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Abstract

In this paper, we introduce a family of rules in minimum cost spanning tree problems with multiple sources called Kruskal sharing rules. This family is characterized by cone-wise additivity and independence of irrelevant trees. We also investigate some subsets of this family and provide axiomatic characterizations of them. The first subset is obtained by adding core selection. The second is obtained by adding core selection and equal treatment of source costs.

Keywords: minimum cost spanning tree problems, multiple sources, Kruskal sharing rules, axiomatic characterizations.

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1 Introduction

Consider a problem where a group of agents is interested in goods provided by several suppliers or sources. To be served, the agents must be connected directly or indirectly to the suppliers which means that they incur a cost. Agents also want to be connected to all sources. This may be motivated by safety issues. For example, suppose that agents are interested in an electricity grid which connects them to several power plants. If they are connected to all power plants, then they are able to safely consume electricity even if one or more plants suddenly fail to support their demands. There could also be situations in which each source provides a different resource (water, electricity, Internet connection, etc.) and agents are interested in all of them.

In our model we first have to find the least costly structure (a tree) that connects all the agents to the sources directly or indirectly, taking into account that agents want to be connected to every source. Kruskal (1956) and Prim (1957) propose algorithms for selecting

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the least costly tree in the classical model, where there is a unique source. Such algorithms can also serve their purpose in the case of multiple sources with the same good characteristics as the polynomial computation time. Some authors have studied related problems that are difficult from a computational point of view. For instance, Granot and Granot (1992) study the fixed cost spanning forest problem; Farley *et al* (2000) study how to compute the spanning tree that minimizes the sum of the distances from each source to every other node, Gouveia and Martins (1999) the capacitated minimal spanning tree problem, and Gouveia *et al* (2014) study hop constrained Steiner trees with multiple root nodes.

Once such a tree is obtained, its associated cost has to be shared among the agents. The many papers that have studied this issue in classical minimum cost spanning tree problems include Bird (1976), Kar (2002), Dutta and Kar (2004), Bergantiños and Vidal-Puga (2007), Bogomolnaia and Moulin (2010),Trudeau (2012), and Bergantiños and Gómez-Rúa (2015). But there are few papers for the case of multiple sources. Rosenthal (1987) assumes that all sources provide the same service and agents want to be connected to at least one of them. He associates a cooperative game with this problem and studies the core. Kuipers (1997) considers situations where each source offers a different service and each agent states which subset of sources he wants to be connected to. He also considers a cooperative game and studies the conditions under which the core is non empty. Bergantiños *et al* (2018) consider the same situation as in this paper. They extend several definitions of the folk rule (Branzei *et al* (2004), Tijs *et al* (2006), Bergantiños and Vidal-Puga (2007), and Bergantiños *et al* (2010, 2011)), one of the most prominent rules in the classical model, to this new framework and prove that they all provide the same allocation. Bergantiños and Navarro-Ramos (2019a, 2019b) extend the definition of the folk rule through a painting procedure studied by Bergantiños *et al* (2014) in the classical model and give a characterization.

Sometimes it is better to study a family of rules rather than focusing on a particular one. In classical minimum cost spanning tree problems there are several sets of rules that contain the folk rule as a particular element of the set. The two sets closely related to our results here are the following: The first is the set of obligation rules introduced in Tijs *et al* (2006) and studied later in Lorenzo and Lorenzo-Freire (2009) and Bergantiños and Kar (2010). The second is the set of generalized obligation rules studied in Bergantiños *et al* (2010, 2011), which contains the set of obligation rules as a subfamily. It should be mentioned that both these sets are defined through Kruskal's algorithm and characterized using different properties.

In this paper we consider several families of rules, all of them containing the folk rule, and study which properties they satisfy. The property of independence of irrelevant trees says that if two problems share a common minimal tree with the same cost in all arcs, then the rule should allocate the same cost to each agent in both problems. Cone-wise additivity demands that the rule should be additive on cones. Both these properties have been widely studied in classical minimum cost spanning tree problems. In Theorem 1 we characterize the set of rules that satisfy both properties, which we call Kruskal sharing rules because they are computed through Kruskal's algorithm. A sharing function states the number of arcs that each agent has to pay for each partition of the set of agents and sources. We associate a Kruskal sharing rule with each sharing function as follows. At each Stage s of Kruskal's algorithm an arc is selected. The cost of that arc is divided between the agents according to the sharing function. Each agent must pay the proportion of the cost of the arc selected at Stage s given by the difference between the sharing function in the partition induced by the arcs selected by the algorithm before Stage s, minus the sharing function in the partition induced by the arcs selected at Stage s. When the issue is restricted to classical minimum cost spanning tree problems the family of Kruskal sharing rules contains the family of generalized obligation rules, and hence the family of obligation rules.

