully by the models incorporating different technologies across countries. The prediction accuracies were improved a lot. The popular sign prediction criteria by equations (4-1), (4-2), and (4-3) methodologically include all of the three trade patterns of this paper. They are not sufficient to deny the Leontief trade although their explanations and presentations of the trade theories incorporating different technologies are right. From this view, no study neither theoretically or empirically really confirmed that the Leontief trade or the Leontief paradox is not right.

The factor content reversal displays a new kind of comparative advantage from the consumption side. Both countries consume more on effective (virtual) scarce factor. Trade converts the globally effective (virtual) abundant factor into a globally effective (virtual) scarce factor, all of which are bundled in the trade flows. The factor conversion trade's behavior is very like the "black hole" [9] in astronomy. The trade has a mechanism that the effective abundant factor cannot "escape" from the market (or it is absorbed by the market). Meanwhile, the trade has also a "white hole"[10] function that the effective scarce factor can only leave the market to go toward to each country (The market does not absorb effective scare factor). It displays a different kind of gains from trade, the gains from the consumption qualities.

Some studies explained the Leontief paradox by identifying data errors of the US economy in the time. Data errors may exist or may not exist. We are more interested in the trade pattern that the Leontief test explored. Even it may be an accident result. Leontief's discovery is of much influence on international economics. We finally realize that his finding as a new trade pattern is right.

We demonstrated that both countries may be effective (or virtual) factor abundant at the same factors. However, we cannot predict or derive that both sectors in a country are effective (or virtual) factor abundant at both factors for $2 \times 2 \times 2$ economy. It is odd for the definition of virtual factor abundance.

Conclusion

[9] Black hole in astronomy is defined as that a region of space having a gravitational field so intense that no matter or radiation can escape.

[10] In general relativity, a white hole is a hypothetical region of spacetime, which cannot be entered from the outside, although matter and light can escape from it. In this sense, it is the reverse of a black hole, which can only be entered from the outside and from which matter and light cannot escape.

The three trade patterns hang in there within the Heckscher-Ohlin framework with different technologies. Understanding them is helpful to know world price formation and to review international trade policies. The Leontief trade and the factor conversion trade counter common understanding of international economics somehow. Actually, they are rooted in the Heckscher-Ohlin theories. We demonstrated it.

The new understanding for factor intensity reversal is that it causes factor price reversal, factor content reversals, and effective (virtual) factor abundance reversal. Another new understanding of the Leontief trade is that it can occur either with the presence of FIRs or with no presence of FIRs.

There may be different formats of trade patterns when fulfilling a more complicated higher dimension's analysis. International trades benefit countries by the diversifications of gains.

Appendix A – The Proof of Leontief Trade by Trefler Model

Trefler implemented the different productivities across countries by

$$A^H = \Pi A^F = \begin{bmatrix} \pi_K & 0 \\ 0 & \pi_L \end{bmatrix} A^F \tag{A-1}$$

where Π is a 2×2 diagonal matrix, its element π_k is factor productivity-argument parameter, $k = K, L$. The Trefler $2 \times 2 \times 2$ model can be denoted as

$$A^H X^H = V^H, \qquad (A^H)' W^H = P^H \tag{A-2}$$

$$\Pi^{-1} A^H X^F = V^F, \qquad (\Pi^{-1} A^H)' W^F = P^F \tag{A-3}$$

The world effective factor endowments by referring to the home technologies are

$$K^{WH} = K^H + \pi_K K^F \quad , \quad L^{WH} = L^H + \pi_L L^F \tag{A-4}$$

The world effective factor endowments by referring to foreign technologies are

$$K^{WF} = K^F + K^H / \pi_K \quad , \quad L^{WF} = L^F + L^H / \pi_L \tag{A-5}$$

Guo (2019) provides an equilibrium solution for the Trefler model. The equilibrium shares of GNP of the two countries are

$$s^H = \frac{1}{2}\left(\frac{K^H}{K^{WH}} + \frac{L^H}{L^{WH}}\right) \tag{A-6}$$

$$s^F = \frac{1}{2}\left(\frac{K^F}{K^{WF}} + \frac{L^F}{L^{WF}}\right) \tag{A-7}$$

Using them, we obtain the following trade-price equilibrium of the model. The prices are

$$W^{H*} = \begin{bmatrix} \frac{L^{WH}}{K^{WH}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{L^H + \pi_L L^F}{K^H + \pi_K K^F} \\ 1 \end{bmatrix} \tag{A-8}$$

$$P^* = (A^H)' W^{H*} \tag{A-9}$$

$$W^{F*} = \Pi W^{H*} \tag{A-10}$$

Factor content of trade for the two countries are

$$F_K^h = K^h - s^h K^{Wh} = \frac{1}{2}\frac{K^h L^{Wh} - K^{Wh} L^h}{L^{Wh}} \qquad (h = H, F) \tag{A-11}$$

$$F_L^h = L^h - s^h L^{Wh} = -\frac{1}{2}\frac{K^h L^{Wh} - K^{Wh} L^h}{K^{Wh}} \qquad (h = H, F) \tag{A-12}$$

We demonstrate the trade pattern that the actual capital abundant country exports the labor-intensive commodity.

