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The Location of the Value Theories in the Complex Plane and the Degree of Regularity-Controllability of Actual Economies

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ABSTRACT
This paper expands the spectral analysis of the Sraffian value system, and shows that: (i) the hitherto alternative value theories can be conceived of as "perturbations" of the so-called pure labour value theory; (ii) these theories correspond to specific complex plane locations of the eigenvalues of the vertically integrated technical coefficients matrix; and (iii) the actual economies cannot be coherently analyzed in terms of the traditional value theories, despite the fact that their Krylov (or controllability) matrices are characterized by rather low degrees of regularity-controllability and relatively low numerical ranks. Hence, on the one hand, the Sraffian value theory is not only the most general one but also provides a sound empirical basis, while on the other hand, real-world economies constitute almost irregular-uncontrollable systems, and this explains the specific shape features of the empirical value-wage-profit rate curves.

Key words: Almost irregular-uncontrollable system; Characteristic value distributions; Circulant matrices; Degree of regularity-controllability; Numerical rank; Value theories
JEL classification: B51, C67, D46, D57

1. Introduction

In spite of their fundamental conceptual differences, the value theories of the traditional political economy (classical, Marxian, Austrian, and neoclassical) reduce, formally, to the existence of an unambiguous relationship between, on the one hand, the movement of the long-period relative price of two commodities arising from changes in income distribution and, on the other hand, the difference in the capital intensities of the industries producing these commodities. Since Sraffa's (1960) contribution, it has been gradually recognized, however, that such a relationship does not necessarily exist: Even in a world of fixed input-output coefficients and at least three commodities, produced by means of themselves and homogeneous labour, long-

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period relative prices (or values; Sraffa, 1960, pp. 8-9) can change in a complicated way as income distribution changes. This phenomenon has crucial implications for all the traditional theories of value, capital, distribution and international trade, while its investigation led to the formation of a *new* value theory, namely, the "modern classical or Sraffian value theory", which includes the abovementioned relationship between value variation and capital intensity difference as its special or limit case.

Following a research line that combines the Sraffian analysis and the spectral representation of linear systems (Schefold, 2008; Mariolis and Tsoulfidis, 2009), this paper develops a unified treatment of the value problem on both theoretical and empirical grounds. In particular, it shows that the hitherto alternative value theories correspond to specific complex plane locations of the eigenvalues of the vertically integrated technical coefficients matrix, identifies new aspects of the Sraffian value theory and, finally, zeroes in on the spectral "imprint" of actual value-wage-profit rate systems by detecting the singular value configuration of the relevant Krylov (or controllability) matrices. Hence, it also supports the recently proposed link between the tendency of actual economies to respond as irregular-uncontrollable systems and the specific shape features of the empirical value-wage-profit rate curves (Mariolis and Tsoulfidis, 2018).

The remainder of the paper is structured as follows. Section 2 analyzes the value system and determines the complex plane location of the value theories. Section 3 provides evidence indicating the empirical relevance of the Sraffian value theory and, at the same time, the almost irregularity-uncontrollability of actual economies. Finally, Section 4 concludes the paper.

2. Value Theories and Non-Dominant Eigenvalues

2.1. The value system

Consider a closed, linear and viable economy involving only single products, "*basic*" commodities (in the sense of Sraffa, 1960, pp. 7-8), circulating capital and homogeneous labour. Furthermore, assume that: (i) wages are paid at the end of the common production period; and (ii) the matrix of direct technical coefficients is diagonalizable, i.e. it has a complete set of linearly independent eigenvectors. The value side of the economy is described by[1]

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{l}^{\mathrm{T}} + (1+r)\mathbf{p}^{\mathrm{T}}\mathbf{A} \tag{1}$$

where $\mathbf{p}^{\mathrm{T}}$ denotes a $1xn$ vector of production prices-values, w the money wage rate, r the uniform profit rate, $\mathbf{l}^{\mathrm{T}}$ ($> \mathbf{0}^{\mathrm{T}}$) the $1xn$ vector of direct labour coefficients, and $\mathbf{A}$ the irreducible nxn matrix of direct technical coefficients, with $\lambda_{\mathbf{A}1} < 1$.

After rearrangement, equation (1) becomes

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{v}^{\mathrm{T}} + \rho\mathbf{p}^{\mathrm{T}}\mathbf{J} \tag{2}$$

where $\mathbf{v}^{\mathrm{T}} \equiv \mathbf{l}^{\mathrm{T}}[\mathbf{I}-\mathbf{A}]^{-1}$ ($> \mathbf{0}^{\mathrm{T}}$) denotes the vector of vertically integrated labour coefficients, or the – traditionally – so-called labour values, and $\mathbf{H} \equiv \mathbf{A}[\mathbf{I}-\mathbf{A}]^{-1}$ ($> \mathbf{0}$) the vertically integrated technical coefficients matrix. Moreover, $\rho \equiv rR^{-1}$, $0 \le \rho \le 1$, denotes the relative profit rate, which equals the share of profits in the Sraffian Standard system (SSS), and $R \equiv \lambda_{\mathbf{A}1}^{-1} - 1 = \lambda_{\mathbf{H}1}^{-1}$ the maximum possible profit rate (i.e. the profit rate corresponding to $w = 0$ and $\mathbf{p} > \mathbf{0}$), which equals the ratio of the net product to the means of production in the SSS (Sraffa, 1960, pp. 21-23). Finally, $\mathbf{J} \equiv R\mathbf{H}$ denotes the normalized vertically integrated technical coefficients matrix, $\lambda_{\mathbf{J}1} = R\lambda_{\mathbf{H}1} = 1$, and the moduli of the normalized non-dominant eigenvalues of system (2) are less than those of system (1), i.e. $|\lambda_{\mathbf{J}k}| < |\lambda_{\mathbf{A}k}|\lambda_{\mathbf{A}1}^{-1}$ holds for all k (see, e.g., Mariolis and Tsoulfidis, 2016a, p. 22).

If Sraffa's Standard commodity (SSC) is chosen as the *numéraire*, i.e. $\mathbf{p}^{\mathrm{T}}\mathbf{z} = 1$, where $\mathbf{z} \equiv [\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}1}$ and $\mathbf{l}^{\mathrm{T}}\mathbf{x}_{\mathbf{A}1} = 1$, then equation (1) implies that the "wage-relative profit rate curve" is the following linear relation

$$w = 1 - \rho \tag{3}$$

with $w(0) = 1$ and $w(1) = 0$. Substituting equation (3) in equation (2) yields

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[1] The transpose of an $n \times 1$ vector $\mathbf{x}$ is denoted by $\mathbf{x}^{\mathrm{T}}$, and the diagonal matrix formed from the elements of $\mathbf{x}$ will be denoted by $\hat{\mathbf{x}}$. Furthermore, $\lambda_{\mathbf{A}1}$ will denote the Perron-Frobenius (P-F) eigenvalue of a semi-positive $n \times n$ matrix $\mathbf{A} \equiv [a_{ij}]$, and $(\mathbf{x}_{\mathbf{A}1}, \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}})$ the corresponding eigenvectors, while $\lambda_{\mathbf{A}k}$, $k = 2,...,n$ and $|\lambda_{\mathbf{A}2}| \ge |\lambda_{\mathbf{A}3}| \ge ... \ge |\lambda_{\mathbf{A}n}|$, will denote the non-dominant eigenvalues, and $(\mathbf{x}_{\mathbf{A}k}, \mathbf{y}_{\mathbf{A}k}^{\mathrm{T}})$ the corresponding eigenvectors. Finally, $\mathbf{e}$ will denote the summation vector, i.e. $\mathbf{e} \equiv [1,1,...,1]^{\mathrm{T}}$, $\mathbf{e}_i$ the i th unit vector, and $\mathbf{I}$ the nxn identity matrix.

$$\mathbf{p}^{\mathrm{T}} = (1-\rho)\mathbf{v}^{\mathrm{T}} + \rho\mathbf{p}^{\mathrm{T}}\mathbf{J} \tag{4}$$

or, if $\rho < 1$,

$$\mathbf{p}^{\mathrm{T}} = (1-\rho)\mathbf{v}^{\mathrm{T}}[\mathbf{I}-\rho\mathbf{J}]^{-1} = (1-\rho)\mathbf{v}^{\mathrm{T}}[\mathbf{I}+\rho\mathbf{J}+\rho^2\mathbf{J}^2+\rho^3\mathbf{J}^3+...] \tag{5}$$

which gives the commodity values, expressed in terms of SSC, as polynomial functions of ρ.

Equation (4) indicates that p_j is a convex combination of v_j and $\mathbf{p}^{\mathrm{T}}\mathbf{J}\mathbf{e}_j$, where the latter equals the ratio of means of production in the vertically integrated industry producing commodity j to means of production in the SSS. From this equation it follows that $\mathbf{p}^{\mathrm{T}}(0) = \mathbf{v}^{\mathrm{T}}$ and $\mathbf{p}^{\mathrm{T}}(1)$ is the left P-F eigenvector of $\mathbf{J}$, expressed in terms of SSC, i.e.

$$\mathbf{p}^{\mathrm{T}}(1) = (\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{z})^{-1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}} = (\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}1})^{-1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}$$

or, since $[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}1} = (1-\lambda_{\mathbf{A}1})\mathbf{x}_{\mathbf{A}1}$ and matrices $\mathbf{A}$ and $\mathbf{J}$ have the same set of eigenvectors,

$$\mathbf{p}^{\mathrm{T}}(1) = [(1-\lambda_{\mathbf{A}1})\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1}]^{-1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}} \tag{6}$$

Equation (5) is the reduction of commodity values to “dated quantities” of labour value in terms of SSC. In the general case, therefore, commodity values are ratios of polynomials of degree $n-1$ in ρ and, therefore, may admit up to $2n-4$ extreme points.

Finally, it should be added that:[2]

(i). Non-diagonalizable systems are of measure zero in the set of all systems and, thus, not generic, while given any $\mathbf{A}$ and an arbitrary $\varepsilon \neq 0$, it is possible to perturb the entries of $\mathbf{A}$ by an amount less than $|\varepsilon|$ so that the resulting matrix is diagonalizable.

(ii). If wages are paid *ex ante*, then the wage-relative profit rate curve is non-linear, i.e. $w = (1+R\rho)^{-1}(1-\rho)$, and ρ is no greater than the share of profits in the Sraffian Standard system. Nevertheless, equation (4) holds true.

(iii). These fundamental value relationships remain valid for the cases of (a) fixed capital *à la* Leontief-Bródy; and (b) differential profit and wage rates (provided that these variables exhibit a stable structure in relative terms). For instance, in the former case, $\mathbf{v}^{\mathrm{T}}$ and $\mathbf{H}$ should be replaced by $\mathbf{l}^{\mathrm{T}}[\mathbf{I}-(\mathbf{A}+\mathbf{A}^{\mathrm{D}})]^{-1}$ and $\mathbf{A}^{\mathrm{C}}[\mathbf{I}-(\mathbf{A}+\mathbf{A}^{\mathrm{D}})]^{-1}$,

[2] See Aruka (1991, pp. 74-76); Kurz and Salvadori (1995, chaps. 7, 8 and 11); Mariolis and Tsoulfidis (2016a, pp. 22-32); Schefold (1971); Sraffa (1960, Part 2).

respectively, where $\mathbf{A}^{\mathrm{D}}$ denotes the matrix of depreciation coefficients, and $\mathbf{A}^{\mathrm{C}}$ the matrix of capital stock coefficients. Nevertheless, the said value relationships do not necessarily remain valid for the joint production case.

2.2. Regular-controllable and irregular-uncontrollable economies

An $n-$ economy is said to be "regular of rank n" or "completely regular" iff the nxn Krylov matrix

$$\mathbf{K} \equiv [\mathbf{p}(0), \mathbf{J}^{\mathrm{T}}\mathbf{p}(0), ..., (\mathbf{J}^{\mathrm{T}})^{n-1}\mathbf{p}(0)]^{\mathrm{T}}$$

has rank equal to n or, equivalently, iff $\mathbf{p}^{\mathrm{T}}(0)$ is not orthogonal to any (real or complex) right eigenvector of $\mathbf{J}$. In that case, the value vectors relative to any n distinct values of the profit rate are linearly independent (see Bidard and Salvadori, 1995; Kurz and Salvadori, 1995, chap. 6).

By contrast, iff $rank[\mathbf{K}] = m < n$, then the economy is said to be "irregular" or, more specifically, "regular of rank m". This means that the value vectors relative to any $m+1$ distinct values of the profit rate are linearly dependent. In that case, there is a vector $\mathbf{z}'$ such that $\mathbf{Kz}' = \mathbf{0}$ and, therefore, $\mathbf{p}^{\mathrm{T}}\mathbf{z}' = 0$ (see equation (5)). Hence, a change of *numéraire*, from $\mathbf{z}$ to $\mathbf{z} + \zeta\mathbf{z}'$, where ζ is a given scalar, has no effect on the wage rate and commodity values (Miyao, 1977). Iff the dimension of an eigenspace associated with an eigenvalue of $\mathbf{J}$ is larger than 1 or, equivalently, iff $\mathbf{J}$ satisfies a polynomial equation of degree less than n, then the economy is irregular whatever $\mathbf{p}^{\mathrm{T}}(0)$ is (see Ford and Johnson, 1968).