The set of Kruskal sharing rules is so large that some rules belonging to it may not be very appealing. For example, a rule where the cost of the minimal tree is paid entirely by a single agent would belong to this set. Thus, we add new properties in order to narrow the set of rules and omit those unappealing rules. The next property considered is core selection (the rule should select an element in the core). Even though the core could in general be empty, in our problem it is always non-empty, so we can claim this property. Theorem 2 characterizes the set of rules satisfying all three properties. We obtain that such rules are associated with sharing functions in which each group of agents has to pay, at most, the cost of the arcs that they need to be connected to all sources. Restricting to classical minimum cost spanning tree problems leads to this set of rules containing the family of obligation rules.

The three properties considered above are extensions of well known properties of classical minimum cost spanning tree problems to our setting. The next property that we consider is equal treatment of source costs. This is specifically defined for our setting. It says that if the cost between two sources increases, then all agents' payments should be affected by the same amount. Theorem 3 characterizes the set of rules satisfying all four properties. We obtain that such rules are associated with sharing functions defined as follows: Given a partition of the set of agents and sources, the arcs needed to join two elements in the partition that contain sources are divided equally among all agents, while the arcs needed to join elements in the partition with sources with those without sources are divided among the agents in the element without sources. When we restrict the issue to classical minimum cost spanning tree problems it emerges that this set of rules contains the family of obligation rules.

The paper is structured as follows. Section 2 introduces minimum cost spanning tree problems with multiple sources. Section 3 defines the family of Kruskal sharing rules and provides an axiomatic characterization of it. Section 4 presents and characterizes several subsets of the family of Kruskal sharing rules.

2 The model

Consider a network whose nodes are elements of a set $N \cup M$, where $N = \{1, ..., |N|\}$ is the set of agents and $M = \{s_1, ..., s_{|M|}\}$ is the set of sources. Respectively, $|N|$ and $|M|$ denote the cardinality of N and M. For each N and M, a *cost matrix* $C = (c_{ij})_{i,j \in N \cup M}$ represents the cost of a direct link between any pair of nodes. It is assumed that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N \cup M$ and $c_{ii} = 0$ for each $i \in N \cup M$. Since $c_{ij} = c_{ji}$ for each $i, j \in N \cup M$, we will work with undirected arcs $\{i, j\}$. Let $\mathcal{C}^{N \cup M}$ be the set of all cost matrices over $N \cup M$. Given $C, C' \in \mathcal{C}^{N \cup M}$, $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N \cup M$. Similarly, given $x, y \in \mathbb{R}^N$, $x \leq y$ if $x_i \leq y_i$ for each $i \in N$.

A *minimum cost spanning tree problem with multiple sources*, or a *problem*, is defined by a triple (N, M, C), where N is the set of agents, M is the set of sources, and C is the cost matrix in $\mathcal{C}^{N \cup M}$. Given a subset $S \subset N$, we denote by (S, M, C) the restriction of the problem (N, M, C) to the subset of agents in S. The classical minimum cost spanning tree problem, or the *classical problem*, corresponds to the case where M has a single element, which is denoted by 0.

Given a network g and a pair of distinct nodes $i, j \in N \cup M$, a *path from i to j in g* is a sequence of distinct arcs $g_{ij} = \{\{i_{s-1}, i_s\}\}_{s=1}^{p}$ such that $\{i_{s-1}, i_s\} \in g$ for each $s \in \{1, 2, ..., p\}$, $i = i_0$, and $j = i_p$. For each $i, j \in N \cup M$, i and j are connected in g if there exists a path from i to j in g. A *cycle* is a path from i to i with at least two arcs. A *tree* is a network where there exists a unique path from i to j for any $i, j \in N \cup M$.