We assume that both countries are capital-intensity in commodity 1. We also assume that the home country is capital abundance.

Substituting (A-4) into (A-11) yields

$$F_K^H = \frac{1}{2}\frac{K^H \pi_L L^F - L^H \pi_K K^F}{L^H + \pi_L L^F} \tag{A-13}$$

If the numerator of F_K^H is less than zero, It means that its numerator is less than zero as

$$\frac{K^H L^F}{L^H K^F} < \frac{\pi_K}{\pi_L} \tag{A-14}$$

Rewrite it as

$$\frac{K^H}{L^H} < \frac{\pi_K K^F}{\pi_L L^F} = \frac{K^{FH}}{L^{FH}} \tag{A-15}$$

It means that the home country is virtual labor abundance. It implies that actual capital abundant country exports the services of labor under the condition (A-14).

Appendix B – The Proof of Conversion Trade by Trefler Model

We demonstrate that F_K^H and F_K^F are at the same sign. Using Cross-Sector Hicks-Neutral FIRs" model in section 3 and substituting (4-17) into (A-11) yields

$$F_K^H = \frac{1}{2}\frac{K^H K^F \theta_K - L^H L^F \theta_L}{L^H + \theta_K K^F} \tag{B-1}$$

If the numerator of F_K^H is greater than zero, It means

$$\frac{K^H K^F}{L^H L^F} > \frac{\theta_L}{\theta_K} \tag{B-2}$$

Similarly, substituting (4-18) into (A-11) yields

$$F_K^F = \frac{1}{2}\frac{K^H K^F /\theta_L - L^H L^F /\theta_K}{L^F + K^F /\theta_K} \tag{B-3}$$

If the numerator of F_K^H is greater than zero, it means,

$$\frac{K^H K^F}{L^H L^F} > \frac{\theta_L}{\theta_K} \tag{B-4}$$

Therefore, F_K^H and F_K^F are at the same sign always.

Appendix C – Numerical Examples

Numerical example 1- Conversion trade by Cross-Sector Hicks-Neutral FIRs model

The technological matrix for the home country is

$$A^H = \begin{bmatrix} 3.0 & 1.0 \\ 1.5 & 2.0 \end{bmatrix}$$

The technological matrix for the foreign country is

$$A^F = \begin{bmatrix} 0.0 & 1/0.9 \\ 1/0.8 & 1.0 \end{bmatrix} \begin{bmatrix} 3.0 & 1.0 \\ 1.5 & 2.0 \end{bmatrix}$$

The factor intensities of the two countries are

$$a_{K1}^H / a_{L1}^H = 2.0 > \ a_{K2}^H / a_{L2}^H = 0.5$$
$$a_{K1}^F / a_{L1}^F = \ \ 0.562 < a_{K2}^F / a_{L2}^F = 2.25$$

The home country is capital intensive in sector 1, and the foreign country is capital intensive in sector 2. This is a case with the presence of FIRs. We take the factor endowments for the two countries as

$$\begin{bmatrix} K^H \\ L^H \end{bmatrix} = \begin{bmatrix} 4200 \\ 3000 \end{bmatrix}, \qquad \begin{bmatrix} K^F \\ L^F \end{bmatrix} = \begin{bmatrix} 3187.5 \\ 2666.6 \end{bmatrix}$$

The home country is actual labor abundant as

$$\frac{K^H}{L^H} = \frac{4200}{3000} = 1.4 < \ \frac{K^F}{L^F} = \frac{3187.5}{2666.6} = 1.19$$

However, the home country is effective capital abundant as

$$\frac{K^H}{L^H} = \frac{4200}{3000} = 1.4 > \ \frac{K^{FH}}{L^{FH}} = \frac{2400}{2550} = 0.94$$

Therefore, the home country exports commodity 1 and is with net excess of capital, since commodity 1 uses the capital intensively.

The foreign country is effective capital abundant as

$$\frac{K^F}{L^F} = \frac{318.75}{2666.6} = 1.19 > \ \frac{K^{HF}}{L^{HF}} = \frac{3750}{4666} = 0.80$$

Therefore, the foreign country exports commodity 2 and is with net excess of capital services since commodity 2 uses the capital intensively. The home country is in the Leontief trade and the foreign country is in the Heckscher-Ohlin trade.

