Consumer information and the limits to competition

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Abstract

   This paper studies competition between firms when consumers observe a private signal of their preferences over products. Within the class of signal structures which allow pure-strategy pricing equilibria, we derive signal structures which are optimal for firms and those which are optimal for consumers. The firm-optimal signal structure amplifies the underlying product differentiation, thereby relaxing competition, while ensuring that consumers purchase their preferred product, thereby maximizing total welfare. The consumer-optimal structure dampens differentiation, which intensifies competition, but induces some consumers with weak preferences between products to buy their less-preferred product. The analysis sheds light on the limits to competition when the information possessed by consumers can be designed flexibly.

Keywords: Information design, Bertrand competition, product differentiation, online platforms.

1   Introduction

Information flows between firms and consumers affect firm competition and market performance. Information travels in both directions between the two sides of the market. Firms are able to obtain information about consumer preferences from data brokers, social media, past interaction with customers, and so on. This enables them (if permitted) to make personalized offers and price discriminate in a targeted manner. On

the other side, consumers have several ways to gather information about the products they might buy, and they often rely on online platforms such as search engines and product comparison websites to obtain information about their likely preferences over the various products. The precision of their information about products affects both the quality of the consumer-product match and the intensity of competition between firms. In this paper we study the second of these information flows.

In more detail, we study a symmetric duopoly market where two firms each costlessly supply a single variety of a product and compete in prices in one-shot Bertrand fashion. A consumer is initially uncertain about her preferences for the varieties, but before purchase she receives a private signal of these preferences. For example, she might be informed which product she will prefer (but not the precise valuation of either product), so that the products are *ranked*. The consumer then updates her beliefs about her preferences and makes her choice given the pair of prices offered by firms. The signal structure, which governs the relationship between the consumer's true preferences and the signal she receives, is common knowledge. We wish to understand how the signal structure affects competition and welfare. In particular, we explore the limits to competition in this market: which signals induce the highest profit for firms and which generate the highest surplus for consumers?

One interpretation of the model is that consumers gather information and make purchases via a platform which provides them with product information. The platform can choose several aspects of its information disclosure to consumers, such as how detailed is the product information it displays, whether to post customer reviews or its own reviews, whether to offer personalized recommendations, and how flexibly consumers can filter and compare products. Some platforms choose to reveal little information about products, as when for instance they offer consumers a list of hotels of specified type for a given price, but the consumer only discovers which hotel she will be allocated once she has paid. Given this flexibility over the information released to consumers, we impose few restrictions on the signal structure. Whether a platform operates to maximize firm profit, consumer surplus, or total welfare will depend partly on which sides of the market it can levy fees, which in turn will depend on the relative intensity of platform competition across the two sides of the market (which is something we do

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not consider in this paper).[1]

Section 2 introduces the model, and in section 3 we show that the *rank* signal allows firms to obtain first-best profit whenever preferences are sufficiently dispersed *ex ante*. That is, the rank signal enables consumers to buy their preferred product, which maximizes total welfare, and with dispersed preferences it is an equilibrium for firms to charge consumers their (expected) valuation of the preferred product. Except for the trivial case where products are perfect substitutes *ex ante*, though, it is not possible for consumers to obtain first-best surplus. Information which enables consumers to buy their preferred product also gives market power to firms and induces prices above marginal cost, and consumers face a trade-off between low prices and the ability to buy the better product.

Beyond situations where first-best profit is feasible, attempts to derive an optimal second-best signal structure encounter two problems. First, consumer preferences are generally two-dimensional, and current understanding of optimal information design in such cases is limited. For that reason, from section 4 onwards we find ways to simplify the model so that relevant consumer heterogeneity is scalar. Second, even with scalar heterogeneity some posterior distributions are such that the only equilibria in the pricing game between firms involve mixed strategies, which can be hard to characterize. For this reason, until section 6 we focus on signal structures which induce a pure strategy equilibrium in the pricing game.

In section 4 we simplify the model so that the outside option for consumers is not relevant for the analysis. (In essence this requires that valuations be sufficiently concentrated *ex ante*.) This implies that only the difference in valuations for the two products—a scalar variable—matters for consumer decisions. Our approach is to find which signal structures (if any) can support given prices as equilibrium prices. It turns out that a price pair can be supported in equilibrium if the posterior preference distribution induced by the signal structure lies between two *bounds*. Posterior distributions which correspond to the upper bound are relevant for the consumer-optimal policy, while the lower bound is what determines the firm-optimal policy. In section 4.2 we re-

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[1]An alternative interpretation is that consumers can commit to how much product information to acquire before firms make their pricing decisions. For instance, a consumer could strategically delegate her purchase decision to an agent who commits to focus more on price than other product characteristics in order to stimulate price competition among suppliers.

strict attention to symmetric signals which induce the two firms to offer the same price in equilibrium, and in section 4.3 we demonstrate that in regular cases neither firms nor consumers can do better if asymmetric signals and prices are implemented. (Surprisingly, asymmetric signals which favour one firm cannot improve that firm's profit relative to the firm-optimal symmetric policy.)

The firm-optimal signal structure amplifies "perceived" product differentiation in order to relax competition, and does so by reducing the likelihood that consumers *ex post* are near-indifferent between products. (The *rank* signal which sometimes yields first-best profit is an extreme instance of this.) In regular cases, the firm-optimal policy allows consumers to buy their preferred product for sure, in which case total welfare is also maximized. The consumer-optimal signal structure by contrast dampens product differentiation in order to stimulate competition, and does so by increasing the number of consumers who are near-indifferent between products. In the consumer-optimal policy, a consumer with strong preferences can buy her preferred product for sure, but a less choosy consumer receives less precise information and may end up with the inferior product. In contrast to the firm-optimal policy, product mismatch means that the consumer-optimal policy does not maximize total welfare.

In section 4 the outside option was not relevant for consumers, and so the constraint on a firm raising its price was that the consumer would buy from its rival. By contrast, when the first-best profit was feasible in section 3 the constraint on raising price was that the consumer would exit the market. We bridge the gap between these extreme situations in section 5 using a framework with scalar consumer heterogeneity in which both constraints play a role. Since the consumer-optimal policy induces low prices in the market, the presence of an outside option has relatively little impact on the design of that policy. However, the high prices typically seen with the firm-optimal policy are often constrained by the outside option, and the optimal policy then induces a posterior distribution such that *no* consumers regard products as close substitutes.

Section 6 discusses whether considering the wider class of signals which allow mixed-strategy pricing equilibria can improve outcomes. By modifying the "bounds" approach we used with pure pricing strategies, we show that allowing mixed-strategy equilibria could at best improve consumer surplus only slightly. We conclude with some comments about how this analysis could usefully be extended in future work.

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Related literature. One strand of the relevant literature considers a monopolist's incentive to provide product information to enable consumers to discover their valuation for its product. An early paper on this topic is Lewis and Sappington (1994). They study a monopoly market and show that, within the class of "truth or noise" signal structures, it is optimal for the firm either to disclose no information or all information. Johnson and Myatt (2006) derive a similar result for a more general class of information structures which induce rotations of the demand curve. Anderson and Renault (2006) argue that partial information disclosure before consumers search can be optimal for a monopolist if consumers need to pay a search cost to buy the product (in which case they learn their valuation automatically). Importantly, they allow for flexible signal structures as in the later Bayesian persuasion literature and show that firm-optimal information disclosure takes the coarse form whereby a consumer is informed merely whether her valuation lies above a threshold.

Roesler and Szentes (2017) study the signal structure which is best for consumers (rather than the firm) in a monopoly model. They show that the optimal signal structure can be found within the class of posterior distributions which induce unit-elastic demand functions. (These unit-elastic demand functions play a similar role as the bounds in our analysis.) They show that partial rather than complete learning is optimal for consumers, and that the optimal information structure induces *ex ante* efficient trade and maximizes total welfare.[2] In their setup, where trade is always efficient, the firm-optimal signal structure is to disclose no information at all, in which case the firm can extract all surplus by charging a price equal to the expected valuation. With competition, however, this is no longer true since without any information consumers regard the firms' products as perfect substitutes and firms earn zero profit. (Indeed in our duopoly model we will show that disclosing no information is nearly optimal for *consumers*, rather than firms.) Therefore, the firm-oriented problem is more interesting and challenging in our setting with competition than it is with monopoly. With compe-

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[2]Condorelli and Szentes (2018) study the related problem of how to choose the demand curve to maximize consumer surplus, given the monopolist chooses its price optimally in response. (The consumer accurately observes her realized valuation in this model.) Choi, Kim, and Pease (2019) extend Roesler and Szentes (2017) to the set-up of Anderson and Renault (2006), and derive the consumer-optimal policy in the context of a search good.

tition, the consumer-optimal policy also exhibits some significant differences with the monopoly case in Roesler and Szentes. For example, it usually causes product mismatch so that the allocation is not efficient, the induced residual demand for each firm is unit-elastic for upward but not downward price deviations, and the consumer-optimal signal structure is no longer the least profitable policy for firms.

Our paper concerns information design in an oligopoly setting. Most of the previous research on this topic studies the "decentralized" disclosure policies of individual firms. For example, Ivanov (2013) studies disclosure in a random-utility model where each firm decides how much information about its own product to release and what price to charge. He focuses on information structures which rotate demand as in Johnson and Myatt (2006), and shows that full disclosure is the only symmetric equilibrium when there are many firms.[3] Hwang, Kim, and Boleslavsky (2019) show that the same result holds if general signal structures are allowed (and more generally are able to show that increasing the number of firms induces each firm to reveal more information). Intuitively, with many firms, a consumer's valuation for the best rival product (if other firms fully disclose their information) is high. To compete for the consumer, a firm discloses all information as that is the policy which maximizes the posterior probability she has a high valuation.[4]

Instead of studying equilibrium disclosure by individual firms, though, we focus on a "centralized" design problem (e.g., where a platform mediates the information flow from products to consumers), which allows us to discuss signals which reflect *relative* valuations across products (e.g., which rank the products). This more general signal structure introduces a number of additional features. In our framework, for instance, full information disclosure is not the firm-optimal design even with many firms, and the rank policy sometimes yields first-best profit for firms (which can never be achieved with a decentralized design). In addition, when decentralized signals are not fully revealing there are welfare losses since consumers sometimes choose a less preferred product, while with a centralized structure it is possible to have a coarse signal structure

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[3]Bar-Isaac, Caruana, and Cuñat (2012) study competitive product design, which can also be interpreted as information design, within a sequential search market. They consider designs which rotate demand, and show that a reduction in the search cost induces more firms to choose niche product design (i.e., full information disclosure).

[4]A similar result appears in other recent works which study competitive disclosure but without price competition, such as Board and Lu (2018).

(e.g., the rank signal) which maintains efficiency. Finally, Hwang *et al.* (2019) show that the equilibrium price (and hence profit) often falls relative to the full-information price when individual firms choose their disclosure policy, while by construction the centralized firm-optimal policy must boost profit.

Other papers have also studied "centralized" aspects of the design of consumer information. Anderson and Renault (2009) study comparative advertising in a duopoly model where each firm unilaterally chooses between fully disclosing its own product information, fully disclosing information about both products, or disclosing nothing. Among other results, they make the point that disclosing more information improves match quality but also softens price competition. (Jullien and Pavan (2019) make a similar point in a model of two-sided markets.) Dogan and Hu (2019) study the consumer-optimal disclosure policy in a sequential search framework with many firms. Consumers receive a signal of their valuation for a particular product only when they visit its seller. Because the reservation value in this search framework is static, their problem is related to the monopoly problem with a deterministic outside option studied by Roesler and Szentes (2017). Moscarini and Ottaviani (2001) study a duopoly model of price competition similar to ours, where the consumer receives a signal of her relative valuation for the two products. A major difference, however, is that they assume both the relative valuation and the signal are binary variables, in which case the pricing equilibrium often involves mixed strategies.

More broadly, our paper belongs to the recent literature on Bayesian persuasion and information design. See Kamenica and Gentzkow (2011) for a pioneering paper in this literature, and Bergemann and Morris (2019) and Kamenica (2019) for recent surveys. Among its many applications, its method and insights have been used to revisit classic problems within Industrial Organization. For instance, Bergemann, Brooks, and Morris (2015) study third-degree price discrimination by a monopolist. (In contrast to our model, their paper considers signals sent to the firm about consumer preferences, and consumers accurately know their valuation from the start. A given signal structure corresponds to a particular partition of consumers.) If all ways to partition consumers are possible, the paper shows that any combination of profit (above the no-discrimination benchmark) and consumer surplus which sum to no more than maximum welfare can

be implemented by means of some partition of consumers.[5]

2 The model

A risk-neutral consumer wishes to buy a single unit of a differentiated product costlessly supplied by two risk-neutral firms, 1 and 2. The consumer's valuation for the unit from firm $i = 1, 2$ is denoted $v_i \geq 0$, and her outside option is sure to have payoff zero. If p_i is firm i's price, the consumer wishes to buy from firm i if $v_i - p_i \geq \max\{v_j - p_j, 0\}$, so that she prefers the net surplus from firm i to that available from firm j or from the outside option. (She wishes to consume the outside option if $v_1 < p_1$ and $v_2 < p_2$.)

The consumer is initially uncertain about her valuations and holds some prior belief about the distribution of (v_1, v_2). Throughout the paper we assume that firms are symmetric *ex ante*, in the sense that the prior distribution for consumer preferences $v = (v_1, v_2)$ is symmetric between v_1 and v_2. We assume that the support of v lies inside the square $[V, V + \Delta]^2$. Here, $V \geq 0$ represents the "basic utility" from any product, while $\Delta \geq 0$ captures the extra utility a consumer might obtain from the ideal product. (If Δ is small then the products are nearly perfect substitutes.) Let $\mu = \mathbb{E}[v_i]$ denote the expected valuation of either product, and write

$$\mu_H = \mathbb{E}[\max\{v_1, v_2\}] \; ; \; \mu_L = \mathbb{E}[\min\{v_1, v_2\}] . \tag{1}$$

Thus μ_H denotes the expected valuation of the preferred product, while μ_L is the expected valuation of the less preferred product. (They are related by $\mu_L + \mu_H = 2\mu$.) Finally, write

$$\delta = \mu_H - \mu = \mathbb{E}[\max\{v_1 - v_2, 0\}] \tag{2}$$

for the incremental expected surplus from choosing the consumer's preferred product rather than a random product.