For each network g, $S \subset N \cup M$ is a *connected component* if (1) for each $i, j \in S$, i and j are connected in g and (2) S is maximal, *i.e.*, for each $i \in S$ and each $j \notin S$, i and j are not connected in g. We define $P(g) = \{S_k(g)\}_{k=1}^{n(g)}$ as the partition of $N \cup M$ in connected components induced by g. Given $i \in N \cup M$ we denote by $S(P(g), i)$ the element of $P(g)$ to which i belongs to. Let $P(N \cup M)$ be the set of all partitions over $N \cup M$ and $P = \{S_1, ..., S_{|P|}\}$ be a generic element of $P(N \cup M)$. For each $P, P' \in P(N \cup M)$, P is *finer* than P' if for each $S \in P$, there exists $T \in P'$ such that $S \subset T$. For each $P, P' \in P(N \cup M)$, P is *1-finer* than P' if P' is obtained by joining two elements of P, *i.e.*, if $P = \{S_1, ..., S_{|P|}\}$ and P is 1-finer than P', then there exists $S_k, S_{k'} \in P$ such that $P' = P \setminus \{S_k, S_{k'}\} \cup \{S_k \cup S_{k'}\}$.

For each problem (N, M, C) and each network g, the cost associated with g is defined as $c(N, M, C, g) = \sum_{\{i,j\} \in g} c_{ij}$. When there is no ambiguity, we denote it by $c(g)$ or $c(C, g)$. A *minimal tree* for a problem (N, M, C) is a tree t such that $c(t) = \min\{c(g) : g \text{ is a tree}\}$. A minimal tree does not have to be unique, but it always exists for any problem. Such tree can be obtained, for instance, through Kruskal's algorithm (1956). Let $m(N, M, C)$ be the cost of any minimal tree in (N, M, C).

We define the irreducible problem associated with a problem (N, M, C) following Bird (1976). Let (N, M, C) be a problem and t be a minimal tree in (N, M, C). We define the *minimal network* (N, M, C^t) associated with t where $c^t_{ij} = \max\limits_{\{k,l\} \in g^t_{ij}} \{c_{kl}\}$ and g^t_{ij} denotes the unique path in t from i to j. It is well known that C^t is independent of the chosen t. Then, we can define the *irreducible problem* (N, M, C^*) of (N, M, C) as the minimal network (N, M, C^t) associated with any minimal tree t. We say that C^* is the *irreducible matrix*.

After obtaining a minimal tree, the second issue addressed is how to divide its cost $m(N, M, C)$ among the agents. A (cost allocation) *rule* is a map f that associates with each problem (N, M, C) a vector of cost shares $f(N, M, C) \in \mathbb{R}^N$ such that $\sum\limits_{i \in N} f_i(N, M, C) = m(N, M, C)$.

3 Kruskal sharing rules

This section defines a family of rules obtained by means of Kruskal's algorithm. At each stage the cost of the arc selected by Kruskal's algorithm is paid by the agents following the so called sharing functions. This family of rules is inspired by the family of generalized obligation rules for classical problems introduced in Bergantiños *et al* (2011).

Kruskal's algorithm constructs a minimal tree by sequentially adding the cheapest arc in the network taking care not to form cycles. Formally, let $A^0(C) = \{\{i, j\} : i, j \in N \cup M \text{ and } i \neq j\}$ and $g^0(C) = \emptyset$.

Step 1: Take an arc $\{i, j\} \in A^0(C)$ such that $c_{ij} = \min\limits_{\{k,\ell\} \in A^0(C)} \{c_{k\ell}\}$. If there are several arcs satisfying this condition, select one of them. Let $\{i^1(C), j^1(C)\} = \{i, j\}$, $A^1(C) = A^0(C) \setminus \{i, j\}$ and $g^1(C) = \{i^1(C), j^1(C)\}$.

Step $p+1$ ($p=1,\ldots,|N|+|M|-2$): Take an arc $\{i,j\}\in A^p(C)$ such that $c_{ij}=\min_{\{k,\ell\}\in A^p(C)}\{c_{k\ell}\}$. If there are several arcs satisfying this condition, select one as before. Two cases are possible:

1. If $g^p(C)\cup\{i,j\}$ has a cycle, then go to the beginning of Step $p+1$ with $A^p(C)$ obtained from $A^p(C)$ by deleting $\{i,j\}$, that is, $A^p(C)=A^p(C)\setminus\{i,j\}$, and $g^p(C)$ the same.

2. If $g^p(C)\cup\{i,j\}$ has no cycles, then take $\left\{i^{p+1}(C),j^{p+1}(C)\right\}=\{i,j\}$, $A^{p+1}(C)=A^p(C)\setminus\{i,j\}$, $g^{p+1}(C)=g^p(C)\cup\left\{i^{p+1}(C),j^{p+1}(C)\right\}$, and go to Step $p+2$.