The concepts of "regularity/irregularity" (introduced by Schefold, 1971) are algebraically equivalent to those of "controllability/uncontrollability" (introduced by Kalman, 1960). The latter concepts apply to the following dynamic version of the value system:

$$\mathbf{p}_{t+1}^{\mathrm{T}} = w_t\mathbf{p}^{\mathrm{T}}(0) + \bar{\rho}\mathbf{p}_t^{\mathrm{T}}\mathbf{J},\ t = 0,1,...$$

where $\bar{\rho}$ denotes the exogenously given nominal relative profit rate, and $\mathbf{p}_0 = \mathbf{0}$ (Mariolis, 2003). Iff $rank[\mathbf{K}] = n$, then this dynamic system is said to be "completely controllable", which means that the initial state $\mathbf{p}_0$ can be transferred, by application of w_t, to any state, in some finite time.

This approach provides only a *yes/no* criterion for complete regularity-controllability, while irregular-uncontrollable systems are of measure zero in the set of all systems and, thus, not generic or, in other words, systems are *always* almost regular-controllable (Kalman et al., 1963; for a recent discussion, see Cowan et al., 2012). However,

> [i]n the *real* world […] it may not be possible to make such sharp distinctions. The problem with the standard definition of controllability […] is that it can lead to discontinuous functions of the system parameters: an arbitrarily small change in a system parameter can cause an abrupt change in the rank of the matrix by which controllability […] is determined. It would be desirable to have definitions which can vary continuously with the parameters of the system and thus can reflect the *degree of controllability* of the system. Kalman et a1. (1963) recognized the need and suggested using the determinant of the corresponding test matrix [$\mathbf{K}$] as a measure of the degree of controllability […]. Friedland (1975), noting that basing the degree of controllability […] on the determinant of the test matrix suffers from sensitivity to the scaling of the state variables, suggested using the ratio of the smallest of the singular values to the largest as a preferable measure. (Friedland, 1986, p. 220; emphasis added)

In this connection, therefore, matrix $\mathbf{J}$ can be decomposed as ("spectral decomposition"; see, e.g. Meyer 2001, 517-518)

$$\mathbf{J} = (\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1})^{-1}\mathbf{x}_{\mathbf{J}1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}} + \sum_{k=2}^{n}\lambda_{\mathbf{J}k}(\mathbf{y}_{\mathbf{J}k}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}k})^{-1}\mathbf{x}_{\mathbf{J}k}\mathbf{y}_{\mathbf{J}k}^{\mathrm{T}} \tag{7}$$

or $\mathbf{J} = \mathbf{X}_{\mathbf{J}}\hat{\boldsymbol{\lambda}}_{\mathbf{J}}\mathbf{X}_{\mathbf{J}}^{-1}$, where $\mathbf{X}_{\mathbf{J}}$ and the diagonal matrix $\hat{\boldsymbol{\lambda}}_{\mathbf{J}}$ are matrices formed from the right eigenvectors and the eigenvalues of $\mathbf{J}$, respectively, while $\mathbf{X}_{\mathbf{J}}^{-1}$ equals the matrix formed from the left eigenvectors of $\mathbf{J}$. Equation (7) implies, in turn, that the Krylov matrix can be expressed as a product of three matrices:

$$\mathbf{K} = \mathbf{V}\hat{\boldsymbol{\omega}}^{\mathrm{T}}\mathbf{X}_{\mathbf{J}}^{-1}$$

where $\mathbf{V} \equiv (\lambda_{\mathbf{J}j})^{i-1}$ denotes the Vandermonde matrix of the eigenvalues of $\mathbf{J}$, and $\hat{\boldsymbol{\omega}}$ the diagonal matrix formed from the elements of $\boldsymbol{\omega}^{\mathrm{T}} \equiv \mathbf{p}^{\mathrm{T}}(0)\mathbf{X}_{\mathbf{J}}$. Consequently, the determinant of $\mathbf{K}$ is given by

$$\det[\mathbf{K}] = \det[\mathbf{V}]\det[\hat{\boldsymbol{\omega}}^{\mathrm{T}}]\det[\mathbf{X}_{\mathbf{J}}^{-1}] \tag{8}$$

where $\det[\mathbf{V}] = \prod_{1\le i<j\le n}(\lambda_{\mathbf{J}j} - \lambda_{\mathbf{J}i})$. Finally, the "degree of regularity-controllability" is defined as

$$DR \equiv \sigma_{\mathbf{K}n}\sigma_{\mathbf{K}1}^{-1} \tag{9}$$

where $0 \leq DR < 1$, and $\sigma_{\mathbf{K}1}$, $\sigma_{\mathbf{K}n}$ denote the largest and the smallest singular values of $\mathbf{K}$, respectively, while DR^{-1} equals the "condition number" of $\mathbf{K}$. When $DR = 0$, the economy is irregular-uncontrollable; otherwise, it is completely regular-controllable. Nevertheless, when the value of DR is "very small", the regularity-controllability is "weak" or "poor"; in other words, the economy is said to be "almost irregular-uncontrollable".[3]

2.3. Value theories

In the Ricardo-Marx-Dmitriev-Samuelson "equal value compositions of capital" case, $\mathbf{l}^{\mathrm{T}}$ ($\mathbf{v}^{\mathrm{T}}$) is the left P-F eigenvector of $\mathbf{A}$ (of $\mathbf{J}$). Therefore, commodity values are independent of income distribution, and equal to the labour values, i.e. $\mathbf{p}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}(0) = \mathbf{p}^{\mathrm{T}}(1)$, or, in other words, the "pure labour value theory" (PLVT) appears to hold true. In that case, the economy is regular of rank 1 irrespective of the rank of $\mathbf{J}$.

In the two-industry case, the elements of the value vector as functions of the relative profit rate, $\mathbf{p}^{\mathrm{T}}(\rho) \equiv [p_j(\rho)]$, are necessarily monotonic and, therefore, the direction of relative value movement is governed only by the differences in the relevant capital intensities ("capital-intensity effect"; see Kurz and Salvadori, 1995, chap. 3; Pasinetti 1977, pp. 82-84), as in the various versions of the "traditional value theory" (TVT), i.e. classical (Ricardo, 1951, p. 46), Marxian (Marx, [1894] 1959, chaps. 11 and 12), Austrian (Böhm-Bawerk, [1889] 1959, vol. 2, pp. 86 and 356-358; von Weizsäcker, 1977), and neoclassical (see, e.g., Stolper and Samuelson, 1941; Kemp, 1973).

However, as Sraffa (1960) pointed out, leaving aside these two restrictive cases, changes in income distribution can activate complex capital revaluation effects, which imply that the direction of relative value movement is governed not only by the differences in the relevant capital intensities but also by the movement of the relevant capital intensities ("value effect") arising from changes in relative commodity values:

[3] According to an alternative approach, the largest difference (or "gap") between consecutive singular values of $\mathbf{K}$ provides a measure of the distance of a regular-controllable pair $[\mathbf{J}, \mathbf{p}^{\mathrm{T}}(0)]$ to the nearest irregular-uncontrollable pair or, in other words, the order of perturbation needed to transform a regular-controllable system into an irregular-uncontrollable one (Boley and Lu, 1986).

> [T]he means of production of an industry are themselves the product of one or more industries which may in their turn employ a still lower proportion of labour to means of production (and the same may be the case with these latter means of production; and so on) (pp. 14-15). […] [A]s the wages fall the price of the product of a low-proportion […] industry may rise or it may fall, or it may even alternate in rising and falling, relative to its means of production (p. 15). […] The reversals in the direction of the movement of relative prices, in the face of unchanged methods of production, cannot be reconciled with *any* notion of capital as measurable quantity independent of distribution and prices. (p. 38; Sraffa, 1960)

Indeed, differentiation of equation (4) with respect to ρ finally gives (for a detailed analysis, see Mariolis and Tsoulfidis, 2016a, pp. 40-45)

$$\dot{p}_j \equiv dp_j / d\rho = Rv_j(\mathrm{CIE} + \rho\mathrm{VE})$$

where $\mathrm{CIE} \equiv \kappa_j - R^{-1}$ denotes the traditional or capital-intensity effect, $\kappa_j \equiv (\mathbf{p}^{\mathrm{T}}\mathbf{H}\mathbf{e}_j)v_j^{-1}$ the capital-intensity of the vertically integrated industry producing commodity j, R^{-1} the capital-intensity of the SSS, which is independent of prices and income distribution (since, in our case, SSC is the *numéraire*), and $\mathrm{VE} \equiv \dot{\kappa}_j = (\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{H}\mathbf{e}_j)v_j^{-1}$ the Sraffian or value effect, which depends on the *entire* economic system and, therefore, is not predictable at the level of any single industry.

Hence, when these two effects have opposite signs, i.e. $\mathrm{CIE} > (<)\ 0$ and $\mathrm{VE} < (>)\ 0$, the traditional statement about the direction of relative value movements does not necessarily hold true, while the underlying phenomena call for a new approach to value theory and, therefore, form the basis of the "Sraffian value theory" (SVT). In effect, all statements and relationships derived from the TVT framework cannot, in general, be extended beyond a world where: (i) there are no produced means of production; or (ii) there are produced means of production, while the profit rate on the value of those means of production is zero; or, finally, (iii) that profit rate is positive, while the economy produces one and only one, single or composite, commodity (Garegnani, 1970; Salvadori and Steedman, 1985). Consequently, it can be stated that the difficulties of the TVT arise from the existence of complex interindustry linkages in the realistic case of production of commodities *and* positive profits by means of commodities.

2.4. "Perturbing" the pure labour value theory

Equations (4) through (7) imply that, from a value theory viewpoint, it suffices to focus on the following seven *ideal-type* (in the Weberian sense) cases:[4]

Case 1: The economy *tends* to be decomposed into n quasi-similar self-reproducing vertically integrated industries, i.e. $\mathbf{J} \approx \mathbf{I}$. It follows that $\lambda_{\mathbf{J}k} \approx 1$ and $\mathbf{p}^{\mathrm{T}} \approx \mathbf{p}^{\mathrm{T}}(0)$. Hence, the economy tends to behave as a one-industry economy, and the PLVT tends to hold true. When $\mathbf{J} = \mathbf{I}$, the economy is regular of rank 1, irrespective of the direction of the labour value vector, $\mathbf{p}^{\mathrm{T}}(0)$.

Case 2: There are strong quasi-linear dependencies amongst the technical conditions of production in all the vertically integrated industries, i.e. $|\lambda_{\mathbf{J}k}| \approx 0$ or $\mathbf{J} \approx (\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1})^{-1}\mathbf{x}_{\mathbf{J}1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}$. It follows that

$$\mathbf{p}^{\mathrm{T}} \approx (1-\rho)\mathbf{p}^{\mathrm{T}}(0) + \rho\mathbf{p}^{\mathrm{T}}(1)$$

namely, $\mathbf{p}^{\mathrm{T}}(\rho)$ tends to be a convex combination of the extreme, economically significant, values of the value vector, $\mathbf{p}^{\mathrm{T}}(0)$ and $\mathbf{p}^{\mathrm{T}}(1)$, and, therefore, linear.

When $|\lambda_{\mathbf{J}k}| = 0$, we obtain a "rank-one economy", i.e. $rank[\mathbf{J}] = 1$, which exhibits the following two essential characteristics:

(i). Irrespective of the direction of $\mathbf{p}^{\mathrm{T}}(0)$, it holds that

$$\mathbf{p}^{\mathrm{T}}(0)\mathbf{J}^{h} = [(1-\lambda_{\mathbf{A}1})\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1}]^{-1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}(1),\ h = 1,2,...$$

since

$$\mathbf{J}^{h} = (\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1})^{-h}(\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1})^{h-1}\mathbf{x}_{\mathbf{J}1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}} = \mathbf{J}$$

Hence, $rank[\mathbf{K}] = 2$ and, therefore, the economy is irregular, i.e. regular of rank 2.

(ii). It is equivalent, via Schur's triangularization (see, e.g., Meyer, 2001, pp. 508-509), to an *economically* significant and generalized ($1 \times n-1$) Marx-Fel'dman-Mahalanobis (or, in more traditional terms, "corn-tractor") economy (Mariolis, 2015, p. 270). Hence, it behaves as a reducible two-industry economy without "self-reproducing non-basics" (in the sense of Sraffa, 1960, Appendix B).

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[4] The first five cases have been extensively analyzed in the literature: Schefold (2008, 2013), Mariolis and Tsoulfidis (2009, 2016a, pp. 154-157, 2018). Thus, here we report, without detailed proofs, the main findings that are directly relevant for our present purposes. To the best of our knowledge, the other two cases have not been addressed in the literature. Examples illustrating these two cases are given in the Appendix.