The outputs of the two countries are

$$\begin{bmatrix} x_1^H \\ x_2^H \end{bmatrix} = \begin{bmatrix} 1200.0 \\ 600.0 \end{bmatrix}, \qquad \begin{bmatrix} x_1^F \\ x_2^F \end{bmatrix} = \begin{bmatrix} 500.0 \\ 900.0 \end{bmatrix}$$

The ranks of cost requirement ratios are

$$\frac{a_{K1}^H}{a_{K2}^H} = \frac{a_{L1}^F}{a_{L2}^F} = 3 \quad > \quad \frac{a_{L1}^H}{a_{L2}^H} = \frac{a_{K1}^F}{a_{K2}^F} = 0.75$$

The shares of GNP of the home country, corresponding to the rays of the intersection cone, are

$$s_b^H \left(p_a \left(\frac{a_{K1}^F}{a_{K2}^F}, 1 \right) \right) = 0.636$$
$$s_a^H \left(p_b \left(\frac{a_{L1}^H}{a_{L2}^H}, 1 \right) \right) = 0.540$$

The middle of the range of the share of GNP is $s_m^H = 0.5884$.

The export volumes and the factor contents of trades by the share of GNP above are:

$$\begin{bmatrix} T_1^H \\ T_2^H \end{bmatrix} = \begin{bmatrix} 199.6 \\ -282.6 \end{bmatrix}, \qquad \begin{bmatrix} T_1^F \\ T_2^F \end{bmatrix} = \begin{bmatrix} -199.6 \\ 282.6 \end{bmatrix}$$
$$\begin{bmatrix} F_K^H \\ F_L^H \end{bmatrix} = \begin{bmatrix} 316.2 \\ -265.9 \end{bmatrix}, \qquad \begin{bmatrix} F_K^F \\ F_L^F \end{bmatrix} = \begin{bmatrix} 332.8 \\ -351.3 \end{bmatrix}$$

We see that both countries export capital services and import labor services. The trade converts the globally effective abundant factor into the globally scarce factor. This is an interesting result.

Appendix D   Conversion Trade for Many Factors and Many Commodities

In multiple-country trade analyses, a trade partner of a country is the rest of the world. So does the analyses of conversion trade and the Leontief trade. When the conversion trade occurs, a country and the rest world export the same factor services and import the same factor services for at least a pair of factors.

The conversion trade occurs also in the context of the models with many commodities and many factors and many countries.

A simple way to specify a FIRs model in high dimensions is by switching a pair of rows in its technology matrix. Row-switching matrix S_{ij}, like the following, switches all matrix elements on row i with their counterparts on row j.

$$S_{ij} = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & 0 & 1 & & & \\ & & & 0 & \ddots & 0 & & & \\ & & & 1 & 0 & 0 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{bmatrix}$$

The corresponding elementary matrix is obtained by swapping row i and row j of the identity matrix. Since the determinant of the identity matrix is unity, $\det[S_{ij}] = -1$. It follows that for any square matrix A (of the correct size), we have $\det[S_{ij}A] = -\det[A]$. Using a row-switching operation, we can implement a FIRs model. This is also available for non-square (not even) technology matrix. The conversion trade not only occurs for even model (factor number equals to commodity number) but also for the non-even model. To specify a non-even FIR model, just use square Row-switching matrix S_{ij}.

We present a numerical example to display a conversion trade for $4 \times 4 \times 2$ model.

The technological matrix for the home country is

$$A^H = \begin{bmatrix} 3.0 & 1.2 & 1.3 & 0.9 \\ 1.1 & 2.0 & 0.9 & 1.4 \\ 0.7 & 1.5 & 2.1 & 1.0 \\ 1.6 & 1.7 & 0.8 & 1.5 \end{bmatrix}$$

The technological matrix for the foreign country is

$$A^F = \psi^{-1} A^H$$

where

$$\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The factor endowments of the two countries are

$$V^H = \begin{bmatrix} 4253 \\ 4189 \\ 3631 \\ 4098 \end{bmatrix}, \qquad V^F = \begin{bmatrix} 3690 \\ 4975 \\ 3865 \\ 4080 \end{bmatrix}$$

The world effective abundant by the home productivities are

$$V^{WbH} = \begin{bmatrix} 8333 \\ 8054 \\ 8606 \\ 7788 \end{bmatrix}$$

The world effective abundant by the foreign productivities are

$$V^{WbF} = \begin{bmatrix} 8333 \\ 8054 \\ 7788 \\ 8606 \end{bmatrix}$$

We see that the values of V_3^{WbF} and V_4^{WbF} are reversals of V_3^{WbH} and V_4^{WbH}. Both countries are effect abundant at factor 4 related to factor 3

$$\frac{v_3^H}{v_4^H} = \frac{3631}{4098} = 0.886 < \frac{v_3^{WbH}}{v_4^{WbH}} = \frac{8606}{7788} = 1.105$$

$$\frac{v_3^F}{v_4^F} = \frac{3864}{4080} = 0.947 < \frac{v_3^{WbH}}{v_4^{WbH}} = \frac{7788}{8606} = 0.949$$

That will cause the factor content reversals between factor 3 and factor 4.