This procedure ends in $|N|+|M|-1$ steps, the minimum number of arcs needed to connect all agents to all sources. The algorithm leads to a tree $g^{|N|+|M|-1}(C)$ which is not necessarily unique. When there is no ambiguity, we write A^p, g^p, and $\{i^p,j^p\}$ instead of $A^p(C)$, $g^p(C)$, and $\{i^p(C),j^p(C)\}$ respectively.

A *sharing function* α is a map that associates with each partition $P=\{S_1,...,S_{|P|}\}\in P(N\cup M)$ a vector $\alpha(P)\in\mathbb{R}^N$ such that $\sum_{i\in N}\alpha_i(P)=|P|-1$.

The interpretation of α is inspired by the generalized obligation functions for classical problems introduced in Bergantiños *et al* (2011). Assume that agents in the same element of P are connected with one another, while agents in different elements of P are not. If all agents want to be connected to all sources, then $|P|-1$ more arcs must be constructed. For each $i\in N$, $\alpha_i(P)$ represents the number of arcs that agent i has to pay and can be interpreted as a measure of the remaining responsibility of agent i.

Given a sharing function α the associated rule f^α is defined as follows. For each problem (N,M,C) and each $i\in N$,

$$f_i^\alpha(N,M,C)=\sum_{p=1}^{|N|+|M|-1}c_{i^pj^p}[\alpha_i(P(g^{p-1}))-\alpha_i(P(g^p))].$$

At each stage of the Kruskal algorithm, an arc (i^p,j^p) is added to the network. Each agent pays the difference between his sharing function before the arc was added and the corresponding function afterwards.

f is a **Kruskal sharing rule** if there is a sharing function α such that $f=f^\alpha$.

In the next proposition, we prove that Kruskal sharing rules are well-defined.

Proposition 1 *Given a sharing function α, f^α is a rule and it does not depend on how the arcs are selected according to Kruskal's algorithm.*

Proof. First[1] it is proved that f^α is a rule, namely $\sum_{i\in N}f_i^\alpha(N,M,C)=m(N,M,C)$. It is possible to prove that the cost of each arc $\{i^p,j^p\}$ is completely allocated among the agents in N at each step of the Kruskal algorithm

Bergantiños *et al* (2017) prove that the rule f^{o^*} (as defined in that paper) is well-defined. Using arguments similar to those used there, it can be proved that f^α does not depend on how the arcs are selected according to Kruskal's algorithm. ■

[1]Because of a suggestion of the Associate Editor and in order to short the paper we only give the idea of the proofs without making the computations. We do the same for the rest of the proofs of the paper.

Remark 1 *As a consequence of Proposition 1, for each sharing function α, and each problem (N, M, C), the allocation $f^{\alpha}(N, M, C)$ can be computed in polynomial time.*

Bergantiños *et al* (2010, 2011) introduce generalized obligation rules in classical problems. Kruskal sharing rules restricted to classical problems induce a family of rules that contains the family of generalized obligation rules. As a consequence, Kruskal sharing rules also contain the family of obligation rules introduced in Tijs *et al* (2006).

Next we provide an axiomatic characterization of the family of Kruskal sharing rules. The following properties are extensions of well-known properties in classical problems.

Independence of irrelevant trees (IIT). Given two different problems (N, M, C) and (N, M, C') that share a common minimal tree t such that $c_{ij} = c'_{ij}$ for each $\{i, j\} \in t$, then $f(N, M, C) = f(N, M, C')$.

This property requires the cost allocation chosen by a rule to depend only on the arcs that belong to a minimal tree. This axiom is introduced in Bergantiños and Vidal-Puga (2007) and also studied in Bogomolnaia and Moulin (2010) under the name of reductionism. An equivalent definition for this property is that for each problem (N, M, C), $f(N, M, C) = f(N, M, C^*)$.

Additivity is a standard property in the literature. In our case additivity says that if there are two problems (N, M, C) and (N, M, C') then, $f(N, M, C+C') = f(N, M, C)+f(N, M, C')$. There is no rule in classical problems that satisfies this property, so there is no rule in the multiple source case that satisfies this property either.

In classical problems there are two versions of additivity. Cone-wise additivity studied, for instance, in Norde *et al* (2004), Bergantiños and Kar (2010), and Bogomolnaia and Moulin (2010), and constrained additivity studied, for instance, in Bergantiños and Vidal-Puga (2009) and Lorenzo and Lorenzo-Freire (2009). Under independence of irrelevant trees, the two properties are equivalent. We use independence of irrelevant trees in all our characterizations here, so our results can be obtained under both additivity properties. Thus, we only present one of them.