Consequently, on the one hand, the value side of a rank-one economy is “a little” more complex than that of the PLVT economy ($\mathbf{J} = \mathbf{I}$) and, at the same time, much simpler than that of a completely regular economy. In fact, its value side corresponds to that of the TVT. On the other hand, a rank-one economy can be fully described by a triangular matrix with only n positive technical coefficients and, therefore, its production structure is “a little” more complex than that of Austrian-type economies, where the technical coefficients matrix is, by assumption, strictly triangular (see, e.g., Burmeister, 1974).

Case 3: Consider a rank-one perturbation of the PLVT economy, i.e. $\mathbf{J} \approx (1+\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi})^{-1}[\mathbf{I}+\boldsymbol{\chi}\boldsymbol{\psi}^{\mathrm{T}}]$ $(\geq \mathbf{0})$. It follows that $\lambda_{\mathbf{J}k} \approx (1+\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi})^{-1}$ and

$$\mathbf{p}^{\mathrm{T}} \approx (1-\rho\lambda_{\mathbf{J}k})^{-1}[(1-\rho)\mathbf{p}^{\mathrm{T}}(0)+\rho(1-\lambda_{\mathbf{J}k})\mathbf{p}^{\mathrm{T}}(1)]$$

namely, $p_j(\rho)$ tend to be rational functions of degree 1 and, therefore, monotonic. Hence, for $-\infty << \boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi} << 0$ or $0 << \boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi} << +\infty$, the economy tends to behave as a two-industry economy with only basic commodities, and the TVT tends to hold true. As $\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi} \to 0$ ($\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi} \to \pm\infty$), we obtain Case 1 (Case 2).

Case 4: Consider a rank-two perturbation of the PLVT economy, i.e. $\mathbf{J} \approx (1+\lambda_{\boldsymbol{\Psi}1})^{-1}[\mathbf{I}+\sum_{\kappa=1}^{2}\boldsymbol{\chi}_{\kappa}\boldsymbol{\psi}_{\kappa}^{\mathrm{T}}]$, where $\boldsymbol{\chi}_{\kappa}$, $\boldsymbol{\psi}_{\kappa}^{\mathrm{T}}$, are semi-positive vectors (or two pairs of complex conjugate vectors), and $\boldsymbol{\Psi} \equiv [\boldsymbol{\psi}_1, \boldsymbol{\psi}_2]^{\mathrm{T}}[\boldsymbol{\chi}_1, \boldsymbol{\chi}_2]$ (in either case, $\boldsymbol{\Psi}$ is a 2×2 matrix with only real eigenvalues). It follows that $n-2$ non-dominant eigenvalues of $\mathbf{J}$ tend to equal $(1+\lambda_{\boldsymbol{\Psi}1})^{-1}$, and the remaining tends to equal $(1+\lambda_{\boldsymbol{\Psi}2})(1+\lambda_{\boldsymbol{\Psi}1})^{-1}$. Hence, the economy tends to behave as a three-industry economy; and the same holds true when $\lambda_{\mathbf{J}k} \approx \alpha_k \pm i\beta_k$, where $i \equiv \sqrt{-1}$ and $0 << |\beta_k|$, for all k.[5]

Case 5: The subdominant eigenvalues are complex, $\lambda_{\mathbf{J}2,3} \approx \alpha_{2,3} \pm i\beta_{2,3}$, where $0 << |\beta_{2,3}|$, and $\lambda_{\mathbf{J}4} \approx ... \approx \lambda_{\mathbf{J}n} \approx 0$. Hence, the economy tends to behave as a reducible four-industry economy without self-reproducing non-basics. Both in Cases 4 and 5, the value functions *may* be non-monotonic.

Case 6: Matrix $\mathbf{J}$ is *doubly* stochastic, i.e. $\mathbf{e}^{\mathrm{T}}\mathbf{J} = \mathbf{e}^{\mathrm{T}}$ and $\mathbf{J}\mathbf{e} = \mathbf{e}$. From equation (6) it follows that

[5] Any complex number is an eigenvalue of a positive 3×3 circulant matrix (Minc 1988, p. 167). For the properties of the circulant matrices, see Davis (1979).

$$\mathbf{p}^{\mathrm{T}}(1) = [(1-\lambda_{\mathbf{A}1})n]^{-1}(\mathbf{l}^{\mathrm{T}}\mathbf{e})\mathbf{e}^{\mathrm{T}}$$

or, since $\mathbf{v}^{\mathrm{T}}\mathbf{e} = (1-\lambda_{\mathbf{A}1})^{-1}(\mathbf{l}^{\mathrm{T}}\mathbf{e})$ and $\mathbf{p}^{\mathrm{T}}(0) = \mathbf{v}^{\mathrm{T}}$,

$$\mathbf{p}^{\mathrm{T}}(1) = \overline{p}(0)\mathbf{e}^{\mathrm{T}} \tag{10}$$

where $\overline{p}(0) \equiv n^{-1}(\mathbf{p}^{\mathrm{T}}(0)\mathbf{e})$ equals the arithmetic mean of the elements of the labour value vector. Hence, if there is a commodity whose labour value equals the arithmetic mean of the labour values, then, by Rolle's theorem, its value curve will necessarily have at least one stationary point in the economically relevant interval of the profit rate.[6]

Case 7: Since $\mathbf{A} = [\mathbf{I}+\mathbf{H}]^{-1}\mathbf{H}$, there is no good economic reason for supposing that $\mathbf{J}$ is doubly stochastic. It should be noted, however, that:

(i). Any doubly stochastic matrix can be expressed as a convex combination of at most $(n-1)^2+1$ permutation matrices (see, e.g., Minc, 1988, pp. 117-122).

(ii). Matrix $\mathbf{J}$ is similar to the *column* stochastic matrix $\mathbf{M} \equiv \hat{\mathbf{y}}_{\mathbf{J}1}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{J}1}^{-1}$:

$$\mathbf{e}^{\mathrm{T}}\mathbf{M} = \mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{J}1}^{-1} = \mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\hat{\mathbf{y}}_{\mathbf{J}1}^{-1} = \mathbf{e}^{\mathrm{T}}$$

The elements of $\mathbf{M}$ are independent of both the choice of physical measurement units and the normalization of $\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}$. Matrix $\mathbf{M}$ can be conceived of as a matrix of the relative shares of the capital goods in the cost of outputs, evaluated at $\rho = 1$, or, alternatively, as derived from $\mathbf{J}$ by changing the units in which the various commodity quantities are measured (see Ara, 1963).[7] Moreover, the Dmitriev and Dynkin (1946) and Karpelevich (1951) inequalities for stochastic matrices imply that

$$\alpha_k + |\beta_k|\tan(\pi n^{-1}) \le 1 \tag{11}$$

for each eigenvalue $\lambda_{\mathbf{M}k}\ (= \lambda_{\mathbf{J}k}) = \alpha_k \pm i\beta_k$.

(iii). Finally, when there is only one commodity input in each industry (i.e. industry κ, $\kappa = 1,2,...,n-1$, produces the input for industry $\kappa+1$, and industry n produces the input for industry 1), $\mathbf{A}$ is imprimitive or "cyclic" (see Solow, 1952, pp. 35-36 and 40-41; Schefold, 2008, pp. 8-14). Therefore, $\mathbf{M}$ is circulant and doubly stochastic (see Mariolis and Tsoulfidis, 2016a, pp. 165-167).

[6] See Example 1 in the Appendix.

[7] When $rank[\mathbf{J}] = 1$, all the columns of $\mathbf{M}$ are equal to each other (see Mariolis and Tsoulfidis, 2016a, p. 48).

Thus, hereafter, we consider a "*basic* circulant" perturbation of the PLVT economy, i.e.

$$\mathbf{J} \approx \mathbf{C} \equiv c\mathbf{I} + (1-c)\mathbf{\Pi}$$

where $0 \leq c < 1$, $\mathbf{\Pi} \equiv \text{circ}[0,1,0,...,0]$ is the basic circulant permutation (or shift) matrix (post-multiplying any matrix by $\mathbf{\Pi}$ shifts its columns one place to the right), and $\mathbf{\Pi}^n = \mathbf{I}$.

The eigenvalues of the circulant doubly stochastic matrix $\mathbf{C}$ are $c + (1-c)\theta^{\kappa}$, where $\kappa = 0,1,...,n-1$, $\theta \equiv \exp(2\pi i n^{-1})$, and

$$\theta^{\kappa} = \cos(2\pi\kappa n^{-1}) + i\sin(2\pi\kappa n^{-1})$$

are the n distinct roots of unity. It then follows that:

(i). The eigenvalues of $\mathbf{C}$ are the vertices of a regular $n-$ gon, and $\mathbf{C}$ is that stochastic matrix that has an "extremal eigenvalue" on the segment joining the points 1 and θ (Dmitriev and Dynkin, 1946; Karplevich, 1951).[8] This eigenvalue is a subdominant eigenvalue, which satisfies relation (11) as an *equality*.

(ii). For $0 < c < 1$, the moduli of the eigenvalues of $\mathbf{C}$ are given by

$$\sqrt{c^2 + 2c(1-c)\cos(2\pi\kappa n^{-1}) + (1-c)^2}$$

or, equivalently,

$$\sqrt{1 + 2c(1-c)[\cos(2\pi\kappa n^{-1}) - 1]}$$

which is a symmetric function with respect to $c = 0.5$ and κ', κ'', where $\kappa' + \kappa'' = n$ (Davis, 1979, pp. 119-120). The modulus of the subdominant eigenvalues occurs for $\kappa = 1,\ n-1$. When n is even, i.e. $n = 2\mu$, the smallest modulus occurs for $\kappa = \mu$, and equals $|1-2c|$, while when n is odd, $n = 2\mu + 1$, the smallest modulus occurs for $\kappa = \mu,\ \mu{+}1$. Finally, $\mathbf{C}$ has rank $n-1$ iff n is even and $c = 0.5$ (Davis, 1979, p. 147). For instance, Figure 1 displays the location of the eigenvalues of $\mathbf{C}$ in the complex plane, for $n = 3,\ 6$ and $c = 0, 0.25, 0.75$.[9]

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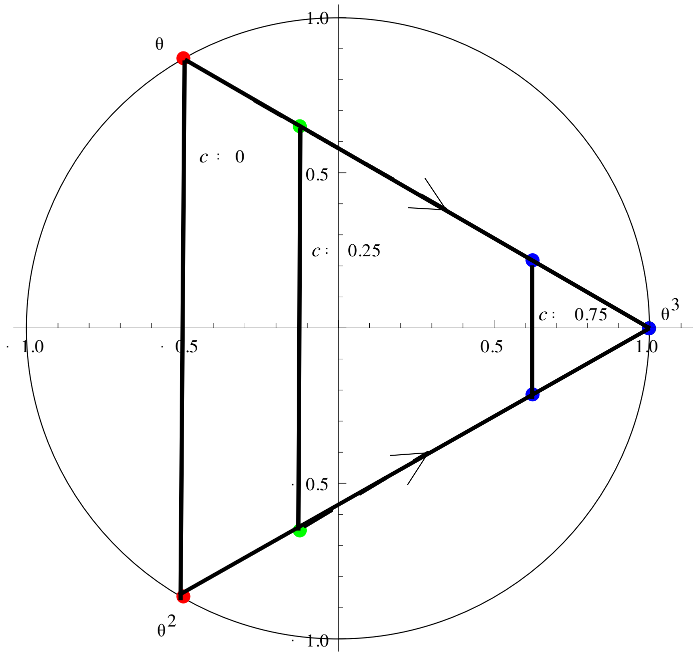
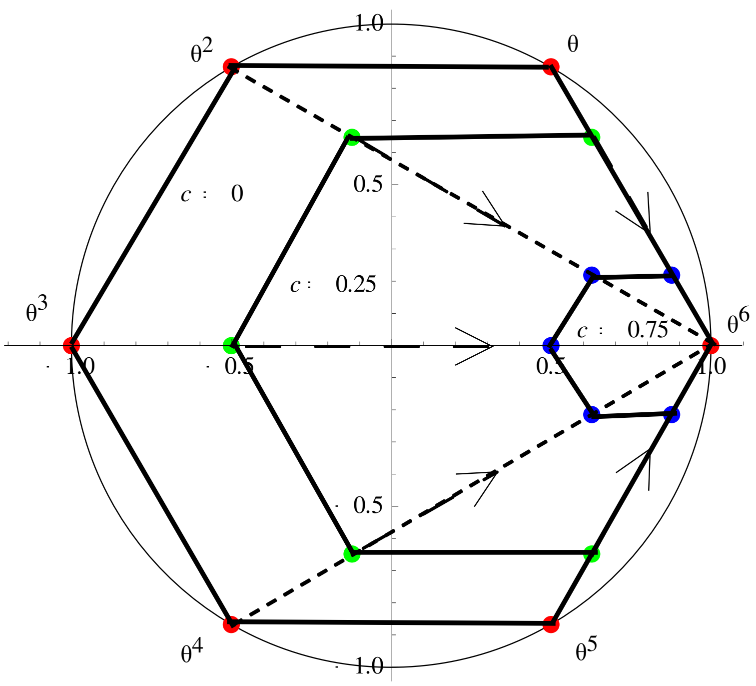
[8] A number λ is called extremal eigenvalue if (i) it belongs to the set of eigenvalues of a stochastic matrix; and (ii) $\alpha\lambda$ does not belong to this set, whenever $\alpha > 1$.