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[5]There are various papers which extend Bergemann *et al.* to different settings. For instance, Elliot and Galeotti (2019) consider a duopoly version of Bergemann *et al.* and assume each firm has some "captive" consumers who buy only from the firm they know. They show that if each firm has enough captive consumers, information design can earn firms the first-best profit. Ali, Lewis, and Vasserman (2019) depart from the information design approach in Bergemann *et al.* by considering a disclosure game with verifiable information where consumers choose how much information about their preferences they disclose to firms. They show that consumer control of their information tends to improve their welfare relative to full or no information disclosure.

We study situations where before purchase the consumer observes a private signal of her preferences rather than the preferences themselves. The signal is generated according to a signal structure $\{\sigma(s|v), S\}$, where S is a (sufficiently rich) signal space and $\sigma(s|v)$ specifies the distribution of signal s when the true preference parameter is v. We assume the signal structure is common knowledge to the consumer and to both firms, and determined before firms choose prices.[6] After observing a signal s, the consumer updates her beliefs about her preferences v. Risk neutrality implies that only the expected v given s matters for the consumer's choice. The prior distribution for v and the signal structure jointly determine a new posterior distribution for (expected) v for the consumer. Since firms do not observe the consumer's private signal, they each choose a single price regardless of the signal received, and only the posterior distribution for v matters for their pricing decisions. Firms set prices simultaneously, and we use Bertrand-Nash equilibrium as the solution concept of the pricing game. Note that prices are accurately observed by the consumer in all cases, so that uncertainty concerns only the consumer's preferences.

To illustrate, consider these simple signal structures:

- Full information disclosure: here the signal perfectly reveals the true preferences, e.g., where $s \equiv v$, and so the posterior and prior distributions for v coincide.

- No information disclosure: here the signal is completely uninformative (i.e., the distribution of s does not depend on v) and the posterior distribution is a single point, $v = (\mu, \mu)$. In particular, the consumer views the two products as perfect substitutes and will choose to buy from the firm with the lower price (if that price is no higher than μ).

- Rank signal structure: here the signal informs the consumer which product she prefers but nothing else, so that $s \in \{s_1, s_2\}$ and she observes $s = s_1$ if $v_1 > v_2$ and $s = s_2$ if $v_2 > v_1$. (She sees each signal with equal probability in the knife-edge

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[6]In the platform context, a platform might know a consumer's preferences (e.g., due to its past interactions with the consumer or using information from data brokers) and sends the consumer a signal contingent on those preferences. A less informationally demanding situation is that the platform does not know idiosyncratic preferences, but consumers with different preferences use the platform's information in different ways. For example, if some consumers care more about attribute A while others care more about attribute B, and if the platform reveals more information about attribute A, then the first group of consumers will learn more about their true valuations than the other group.

case $v_1 = v_2$.) In this case, the posterior distribution divides consumers into two groups: all consumers who see s_1 have expected valuation μ_H for product 1 and μ_L for product 2, while consumers who see s_2 have the reverse valuations.

In all these cases the signal structure is symmetric in the sense that firms are treated equally, but we also allow for asymmetric signal structures in which one firm is systematically favored. For example, the consumer might be informed whether or not v_1 exceeds v_2 by a specified margin.

We aim to investigate how the signal structure affects competition and the ability of consumers to buy their preferred product. In particular, we search for those signal structures which maximize industry profit and those which maximize consumer surplus. At a general level this appears to be an intractable problem, and in the following analysis we study special cases of this framework which are tractable. In the next section we discuss the easiest case to analyze, which is when the first-best profit can be achieved.

3   First-best outcomes

Suppose we find a signal structure which (i) maximizes total surplus (profit plus consumer surplus) and (ii) allocates all of that surplus to the firms in equilibrium. Then clearly no other signal structure can do better for firms (or do worse for consumers). If such a signal structure exists, its form is straightforward to derive. Since total surplus is maximized the consumer must always buy her preferred product, and since her surplus is fully extracted, she must only learn her expected valuation of the preferred product and pay a price equal to that valuation, i.e., $p_1 = p_2 = \mu_H$. For this to constitute an equilibrium, however, a firm cannot obtain more profit by deviating to a low enough price to attract those consumers who prefer the rival product, which requires deviating to price $p = \mu_L$, and thereby serving all consumers. Thus, if $\frac{1}{2}\mu_H \geq \mu_L$ the rank signal structure has equilibrium prices $p_1 = p_2 = \mu_H$ and fully extracts maximum surplus for firms.[7] This discussion is formally stated in the following result.

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[7] If instead of duopoly the two products were jointly supplied by a multiproduct monopolist, the rank signal structure with associated prices $p_1 = p_2 = \mu_H$ allows the firm to fully extract surplus, and so is always the most profitable signal structure for the firm, regardless of whether (3) holds.

Proposition 1 *If*

$$\mu_H \geq 2\mu_L \tag{3}$$

then the rank signal structure leads to an equilibrium which fully extracts maximum surplus for firms, and is therefore the signal structure which maximizes industry profit.

Since $\mu_L \geq V$ and $\mu_H \leq V + \Delta$, condition (3) requires $\Delta \geq V$, so that the range of valuations is large relative to the basic utility. Using (2), condition (3) can be written equivalently as

$$3\delta \geq \mu \ , \tag{4}$$

so that for a given mean μ first-best profit is more likely to be feasible when the difference $v_1 - v_2$ is more dispersed. There are at least two ways in which the valuation difference might be made more dispersed. First, for a given marginal distribution for v_i if the joint distribution becomes less positively correlated then δ rises.[8] Second, if the two valuations are independently distributed, then one can show that a mean-preserving spread in this distribution implies that δ rises.

If v is uniformly distributed on the square $[0, \Delta]^2$ then $\mu_H = \frac{2}{3}\Delta$ and $\mu_L = \frac{1}{3}\Delta$ and so (3) is (just) satisfied. More generally, the following result shows that (3) holds when the density is weakly decreasing on $[0, \Delta]$. (All omitted proofs can be found in the appendix.)

Corollary 1 *Suppose that v_1 and v_2 are independently distributed with density which weakly decreases over the support $[0, \Delta]$. Then condition (3) is satisfied and the rank signal structure generates first-best profit.*

One can also consider whether there exists a signal structure which maximizes total surplus and allocates it all to consumers in equilibrium. This would require that $p_1 = p_2 = 0$ are equilibrium prices, which in turn requires that consumers regard the two products as perfect substitutes. Except in the trivial case where the underlying products *are* perfect substitutes, i.e., when the prior distribution has $v_1 \equiv v_2$, if a signal structure

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[8]More precisely, let $F(v_1, v_2)$ and $\hat{F}(v_1, v_2)$ denote two joint CDFs for valuations with the same marginal distribution for v_i (and hence with the same mean μ), such that F is more correlated than $\hat{F}$ in the sense of Epstein and Tanny (1980), i.e., that $F \geq \hat{F}$. Since the function $\max\{v_1 - v_2, 0\}$ is "correlation averse" as defined in Epstein and Tanny, its expectation δ is higher with $\hat{F}$ than with F, and so if first-best profit is feasible with F it is also feasible with the less correlated distribution $\hat{F}$.

induces consumers to view the products as identical, they are unable to choose the preferred product more than half the time. Therefore, there is a trade off for consumers between paying low prices and being able to buy their preferred product, and thus no signal structure can implement the first-best outcome for consumers.

In this section we have derived the firm-optimal signal structure—which is the rank signal structure—when valuations are sufficiently dispersed that (3) holds. In this case, firms fully extract consumer surplus and the constraint that consumers not wish to consume their outside option always binds. In the remainder of the paper we discuss optimal signal structures when the first-best is not feasible, which is a considerably harder problem. In the next section we examine the next simplest case, which is when valuations are sufficiently concentrated, in which case we can ignore the consumer participation constraint altogether.

4   The market without an outside option

As discussed in the introduction, when first-best profit is not achievable (i.e., when (3) does not hold) attempts to derive an optimal second-best signal structure face two problems. First, preferences $v = (v_1, v_2)$ are generally two-dimensional, and current understanding of optimal information design in such cases is limited.[9] For that reason, in the remainder of the paper we simplify the model so that relevant consumer information is only scalar. Second, even with scalar heterogeneity some posterior distributions are such that the only equilibria in the pricing game between firms involve complicated mixed strategies.[10] For this reason, until section 6 we focus on signal structures which induce a pure strategy equilibrium in the pricing game.

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[9]It is well known that the posterior (expected) consumer valuation distribution induced by any information structure is a mean-preserving contraction of the underlying prior distribution. However, unlike the scalar case, a mean-preserving contraction has no simple characterization when consumer heterogeneity is multidimensional. See section 7.2 of Dworczak and Martini (2019) for discussion of this point.

[10]For example, when the first-best profit is *not* feasible, the rank signal structure induces a binary posterior distribution for valuations with which there is no pure strategy Bertrand equilibrium. For particular distributions with several mass points, it can be difficult even to verify whether there exists a mixed strategy equilibrium.

4.1 Preliminaries

Throughout section 4 we suppose that consumer preferences are such that the outside option is never relevant for consumers. The advantage of this assumption is that only the difference in valuations, $x \equiv v_1 - v_2$, matters for consumer decisions and welfare, and so the relevant consumer heterogeneity is captured by this scalar variable. The following result provides a sufficient condition for the outside option to be irrelevant, which is that valuations have a concentrated distribution in the sense that the range of valuations is small relative to the minimum valuation. (Note that we allow signal structures which make the posterior market asymmetric.)

Lemma 1 *If $V > 3\Delta$, then under any signal structure which induces a pure strategy equilibrium, equilibrium prices are below V and all consumers obtain positive surplus from both firms.*

Proof. With any signal structure the maximum posterior valuation for a product does not exceed $V + \Delta$ and the minimum posterior valuation is at least V. Suppose a signal structure induces firms $i = 1, 2$ to offer respective prices p_1 and p_2 and to obtain profits π_1 and π_2. It is clear that neither p_1 nor p_2 can exceed $V + \Delta$.[11] Firm j will serve all consumers if it deviates to a low price p such that $V - p \geq V + \Delta - p_i$, i.e., if $p \leq p_i - \Delta$. (Since $p_i \leq V + \Delta$ the inequality $V - p \geq V + \Delta - p_i$ ensures that $V - p \geq 0$ so that all consumers prefer to buy from firm j than to buy nothing.) Therefore, we must have

$$p_i - \Delta \leq \pi_j \ . \tag{5}$$

If firms are labelled so $p_1 \geq p_2$, then the above inequality implies $p_1 - \Delta \leq \pi_2 \leq p_2$ so that the price difference $p_1 - p_2$ cannot exceed Δ. Adding the pair of inequalities (5) implies that

$$p_1 + p_2 - 2\Delta \leq \pi_1 + \pi_2 \leq p_1 \ ,$$

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[11] If firm j did choose price $p_j > V + \Delta$ in equilibrium, then no consumer will buy from it, and firm i acts as a monopolist and its optimal price must be $p_i \geq V > 0$. Then firm j can earn a strictly positive profit by deviating to a price slightly below p_i, as under any signal structure there must be a positive measure of consumers who weakly prefer product j over product i given the two products are symmetric *ex ante*.

where the second inequality follows since industry profit cannot exceed the maximum price p_1, and so $p_2 \leq 2\Delta$. Since $p_1 - p_2 \leq \Delta$ it follows that $p_1 \leq 3\Delta$, and so when $3\Delta < V$ we must have $\max\{p_1, p_2\} < V$. ∎

Note that in the case of a symmetric equilibrium, the proof shows that $p_1 = p_2 \leq 2\Delta$, so that feasible symmetric prices lie in the interval $[0, 2\Delta]$. Note also that in the consumer-optimal policy we derive below the induced prices are sufficiently low that the condition $V > 3\Delta$ can be considerably weakened.

For the remainder of section 4 suppose that both prices in any pure strategy equilibrium are less than V, so that the outside option is irrelevant in the sense that if at most one firm deviates from equilibrium all consumers continue to participate. The consumer prefers to buy from firm 1 if $x \equiv v_1 - v_2 > p_1 - p_2$, prefers to buy from firm 2 if $x < p_1 - p_2$ (and is indifferent when $x = p_1 - p_2$). Since in the underlying market v_1 and v_2 are symmetrically distributed, the scalar variable x is symmetrically distributed within the line segment $[-\Delta, \Delta]$, with a CDF denoted by $F(x)$ say. Then δ in (2) takes the form

$$\delta = \mathbb{E}[\max\{v_1 - v_2, 0\}] = \int_0^\Delta x dF(x) = \int_{-\Delta}^0 F(x) dx \qquad (6)$$

where the final expression follows after integration by parts (which remains valid even if F is not continuous) and uses the fact that x has a zero mean (which implies $\int_{-\Delta}^\Delta F(x) dx = \Delta$). Clearly $\delta \leq \frac{1}{2}\Delta$ for any symmetric F, while $\delta \leq \frac{1}{4}\Delta$ if F is convex in the range $[-\Delta, 0]$ and has no mass point at $x = -\Delta$.

Clearly, any signal of the two-dimensional preference parameter v in section 2 induces a signal s of the scalar preference parameter x (while any additional information in the signal plays no role for consumers or welfare). After observing signal s, the consumer updates her belief about her expected x. The prior distribution F and the signal structure jointly determine a signal distribution for the consumer, which further determines a posterior distribution for (expected) x which has CDF $G(x)$, say. For a given prior F, the only restriction on the posterior G imposed by Bayesian consistency is that it is a *mean-preserving contraction (MPC)* of F, i.e.,

$$\int_{-\Delta}^x G(\tilde{x}) d\tilde{x} \leq \int_{-\Delta}^x F(\tilde{x}) d\tilde{x} \text{ for } x \in [-\Delta, \Delta], \text{ with equality at } x = \Delta . \qquad (7)$$

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Moreover, any G which is an MPC of F can be generated by some signal structure (which can be based simply on the scalar preference parameter x instead of (v_1, v_2)).[12] Therefore, instead of analyzing the signal structure directly (as we did with the first-best analysis), we work with the posterior distribution G subject only to the MPC constraint.