Cone-wise additivity (CA). Let (N, M, C) and (N, M, C') be two problems satisfying that there exists an order $\sigma : \{\{i, j\}\}_{i,j \in N\cup M, i<j} \to \left\{1, 2, ..., \frac{|N\cup M|(|N\cup M|-1)}{2}\right\}$ such that for all $i, j, k, l \in N \cup M$ satisfying that $\sigma\{i, j\} \leq \sigma\{k, l\}$, then $c_{ij} \leq c_{kl}$ and $c'_{ij} \leq c'_{kl}$. Thus,

$$f(N, M, C + C') = f(N, M, C) + f(N, M, C').$$

Norde *et al* (2004) prove that every classical problem can be written as a non-negative combination of classical problems where the costs of the arcs are 0 or 1. Even this result it is not mentioned in the text because proofs have been simplified, it will be used in several parts for making such computations.

We now present the main result of this section.

Theorem 1 *A rule f satisfies CA and IIT if and only if f is a Kruskal sharing rule.*

Proof. Let f be a Kruskal sharing rule, then there exists a Kruskal sharing function α such that $f = f^{\alpha}$. Making some computations we can prove that f^{α} satisfies CA and IIT.

We now prove the reciprocal. Let us consider an allocation rule f that satisfies CA and IIT. Given a partition $P = \{S_1, S_2, ..., S_{|P|}\} \in P(N \cup M)$, we define the function $\alpha(P) = f(N, M, C^P)$ where $c_{ij}^P = 0$ if $i, j \in S_k$ for some $k \in \{1, 2, ..., |P|\}$ and $c_{ij}^P = 1$ otherwise. Making some computations it is possible to prove that α is a sharing function satisfying $f = f^\alpha$. ■

The axioms used in this theorem and the other theorems of the paper are independent[2].

4 Characterizations of other families of rules

The set of Kruskal sharing rules is quite large and contains some rules, such as f^{α^1}, which are not very appealing. In this section we characterize several subsets of Kruskal sharing rules by demanding properties in addition to the original family. We first characterize the set of Kruskal sharing rules that satisfy core selection. Then we characterize the set of rules that satisfy core selection and equal treatment of source costs (if the connection cost between two sources increases, all agents must be affected by the same amount). Finally, we show that if symmetry is added to the previous characterization the folk rule is obtained.

We now introduce the well known property of core selection, which is a stability property that states that no group of agents should pay more than the minimal cost of connecting them to all the sources using only their locations. This means that agents in the coalition have no incentives to build their own minimal tree. Formally,

Core selection (CS). Given a problem (N, M, C), a rule f satisfies CS if for each $S \subset N$

$$\sum_{i \in S} f_i(N, M, C) \leq m(S, M, C).$$

Next we prove that a Kruskal sharing rule f^α satisfies CS if and only if, given a partition P and a coalition $S \subset N$, the sum of the sharing function of the members of S is less than or equal to the number of arcs that they need to construct to connect all their components in P to all the sources.

Theorem 2 *A rule f satisfies CA, IIT, and CS if and only if f is a Kruskal sharing rule f^α where α is a sharing function that satisfies the requirement that for each $P \in P(N \cup M)$ and each $S \subset N$,*

$$\sum_{i \in S} \alpha_i(P) \leq |S_k \in P : S_k \cap (S \cup M) \neq \emptyset| - 1.$$

Proof. We first prove "⇒". By Theorem 1 we know that there exists a sharing function α such that $f = f^\alpha$. Since f satisfies CS, making some computations it is possible to prove that the statement of the this theorem holds.

[2]Because of a suggestion of the Associate Editor and in order to short the paper we have removed the proof of this statement.

We now prove "$\Leftarrow$". By Theorem 1 we know that f^α satisfies CA and IIT. Making some computations it is possible to prove that f^α satisfies CS. ■

The next corollary is a straightforward consequence of the previous result.

Corollary 1 *The core of a minimum cost spanning tree with multiple sources is non empty.*

Tijs *et al* (2006) define the family of obligation rules in classical problems. It is straightforward to show that, in classical problems, obligation rules are a subset of the family of rules characterized in Theorem 2.