[9] Furthermore, see Example 2 in the Appendix.



$n = 3$, $c = 0$, 0.25, 0.75



$n = 6$, $c = 0$, 0.25, 0.75

Figure 1. The location of the eigenvalues of **C** in the complex plane; $n = 3$, 6 and $c = 0, 0.25, 0.75$

Now we turn to the value side of the economies $[\mathbf{C},\ \mathbf{p}^{\mathrm{T}}(0)]$, $0 \le c < 1$. Ignoring the approximation error, equation (4) reduces to

$$\mathbf{p}^{\mathrm{T}} = (1-\gamma)\mathbf{p}^{\mathrm{T}}(0) + \gamma\mathbf{p}^{\mathrm{T}}\mathbf{\Pi} \tag{12}$$

where $\gamma \equiv (1-c)\rho(1-\rho c)^{-1}$, $0 \le \gamma \le 1$. Hence, since $\mathbf{\Pi}^n = \mathbf{I}$, equation (5) reduces to

$$\mathbf{p}^{\mathrm{T}} = (1-\gamma)(1-\gamma^n)^{-1}\mathbf{p}^{\mathrm{T}}(0)[\mathbf{I} + \gamma\mathbf{\Pi} + \gamma^2\mathbf{\Pi}^2 + ... + \gamma^{n-1}\mathbf{\Pi}^{n-1}]$$

or, since $(1-\gamma)(1-\gamma^n)^{-1} = (1+\gamma+\gamma^2+...+\gamma^{n-1})^{-1}$,

$$\mathbf{p}^{\mathrm{T}} = (1+\gamma+\gamma^2+...+\gamma^{n-1})^{-1}\mathbf{p}^{\mathrm{T}}(0)[\mathbf{I} + \gamma\mathbf{\Pi} + \gamma^2\mathbf{\Pi}^2 + ... + \gamma^{n-1}\mathbf{\Pi}^{n-1}] \tag{13}$$

From equations (12) and (13) it follows that:

(i). Although matrix **C** is irreducible, commodity values reduce to a *finite* series of dated quantities of vertically integrated labour. Hence, it can be stated that these "basic circulant economies" bear some characteristic similarities with the "wine-oak" economy example constructed by Sraffa (1960, pp. 37-38). And "this example is a crucial test for the [traditional] ideas of a quantity of capital and of [an average] period of production." (Sraffa, 1962, p. 478).

(ii). In fact, because of the structure of the economies' matrices, commodity values are governed by the terms

$$\delta_\kappa \equiv (1+\gamma+\gamma^2+...+\gamma^{n-1})^{-1}\gamma^\kappa,\ \kappa = 0,1,...,n-1$$

where the denominator has either no real roots (when n is odd) or one real root (i.e. -1, when n is even). The first derivative of δ_κ with respect to γ is

$$\dot{\delta}_\kappa = (1-\gamma^n)^{-2}\gamma^{\kappa-1}(\varepsilon_\kappa + \zeta_\kappa)$$

where $\varepsilon_\kappa \equiv \kappa - (1+\kappa)\gamma$ defines a linear function, and $\zeta_\kappa \equiv [(n-\kappa)-(n-\kappa-1)\gamma]\gamma^n$ defines a polynomial function. Hence, we get $\dot{\delta}_0(0) = -1$, $\dot{\delta}_1(0) = 1$ and $\dot{\delta}_\kappa(0) = 0$, for $\kappa \ge 2$, while $\dot{\delta}_\kappa(1) = (2n)^{-1}(1+2\kappa-n)$. Moreover, when $\kappa \ge 2$ is even (is odd), δ_κ has a minimum (an inflection point) at $\gamma = 0$. Finally, iff $1 \le \kappa < 2^{-1}(n-1)$ and $n \ge 4$, then the equation $\varepsilon_\kappa + \zeta_\kappa = 0$ has two roots in the interval $[0,1]$, i.e. $\gamma = \gamma_\kappa^*$ (unique), where $0 < \gamma_\kappa^* < 1$, at which δ_κ is maximized, and $\gamma = 1$ (repeated), where $\dot{\varepsilon}_\kappa(1) + \dot{\zeta}_\kappa(1) = 0$ (in all other cases, it has, in the said interval, the roots 0 and/or 1). For instance, Figure 2 displays the terms δ_κ as functions of γ, for $n = 7$: $\gamma_1^* \cong 0.517$, $\gamma_2^* \cong 0.768$, and δ_3 has a maximum at $\gamma = 1$. The values γ_κ^* tend to the values of the sequence $(1+\kappa)^{-1}\kappa$ as n tends to infinity and, therefore, the maximum values of δ_κ tend to the values of the sequence $(1+\kappa)^{-(1+\kappa)}\kappa^\kappa$.

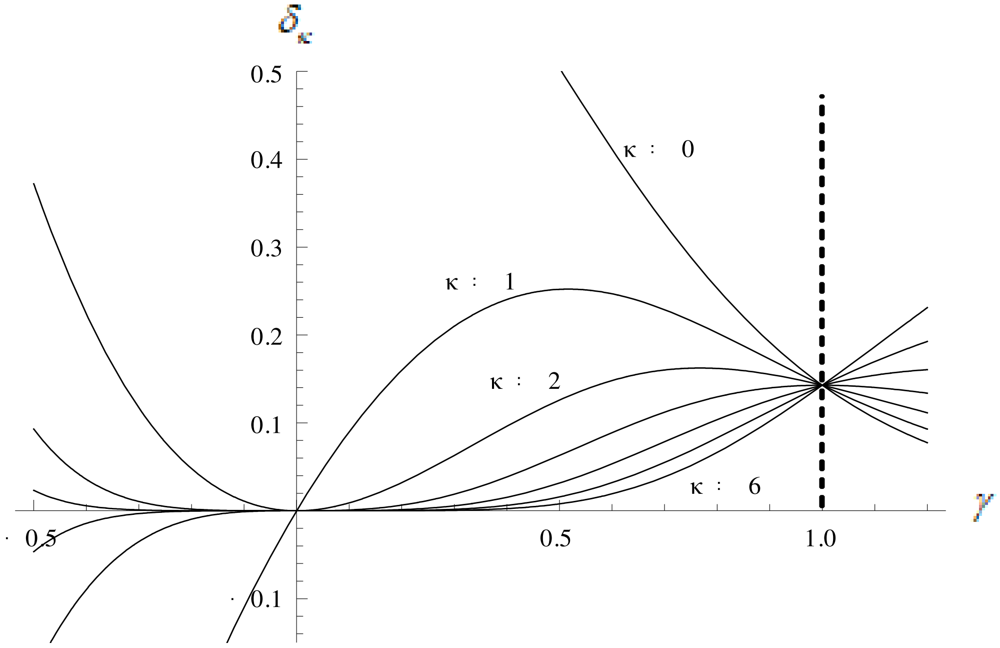


Figure 2. The rational function terms that govern the commodity values in basic circulant economies; $n = 7$, $-0.5 \leq \gamma \leq 1.2$

(iii). Commodity values tend to $\mathbf{p}^{\mathrm{T}}(0)\mathbf{\Pi}^{n-1}$ as γ tends to plus or minus infinity. Iff there exists a non-zero value of γ, say γ^{**}, such that $p_j(\gamma^{**}) = p_j(0)$, then

$$p_j(\gamma^{**}) = \mathbf{p}^{\mathrm{T}}(\gamma^{**})\mathbf{\Pi}\mathbf{e}_j = p_{j-1}(\gamma^{**})$$

where $j = 1, 2, ..., n$ and $p_0(\gamma^{**}) \equiv p_n(\gamma^{**})$.

(iv). Finally, differentiation of equation (12) with respect to ρ gives

$$\dot{\mathbf{p}}^{\mathrm{T}} = -\dot{\gamma}(\mathbf{p}^{\mathrm{T}}(0) - \mathbf{p}^{\mathrm{T}}\mathbf{\Pi}) + \gamma\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{\Pi} \tag{14}$$

$$\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{e} = 0 \tag{15}$$

where $\dot{\gamma} \equiv (1-c)(1-\rho c)^{-2} > 0$, the difference $\mathbf{p}^{\mathrm{T}}\mathbf{\Pi} - \mathbf{p}^{\mathrm{T}}(0)$ represents the capital-intensity effect, while the term $\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{\Pi}$ represents the value effect. Now, it suffices to focus on the extreme, economically significant, values of ρ:

(a). At $\rho = 0$ equation (14) reduces to

$$\dot{\mathbf{p}}^{\mathrm{T}}(0) = -(1-c)^{-1}\mathbf{p}^{\mathrm{T}}(0)\mathbf{D} \tag{16}$$

where $\mathbf{D} \equiv \mathbf{I} - \mathbf{\Pi}$ is a circulant double-centered matrix, since all its columns and rows sum to zero, i.e. $\mathbf{e}^{\mathrm{T}}\mathbf{D} = \mathbf{0}^{\mathrm{T}}$, $\mathbf{De} = \mathbf{0}$, and $rank[\mathbf{D}] = n-1$.

(b). At $\rho = 1$ equation (14) reduces to

$$\dot{\mathbf{p}}^{\mathrm{T}}(1) = -(1-c)^{-1}(\mathbf{p}^{\mathrm{T}}(0) - \mathbf{p}^{\mathrm{T}}(1)\mathbf{\Pi}) + \dot{\mathbf{p}}^{\mathrm{T}}(1)\mathbf{\Pi}$$

or, rearranging terms and invoking equations (10) and $\mathbf{e}^{\mathrm{T}}\mathbf{\Pi} = \mathbf{e}^{\mathrm{T}}$,

$$\dot{\mathbf{p}}^{\mathrm{T}}(1)\mathbf{D} = -(1-c)^{-1}\mathbf{p}^{\mathrm{T}}(0)\mathbf{F} \tag{17}$$

where $\mathbf{F} \equiv \mathbf{I} - n^{-1}(\mathbf{e}\mathbf{e}^{\mathrm{T}})$ is the centering matrix, which is symmetric and idempotent (multiplication of any vector by the centering matrix has the effect of subtracting its arithmetic mean from every element). The solution to equations (15) and (17) is given by

$$\dot{\mathbf{p}}^{\mathrm{T}}(1) = -(1-c)^{-1}\mathbf{p}^{\mathrm{T}}(0)\mathbf{F}\mathbf{D}^{+}$$

or

$$\dot{\mathbf{p}}^{\mathrm{T}}(1) = -(1-c)^{-1}\mathbf{p}^{\mathrm{T}}(0)\mathbf{D}^{+} \tag{18}$$

where $\mathbf{D}^{+}$ denotes the Moore-Penrose inverse of $\mathbf{D}$, which is, in our case, a circulant double-centered matrix satisfying $\mathbf{D}\mathbf{D}^{+} = \mathbf{D}^{+}\mathbf{D} = \mathbf{F}$.[10] Moreover, when n is even, $n = 2\mu$, the explicit expression for matrix $\mathbf{D}^{+}$ can be written as

$$\mathbf{D}^{+} = (4\mu)^{-1}\mathrm{circ}[2\mu-1, 2\mu-3, 2\mu-5, ..., -(2\mu-3), -(2\mu-1)] \tag{19}$$

while when n is odd, $n = 2\mu+1$, it can be written as

$$\mathbf{D}^{+} = (2\mu+1)^{-1}\mathrm{circ}[\mu, \mu-1, \mu-2, ..., -(\mu-1), -\mu] \tag{20}$$

(consider Davis, 1979, pp. 148-149). The elements of the first row of $-\mathbf{D}^{+}$ are equal to $\dot{\delta}_{\kappa}(1) = (2n)^{-1}(1+2\kappa-n)$, $\kappa = 0,1,...,n-1$.

Hence, it is easy to check that equations (16), (18), (19) and (20) imply that, when $n \geq 3$ and $p_j(0) < p_{j+1}(0)$, $j = 1,2,...,n-1$, there is at least one element of $\dot{\mathbf{p}}^{\mathrm{T}}$, say $\dot{p}_h$, such that $\dot{p}_h(0)\dot{p}_h(1) < 0$, *irrespective* of the direction of $\mathbf{p}^{\mathrm{T}}(0)$. Then, by Bolzano's theorem, it follows that p_h necessarily has at least one extreme point in the interval $(0,\ 1)$.[11]

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[10] There is an algebraic analogue of equations (15) and (17) in electrical network theory: $\dot{\mathbf{p}}^{\mathrm{T}}(1)$ and $-(1-c)^{-1}\mathbf{p}^{\mathrm{T}}(0)\mathbf{F}$ correspond to the vectors of voltages and currents, respectively; equations (15) and (17) correspond to Kirchhoff's voltage law and Ohm's law, respectively; $\mathbf{D}$ and $\mathbf{D}^{+}$ correspond to the matrices of admittance and impedance, respectively (see Sharpe and Styan, 1965).