To fix ideas, Figure 1 depicts various kinds of posterior distributions for x which are an MPC of the prior distribution (marked as dashed lines), here taken to be a uniform distribution on $[-1, 1]$. When G crosses F once and from below on $(-\Delta, \Delta)$, as in Figures 1a and 1c, then the necessary condition that G has the same mean as F, i.e., that there is equality at $x = \Delta$ in (7), is also sufficient for G to be an MPC of F. If G is symmetric and crosses F at most once and from below in the negative range $x \in (-\Delta, 0)$, as on Figures 1a and 1b, then the necessary condition

$$\int_{-\Delta}^{0} G(x)dx \leq \int_{-\Delta}^{0} F(x)dx \equiv \delta \qquad (8)$$
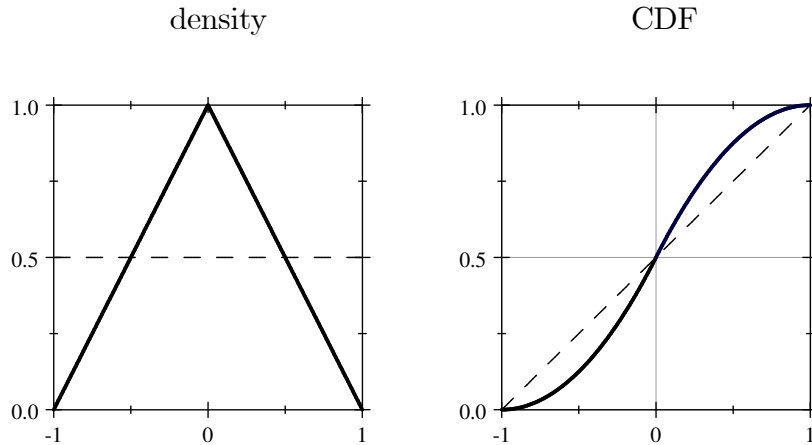
is sufficient for G to be an MPC of F.

density CDF

Figure 1a: MPC which increases density of consumers at $x = 0$

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[12]See, for example, Blackwell (1953), Rothschild and Stiglitz (1970), Gentzkow and Kamenica (2016), and Roesler and Szentes (2017).
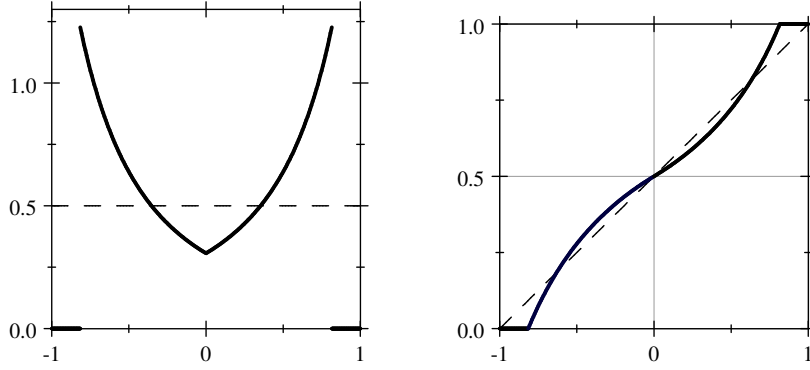
Figure 1b: MPC which reduces density of consumers at $x = 0$
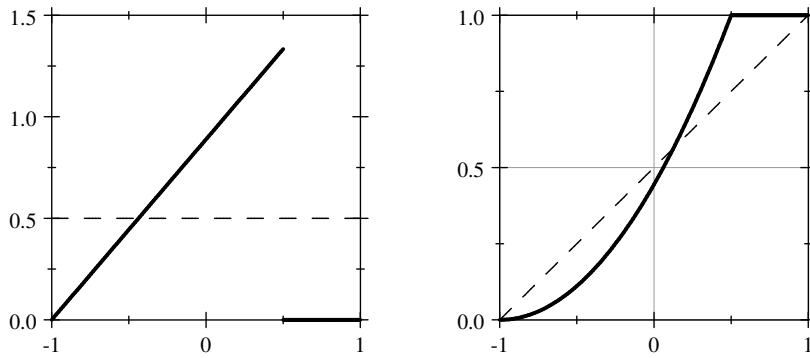

Figure 1c: Asymmetric MPC which shifts demand to one firm

It is useful to have a measure of the efficiency of product choice with a given signal structure corresponding to posterior G. With symmetric prices and full consumer participation, total surplus is the expected value of $\max\{v_1, v_2\}$ given G, which can be written as

$$W_G = \mathbb{E}_G[\max\{v_1, v_2\}] = \mu + \mathbb{E}_G[\max\{v_1 - v_2, 0\}] = \mu + \int_{-\Delta}^{0} G(x)dx \ , \qquad (9)$$

where the final equality follows with similar logic to (6). Since G is an MPC of F, the necessary condition (8) shows that match efficiency cannot increase when the consumer observes a noisy signal of her preferences rather than her actual preferences, as is intuitive. When the equality (8) is strict—as in Figures 1a and 1c but not Figure 1b— then there is mismatch with the posterior G, due to the consumer sometimes buying

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the wrong product. In the extreme case where the signal is completely uninformative, we have $G = 0$ for $x < 0$ and $W_G = \mu$, the expected surplus from consuming a random product. There is no product mismatch when there is equality in (8). In terms of the signal structure this is the case when the range of signals seen when $x > 0$ does not overlap with the range of signals seen when $x < 0$, so that the consumer is fully informed about whether $x > 0$ or $x < 0$, even though she may not be fully informed about the magnitude of x.

Some of the most frequently-used signal structures induce consumers to become more concentrated around $x = 0$, similarly to Figure 1a. This is so with a "truth-or-noise" structure, whereby the signal s is equal to the true x with some probability and otherwise the signal is a random realization of x, or more generally when the distribution for x is "rotated" about $x = 0$ as studied by Johnson and Myatt (2006). Such signals induce a degree of mismatch. By contrast, a signal which accurately reveals to a consumer which product she prefers (so that (8) holds with equality) will necessarily induce weakly *fewer* consumers around $x = 0$ *ex post*, as on Figure 1b.[13]

A leading case, which simplifies the following analysis and ensures all our major results, is when the prior distribution for x has a (symmetric) density which is *log-concave* on $[-\Delta, \Delta]$.[14] In this case, as shown by Bagnoli and Bergstrom (2005), both $F(\cdot)$ and $1 - F(\cdot)$ are log-concave on $[-\Delta, \Delta]$, and in addition the underlying market (where consumers are fully informed about valuations) has a symmetric pure strategy equilibrium where the equilibrium price is $p_F = 1/(2f(0))$. A useful observation for later is the following:

Lemma 2 *If the prior density $f(x)$ is log-concave on $[-\Delta, \Delta]$, then the full-information price $p_F = 1/(2f(0))$ satisfies*

$$2\delta \leq p_F \leq 4\delta \ . \tag{10}$$

Since δ in (6) measures the extent of product differentiation when consumers are fully informed, this result shows that the corresponding price moves roughly in step with this product differentiation.

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[13] This is because when (8) binds, the MPC constraint (7) requires that G lie weakly above F for x just below zero, in which case G is weakly flatter than F at $x = 0$.

[14] For example, if (v_1, v_2) have a log-concave joint density (which includes the case when v_1 and v_2 are i.i.d. with a log-concave density), their difference x also has a log-concave density.

Before the general analysis, we first compare the performance of several simple signal structures, related to the signal structures mentioned in section 2, when the prior density for x is log-concave:

- Full information disclosure: in this case industry profit is p_F, and consumer surplus is $\mu + \delta - p_F \leq \mu - \delta$, where the inequality follows from (10). Consumers always buy their preferred product but the market price is relatively high.

- No information disclosure: in this case the two products are perceived to be perfect substitutes. Equilibrium price and industry profit are zero, while consumer surplus is μ which is higher than with full information disclosure. No information leads to a random product match but also the lowest possible price, and the benefit of the low price dominates.

- "Truth or rank" signal structure: suppose the consumer learns her x perfectly with probability $\theta < 1$ and otherwise learns only whether $x > 0$ or $x < 0$. Then the posterior distribution has two mass points at -2δ and 2δ (with mass $\frac{1-\theta}{2}$ at each) and is otherwise the same as the prior but with a reduced density $\theta f(x)$. In this case, if θ is sufficiently close to 1 (so the two mass points do not have too much weight), a symmetric equilibrium price exists and is equal to $\frac{p_F}{\theta} > p_F$, and so firms earn more than they did with full disclosure.[15] (Consumer surplus, however, is lower than with full disclosure because in either case the consumer buys the product she prefers.)

These *ad hoc* signals illustrate that with a regular prior full information disclosure is optimal neither for firms nor consumers. In the next section we derive the signal structures which are optimal for firms and for consumers.

4.2 Optimal symmetric signal structures

In this section, we focus on the relatively simple case of symmetric signal structures, where the posterior distribution G is symmetric, and study which symmetric prices

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[15] For example, when the prior distribution is uniform on $[-1, 1]$, one can check, by using the posterior bounds derived in section 4.2, this is the case for $\theta \geq 3 - \sqrt{5} \approx 0.76$. If θ is below this threshold, the incentive for each firm to undercut and steal the rival's consumers on the mass points becomes so strong that there is no pure strategy equilibrium.

can be implemented and which symmetric signal structures are best for firms and for consumers.[16] When the underlying market has a log-concave density, we will show in section 4.3 that no individual firm nor consumers in aggregate can do better using asymmetric signals and prices.

Having discussed the constraints on G imposed by Bayesian consistency in section 4.1, we turn next to the constraints on G needed to achieve a target symmetric price in pure strategy equilibrium. For $p = 0$ to be an equilibrium price, the consumer must regard the two varieties as identical, and this can happen only if G is degenerate at $x = 0$. In the following, we focus on positive prices. First note that to have a positive symmetric equilibrium price the distribution G cannot have an atom at $x = 0$, i.e., we must have $G(0) = \frac{1}{2}$, for otherwise a firm obtains a discrete jump in demand if it slightly undercuts its rival. Recall also that any symmetric equilibrium price satisfies $p \leq 2\Delta$.

Consider a candidate symmetric equilibrium price $p > 0$. If firm 2 deviates to price $p' \neq p$ the consumer buys from firm 2 if $x \leq p - p'$. (Thus we suppose that if G has a mass point at $x = p - p'$, firm 2 serves all consumers at that mass point. This is the natural tie-breaking rule given that the firm can achieve this outcome by charging a price slightly below p'.) Therefore, firm 2 has no incentive to deviate if and only if

$$p'G(p - p') \leq \tfrac{1}{2}p$$

holds for all p'. (The inequality holds with equality at $p' = p$ since $G(0) = \frac{1}{2}$.) By changing variables from p' to $x = p - p'$, we can write this requirement as

$$G(x) \leq U_p(x) \equiv \min\left\{1, \frac{p}{2\max\{0, p-x\}}\right\}. \tag{11}$$

(It is unprofitable for firm 2 to set a negative price p', and so there are restrictions on G only in the range where $p' = p - x > 0$, which is why there is $\max\{0, \cdot\}$ in the denominator. In addition, a CDF cannot exceed 1 which is why there is $\min\{1, \cdot\}$ in (11).) Likewise, for firm 1 to have no incentive to deviate we require $p'(1 - G(p' - p)) \leq \frac{1}{2}p$ for all (positive) p'.[17] Following the parallel argument to that for firm 2, this

    [16]In section 4.3 we show that equilibrium prices induced by a symmetric posterior G, or any G such that $G(0) = 1/2$, must be symmetric.

    [17]As with firm 2, if x has an atom at $x = p' - p$ the natural tie-breaking assumption is that firm 1

constraint can be written as

$$G(x) \geq L_p(x) \equiv \max\left\{0, 1 - \frac{p}{2\max\{0, p + x\}}\right\}. \tag{12}$$

This analysis shows that $p \in (0, 2\Delta]$ can be supported as an equilibrium price by G if and only if G lies between the two bounds L_p and U_p. (This bounds condition also ensures $G(0) = \frac{1}{2}$ as shown below.) Notice also that the two bounds are mirror images of each other, in the sense that $L_p(x) \equiv 1 - U_p(-x)$. Therefore, if a symmetric G lies between the bounds in the negative range $x \in [-\Delta, 0]$ it will lie between the bounds over the whole range $[-\Delta, \Delta]$.
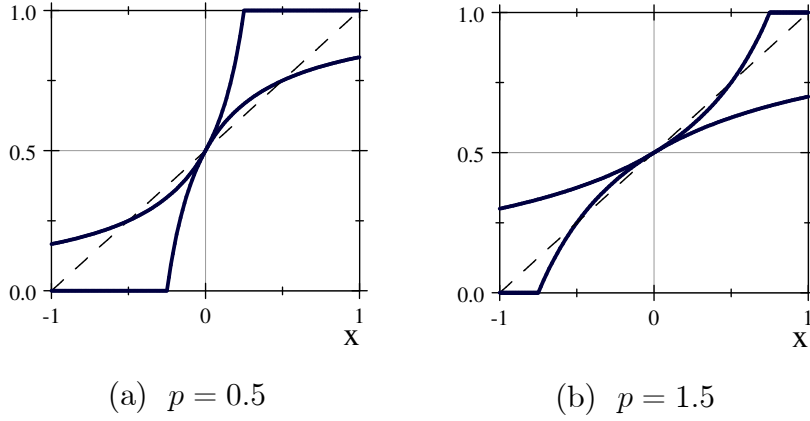

(a) $p = 0.5$ (b) $p = 1.5$

Figure 2: Bounds on G to implement prices $p = 0.5$ and $p = 1.5$

Figure 2 illustrates the two bounds (depicted as bold curves) for target prices $p = 0.5$ and $p = 1.5$. The lower bound L_p is increasing in x and begins to be positive at $x = -\frac{1}{2}p$ which exceeds $-\Delta$ given $p \leq 2\Delta$. Moreover, L_p is concave whenever L_p is positive. The upper bound U_p is increasing in x and reaches 1 at $x = \frac{1}{2}p \leq \Delta$, and it is convex when below 1. These two bounds coincide and equal $\frac{1}{2}$, and have the same slope $1/(2p)$, when $x = 0$. It follows that the lower bound is always below the upper bound, and so the set of G's lying between the bounds L_p and U_p is non-empty for $0 < p \leq 2\Delta$. Note also that both L_p and U_p rotate clockwise about the point $(0, \frac{1}{2})$ as p increases, and in

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serves all customers at x. Strictly speaking, as a CDF is right-continuous, here we need to interpret $1 - G(p' - p)$ as including any atom at $p' - p$. Since the bounds we derive are continuous functions, this point makes no difference to the analysis.

particular the bounds increase with p for negative x. (Intuitively, to induce a higher price we need fewer consumers around $x = 0$, which requires the bounds to be flatter.) Another observation used in the subsequent analysis is that U_p is log-convex in x when it is below 1, and $1 - L_p$ is log-convex when L_p is positive.[18]

Recall that a posterior is only feasible if it is an MPC of the prior. Therefore, price p can be implemented with some signal structure provided a G can be found within the bounds (11)–(12) which is an MPC of the prior. Figure 2 illustrates the case when the prior is uniform on $[-1, 1]$, where the prior CDF is marked as the dashed line and where the full-information price is $p_F = 1$. In either case, if the posterior G is chosen to be the lower bound for $x \leq 0$ and the upper bound for $x \geq 0$, then this G is an MPC of F and (by construction) lies between the bounds. Therefore, both prices can be implemented with a suitable signal structure. The latter price is already 50% higher than the full-information price.