Appendix E

Figure 4 is a generalized IWE diagram for the Heckscher-Ohlin-Ricardo model. It draws a multiscale diagram that merges the two diagrams together. The densities of each diagram's scales are different. The lower-left corner is two origins for the home country. The upper-right corner is the two origins of the foreign country. Dimension O^1O^{1*} is for two countries' virtual factor endowments measured by home technology. Dimension O^2O^{2*} is for two countries' virtual factor endowments measured by foreign technology. The diagram dimension just fits V^{WbH} and V^{WbF}, although $V^{WbH} \neq V^{WbF}$. The goal is to make subtle changes to the feature density of each scale to avoid distortion of the factor content of trade and overall message.

Giving factor endowments of two countries V^H and V^F, there are two respective allocations of virtual factor endowments E^H and E^{F*}. Allocation E^H is the vector from origin O^1. Allocation E^{F*} is the vector from origin O^{2*}. There are two factor-content vectors, $F^H$and F^F. Both of them point at C and reach the same point of share of GNP. Point C represents the trade equilibrium point. It indicates the sizes of the consumption of the two countries.

Figure 4 also draws two trade boxes by the boundaries of shares of GNP (4-11) through (4-14). The solid-line box is for the home country; the dash-line box is for the foreign country. The share of GNP is a convex function of commodity prices. The intersection of the two trade boxes, indicated by the diagonal line C^2C^3, reflects the intersection cone of commodity price cones. Point C will changes when giving different commodity prices. However the signs of F_L^H, F_K^H, F_L^F and F_K^F will not change within the diagonal line C^2C^3 . Such as F_L^H is always negative, which means import the services of labor. F^Hcan end at any point within C^2C^3, No matter which point it ends at, the trade direction F_L^H and F_K^H remain the same.

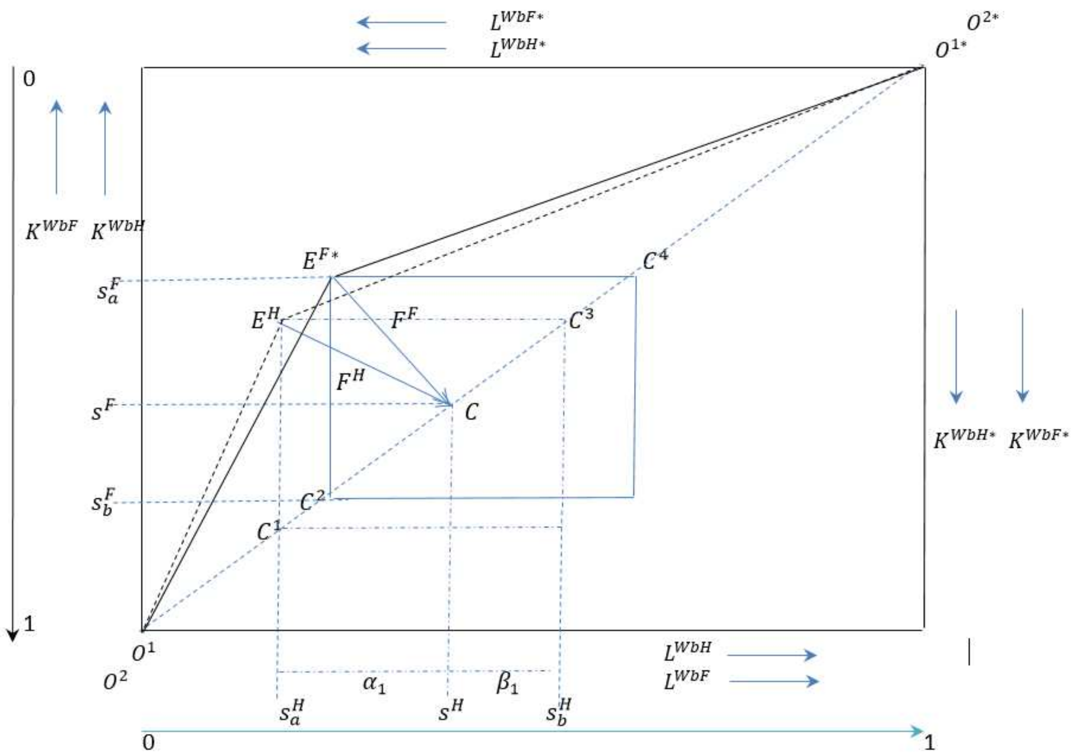

Figure 4 Multiscale IWE Diagram with the Two Trade Boxes

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