[11] See Examples 3, 4 and 5 in the Appendix.

2.5. The complex plane location of the polar value theories

These seven ideal-type cases (and their possible combinations) indicate that the location of the non-dominant eigenvalues in the complex plane could be considered as an index for the underlying interindustry linkages. Case 1 corresponds to the PLVT, while Cases 2 and 3 correspond to the TVT. Cases 4, 5 and 6 fall into the SVT. Finally, it could be said that Case 7, i.e. the basic circulant perturbation of the PLVT economy, corresponds to the "Sraffian polar value theory" (SPVT), since in that case the value-profit rate relationship is non-monotonic whatever the labour value vector is. Hence, Figure 3 displays the location of the *polar* value theories, i.e. PLVT, TVT, and SPVT, in the complex plane.

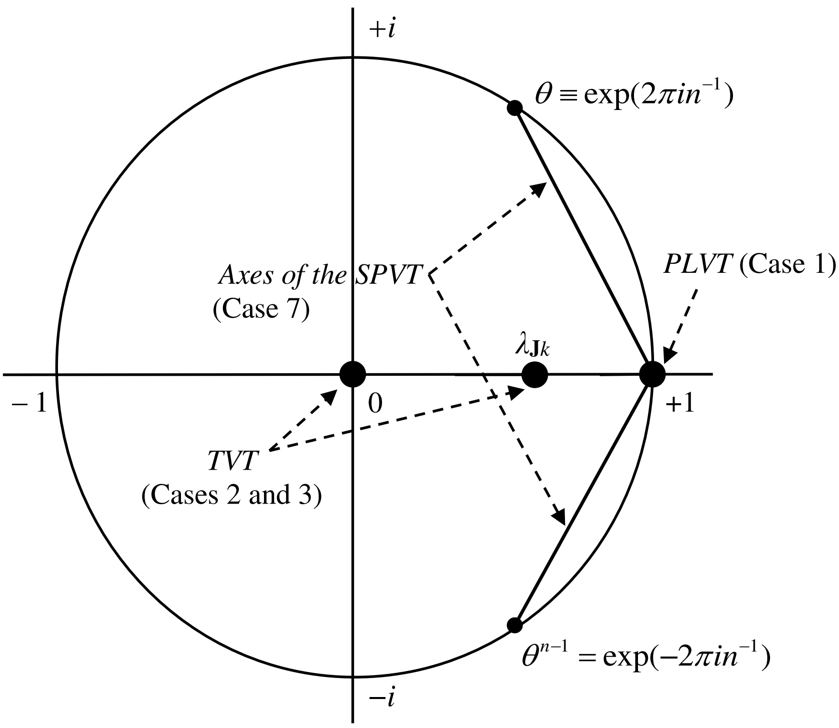

Figure 3. The complex plane location of the polar value theories

3. The Degree of Regularity-Controllability of Actual Economies

The value-wage-profit rate system of quite diverse actual economies (but, *ex hypothesis*, linear and single-product) has been examined in a relatively large number of studies. The key stylized findings were that, in the economically relevant interval of the profit rate:[12]

(i). Non-monotonic value curves *do* exist. Nevertheless, they are not significantly more than 20% of the tested cases, while, expressed in terms of SSC, they have no more than one extreme point. Cases of reversal in the direction of deviation between values and labour values are rarer.

(ii). Despite the presence of considerable deviations from the "equal value compositions of capital" case, the wage-profit rate curves are near-linear, i.e. the correlation coefficients between the distributive variables tend to be above 99%, and their second derivatives change sign no more than once or, very rarely, twice, irrespective of the *numéraire* chosen.

(iii). The approximation of the value-wage-profit rate curves through low-order formulae (ranging from linear to quadratic) works well.

(iv). The aforementioned shapes of the value-wage-profit rate curves can be explained by the fact that, across countries and over time, both the moduli of the first non-dominant eigenvalues and the first non-dominant singular values of matrices $\mathbf{J}$ fall quite rapidly, whereas the rest constellate in much lower values forming "long tails". More specifically:

(a). The majority of the non-dominant eigenvalues are crowded at very low values and bounded in a relatively small region of the unit circle. In point of fact, both the eigenvalue moduli and the singular values follow exponentially decaying trends, in the case of circulating capital, or a nearly "L-shaped form", in the case of the presence of fixed capital stocks (treated, however, in terms of the Leontief-Bródy approach). Hence, although $rank[\mathbf{J}] = n$ holds true, the "effective rank (or dimensions)" of $\mathbf{J}$ is much lower than n.

(b). The complex (as well as the negative) eigenvalues tend to appear in the lower ranks, i.e. their modulus is relatively small. However, even in the cases that they appear in the higher ranks, i.e. second or third rank, the real part is much larger than the imaginary part. In the fewer cases that the imaginary part of an eigenvalue exceeds the real one, not only their ratio is relatively small but also the modulus of the eigenvalue can be considered as a negligible quantity. In general, the imaginary part

[12] See Mariolis and Tsoulfidis (2016a, chaps. 3, 5, and 6, 2016b, 2018), Mariolis et al. (2019) and the references therein.

gets progressively smaller. Consequently, the distribution of the moduli is a fair representation of the eigenvalue distribution, while the complex eigenvalues play no decisive role in determining the shapes of the empirical value-wage-profit rate relationships.

The aforementioned stylized findings in combination with the theoretical analysis developed in this paper suggest that the actual single-product economies *tend* to behave as three-industry irregular systems. To look deeper into this interesting and important phenomenon, we will deal with data from ten flow Symmetric Input-Output Tables (SIOTs) of five European economies, i.e. Denmark (for the years 2000 and 2004; $n=56$), Finland (for the years 1995 and 2004; $n=57$), France (for the years 1995, $n=58$, and 2005, $n=57$), Germany (for the years 2000 and 2002; $n=57$) and Sweden (for the years 1995, $n=53$, and 2005, $n=51$). These SIOTs have been firstly used by Iliadi et al. (2014), and their findings (for instance, non-monotonic value curves, expressed in terms of SSC, are observed in about 105/559 or 19% of the tested cases) are absolutely consistent with those of *all* other studies of actual value-wage-profit rate systems. Hence, this data sample could be considered as sufficiently representative. Table 1 reports:

(i). $|\lambda_{\mathbf{J}2}|$, $|\lambda_{\mathbf{J}3}|$, $|\lambda_{\mathbf{J}n}|$ and the geometric mean, GM, of the moduli of the non-dominant eigenvalues of $\mathbf{J}$ (reproduced from Iliadi et al., 2014, p. 43), which can be written, in our case, as

$$GM = |\det[\mathbf{J}]|^{(n-1)^{-1}} = (\prod_{i=1}^{n} \sigma_{\mathbf{J}i})^{(n-1)^{-1}}$$

As is well known, the geometric mean is rather appropriate for detecting the central tendency of an exponential set of numbers.

(ii). The ratio between the smallest and the largest singular values, $\sigma_{\mathbf{J}n}\sigma_{\mathbf{J}1}^{-1}$, of $\mathbf{J}$.

(iii). The absolute values of the determinant of the Krylov matrices and of the determinant of the Vandermonde matrices of the eigenvalues of $\mathbf{J}$ (see equation (8)).

(iv). The degree of regularity, DR (see equation (9)).

(v). The “numerical rank”, NR, of $\mathbf{K}$, defined as the number of singular values of $\mathbf{K}$ that are larger than $\tau\sigma_{\mathbf{K}1}$, where τ is a positive tolerance.

Finally, Figure 4 (reproduced from Iliadi et al., 2014, p. 45) displays the location of the eigenvalues of all matrices $\mathbf{J}$ in the complex plane, while Figure 5 (the horizontal axis is plotted in logarithmic scale) displays the normalized singular values, $\sigma_{\mathbf{K}j}\sigma_{\mathbf{K}1}^{-1}$, of all matrices $\mathbf{K}$.

Table 1. Spectral and regularity characteristics of actual economies; five European economies – ten SIOTs

	Denmark		Finland		France		Germany		Sweden	
	2000 $n=56$	2004 $n=56$	1995 $n=57$	2004 $n=57$	1995 $n=58$	2005 $n=57$	2000 $n=57$	2002 $n=57$	1995 $n=53$	2005 $n=51$
$\lvert\lambda_{\mathbf{J}2}\rvert$	0.53	0.62	0.59	0.83	0.63	0.59	0.56	0.63	0.53	0.42
$\lvert\lambda_{\mathbf{J}3}\rvert$	0.48	0.50	0.43	0.50	0.53	0.43	0.50	0.53	0.43	0.38
$\lvert\lambda_{\mathbf{J}n}\rvert$	6×10^{-4}	1×10^{-3}	1×10^{-3}	3×10^{-4}	3×10^{-5}	1×10^{-3}	1×10^{-3}	6×10^{-3}	3×10^{-3}	9×10^{-4}
GM	0.07	0.07	0.05	0.05	0.06	0.08	0.11	0.11	0.05	0.05
$\sigma_{\mathbf{J}n}\sigma_{\mathbf{J}1}^{-1}$	9×10^{-5}	1×10^{-3}	1×10^{-3}	5×10^{-5}	10^{-5}	2×10^{-4}	9×10^{-5}	4×10^{-4}	5×10^{-4}	10^{-4}
$\lvert\det[\mathbf{K}]\rvert$	10^{-782}	10^{-793}	10^{-727}	10^{-740}	10^{-741}	4×10^{-744}	10^{-728}	10^{-722}	2×10^{-727}	6×10^{-721}
$\lvert\det[\mathbf{V}]\rvert$	2×10^{-647}	3×10^{-657}	3×10^{-674}	2×10^{-674}	3×10^{-681}	2×10^{-672}	1×10^{-659}	4×10^{-646}	3×10^{-610}	3×10^{-591}
DR	6×10^{-19}	6×10^{-19}	8×10^{-19}	1×10^{-19}	5×10^{-19}	8×10^{-19}	9×10^{-20}	6×10^{-19}	5×10^{-19}	3×10^{-20}
NR , $\tau=10^{-4}$	5	6	6	6	6	6	6	6	5	5
NR , $\tau=10^{-2}$	3	3	3	4	3	3	3	3	3	3

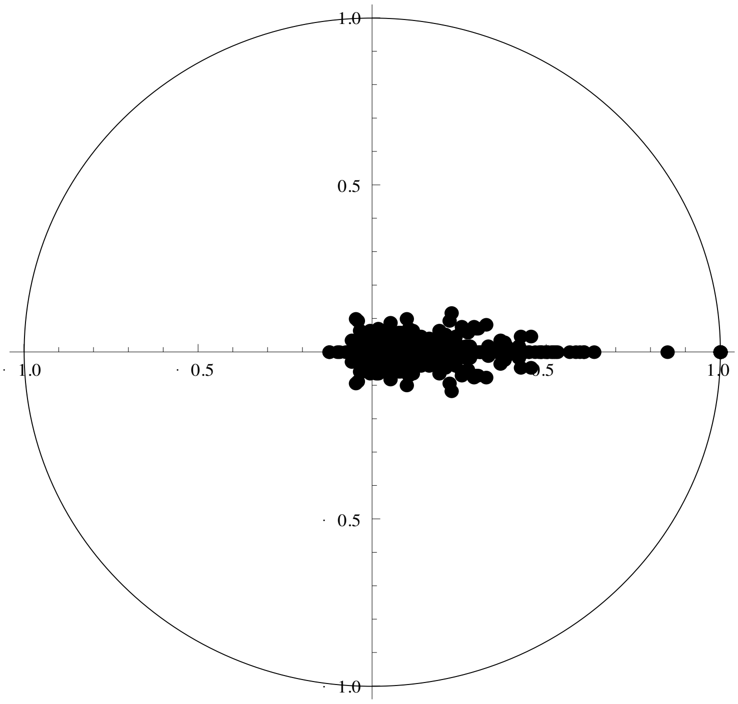

Figure 4. The complex plane location of the eigenvalues of all normalized vertically integrated technical coefficients matrices; five European economies – ten SIOTs