Bergantiños *et al* (2011) characterize obligation functions with CA, CS, and strong cost monotonicity (if some connection costs increase, no agent ends up better off) in classical problems (Theorem 2 (a)). Strong cost monotonicity implies IIT, so Theorem 2 can be seen as a generalization of that result.

The three properties considered so far (IIT, CA, and CS) are extensions of well known properties of classical problems to our setting. We now consider a property specifically defined for our setting. The idea is very simple: if the cost of the connection between two sources increases, then all agents should be affected by the same amount.

Equal treatment of source costs ($ETSC$). For each pair of problems (N, M, C) and (N, M, C') such that there exist $a, b \in M$ satisfying that $c_{ab} < c'_{ab}$ and $c_{ij} = c'_{ij}$ otherwise, then for each $i, j \in N$

$$f_i(N, M, C') - f_i(N, M, C) = f_j(N, M, C') - f_j(N, M, C).$$

The next theoerem characterizes the set of rules satisfying $ETSC$ and the properties considered in Theorem 2.

Theorem 3 *A rule f satisfies CA, IIT, CS, and $ETSC$ if and only if f is a Kruskal sharing rule f^α where α is a sharing function that satisfies that for each $P \in P(N \cup M)$ and each $i \in N$*

$$\alpha_i(P) = \begin{cases} \dfrac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|} & \text{if } S(P,i) \cap M \neq \emptyset \\ \dfrac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|} + x_i & \text{if } S(P,i) \subset N \end{cases}$$

where for each $S_k \in P$ such that $S_k \subset N$, $\sum_{j \in S_k} x_j = 1$.

Proof. We first prove "$\Rightarrow$". By Theorem 1 we know that $f(N, M, C) = f^\alpha(N, M, C)$. Consider a partition $P = \{S_1, ..., S_\mu, .., S_{|P|}\}$ such that $S_k \cap M \neq \emptyset$ when $k \leq \mu$ and $S_k \subset N$ when $k > \mu$. Note that $|S_k \in P : S_k \cap M \neq \emptyset| = \mu$.

Let us define the following sequence of problems $\{(N, M, C^r)\}_{r=1,2,...,\mu}$ where $C^1 = C^P$ and for each $r > 1$, C^r is obtained from C^{r-1} by decreasing the connection cost between two sources in the following way. Consider $a^{r-1} \in S_{r-1} \cap M$ and $a^r \in S_r \cap M$. Then $c^r_{a^{r-1}a^r} = 0$ and $c^r_{ij} = c^{r-1}_{ij}$ otherwise. Making some computations over the sequence of problems it is possible to prove that the statement of this theorem holds.

We now prove "$\Leftarrow$". Since α satisfies the conditions of Theorem 2, we only need to prove that f^α satisfies $ETSC$. Let (N, M, C) and (N, M, C') be in the conditions of the definition of $ETSC$.Assume there exists a minimal tree t in (N, M, C) such that $\{a, b\} \notin t$. Thus, t is also a minimal tree in (N, M, C') with exactly the same costs. Since f^α satisfies IIT we have that $f^\alpha(N, M, C) = f^\alpha(N, M, C')$.

Assume that $\{a, b\} \in t$ for every minimal tree t in (N, M, C). Let us denote by T the set of trees in (N, M, C) that do not contain $\{a, b\}$. We define,

$$\begin{aligned} x &= \min_{t \in T} c(N, M, C, t) - m(N, M, C) \text{ and} \\ A &= \{\{i, j\} \in t : c_{ab} < c_{ij} < c'_{ab}\}. \end{aligned}$$

Making some computations it is possible to prove that f^α satisfies $ETSC$ by considering the following cases: $c'_{ab} - c_{ab} \leq x$ and $A = \emptyset$; $c'_{ab} - c_{ab} \leq x$ and $A \neq \emptyset$; and $c'_{ab} - c_{ab} > x$. ■

Bergantiños *et al* (2018) prove that the folk rule in the multiple source case is the only rule that satisfies CA, IIT, CS, $ETSC$, and symmetry (symmetric agents, in terms of their connection costs to the rest of the agents and the sources, should pay the same). Thus, the folk rule is the only symmetric rule in the family characterized in Theorem 3. Besides it coincides with the Kruskal sharing rule associated with the following α. For each $i \in N$,

$$\alpha_i(P) = \begin{cases} \frac{\mu-1}{|N|} + \frac{1}{|S_k|} & \text{if } i \in S_k \text{ with } k > \mu \\ \frac{\mu-1}{|N|} & \text{otherwise.} \end{cases}$$

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