We say that (G, p) is a symmetric *outcome* if G is a symmetric MPC of F and p is a symmetric equilibrium price given G. Note that if the posterior G is symmetric about $x = 0$ then it automatically has the same mean as F, and the condition for G to be an MPC of F only requires that (7) hold in the lower range $x \in [-\Delta, 0]$. We summarize this discussion in the following:

Lemma 3 (G, p) *is a symmetric outcome if and only if G is a symmetric distribution such that for $x \in [-\Delta, 0]$ condition (7) holds and*

$$L_p(x) \leq G(x) \leq U_p(x) .\tag{13}$$

In the following we will mainly focus on the case when the prior distribution has a density $f(x)$ which is log-concave on $[-\Delta, \Delta]$, although as we will discuss this analysis can be generalized to more general prior distributions.

Firm-optimal policy: Using the posterior bounds it is straightforward to derive the symmetric signal structure which maximizes profit in this market. Since there is full

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[18]This analysis may have some independent interest for studying price competition. For a given distribution $G(x)$ which is differentiable at $x = 0$ (which is required by the bounds condition), one can calculate a potential symmetric equilibrium price p from the usual first-order condition. If G is verified to lie between the bounds L_p and U_p then p is indeed the equilibrium price. This approach is more general than the usual approach of checking the quasi-concavity of each firm's profit function with G.

consumer participation, maximizing profit corresponds to finding the maximum price which can be implemented with an MPC of the prior. Lemma 3 implies that if (G, p) is a symmetric outcome then so is $(\tilde{G}, p)$, where $\tilde{G}(x) = L_p(x)$ for $x \in [-\Delta, 0]$ and $\tilde{G}(x) = U_p(x)$ for $x \in [0, \Delta]$. Therefore, to find the firm-optimal price we can restrict attention to symmetric posteriors which take the form L_p for $x \in [-\Delta, 0]$.

Consider first the example with a uniform prior, as on Figure 2. Given that L_p is concave whenever it is positive, it crosses the (linear) prior CDF at most once and from below in the range of $(-1, 0)$. Therefore, from (8) a symmetric G which is equal to L_p for $x \leq 0$ is an MPC of the prior if and only if L_p has integral on $[-1, 0]$ no greater than δ, where $\delta = 1/4$ in this uniform example. Since L_p increases with p for $x \leq 0$, it is optimal to make this integral constraint bind, so that the optimal price p^* solves

$$\tfrac{1}{4} = \int_{-1}^{0} L_p(x)dx = \int_{-\frac{1}{2}p}^{0} \left(1 - \frac{p}{2(p+x)}\right) dx = \tfrac{1}{2}(1 - \log 2)p$$

and so

$$p^* = \frac{1}{2(1 - \log 2)} \approx 1.63 \ .$$

This optimal price is about 63% higher than the full-information price $p_F = 1$. The posterior distribution which implements this optimal price is what we depicted on Figure 1b above where the density is U-shaped. Intuitively, to soften price competition we reduce the number of marginal consumers around $x = 0$, as these consumers are the most price sensitive, and push consumers towards the two extremes insofar as this is feasible given the pure-strategy and the MPC constraints.

This same argument applies more generally whenever the lower bound L_p for each price crosses the prior CDF F at most once and from below in the range $(-\Delta, 0)$. This is true, for example, if F is convex in the range $[-\Delta, 0]$. More generally, since $1 - L_p$ is log-convex when L_p is positive, this is the case if $1 - F$ is log-concave in the range $[-\Delta, 0]$.[19] (A sufficient condition for $1 - F$ to be log-concave on $[-\Delta, 0]$ is that it has a density f which is log-concave on $[-\Delta, \Delta]$.) Then the lower bound is an MPC of the prior if and only if its integral over $[-\Delta, 0]$ is no greater than that of the prior (which is δ). Since the former integral increases in p, at the optimum there is equality in the

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[19]Because $\log(1 - F)$ is concave and decreasing on $[-\Delta, 0]$ and $\log(1 - L_p)$ is (strictly) convex and decreasing on $[-\Delta, 0]$, the former function can cross the latter at most once in $(-\Delta, 0)$.

two integrals (unless p reaches 2Δ first), so that the profit maximizing price p^* is

$$p^* = \frac{2\delta}{1 - \log 2} \ . \tag{14}$$

(This does not exceed 2Δ when the prior density is log-concave since in that case $\delta \leq \frac{1}{4}\Delta$.) In particular, the firm-optimal price p^* in (14) is higher when the prior distribution for x is more dispersed, in the sense that δ is larger. The firm-optimal symmetric G, equal to the lower bound L_{p^*} for $x < 0$, is unique in this case. With this posterior distribution, when a firm deviates to a price lower than p^* its demand is unit-elastic (i.e., its profit is unchanged), although when it deviates to a higher price its profit strictly falls.

There are also two other useful observations. First, since by construction (8) holds with equality, the firm-optimal signal structure induces no mismatch, and total welfare is maximized as well as profit. Second, the firm-optimal price p^* is considerably higher than the price p_F under full information disclosure, so that full information disclosure is not optimal for firms. Indeed, with a log-concave density (10) and (14) together imply that $p^* \geq \frac{1}{2(1-\log 2)}p_F \approx 1.63 \times p_F$, and so relative to full disclosure profits rise by at least 63% using the optimal signal structure.

We summarize this discussion in the following result:

Proposition 2 *Suppose the outside option is not relevant and the prior distribution has a log-concave density. Then:*
(i) the firm-optimal symmetric price p^* is (14), which is at least 63% higher than the full-information price p_F, and it is uniquely implemented by the symmetric posterior which is equal to L_{p^*} in the negative range $x \in [-\Delta, 0]$;
(ii) with the firm-optimal symmetric signal structure there is no mismatch and total welfare is also maximized.

Note that familiar signal structures such as rotations in the distribution around $x = 0$ induce consumers to be *more* concentrated around $x = 0$, and so cannot be used to enhance profit relative to the full-information policy. Therefore, the use of unrestricted signal structures, which allow consumers to buy their preferred product, enables firms to do at least 63% better than they could with these more restricted signals.

Beyond the simple case with a log-concave density, the firm-optimal price which can be implemented with some signal structure is the highest $p \leq 2\Delta$ such that L_p in (12) satisfies

$$\int_{-\Delta}^{x} L_p(\tilde{x})d\tilde{x} \leq \int_{-\Delta}^{x} F(\tilde{x})d\tilde{x} \text{ for } x \in [-\Delta, 0] .$$

In general L_p and F can cross multiple times in the range $(-\Delta, 0)$, in which case solving the optimal price can be less straightforward. Moreover, (8) might hold strictly and so there could be welfare loss associated with the firm-optimal signal structure. Figure 3 illustrates both points, where the prior shown as the dashed curve is initially convex and then concave. The highest price such that L_p is an MPC of the prior is shown as the solid curve, where the integrals of the two curves up to the crossing point a are equal (so with any higher price the MPC constraint would be violated). Here, since L_{p^*} lies below the prior for x above a, (8) holds strictly and there is some mismatch at the optimum.
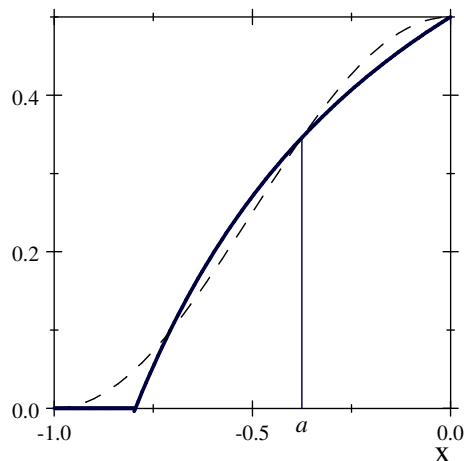

Figure 3: Firm-optimal posterior with a less regular prior

Consumer-optimal policy: We turn next to the problem of finding the best symmetric signal structure for consumers. Unlike firms, consumers do not care solely about the induced price but also about the reliability of the product match, and consumer surplus with posterior G and price p is $W_G - p$, where total welfare W_G is given in (9).

Note first that δ is the incremental consumer benefit from buying the preferred product rather than a random product, and that in the information structure where

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consumers receive no product information they buy a random product at price zero. Therefore, for consumers to do better than the no-information policy, the price they pay cannot exceed δ. With a log-concave prior density, however, (10) shows that the full-information price satisfies $p_F \geq 2\delta$, and so the consumer-optimal price must be less than half of the full-information price. In addition, maximum consumer surplus is at least μ (which is consumer surplus with no information disclosure), and with a log-concave density this is at least δ more than consumer surplus with full information disclosure (which is $\mu + \delta - p_F \leq \mu - \delta$).

To solve the consumer-optimal problem in more detail, we first find the highest possible G to maximize match efficiency for a given price p, subject to the bounds condition (13) and the MPC constraint, and we then identify the optimal price. Consider again the example where the prior distribution is uniform on $[-1, 1]$, where the equilibrium price with full information disclosure is $p_F = 1$. For consumers to do better than this policy, the induced price must be below 1 to counter-act any potential product mismatch, in which case the bounds look similar to Figure 2a and the upper bound is below the prior for x close to zero. (In fact, since this density is log-concave, we know the optimal price is below $\frac{1}{2}$.) Therefore, it is the *upper* bound (11) which will constrain G, rather than the lower bound which was relevant for the firm-optimal policy. Since the upper bound is convex, for any price $p < 1$ the upper bound cuts the prior CDF once and from above. Figure 4 illustrates how to maximize consumer surplus for a given price $p < 1$. Figure 4a shows the two bounds in (13) as bold curves, where the upper bound cuts the prior CDF at $x_p \equiv p - 1$ in this example. Given p, we wish to maximize the integral of G over $[-\Delta, 0]$, subject to lying between these bounds and the MPC constraint.

Two necessary conditions for G are that it satisfy the MPC constraint (7) at the intercept point x_p, i.e.,

$$\int_{-\Delta}^{x_p} G(x)dx \leq \int_{-\Delta}^{x_p} F(x)dx , \tag{15}$$

and that G lies below the upper bound for all x above the intercept point. Clearly, the solution involves setting $G(x) = U_p(x)$ for x above the intercept point, since the MPC constraint (7) is surely satisfied for $x \geq x_p$ if (15) holds for any $G \leq U_p$. In addition, it is clear that (15) must bind. However, there are many ways to choose G such that this constraint binds, all of which yield the same consumer surplus. Figure 4b shows a

convenient way to do this, which is to set G equal to the prior CDF for $x \leq x_p$, so that

$$G(x) = \min\{F(x), U_p(x)\} , \tag{16}$$

while Figure 4c depicts an alternative way to satisfy the constraint. Since there is a strict inequality in (8) for any optimal G, there is welfare loss at the consumer optimum and some consumers buy their less preferred product. However, those consumers with strong preferences, i.e., those with $x \leq x_p$, receive their preferred product for sure.[20]
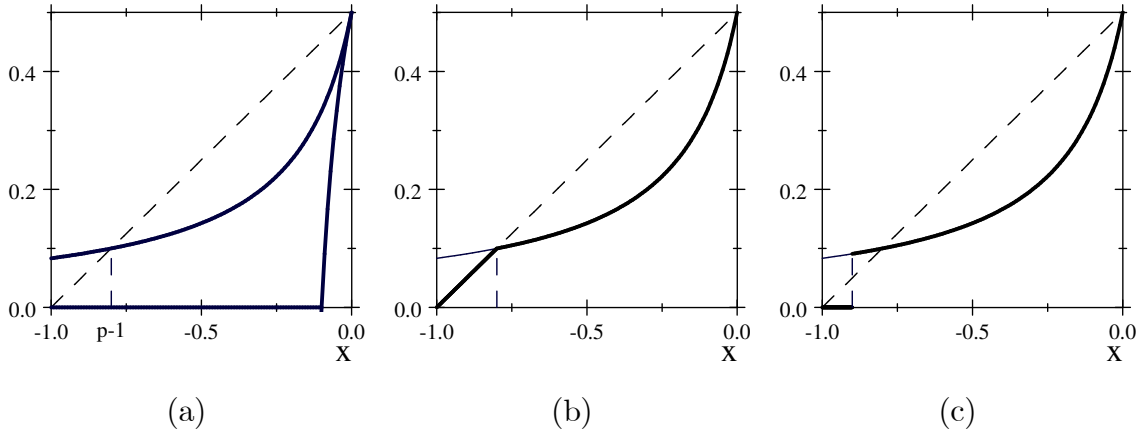

Figure 4: Consumer-optimal G for a given price p

Expression (16) implies that maximum consumer surplus for a given price $p \leq 1$ is

$$W_G - p = \mu + \int_{-\Delta}^{0} \min\{F(x), U_p(x)\}dx - p . \tag{17}$$

The derivative of (17) with respect to p is

$$\int_{x_p}^{0} \frac{\partial U_p(x)}{\partial p}dx - 1 = \frac{1}{2}\left(\frac{p}{p - x_p} - \log\frac{p}{p - x_p} - 3\right) , \tag{18}$$

where x_p is the intercept point of $F(x)$ and $U_p(x)$. In the uniform example where $x_p = p - 1$, (18) becomes $\frac{1}{2}(p - \log p - 3)$ which decreases with p in the range $[0,1]$. Then the optimal price is $p^* = \gamma$, where $\gamma \approx 0.05$ is the root of $\gamma - \log\gamma = 3$ in the range $[0,1]$.