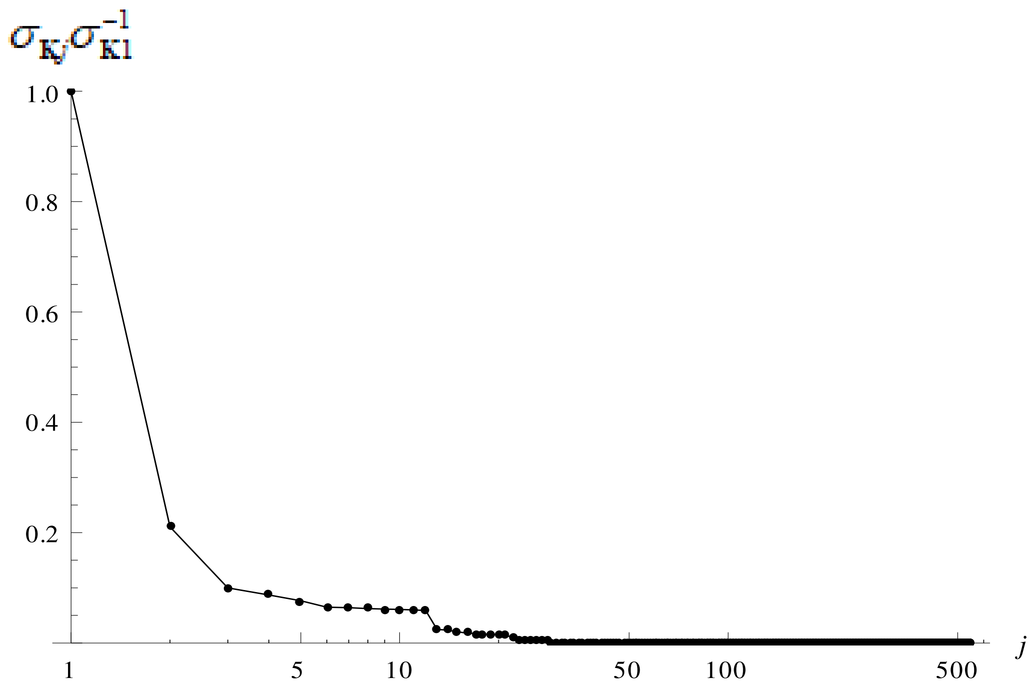

Figure 5. The normalized singular values of all Krylov matrices; five European economies – ten SIOTs

From these representative results it is deduced that the actual single-product economies:

(i). Cannot be coherently analyzed in terms of the TVT, since they exhibit non-monotonic value curves. And it need hardly be said that the existence of fairly good, low-order approximations to these curves is insufficient to restore the TVT. Hence, the SVT provides a sound empirical basis, although the eigenvalue distributions of the actual matrices $\mathbf{J}$ sharply differ from those of the basic circulant economies, which correspond to the SPVT (compare Figure 1 with Figure 4, and see Example 2 in the Appendix). In fact, the actual eigenvalue distributions can be viewed as mixed combinations of the ideal-type Cases 4 and 5 (presented in Section 2.4).

(ii). Are characterized by rather low degrees of regularity and relatively low numerical ranks. This primarily results from the skew characteristic value distributions of the actual matrices $\mathbf{J}$, and indicates that the actual economies constitute almost irregular-uncontrollable systems (see Table 1 and Figure 5; compare with Figure A6 in the Appendix). In this connection, experiments with Krylov matrices formed from pseudo-random[13] vectors $\mathbf{p}^{\mathrm{T}}(0)$ and the abovementioned actual matrices $\mathbf{J}$ lead to similar results, i.e. to degrees of regularity of the order of 10^{-19}.

It should, finally, be added that, regarding actual Krylov matrices, we also experimented with the input-output data used by Soklis (2011), i.e. ten Supply and Use Tables (SUTs) of the Finnish economy (for the years 1995 through 2004; $n = 57$), and the results were similar.[14] For instance, when the Krylov matrix is formed from the vector $\mathbf{l}^{\mathrm{T}}[\mathbf{A}-\mathbf{B}]^{-1}$ and the matrix $\mathbf{A}[\mathbf{A}-\mathbf{B}]^{-1}$, where $\mathbf{B}$ denotes the output coefficients matrix, the degree of regularity is in the range of 6×10^{-93} to 10^{-27}, while, when the Krylov matrix is formed from $\mathbf{l}^{\mathrm{T}}\mathbf{B}^{-1}$ and $\mathbf{AB}^{-1}$, the degree of regularity is in the range of 2×10^{-28} to 2×10^{-20}.

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[13] Generated by *Mathematica*; see https://reference.wolfram.com/language/tutorial/PseudorandomNumbers.html

[14] In these SUTs of the Finnish economy (i) there exists an interval of r (>0), such that the vector of "labour commanded" values, $w^{-1}\mathbf{p}^{\mathrm{T}}$, is positive, for the years 1995 through 1998 and 2000 through 2002, and (ii) the monotonicity of the estimated wage-profit rate curves (for the years 1995, 1997, 2000 and 2001) depends on the numeraire chosen (Soklis, 2011, pp. 553-555). As is well known, in the SUTs there are industries that produce more than one commodity, and commodities that are produced by more than one industry. Therefore, the SUTs could be considered as the empirical counterpart of joint production systems *à la* v. Neumann and Sraffa.

4. Concluding Remarks

The spectral analysis of the value system of economies of production of commodities by means of commodities revealed that the hitherto alternative value theories correspond to specific production structures and, therefore, to specific eigenvalue locations in the complex plane. More specifically, it has been shown that these theories can be conceived of as "perturbations" of the pure labour value theory, which is a polar theory of value that tends to hold true when all the eigenvalues of the vertically integrated technical coefficients matrix tend to be equal to each other.

It has also been shown that, although the existence of value-profit rate curves that are non-monotonic irrespective of the labour value vector direction presupposes eigenvalue distributions sharply different from those appearing in real-world economies, the Sraffian value theory is not only the most general one but also provides a sound empirical basis. On the other hand, empirical evidence suggests that the Krylov matrices of actual economies are characterized by rather low degrees of regularity-controllability and relatively low numerical ranks. This finding results from the skew characteristic value distributions of the actual vertically integrated technical coefficients matrices, and indicates that the actual economies constitute almost irregular-uncontrollable systems. Finally, the almost irregularity-uncontrollability of real-world economies explains, in turn, the specific shape features of the empirical value-wage-profit rate curves.

Future research work should (i) expand the empirical analysis of the joint production economies using data from the Supply and Use Tables; (ii) delve into the proximate determinants of the irregular-uncontrollable aspects of real-world economies, and draw their broader implications for both political economy and economic policy issues; and (iii) heuristically look for eigenvalue locations in the complex plane that could lead to new versions of the value theory.

Appendix: Examples

Example 1

Consider the following 3×3 doubly stochastic and irreducible economies:

$$\mathbf{J}=\begin{bmatrix}\alpha & \beta & 1-\alpha-\beta\\ \delta & \varepsilon & 1-\delta-\varepsilon\\ 1-\alpha-\delta & 1-\beta-\varepsilon & \alpha+\beta+\delta+\varepsilon-1\end{bmatrix}$$

where $\mathbf{p}^{\mathrm{T}}(0)\neq\mathbf{p}^{\mathrm{T}}(1)$.

Iff $\alpha=\varepsilon$ and $\beta=\delta=2^{-1}(1-\alpha)$, then $\mathbf{J}$ is circulant with a repeated eigenvalue, $\lambda_{\mathbf{J}2}=\lambda_{\mathbf{J}3}=2^{-1}(3\alpha-1)$, and this eigenvalue has two linearly independent eigenvectors, i.e. $\mathbf{x}_{\mathbf{J}2}=\chi_2[-1,0,1]^{\mathrm{T}}$ and $\mathbf{x}_{\mathbf{J}3}=\chi_3[-1,1,0]^{\mathrm{T}}$, where χ_2, χ_3 denote arbitrary non-zero scalars. Since $\mathbf{x}_{\mathbf{J}2}+\mathbf{x}_{\mathbf{J}3}$ is also an eigenvector, there exists an eigenvector of $\mathbf{J}$ that is orthogonal to any given $\mathbf{p}^{\mathrm{T}}(0)$. Therefore, $rank[\mathbf{K}]=2$, the economies are irregular whatever $\mathbf{p}^{\mathrm{T}}(0)$ is, and p_j are not rational functions of degree $n-1\ (=2)$ but of degree $n-2\ (=1)$. This reduction in degree (known in control theory as “pole-zero cancellation”) is a characteristic feature of irregular-uncontrollable systems (Kalman, 1960, p. 494). More specifically, when $\mathbf{p}^{\mathrm{T}}(0)\mathbf{x}_{\mathbf{J}2}=0$ or $\mathbf{p}^{\mathrm{T}}(0)\mathbf{x}_{\mathbf{J}3}=0$, equation (4) implies that $p_1=p_3$ or $p_1=p_2$, respectively. When $\mathbf{p}^{\mathrm{T}}(0)\mathbf{x}_{\mathbf{J}k}\neq 0$, it follows that

$$\mathbf{p}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}2}(\mathbf{p}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}3})^{-1}=\mathbf{p}^{\mathrm{T}}(0)\mathbf{x}_{\mathbf{J}2}(\mathbf{p}^{\mathrm{T}}(0)\mathbf{x}_{\mathbf{J}3})^{-1}$$

or

$$p_2=\eta p_1+(1-\eta)p_3$$

where

$$\eta\equiv(p_3(0)-p_2(0))(p_3(0)-p_1(0))^{-1}\qquad\text{(A1)}$$

Iff $p_2(0)=\overline{p}(0)$, then $\eta=2^{-1}$ and, therefore, $p_2=2^{-1}(p_1+p_3)=p_2(0)$.

It should be added that, when $\lambda_{\mathbf{J}2}=\lambda_{\mathbf{J}3}$, the Schur triangularization theorem implies that $\mathbf{J}$ can be transformed, via the semi-positive similarity matrix $\mathbf{T}\equiv[\mathbf{x}_{\mathbf{J}1},\mathbf{e}_2,\mathbf{e}_3]$, into

$$\mathbf{T}^{-1}\mathbf{J}\mathbf{T} = \begin{bmatrix} 1 & 2^{-1}(1-\alpha) & 2^{-1}(1-\alpha) \\ 0 & 2^{-1}(3\alpha-1) & 0 \\ 0 & 0 & 2^{-1}(3\alpha-1) \end{bmatrix}$$

Hence, when $\alpha > 3^{-1}$, the original economies are economically equivalent to a triangular economy involving only one composite basic commodity ("hyper-basic commodity"; Mariolis and Tsoulfidis, 2016a, p. 155), which is no more than SSC, and a diagonal non-basic system. When $\alpha = 3^{-1}$, we obtain a version of the ideal-type Case 2 (presented in Section 2.4).

By contrast, when $\lambda_{\mathbf{J}2} \neq \lambda_{\mathbf{J}3}$ and $p_2(0) = \overline{p}(0)$, p_2 has an extreme point in the economically relevant interval of the profit rate, i.e. at

$$\rho^* \equiv [1+\sqrt{3}\sqrt{(1-\alpha)(1-\varepsilon)-\beta\delta}\,]^{-1}$$

See, for instance, Figure A1, where $\alpha = \varepsilon = 0.5$, $\beta = 0.2$ or 0.4, $\delta = 0.1$, and $\mathbf{p}^{\mathrm{T}}(0) = [1,2,3]$. When $\beta = 0.2$, $\det[\mathbf{K}] = -0.414$ and $DR = 0.010$ (see equations (8) and (9)), while when $\beta = 0.4$, $\det[\mathbf{K}] = 1.134$ and $DR = 0.029$. Also note that, when $p_2(0) \neq \overline{p}(0)$, all six value curves may be monotonic.