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[20]Note that for price $p \leq 1$, the posterior (16) is above L_p. This is because at the full information price $p_F = 1$ we have $L_{p_F} \leq F$, and so the same is true for any lower price.

Figure 5 depicts the consumer-optimal posterior distribution (when it takes the particular form in Figure 4b), where the number of price-sensitive consumers near $x = 0$ is amplified compared with the prior distribution, and this forces firms to reduce their price in equilibrium. Those consumers near $x = 0$ do not have strong preferences about which product they buy, and so there is only limited welfare loss due to product mismatch. Those consumers with very strong preferences, however, are sure to buy their preferred product and at a low price. Such a posterior distribution also implies that when a firm unilaterally increases its price its residual demand is unit-elastic.
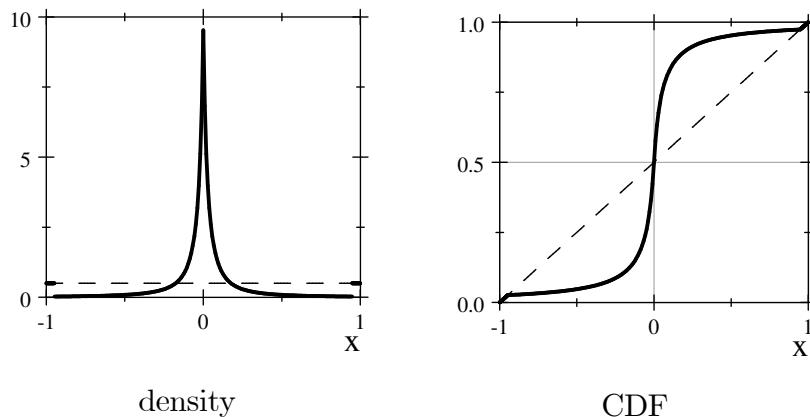

density   CDF

Figure 5: Consumer-optimal information structure

The same argument applies more generally whenever the upper bound for each price below the full-information price crosses the prior CDF once and from the above in the range $[-\Delta, 0]$. A sufficient condition for that is that the prior density is log-concave.

Proposition 3 *Suppose the outside option is not relevant and the prior distribution has a log-concave density. Then:*
(i) the consumer-optimal symmetric price is

$$p^* = \frac{-\gamma}{1 - \gamma} F^{-1}(\tfrac{1}{2}\gamma) \; , \tag{19}$$

which satisfies $\gamma p_F \leq p^* \leq \frac{1}{2} p_F$, and it is implemented by the posterior (16);
(ii) with the consumer-optimal symmetric signal structure, only a fraction γ of consumers are sure to buy their preferred product, so there is mismatch and total welfare is not maximized.

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Because the optimal price is often low, the constraint that G should lie below the upper bound U_p is more important than the constraint that G is an MPC of F. It follows that this consumer-optimal policy is often approximated by the solution to a simpler problem, which is to choose a symmetric distribution G in order to maximize equilibrium consumer surplus. In this alternative scenario, there is no prior and no MPC constraint (or the prior is sufficiently dispersed such that the MPC constraint does not bind). In this "relaxed" problem we wish to choose the distribution for x, say within the support $[-\Delta, \Delta]$, which trades off the benefits of a low equilibrium price (which is implemented by a distribution concentrated around $x = 0$) and the benefits of being able to the choose the better of two products (which is greater when the distribution for x is more dispersed).[21] The above discussion shows that the solution to this relaxed problem is to choose the price p to maximize

$$W_G - p = \mu + \int_{-\Delta}^{0} U_p(x)dx - p \tag{20}$$

instead of (17). Since the difference between (20) and (17) increases with p, the solution to this second problem involves a higher price than in Proposition 3. However, in many cases the difference is tiny.[22]

Another implication of the low price in the consumer-optimal policy is that the conditions required for the validity of ignoring the consumer participation constraint are much less stringent than required for Lemma 1 (which was $V > 3\Delta$). More precisely, we claim that if $V > \delta$ then the consumer-optimal policy is as described in Proposition 3.[23] To see this, note that as we have pointed out before, for consumers to do better than the no-information policy, the industry profit π they generate cannot exceed δ. If some consumers do not participate under the optimal policy, the price p must exceed the minimum valuation V. If one firm deviates to a lower price V, it then sells to

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[21] This relates to the policy concern about the extent to which products such as insurance should be "standardized". More standardized products are close substitutes and so facilitate competition on price, but prevent some consumers with particular preferences from obtaining a product tailored to those preferences. See for instance Ericson and Starc (2016) for a discussion and empirical analysis of this issue.

[22] With support $[-1, 1]$, the price which maximizes (20) is $p \approx 0.055$, compared to $p^* = \gamma \approx 0.052$ for the example with a uniform prior.

[23] Note that $\delta \leq \frac{1}{2}\Delta$, and so a sufficient condition to ignore the participation constraint is $V > \frac{1}{2}\Delta$. If the prior density for $x = v_1 - v_2$ is further log-concave, then F is convex on $[-\Delta, 0]$ and so $\delta \leq \frac{1}{4}\Delta$, in which a weaker sufficient condition is $V > \frac{1}{4}\Delta$.

at least half the consumers and so it obtains deviation profit at least $\frac{1}{2}V$. Therefore, equilibrium industry profit π exceeds V, which is not possible when $\pi \le \delta$ and $\delta < V$. Therefore, there must be full participation at the consumer-optimal policy when $V > \delta$. It follows that the price itself must be below V, in which case the outside option does not then apply (even when one firm deviates to a higher price).

For more general prior distributions, the following result in the spirit of Roesler and Szentes (2017) shows that we can restrict attention to a simple family of posteriors illustrated by Figure 4c above.

Lemma 4 *The consumer-optimal policy can be implemented by a symmetric posterior defined on $[-\Delta, 0]$ of the form*

$$G_p^m(x) = \begin{cases} 0 & \text{if } x < m \\ U_p(x) & \text{if } m \le x \le 0 \end{cases} . \tag{21}$$

Proof. To see why we need only consider this family, suppose a candidate consumer-optimal policy involves the equilibrium price p and the posterior G. For this policy to do better than disclosing no information, it is necessary that

$$p \le \int_{-\Delta}^0 G(x)dx . \tag{22}$$

Since G is below U_p for $x \in [-\Delta, 0]$, there is a unique $m \in [-\Delta, 0]$ which satisfies

$$\int_{-\Delta}^0 G(x)dx = \int_{-\Delta}^0 G_p^m(x)dx = \frac{p}{2} \log \frac{p - m}{p} .$$

Clearly, since it "crosses" G once and from below in $(-\Delta, 0)$, this G_p^m is an MPC of G and hence of F. Expression (22) implies that $m \le -(e^2 - 1)p < -\frac{1}{2}p$. Since the lower bound L_p becomes positive at $x = -\frac{1}{2}p$ it follows that G_p^m lies above L_p for $x \in [-\Delta, 0]$. We deduce that G_p^m lies between the bounds L_p and U_p, and so induces the same equilibrium price p and has the same match efficiency as the original G. Therefore, there is no posterior that does better for consumers than those which take the form (21). ∎

In sum, we can solve the consumer problem by choosing (p, m) to maximize consumer surplus, $\mu + \int_{-\Delta}^0 G_p^m(x)dx - p$, subject to the constraint that G_p^m is an MPC of F. Using this method one can show that it is always optimal to disclose some information to consumers.

Corollary 2 *Except in the degenerate case where products are perfect substitutes (i.e., $x \equiv 0$ under the prior distribution), it is sub-optimal to disclose no information to consumers.*

The welfare limits. Having discussed the signal structures which maximize profit and which maximize consumer surplus, we are in a position to describe the combinations of profit and consumer surplus which are feasible with some choice of symmetric signal structure. First, it is clear that any such combination cannot sum to more than maximum welfare, which is $\mu_H = \mu + \delta$. Thus, any feasible combination lies weakly under the efficient frontier marked as the higher dashed line on Figure 6 (where for convenience we set $\mu = 2$ and $\delta = \frac{1}{4}$). Likewise, the sum cannot be lower than minimum welfare in (9), which is μ as marked as the lower dashed line on the figure, and so feasible combinations lie between these dashed bounds. In particular, feasible combinations cannot be too inefficient, and welfare cannot be further than δ from the efficient frontier. This contrasts with the corresponding figure for monopoly in Roesler and Szentes (2017, Figure 1), where it was feasible to have low consumer surplus and low profit simultaneously.[24]
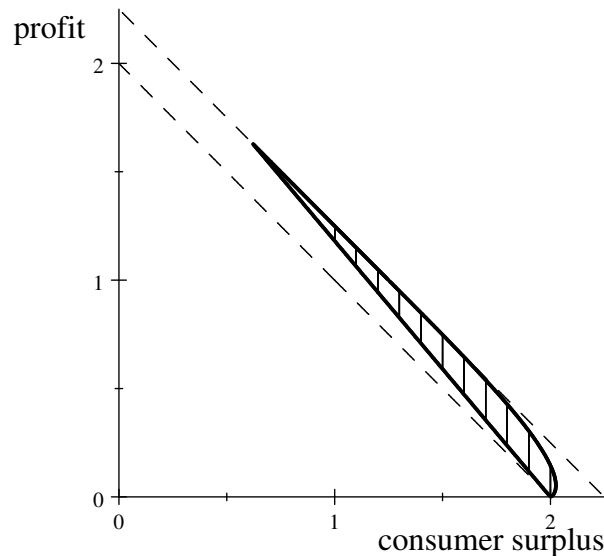

Figure 6: Feasible combinations of profit and consumer surplus

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[24]In Roesler and Szentes the construction for a given profit level is that the firm is indifferent between all prices in the support; the lowest price is best for consumers while the highest price leaves consumers with nothing.

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Suppose that the prior distribution has log-concave density. Then Proposition 2 shows that the maximum possible price/profit is p^* in (14), which in this case with $\delta = \frac{1}{4}$ is $p^* \approx 1.63$. Any price $p \in [0, p^*]$ can be implemented, and given such a price the range of consumer surplus which is possible is determined by the posteriors G which lie within the bounds (13) and which are an MPC of the prior. Since the set of such posterior distributions is convex, we merely need to determine the worst and the best consumer surplus for a given price, as any intermediate surplus can be achieved with a convex combination of the two extreme posteriors.

For a given price $p \in [0, p^*]$ the lowest value of consumer surplus is generated by the lower bound L_p. This is because consumer surplus is lower with a smaller G, and the smallest possible G given p is L_p. (Since L_{p^*} is an MPC of F by construction, so is L_p for any lower price.) The integral of $L_p(x)$ over $[-\Delta, 0]$ is $\frac{1}{2}(1 - \log 2)p$ and so the minimum consumer surplus with price $p \leq p^*$ is $\mu - \frac{1}{2}(1 + \log 2)p \approx \mu - 0.85p$. This minimum frontier is shown as the lower bold line on Figure 6.

The precise shape of outer feasible frontier depends on the details of prior distribution, and to derive the maximum consumer surplus for a price $p \in [0, p^*]$, we need to deal with two cases. When $p \geq p_F$, the full information price, it is possible to find a supporting posterior G between the bounds which is an MPC of F such that (8) holds with equality. This is because given $p < p^*$ the lower bound L_p must have (8) hold with strict inequality, while given $p \geq p_F$ the log-convex upper bound U_p must be above the log-concave F in the range of $x < 0$. One way to construct such a G is a modified version of (21), where $G(x) = L_p(x)$ for $x < m$ and $G(x) = U_p(x)$ for $x \geq m$, and where m is chosen to make (8) bind. Since there is no mismatch with such a posterior, consumer surplus is $\mu + \delta - p$. For these prices, profit and maximum consumer surplus sum to maximum welfare $\mu + \delta$.

When $p < p_F$, however, we have shown that the maximum consumer surplus is given by (17). For these lower prices, profit and maximum consumer surplus sum to strictly less than maximum welfare due to the mismatch needed to achieve lower prices. Proposition 3 showed that the maximum feasible consumer surplus was achieved with a positive price. (Again, this contrasts with the figure in Roesler and Szentes, where consumer surplus was maximized when profit was minimized.)

When the prior distribution is uniform on $[-1, 1]$, the higher bold curve in Figure

6 shows the feasible outer frontier and the shaded area is the feasible combinations of profit and consumer surplus. Here, the full-information price is $p_F = 1$, and for prices above p_F the feasible frontier coincides with the efficient frontier. For lower prices, the feasible frontier lies strictly inside the efficient frontier.

4.3 Asymmetric signal structures

We now extend the analysis to allow for asymmetric signal structures, such as illustrated on Figure 1c above, which induce firms to choose distinct prices in equilibrium. This extension is important as, for instance, it could be possible to design consumer information in such a way that a firm obtains higher profit than it did with the firm-optimal symmetric signals presented in Proposition 2. This higher profit might come either at the expense of its rival or as part of a reduction in competitive intensity due to asymmetric rivalry between firms. It might also be possible for consumers to prefer asymmetric prices: for a *fixed* distribution over x consumer surplus is convex and decreasing in the two firms' prices, so that consumers prefer distinct prices to a uniform price equal to the average of the two prices. Clearly, however, maximum welfare cannot be improved with asymmetric signals when firms are symmetric, as welfare is maximized by ensuring consumers buy their preferred product and this requires that prices be equal.

As with the symmetric analysis, our approach is to provide bounds on the posterior distributions $G(\cdot)$ which induce a given pair of positive prices (p_1, p_2) in equilibrium. As before, G can have no atom at $x = p_1 - p_2$, and firm 2's equilibrium market share is $G(p_1 - p_2)$. Since the posterior support of x must lie within $[-\Delta, \Delta]$, if π_i denotes firm i's equilibrium profit then (5) must hold. More generally, firm 2 can make no greater profit with another price p_2', so that

$$p_2' G(p_1 - p_2') \leq \pi_2 .$$

As in section 4.2, changing variable to $x = p_1 - p_2'$ implies

$$G(x) \leq \min\left\{1, \frac{\pi_2}{\max\{0, p_1 - x\}}\right\} \equiv U_{p_1, p_2}(x) . \tag{23}$$

(Here only the range $x \leq p_1$ is relevant, for otherwise firm 2 offers a negative price.)

The parallel argument for firm 1 yields

$$G(x) \geq \max\left\{0, 1 - \frac{\pi_1}{\max\{0, p_2 + x\}}\right\} \equiv L_{p_1,p_2}(x) \ . \tag{24}$$

Note that the lower bound L_{p_1,p_2} is increasing in x and begins to be positive at $x = \pi_1 - p_2$ which exceeds $-\Delta$ from (5). Moreover, similar as in the symmetric case, L_{p_1,p_2} is concave and $1 - L_{p_1,p_2}$ is log-convex in x whenever L_{p_1,p_2} is greater than zero. The upper bound U_{p_1,p_2} is increasing in x and reaches 1 at $x = p_1 - \pi_2$ which is below Δ from (5). Moreover, it is log-convex (and hence convex) when it is less than 1.

Since we must have $U_{p_1,p_2} \geq L_{p_1,p_2}$ to have a chance to implement these prices, and since the bounds coincide and equal firm 2's market share when $x = p_1 - p_2$, the two functions should have the same slope at $x = p_1 - p_2$, i.e., $\pi_2/p_2^2 = \pi_1/p_1^2$. If we write $s = 1 - G(p_1 - p_2)$ for firm 1's market share, so that $\pi_1/p_1 = s$ and $\pi_2/p_2 = 1 - s$, this then implies

$$s = \frac{p_1}{p_1 + p_2} \ , \tag{25}$$

and

$$\pi_i = \frac{p_i^2}{p_1 + p_2} \ . \tag{26}$$

In particular, equilibrium profits and market shares are determined entirely by equilibrium prices and do not depend separately on $G(x)$, and the firm with the higher equilibrium price necessarily has the higher market share (and hence the higher profit).
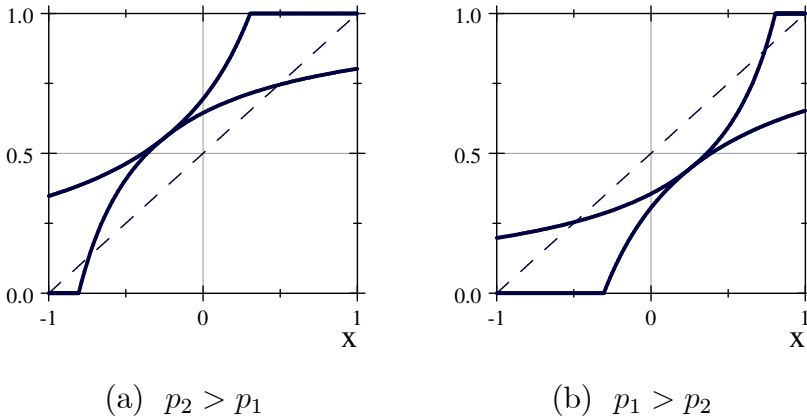

(a) $p_2 > p_1$ (b) $p_1 > p_2$

Figure 7: Asymmetric bounds on G

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In sum, as with Lemma 3 a price pair (p_1, p_2) can be implemented with some signal structure if and only if a posterior G exists which is both (i) an MPC of the prior F and (ii) lies between the bounds (23)–(24), where profits are (26). Figure 7 illustrates this discussion, where a uniform prior F is shown as the dashed line. Note that equilibrium prices induced by a posterior G which favours firm 1 say, so that $G(0) \leq \frac{1}{2}$, necessarily satisfy $p_1 \geq p_2$. In particular, equilibrium prices with G such that $G(0) = \frac{1}{2}$ must be symmetric.

Suppose firms are labelled so that firm 2 has the higher price (as in Figure 7a). Since the posterior G must lie above the lower bound (24), a necessary requirement to implement prices (p_1, p_2) is that this lower bound satisfies (8), so that

$$\delta \geq \int_{\pi_1 - p_2}^{0} \left(1 - \frac{\pi_1}{p_2 + x} \right) dx = p_2 - \pi_1 - \pi_1 \log \frac{p_2}{\pi_1} . \tag{27}$$

(When $p_2 \geq p_1$, the point $\pi_1 - p_2$ where the lower bound reaches zero is negative as in Figure 7a.) The right-hand side of (27) decreases with π_1 for $\pi_1 \in [0, p_2]$, and since π_1 in (26) is lower than $\frac{1}{2}p_2$ it follows from (27) that $\frac{1}{2}p_2(1 - \log 2) \leq \delta$. Since p_2 is the higher of the two prices, we deduce that it is not possible to use asymmetric signals to implement a price for either firm which exceeds p^* in (14). In addition, industry profit cannot exceed p^* if neither price does. When the prior has a log-concave density, Proposition 2 implies that industry profit p^* can be achieved with a symmetric signal structure, in which case we can deduce that the use of asymmetric signals cannot boost industry profit relative to the symmetric firm-optimal policy.

More detailed analysis in the next proposition shows that even an individual firm cannot achieve higher profit than with the firm-optimal symmetric policy, and that consumers also can do no better with asymmetric signals.

Proposition 4 *Suppose the outside option is not relevant and the prior distribution has a log-concave density. Then relative to the optimal symmetric signal structures in Propositions 2 and 3 the use of asymmetric signal structures cannot improve either firm's profit or aggregate consumer surplus.*

Perhaps surprisingly, then, firms have congruent interests when it comes to the design of consumer information. Intuitively, the firm which is treated unfavorably under an asymmetric signal structure has an incentive to set a low price, and this force

turns out to be sufficiently powerful so that the firm which is treated favorably will also reduce its price. Consumers also do not benefit from asymmetric signals, as the resulting mismatch outweighs the possible benefit from lower prices in an asymmetric market.

The analysis in this section is also useful for studying optimal policies when the underlying market is asymmetric. For example, the bounds (23)–(24) continue to apply, as does the expression for equilibrium profit in (26). Figure 7 illustrates the bounds, except that the prior no longer need pass through the point $(0, \frac{1}{2})$. However, the calculation of optimal signals becomes significantly more complicated, as the symmetric benchmark—which played an important role in both the firm and consumer analysis above—is no longer relevant.

5   A market with an outside option

In section 3 we showed how firms earn the first-best profit with the rank signal structure when consumer valuations were sufficiently dispersed, in which case the participation constraint for all consumers was binding. By contrast, section 4 studied the situation where valuations were sufficiently concentrated, in which case the participation constraint was irrelevant and second-best policies could be derived since only the (scalar) valuation difference $x = v_1 - v_2$ mattered. In this section we bridge the gap between these two situations by considering a case where consumer heterogeneity is actually one-dimensional.

There are a number of demand specifications where there is scalar heterogeneity. For instance, we could suppose that the valuation for one of the products is accurately known *ex ante*, while information about a second product (perhaps a new product) might be manipulated. This is a special case of the general setup when consumers are distributed on a vertical or horizontal segment in the valuation space. In this section, however, we maintain the assumption that products are symmetric and suppose that average valuations are the same for all consumers while there is uncertainty about relative preferences, as in a Hotelling-style market. This is a special case of the general setup when consumers are distributed on a diagonal segment of the form $v_1 + v_2 = $ *constant* in valuation space. More precisely, suppose a consumer values product 1 at

$v_1 = 1 + \frac{1}{2}x$ and product 2 at $v_2 = 1 - \frac{1}{2}x$, where $x = v_1 - v_2 \in [-\Delta, \Delta]$ indicates her relative preference for product 1 and each consumer's average valuation $\frac{1}{2}(v_1 + v_2) \equiv 1$ is constant. Assume $\Delta \leq 2$ so that all consumers value both products. The prior distribution for $x \in [-\Delta, \Delta]$ is symmetric about zero and has CDF $F(x)$. For simplicity, in this section we focus on symmetric signal structures which induce a pure strategy pricing equilibrium.

We first show that even if the outside option binds, all consumers purchase in the firm-optimal or consumer-optimal solution.

Lemma 5 *A firm-optimal or consumer-optimal symmetric signal structure induces an equilibrium with full market coverage.*

The argument for the consumer-optimal policy is simple. A consumer-optimal signal structure must be weakly better for consumers than no information disclosure, where consumers buy a random product at price zero and so consumer surplus is 1. Since the match efficiency improvement relative to random match is at most δ (which has the same definition as in (6)), firms cannot earn more than δ in the consumer-optimal solution, where $\delta \leq \frac{\Delta}{2} \leq 1$ given $\Delta \leq 2$. Suppose in contrast to the claim that the market is fully covered that some consumers do not buy, in which case the price must exceed 1 and consumers around $x = 0$ are excluded. Then a feasible unilateral deviation is to charge at 1, in which case at least half of the consumers will buy from the deviating firm. Hence, each firm's equilibrium profit must be greater than $\frac{1}{2}$, and so industry profit exceeds 1 which is a contradiction. The argument for the firm-optimal policy is less straightforward, and we provide the details in the appendix.

When the market is fully covered in equilibrium, the previous bounds analysis can be extended to the situation where the outside option may be relevant. The major difference is that to implement a price $p > 1$ in a symmetric equilibrium, which consumers near $x = 0$ would be unwilling to pay, the posterior distribution should have no consumers near $x = 0$. (The details are provided in the appendix.) Using the adjusted bounds, a similar analysis as in section 4.2 can be done. The consumer-optimal price is low so that the presence of the outside option has little impact on the consumer-optimal policy, as we discuss in the appendix. In the following we focus on how the outside

option might affect the firm-optimal solution.[25]

Proposition 5 *When the prior distribution has a log-concave density, the firm-optimal solution involves no mismatch, and is as follows:*

(i) when $\delta \leq \frac{1}{2}(1 - \log 2)$, the firm-optimal price is p^* in (14), which satisfies $p^* \leq 1$, and is uniquely implemented by L_{p^*};

(ii) when $\frac{1}{2}(1 - \log 2) < \delta < \frac{1}{3}$, the firm-optimal price $p^* \in (1, \frac{4}{3})$ solves $\frac{p}{2}[1 + \log(\frac{4}{p} - 2)] = 1 - \delta$ and is uniquely implemented by a modified posterior lower bound;

(iii) when $\delta \geq \frac{1}{3}$, i.e., when (4) holds, the firm-optimal price is $p^* = \mu_H = 1 + \delta$ which earns firms the first-best profit and is implemented by the rank signal structure.

Intuitively, if the prior distribution is sufficiently concentrated (in the sense that δ is small), the firm-optimal price must be low so that the outside option is irrelevant and the solution is the same as in Proposition 2. In contrast, if the prior distribution is sufficiently dispersed that (4) holds the first-best outcome is achievable. In between, the optimal solution is a mixture of these two cases, and it changes smoothly with δ. In all the three cases, there is no product mismatch and so total welfare is maximized as well.

To illustrate, consider the uniform example with support $[-\Delta, \Delta]$ and $\delta = \frac{\Delta}{4}$. When $\Delta \leq 2(1 - \log 2) \approx 0.61$, case (i) in Proposition 5 applies and the optimal G has a U-shaped density similar to Figure 1b before. When $2(1 - \log 2) \leq \Delta \leq \frac{4}{3}$, case (ii) applies and the optimal G is as described in Figure 8 in the case $\Delta = 1$ (where $p^* \approx 1.235$). The distribution has two symmetric mass points (represented as the dots on the density figure) and no consumers located between them. When Δ is larger, the optimal distribution has more weight on the two mass points, and as Δ approaches $\frac{4}{3}$ it converges to a binary distribution on $\{-\frac{\Delta}{2}, \frac{\Delta}{2}\}$ which is implemented by the rank structure and earns firms the first-best profit.

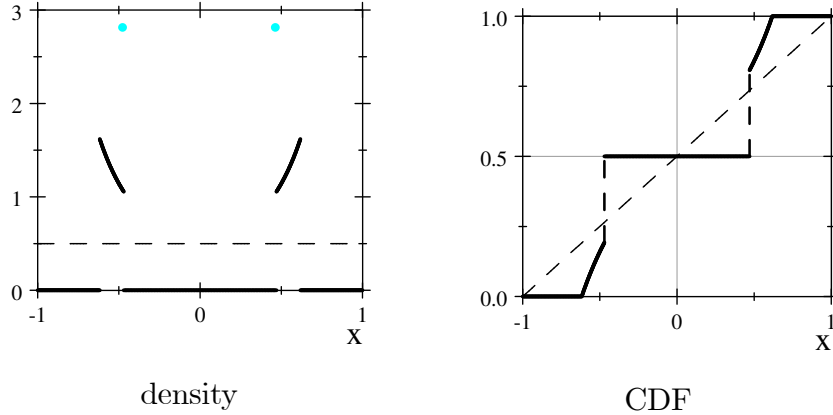[25] As in section 4.2, the analysis can also be extended to the case with a more general prior.

Figure 8: Firm-optimal G when $\Delta = 1$

6 Allowing mixed pricing strategies

It is hard to deal systematically with signal structures which induce mixed strategy pricing equilibrium, when the bounds in section 4 does not apply. Instead, in this section we derive an upper bound for consumer surplus across all symmetric signal structures which induce a symmetric (pure or mixed) strategy equilibrium, and when the prior distribution is regular this upper bound is close to the maximum consumer surplus available with pure strategies. Intuitively, mixed strategy pricing usually does not intensify price competition and the resulting price dispersion further causes product mismatch, in which case it does not benefit consumers.