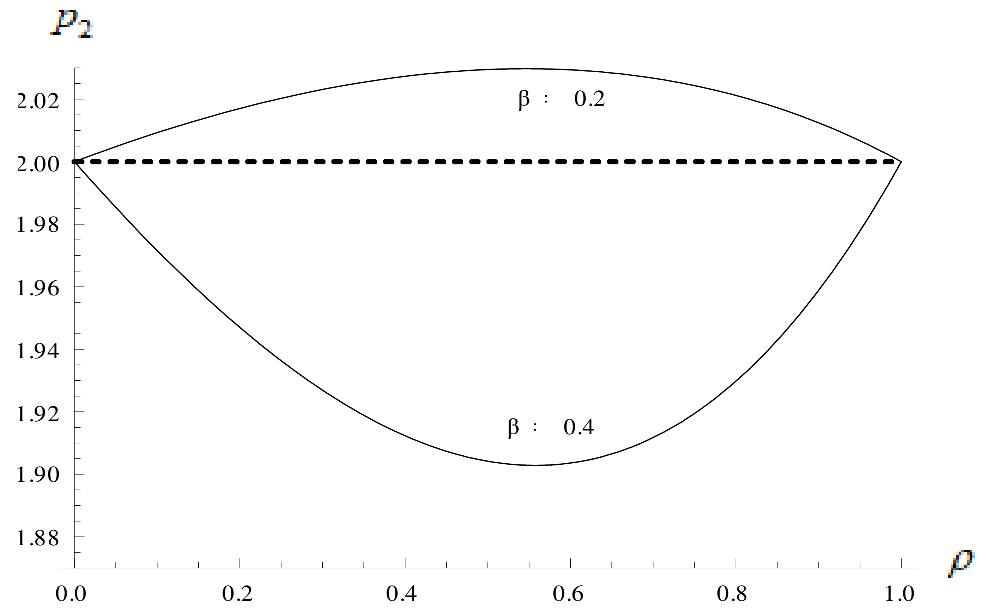
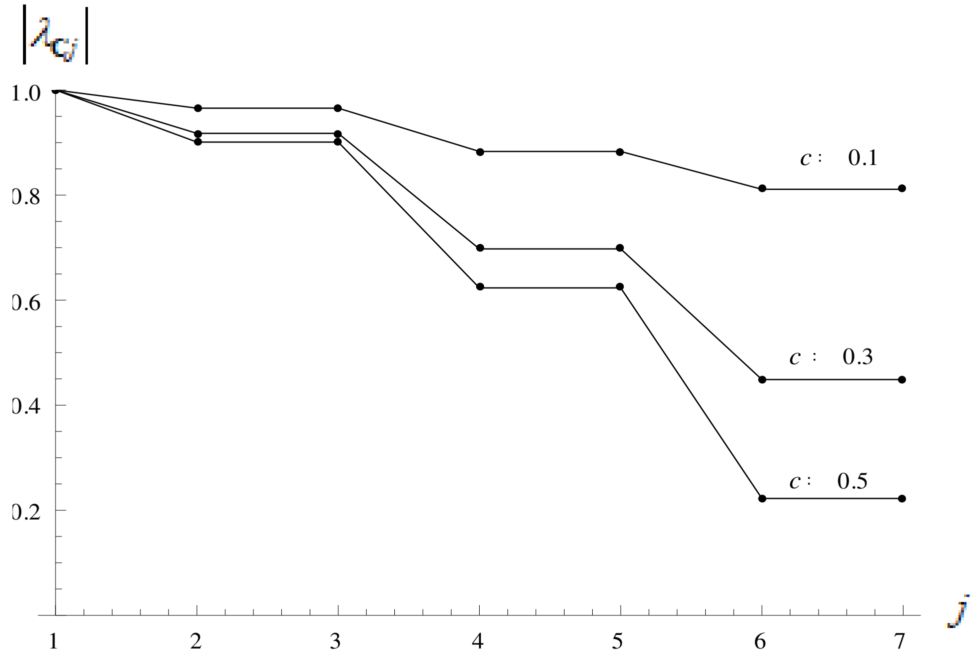


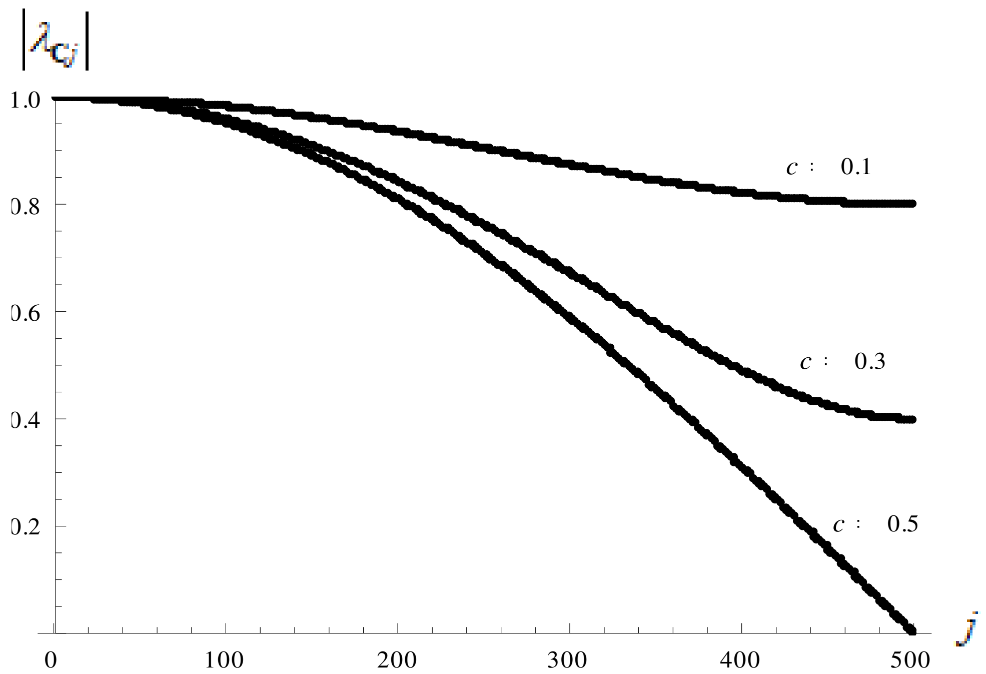
Figure A1. Non-monotonic value curves in 3×3 doubly stochastic economies where $p_2(0) = \overline{p}(0)$

Example 2

Figure A2 displays the moduli of the eigenvalues of $\mathbf{C}$, for $n=7,\ 500$ and $c=0.1,\ 0.3,\ 0.5$. Figure A3 displays the modulus of the subdominant eigenvalues of $\mathbf{C}$ as a function of c, $0<c\le 0.5$, and n, $3\le n\le 50$.



$n=7$, $c=0.1,\ 0.3,\ 0.5$



$n=500$ and $c=0.1,\ 0.3,\ 0.5$

Figure A2. The moduli of the eigenvalues of $\mathbf{C}$; $n=7,\ 500$ and $c=0.1,\ 0.3,\ 0.5$

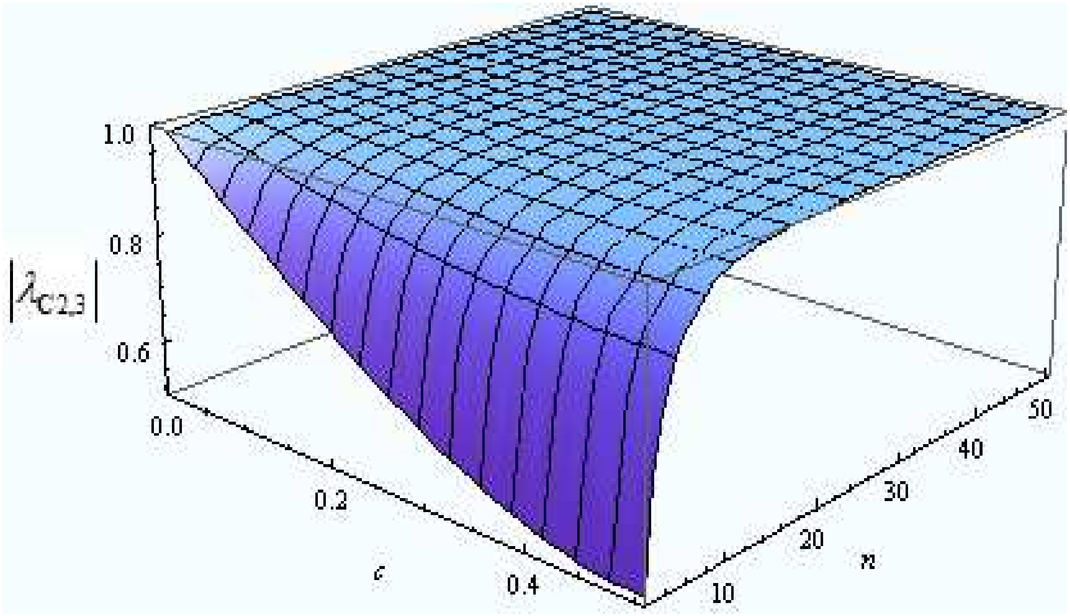


Figure A3. The modulus of the subdominant eigenvalues of **C** as a function of c and n; $0 < c \leq 0.5$ and $3 \leq n \leq 50$

Example 3

Consider the following economies: $\mathbf{J} = \mathbf{C}$, where $p_j(0) < p_{j+1}(0)$. When $n = 6$, i.e. $\mu = 3$, it necessarily follows that $\dot{p}_\mu(0)\dot{p}_\mu(1) < 0$, since

$$\dot{p}_3(0) = (1-c)(p_2(0) - p_3(0)) < 0$$

and

$$\dot{p}_3(1) = [12(1-c)]^{-1}[5(p_4(0) - p_3(0)) + 3(p_5(0) - p_2(0)) + p_6(0) - p_1(0)] > 0$$

When $n = 7$, i.e. $\mu = 3$, it necessarily follows that $\dot{p}_\mu(0)\dot{p}_\mu(1) < 0$ and $\dot{p}_{\mu+1}(0)\dot{p}_{\mu+1}(1) < 0$, since

$$\dot{p}_3(0) = (1-c)(p_2(0) - p_3(0)) < 0$$

$$\dot{p}_4(0) = (1-c)(p_3(0) - p_4(0)) < 0$$

and

$$\dot{p}_3(1) = [7(1-c)]^{-1}[3(p_4(0) - p_3(0)) + 2(p_5(0) - p_2(0)) + p_6(0) - p_1(0)] > 0$$

$$\dot{p}_4(1) = [7(1-c)]^{-1}[3(p_5(0) - p_4(0)) + 2(p_6(0) - p_3(0)) + p_7(0) - p_2(0)] > 0$$

Example 4

Consider the following economies: $\mathbf{J}=\mathbf{C}$, $n=3$, where $p_j(0)<p_{j+1}(0)$. It follows that, in the economically relevant interval of γ, $p_1(\gamma)$ and $p_3(\gamma)$ are monotonic, while $p_2(\gamma)$ is minimized at

$$\gamma^* \equiv -\eta+\sqrt{1-\eta+\eta^2}$$

where $\eta<1$ (see equation (A1)) and $d\gamma^*/d\eta<0$. Since γ increases (decreases) with ρ (with c), the relevant value of the profit rate, i.e.

$$\rho^* = \gamma^*[1-c(1-\gamma^*)]^{-1}$$

increases with c. Finally, $p_2(\gamma)=p_2(0)$ at $\gamma=0$ and at

$$\gamma^{**} \equiv \eta^{-1}-1$$

where $\gamma^{**}>\gamma^*$, while $\gamma^{**}\leq 1$ iff $\eta \geq 2^{-1}$ or, equivalently, $p_2(0)\leq \bar{p}(0)$. See, for instance, Figure A4, where $\mathbf{p}^{\mathrm{T}}(0)=[1,2,4]$ and c is in the range of 0 to 0.995; $\eta=2/3$, $\gamma^*=(\sqrt{7}-2)/3\cong 0.215$, $p_2(\gamma^*)=(11-2\sqrt{7})/3\cong 1.903$ and $\gamma^{**}=0.5$).

It should be noted that $n\times n$ doubly stochastic circulant economies of the form

$$c_1\mathbf{I}+c_2\mathbf{\Pi}+c_3\mathbf{\Pi}^2+...+c_n\mathbf{\Pi}^{n-1},\ (c_2,c_3,...,c_{n-1})>0$$

do not necessarily generate non-monotonic value curves. For instance, consider Example 1 ($\lambda_{\mathbf{J}2}=\lambda_{\mathbf{J}3}$) or the following 3×3 cases: (i) $c_1=0$, $c_2=0.6$, $c_3=0.4$; and (ii) $c_1=0.6$, $c_2=0.25$, $c_3=0.15$, with $\mathbf{p}^{\mathrm{T}}(0)=[1,2,4]$, and take into account the structure of the relevant matrices $\mathbf{D}^+$ (compare with equations (19) and (20)).

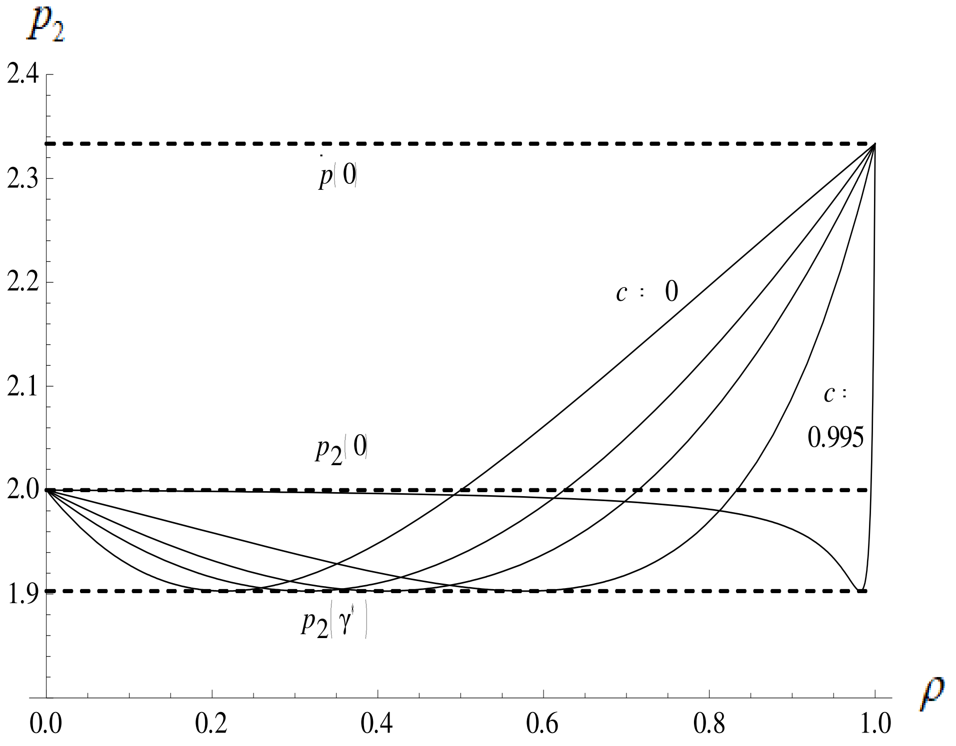


Figure A4. Non-monotonic value curves in 3×3 basic circulant economies

Example 5

Consider the following economies: $\mathbf{J}=\mathbf{C}$, $n=4$, where $\mathbf{p}^{\mathrm{T}}(0)=[1,5,4,p_4(0)]$ and $p_4(0)>5$. It follows that $p_3(\gamma)=p_3(0)$ at $\gamma=0$ and at

$$\gamma_{1,2}^{**}\equiv 2^{-1}(\eta_1^{-1}-1)\mp\sqrt{4^{-1}(\eta_1^{-1}-1)^2+(\eta_2^{-1}-1)}$$

where

$$\eta_1\equiv(p_4(0)-p_3(0))(p_4(0)-p_1(0))^{-1}\ ,\ 0<\eta_1<1$$

$$\eta_2\equiv(p_4(0)-p_3(0))(p_4(0)-p_2(0))^{-1},\ \eta_2>1$$

and, therefore, that $0<\gamma_{1,2}^{**}\leq 1$ for $6\leq p_4(0)\leq 6.25$. More specifically, for $p_4(0)=6$ we get $\gamma_1^{**}=1/2$ and $\gamma_2^{**}=1$, while for $p_4(0)=6.25$ we get $\gamma_1^{**}=\gamma_2^{**}=2/3$.