Consider the model introduced in section 2, where F denotes the symmetric prior distribution of $x = v_1 - v_2$ and G denotes a symmetric posterior distribution. Consumer surplus under G is no greater than $W_G = \mu + \int_{-\Delta}^{0} G(x)dx$ minus industry profit in a symmetric equilibrium with posterior G. We first derive a lower bound on that industry profit:

Lemma 6 *Suppose $V \geq \Delta$ and let G be a symmetric distribution for $x = v_1 - v_2$. Then in any symmetric equilibrium (with pure or mixed strategies) industry profit is no lower than*

$$\max_{x \in [-\Delta, \Delta]} : \frac{-2xG(x)}{1 - G(x)} . \tag{28}$$

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Proof. Note that (28) is zero if and only if the distribution G has $x \equiv 0$, in which case equilibrium profit is also zero and the result holds. Suppose now that (28) is positive, and (slightly abusing notation) denote its value by $p > 0$. Since x which solves (28) must be negative, we have $p \leq 2\Delta$. Suppose in contrast that there is an equilibrium where each firm obtains profit π^* strictly below $p/2$. Firm 1, say, will never choose a price below π^* in this equilibrium (as then it obtains lower profit even if it serves all consumers). Firm 2's profit π^* is then at least equal to the maximum profit it can obtain if firm 1 chooses price π^*. Given $\pi^* < p/2 \leq \Delta \leq V$, if firm 1 chooses price π^* the outside option is not relevant for consumers, regardless of the price chosen by firm 2. Hence firm 2's profit π^* satisfies

$$\pi^* \geq \max_{p'} : p' \times \Pr\{v_2 - p' \geq v_1 - \pi^*\} = \max_{p'} : p'G(\pi^* - p') = \max_{x \in [-\Delta, \Delta]} : (\pi^* - x)G(x) \ ,$$

where the final equality follows after changing to the variable $x = \pi^* - p'$. Thus for any $x \in [-\Delta, \Delta]$ we have $(1 - G(x))\pi^* \geq -xG(x)$, in which case π^* is at least equal to $p/2$. As this contradicts our assumption, the result is proved. ∎

Slightly abusing the notation let p denote (28) for a given G. (The proof of Lemma 6 shows that $p \leq 2\Delta$.) Then an upper bound on consumer surplus with posterior G is $W_G - p$. By construction, for any $x \in [-\Delta, 0)$ we have $G(x)/(1 - G(x)) \leq \frac{p}{-2x}$, or $G(x) \leq \frac{p}{p-2x}$. (If the lower bound p is attained, then G should equal $\frac{p}{p-2x}$ for some $x < 0$.) Since G cannot exceed $\frac{1}{2}$ for $x \in [-\Delta, 0]$, it follows that G in the negative range lies below the upper bound

$$G(x) \leq \hat{U}_p(x) \equiv \min\left\{\frac{1}{2}, \frac{p}{p - 2x}\right\} \ . \tag{29}$$

(Here, the upper bound $\hat{U}_p$ increases with p and x, and reaches $\frac{1}{2}$ at $x = -\frac{1}{2}p \geq -\Delta$.) Without considering the prior, an upper bound on consumers surplus is then $\mu + \max_p(W_{\hat{U}_p} - p)$, which is similar to the relaxed problem in (20). Note that $\hat{U}_p$ lies above the upper bound U_p in (11), which was relevant with the restriction to pure strategies and which for negative x equals $p/(2p - 2x)$. However, for small p, which is usually the relevant case, the two bounds are very close and for this reason the use of mixed strategies cannot significantly benefit consumers.

Considering the MPC constraint from the prior distribution can tighten the consumer surplus upper bound. Given the prior distribution F, let τ_p denote the maximum

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match efficiency when the lower bound on industry profit is p, i.e., $\tau_p =: \max_G \int_{-\Delta}^0 G(x)dx$ subject to (i) G lying below the upper bound $\hat{U}_p(x)$ in (29) (and touching it at some $x < 0$) and (ii) G being a symmetric MPC of F. Therefore, an upper bound on consumer surplus is $\mu + \max_p(\tau_p - p)$.

The remaining task is to calculate τ_p, and this can be done in a manner similar to the way we found the consumer-optimal policy for a given price with pure strategies in section 4.2. If the prior has a log-concave density, then F is log-concave on $[-\Delta, 0]$, while the upper bound $\hat{U}_p$ is log-convex in the range $[-\Delta, -\frac{1}{2}p]$. For relatively small p, which will be the relevant case, the upper bound $\hat{U}_p$ therefore crosses the prior F twice.[26] See Figure 9 for an illustration when the prior is uniform.

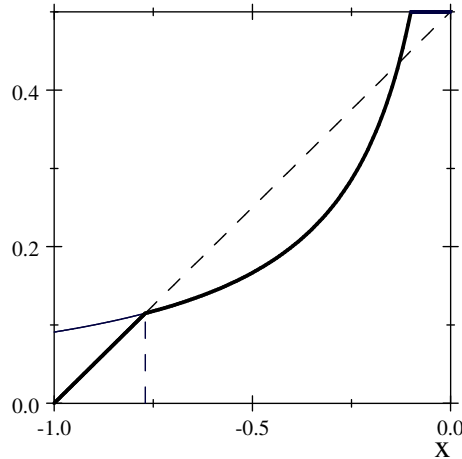

Figure 9: Consumer-optimal way to reach the profit lower bound p

Using this adjusted upper bound on the feasible posterior G, the following result demonstrates that, in regular cases, it is not possible that consumers can do significantly better if the class of signal structures is broadened to permit mixed pricing strategies in equilibrium. (Note, however, we have not found an example where the use of mixed strategies improves consumer surplus at all.)

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[26] To see that with the p which maximizes $\hat{\tau}_p - p$ the upper bound $\hat{U}_p$ crosses the prior F, we argue as follows. For any p we must have $\hat{\tau}_p \leq \delta$ since G is an MPC of F. Let $\tilde{p}$ denote the price such that $\hat{U}_{\tilde{p}}$ just touches F. Then setting $G \equiv F$ solves the stated problem for $\hat{\tau}_{\tilde{p}}$, in which case $\hat{\tau}_{\tilde{p}} = \delta$. For $p > \tilde{p}$, when the upper bound $\hat{U}_p$ lies strictly above F, we must have $\hat{\tau}_p - p \leq \delta - p < \hat{\tau}_{\tilde{p}} - \tilde{p}$. As claimed, then, the p which maximizes $\hat{\tau}_p - p$ is no greater than $\tilde{p}$ and so the upper bound crosses F.

Proposition 6 *Suppose $V \geq \Delta$ and $x = v_1 - v_2$ has a log-concave density. Then the maximum consumer surplus available using only pure strategies attains at least 98.4% of the maximum consumer surplus available across all symmetric signal structures which induce a symmetric pure or mixed strategy equilibrium.*

Ideally we would like to obtain a tight upper bound on profit as well, and see how close the optimal profit under the pure strategy restriction is relative to such an upper bound. This appears to be a harder problem, though, and we leave it for future work.[27]

7 Conclusion

This paper has studied the limits to competition when product information possessed by consumers can be designed flexibly. Among signal structures which induce pure strategy pricing equilibrium, we derived the optimal policy for firms and for consumers. The firm-optimal signal structure amplifies perceived product differentiation by reducing the number of consumers who regard the products as close substitutes. The firm-optimal signal structure typically enables consumers to buy their preferred product, and so it maximizes total welfare as well. In particular, the rank information structure which only informs consumers of which product is a better match can sometimes be optimal for firms. The consumer-optimal policy, in contrast, dampens perceived product differentiation by increasing the number of marginal consumers and so implements a low price. This low price can only be achieved by inducing a degree of product mismatch, however, and so the policy does not maximize welfare.

One interesting extension to this analysis would be to consider situations where firms were asymmetric *ex ante*, including the case of vertical differentiation where one firm was known to provide a higher match utility than its rival. One could investigate whether the optimal information policy maintains, amplifies or reduces this prior asymmetry, and whether firms continue to have aligned interests over the design of consumer information.

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[27]Under the rank signal structure, following Moscarini and Ottaviani (2001) one can characterize a symmetric mixed strategy pricing equilibrium when (3) does not hold. The resulting profit is lower than the optimal profit derived in section 4.2 under the pure strategy restriction if the prior density is log-concave. However, the opposite can be true for an irregular prior when there is no pure strategy equilibrium with full information (e.g., when x is binary on $\{-\Delta, \Delta\}$).

Another, more ambitious, extension would be to investigate how the number of rivals affects optimal information design. With just two firms, consumers face a trade off between low prices and the ability to choose the better product, and often this trade off was resolved by revealing little product information to consumers. With *many* firms, however, this trade-off usually vanishes: with full information disclosure, consumers in such a market can choose their preferred product *and* usually pay a price close to marginal cost, and so this policy achieves the first best for consumers. Of course, the need to choose among many products may bring its own information processing costs for consumers. If this is a concern, a modified version of the "rank" signal structure can also approximate the first best outcome for consumers: if consumers are informed which are their best *two* products (but without ranking them) then there will be marginal-cost pricing, and with many firms there will be a negligible sacrifice of match quality.

Full analysis of markets with more than two firms would require consideration of multi-dimensional consumer heterogeneity, however, rather than the scalar analysis in this paper, even in situations where the outside option could be ignored. Nevertheless, some preliminary observations about the n-firm case are that the signal structure which informs consumers of their most preferred product but nothing else can sometimes enable firms to achieve the first-best outcome under a suitably modified version of (3), and even if it does not achieve first-best profit, the same signal structure bounds industry profit away from zero regardless of the number of firms. This contrasts with the literature discussed in the introduction, where firms disclose only information about their own product, where firms disclose all information and price is close to marginal cost when there are many firms.

A third extension would be to allow consumers to be heterogeneous *ex ante*. For instance, a consumer's valuation v_i for product i might be decomposed as $v_i = a_i + b_i$, where consumers know the vector $(a_1, ...)$ from the start, from other information sources, and there is scope to manipulate information about the vector $(b_1, ...)$. If there was enough heterogeneity in $(a_1, ...)$ then one may be able to rule out mixed pricing strategies in equilibrium, rather than assuming them away as we mostly did in this paper.

Finally, it would be valuable to embed this analysis within a framework in which the "information designer" is modelled explicitly. Platforms typically compete with each other to provide intermediation services. If a profit-maximizing platform chooses what

product information to reveal to consumers, and also chooses its fees to each side of the market, then the relative competitive intensity among platforms on the two sides of the market and the platform's equilibrium fee structure will presumably affect whether its information policy is focussed more on delivering firm profit or consumer surplus.

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Technical Appendix

Proof of Corollary 1. Let $H(v_i)$ be the CDF for each valuation, with weakly decreasing density $h(v_i)$. Then the CDF for the variable $\max\{v_1, v_2\}$ is $H^2(v)$, and so from (2)

$$
\begin{aligned}
3\delta - \mu &= \int_0^\Delta \{3[1 - H^2(v)] - 4[1 - H(v)]\}dv \\
&= \int_0^\Delta \{[1 - H(v)][3H(v) - 1]\}dv \\
&= \int_0^1 \frac{(1 - z)(3z - 1)}{h(H^{-1}(z))}dz \ .
\end{aligned}
$$

Here, the final equality follows by changing variables from v to $z = H(v)$, and $H^{-1}(\cdot)$ is the inverse function to $H(\cdot)$. Noting that the above integrand is negative for $z < \frac{1}{3}$ and positive for $z > \frac{1}{3}$, and that $h(H^{-1}(z))$ weakly decreases with z, it follows that

$$
3\delta - \mu \geq \frac{1}{h(H^{-1}(\frac{1}{3}))} \int_0^1 [(1 - z)(3z - 1)]dz = 0
$$

as claimed. ∎

Proof of Lemma 2. To show the upper bound, note f being symmetric and log-concave on $[-\Delta, \Delta]$ implies the density is single peaked so that F is convex for $x \leq 0$. In this case

$$
\delta = \int_{-\Delta}^0 F(x)dx \geq \int_{\frac{-1}{2f(0)}}^0 (\tfrac{1}{2} + xf(0))dx = \frac{1}{8f(0)} = \tfrac{1}{4}p_F \ ,
$$

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where the inequality follows since F lies above its tangent at $x = 0$ in the range $x \leq 0$. To show the lower bound, note that

$$\tfrac{1}{2} = \int_{-\Delta}^{0} f(x)dx = \int_{-\Delta}^{0} \frac{f(x)}{F(x)} F(x)dx \geq \frac{f(0)}{F(0)} \int_{-\Delta}^{0} F(x)dx = 2f(0)\delta = \frac{\delta}{p_F} \; ,$$

where the inequality follows from F being log-concave (which in turn follows from f being log-concave). ∎

Proof of Proposition 3. With a log-concave density F is log-concave on $[-\Delta, 0]$. Since U_p is log-convex, it follows that for each price below the full information price p_F the upper bound U_p crosses the prior CDF once and from above in the range $[-\Delta, 0]$.[28] Then the same argument as used for the uniform prior shows that an optimal G given $p \leq p_F$ is given by (16).

With G in (16), the derivative of consumer surplus with respect to price is (18). Since $F(x_p) \equiv U_p(x_p)$ it follows that $\frac{p}{p-x_p} = 2F(x_p)$, and substituting this into (18) shows the derivative of consumer surplus with respect to price to be $\frac{1}{2}(2F(x_p) - \log(2F(x_p)) - 3)$. Note that the intercept point x_p increases with p given that the upper bound crosses F from above. The above derivative therefore decreases with p, and so the optimal intercept point x^* satisfies $2F(x^*) = \gamma$, or $x^* = F^{-1}(\frac{1}{2}\gamma)$. The optimal price p^* then satisfies $\frac{p^*}{p^*-x^*} = 2F(x^*) = \gamma$, from which we obtain $p^* = \frac{-\gamma}{1-\gamma}x^*$ and so (19). For reference later, note that optimal consumer surplus is

$$
\begin{aligned}
\mu + \int_{-\Delta}^{x^*} F(x)dx + \int_{x^*}^{0} U_{p^*}(x)dx - p^* &= \mu + \int_{-\Delta}^{x^*} F(x)dx - p^*\left(1 + \tfrac{1}{2}\log \frac{p^*}{p^* - x^*}\right) \\
&= \mu + \int_{-\Delta}^{x^*} F(x)dx - x^* F(x^*) \\
&= \mu - \int_{-\Delta}^{x^*} x dF(x) \; . \quad (30)
\end{aligned}
$$

(Here, the second equality used $\frac{p^*}{p^*-x^*} = 2F(x^*) = \gamma$, the definition of γ, and (19), while the last equality follows by integration by parts.)