Hence, when, for instance, $p_4(0)=6.1$, we get $\gamma_{1,2}^{**}=21^{-1}(15\mp\sqrt{15})$, i.e. $\gamma_1^{**}\cong 0.530$, $\gamma_2^{**}\cong 0.899$; $p_3(\gamma)$ has two extreme points, i.e. at $\gamma_1^{*}\cong 0.183$ and $\gamma_2^{*}\cong 0.712$, while $p_2(\gamma)$ is minimized at $\gamma^{*}\cong 0.689$. The graphs in Figure A.5 display $p_3(\rho)$ for values of c in the range of 0 to 0.90 (note that $\dot{p}_3(0)\dot{p}_3(1)>0$), and the value difference $p_3(\gamma) - p_2(\gamma)$, which equals zero at $\gamma = \gamma_{1,2}^{**}$ and at $\gamma = 1$ (compare with Figure 3 in Sraffa, 1960, p. 38).

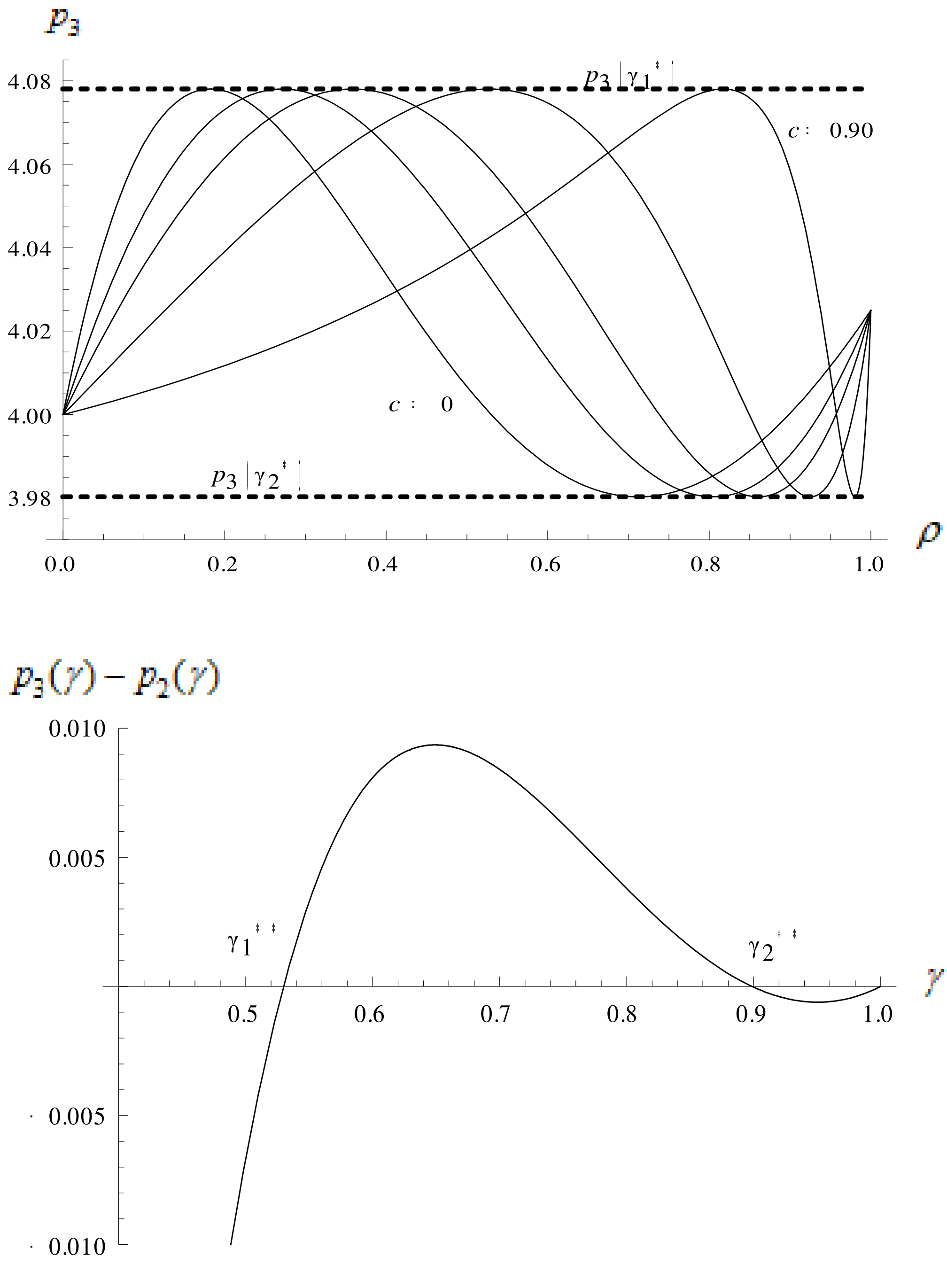


Figure A5. Possible non-monotonic value curves and value difference in 4×4 basic circulant economies

Now, assume that $\mathbf{p}^{\mathrm{T}}(0)$ is arbitrary but $\mathbf{p}^{\mathrm{T}}(0) \neq \mathbf{p}^{\mathrm{T}}(1)$. The determinant of the Krylov matrix is given by

$$\det[\mathbf{K}] = -(1-c)^6 P_0 P_1 P_2$$

where $-(1-c)^6 = \det[\mathbf{V}]\det[\mathbf{X}_{\mathbf{J}}^{-1}]$ (see equation (8)), $P_0 \equiv -\sum_{j=1}^{4} p_j(0)$ and

$$P_1 \equiv p_1(0) - p_2(0) + p_3(0) - p_4(0)$$

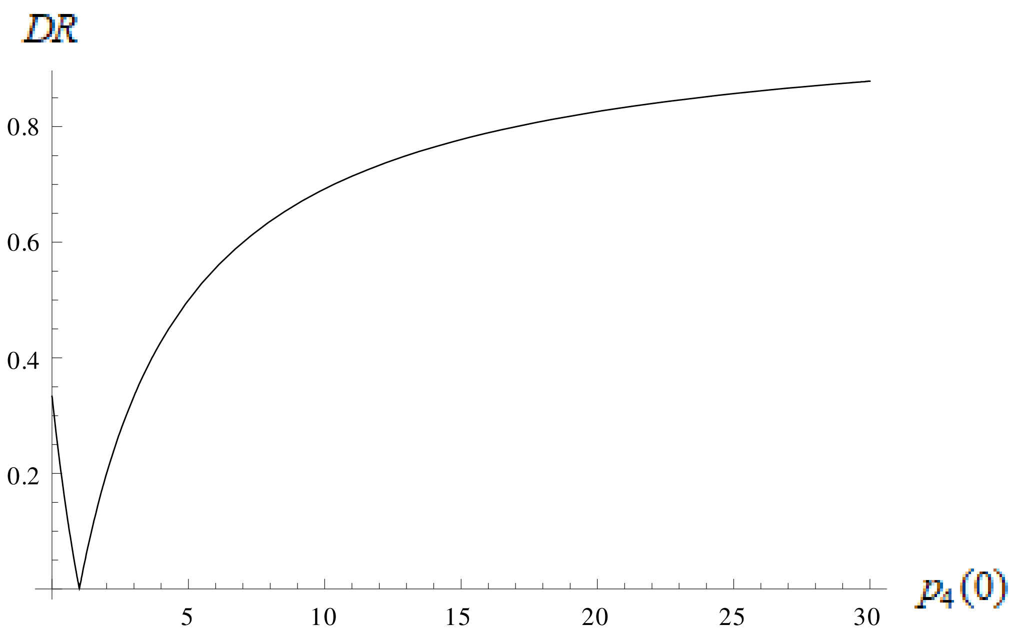
$$P_2 \equiv (p_1(0) - p_3(0))^2 + (p_2(0) - p_4(0))^2$$

Hence, these economies are irregular iff either $P_1 = 0$ ($rank[\mathbf{K}] = 3$) or $P_2 = 0$ ($rank[\mathbf{K}] = 2$). When $P_1 = 0$, $p_j(\gamma)$ are rational functions of degree $n-2$ $(=2)$ and

$$p_1(\gamma) + p_3(\gamma) = p_2(\gamma) + p_4(\gamma) = p_1(0) + p_3(0)$$

When $P_2 = 0$, $p_j(\gamma)$ are rational functions of degree $n-3$ $(=1)$, $p_1(\gamma) = p_3(\gamma)$ and $p_2(\gamma) = p_4(\gamma)$.

Finally, the graphs in Figure A6 display the degree of regularity, DR, as a function of $p_4(0)$, for $c = 0$ and $\mathbf{p}^{\mathrm{T}}(0) = [1,1,1,p_4(0)]$ or, alternatively, $\mathbf{p}^{\mathrm{T}}(0) = [1,5,4,p_4(0)]$: $DR = 0$ at $p_4(0) = 1$ or $DR = 0$ at $p_4(0) = 0$, respectively, while DR tends to 1 as $p_4(0)$ tends to plus infinity.



$\mathbf{p}^{\mathrm{T}}(0) = [1,1,1,p_4(0)]$

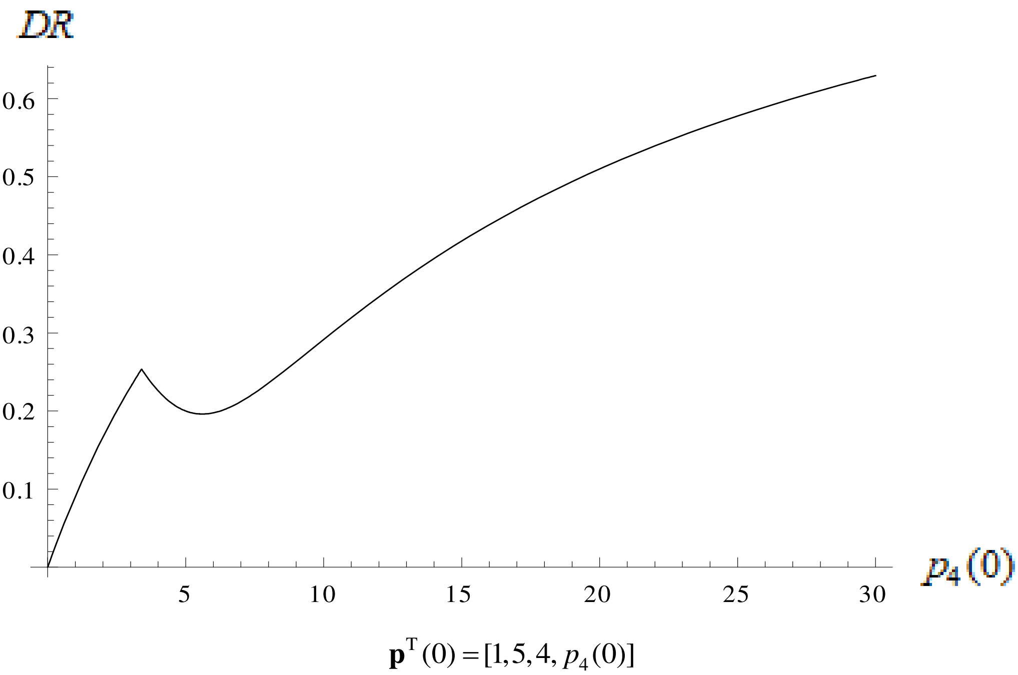


Figure A6. The degree of regularity of a 4×4 basic circulant economy ($c=0$) as a function of $p_4(0)$

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