The fraction of consumers sure to choose their preferred product, which is $2F(x^*)$, is equal to γ regardless of the prior (provided it has log-concave density). Given a

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[28] We also need to check that G in (16) is above the lower bound L_p for prices below p_F, which is ensured if F is above L_p or $1 - F$ is below $1 - L_p$. However, since $1 - L_p$ is log-convex, this is the case for all $p \leq p_F$ when $1 - F$ is log-concave.

log-concave density F is convex in the range $[-\Delta, 0]$, and so $\frac{\gamma}{2} = F(x^*) \geq \frac{1}{2} + x^* f(0) = \frac{1}{2}(1 + \frac{x^*}{p_F})$, or $\frac{-x^*}{p_F} \geq 1 - \gamma$. Using (19), this implies that $p^* \geq \gamma p_F$ as claimed. ∎

Proof of Corollary 2. It suffices to find a signal structure which is strictly better for consumers than no information disclosure. Consider G_p^m defined in (21) where $m \equiv -\kappa p$ as p varies and $\kappa > e^2 - 1$ is a constant . Since F is not degenerate at $x = 0$, G_p^m is an MPC of F when $p > 0$ is sufficiently small. (Since $m < -\frac{1}{2}p$ we have $G_p^m \geq L_p$.) Consumer surplus with this policy is

$$\mu + \int_{-\Delta}^0 G_p^m(x)dx - p = \mu + \int_{-\kappa p}^0 U_p(x)dx - p = \mu + \tfrac{1}{2}p[\log(1 + \kappa) - 2] .$$

The term $[\cdot]$ is positive by assumption, and so this policy is better for consumers than no information disclosure (since the latter corresponds to $p = 0$). ∎

Proof of Proposition 4. Suppose firms are labelled so $p_2 \geq p_1$, in which case $\pi_2 \geq \pi_1$ and (27) must be satisfied. If $r = \pi_2/\pi_1 \geq 1$ denotes the profit ratio, then (27) can be written as

$$\delta \geq \pi_2 \left(\frac{p_2}{\pi_2} - \frac{1}{r} - \frac{1}{r} \log \frac{p_2}{\pi_1} \right) = \pi_2 \left[1 + \frac{1}{\sqrt{r}} - \frac{1}{r} - \frac{1}{r} \log(r + \sqrt{r}) \right] ,$$

where the equality follows after inverting the pair of equations (26) to obtain $p_2 = \pi_2 + \sqrt{\pi_1 \pi_2}$. The term $[\cdot]$ is equal to $1 - \log 2$ when $r = 1$ and is strictly greater than $1 - \log 2$ for $r > 1$. (This can be verified by using the concavity of $\log(\cdot)$ to show that $\log(r + \sqrt{r}) \leq \log 2 + \frac{1}{2}(r + \sqrt{r} - 2)$.) It follows that π_2 cannot exceed $\frac{1}{2}p^*$, and hence that neither firm's profit can exceed the symmetric firm-optimal profit in Proposition 2.

Turning to the consumer problem, let p_1 and p_2 be the equilibrium prices induced with some signal structure. Similarly to (9), consumer surplus with these prices is

$$\mathbb{E}[\max\{v_1 - p_1, v_2 - p_2\}] = \mu - p_1 + \int_{-\Delta}^{p_1-p_2} G(x)dx . \tag{31}$$

Suppose asymmetric prices are implemented, and now suppose firms are labelled so $p_1 > p_2$ as in Figure 7b. As on that figure, the upper bound satisfies $U_{p_1,p_2}(x) < \frac{1}{2}$ for $x \leq p_1 - p_2$. Since the upper bound is log-convex and the prior F is log-concave given its density is log-concave, the upper bound must cross F once and from above in the

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range $[-\Delta, p_1 - p_2]$. Let $\hat{x} < 0$ denote this intercept point. Then a similar argument to that which led to (17) shows that consumer surplus in (31) can be no greater than

$$\mu - p_1 + \int_{-\Delta}^{p_1 - p_2} \min\{F(x), U_{p_1, p_2}(x)\}dx \ . \tag{32}$$

Consider changing prices to a symmetric price pair $p_1 = p_2 = p$ such that the new upper bound crosses F at the *same* point $\hat{x}$. This implies that p satisfies

$$\frac{\pi_2}{p_1 - \hat{x}} = \frac{\frac{1}{2}p}{p - \hat{x}} \ ,$$

or

$$p = \frac{-\hat{x}\pi_2}{\frac{1}{2}(p_1 - \hat{x}) - \pi_2} < 2\pi_2 < p_2 \ .$$

Here, the first inequality follows since $\pi_2 < \frac{1}{2}p_1$ and the second inequality follows since $\pi_2 < \frac{1}{2}p_2$. Therefore, this uniform price is lower than p_2 and hence also lower than p_1.

The difference between expression (32) with the uniform price p and with original prices (p_1, p_2) is

$$p_1 - p + \int_{\hat{x}}^{0} \frac{\frac{1}{2}p}{p - x}dx - \int_{\hat{x}}^{p_1 - p_2} \frac{\pi_2}{p_1 - x}dx \ . \tag{33}$$

Note that the first integrand, $\frac{1}{2}p/(p-x)$, is greater than the second, $\pi_2/(p_1 - x)$, in the range $\hat{x} \le x \le 0$. (This is because there is equality by construction in the two terms when $x = \hat{x}$, and $(p - x)/(p_1 - x)$ decreases with x given that $p < p_1$.) Since we also have $p < p_2$, it follows that (33) is greater than

$$p_1 - p_2 - \int_{0}^{p_1 - p_2} \frac{\pi_2}{p_1 - x}dx > 0$$

where the inequality holds since the integrand (i.e., the upper bound) is less than 1.

We deduce that starting from any distinct prices (p_1, p_2), the upper bound on consumer surplus (32) increases if we instead implement this uniform price p. Since this upper bound is achieved with symmetric prices (given the log-concavity assumption), we see that consumer surplus cannot be increased by using asymmetric signals and prices. ■

The bounds analysis for Section 5. Here we first extend the posterior bounds analysis to the setup in section 5 when the market is fully covered in a symmetric equilibrium with price p. (This analysis will also be used to prove the full-coverage result in Lemma

5.) Consider a symmetric equilibrium price p, and suppose firm 2 deviates to p'. A type-x consumer will buy from firm 2 if and only if $1 - \frac{x}{2} - p' \geq \max\{0, 1 + \frac{x}{2} - p\}$, i.e., if $x \leq \min\{2(1 - p'), p - p'\}$. Hence, p is a full-coverage equilibrium price if and only if

$$p'G(\min\{2(1 - p'), p - p'\}) \leq \tfrac{1}{2}p \tag{34}$$

holds for any p' and with equality at $p' = p$. To implement a price $p \leq 1 - \frac{\Delta}{2}$ (which is the lowest valuation for a product), the extensive margin $2(1 - p')$ does not matter and the bounds are (13) as before. To implement a higher price $p > 1 - \frac{\Delta}{2}$, we need to deal with the extensive margin explicitly. Note that if $p > 1$ then any consumers with posterior $x \approx 0$ will not participate. However, as with the rank signal, the signal could induce a gap in the posterior distribution around $x = 0$, in which case it is possible to have full coverage with a price $p > 1$.

For convenience, define

$$U_p^M(x) = \min\left\{1, \frac{p}{\max\{0, 2 - x\}}\right\} \;\; ; \;\; U_p(x) = \min\left\{1, \frac{p}{2\max\{0, p - x\}}\right\} . \tag{35}$$

Here, $U_p(x)$ is the same upper bound as before, and $U_p^M(x)$ is the upper bound when the outside option binds. Notice that U_p^M and U_p intersect only once at $\tilde{x}_p$ and $U_p^M > U_p$ if and only if $x < \tilde{x}_p$. (Note that $\tilde{x}_p \leq \Delta$ given p never exceeds $1 + \frac{\Delta}{2}$, the highest valuation for a product.) Using this notation, condition (34) can be written as

$$G(x) \leq \max\{U_p^M(x), U_p(x)\}$$

and

$$G(\min\{-\tilde{x}_p, 0\}) = \tfrac{1}{2} . \tag{36}$$

The qualitative form of the bounds depend on the size of p as shown in Figure 10 below. (Recall that when G is symmetric, the lower bound is the mirror image of the upper bound.) For price $1 - \frac{\Delta}{2} < p \leq 1$, we have $\tilde{x}_p \in (-\Delta, 0]$ so the upper bound takes the form of U_p^M for $x < \tilde{x}_p$ as illustrated in Figure 10a. The upper bound passes through the point $(0, \frac{1}{2})$, and the bounds conditions imply (36) which is now $G(0) = \frac{1}{2}$. In particular, the lower bound in the range $x \in [-\Delta, 0]$ is unchanged from (13). For a price $1 < p < \frac{4}{3}$, the bounds are shown in Figure 10b. We have $\tilde{x}_p \in (0, \frac{p}{2})$ where $\frac{p}{2}$ is the value of x where U_p reaches 1. The crucial difference is that now (36) implies

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$G(-\tilde{x}_p) = \frac{1}{2}$. This requires $G(x) = \frac{1}{2}$ for $x \in [-\tilde{x}_p, \tilde{x}_p]$, and so in this middle range there are no consumers and the upper bound and the lower bound coincide. Finally, for price $p \geq \frac{4}{3}$, we have $\tilde{x}_p > \frac{p}{2}$ and the middle range is so large that the bounds are as shown in Figure 10c. In particular, the lower bound for negative x is a step function with discontinuity at $-\tilde{x}_p$.

(a) $1 - \frac{\Delta}{2} < p \leq 1$ (b) $1 < p < \frac{4}{3}$ (c) $p \geq \frac{4}{3}$
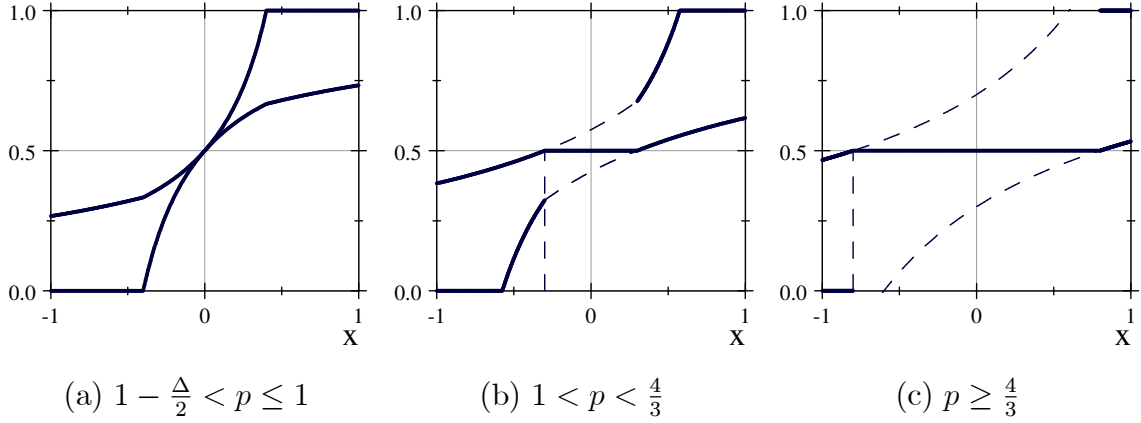
Figure 10: Bounds on G to implement price $p > 1 - \frac{\Delta}{2}$

Proof of Lemma 5. Here we prove that the market is fully covered in the firm-optimal solution. It suffices to show that for any signal structure which induces a partial-coverage equilibrium, there exists another signal structure which induces a full-coverage equilibrium with a strictly higher industry profit.

Consider a symmetric posterior distribution G which is an MPC of F and induces an equilibrium where each firm charges $p > 1$ and only a fraction $\alpha < 1$ of consumers buy. (If $p \leq 1$ all consumers would buy in equilibrium.) Notice that $\tilde{x}_p \equiv 2(p-1) > 0$ solves $1 + \frac{x}{2} = p$, so consumers with $x \geq \tilde{x}_p$ buy from firm 1 and those with $x \leq -\tilde{x}_p$ buy from firm 2. Other consumers in the range of $(-\tilde{x}_p, \tilde{x}_p)$ are excluded from the market. Industry profit in this equilibrium must be no less than one, i.e., $\alpha p \geq 1$, since each firm could attract half the consumers by charging price 1.

Suppose firm 1 charges the equilibrium price p but firm 2 deviates to p'. A consumer of type x will buy from firm 2 if and only if $1 - \frac{x}{2} - p' \geq \max\{0, 1 + \frac{x}{2} - p\}$. This requires $x \leq \min\{2(1-p'), p-p'\}$. The no-deviation condition for the partial-coverage

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equilibrium is then $p'G(\min\{2(1-p'), p-p'\}) \leq \frac{1}{2}\alpha p$ for any p', with equality at $p' = p$. Changing variables yields

$$G(x) \leq \alpha \max\{U_p^M(x), U_p(x)\} \quad \text{and} \quad G(-\tilde{x}_p) = \frac{\alpha}{2} \ ,$$

where U_p^M and U_p are given in (35). Here, $U_p^M > U_p$ if and only if $x < \tilde{x}_p$. The upper bound passes through the point $(-\tilde{x}_p, \frac{\alpha}{2})$. For our purpose, we only need the lower bound which is the mirror image of the upper bound:

$$L_{\alpha,p}(x) = \begin{cases} 1 - \alpha U_p(-x) & \text{if } x < -\tilde{x}_p \\ \max\{\frac{\alpha}{2}, 1 - \alpha U_p^M(-x)\} & \text{if } -\tilde{x}_p \leq x < 0 \end{cases} \ .$$

In the following, we will use

$$L_{\alpha,p}^-(x) = \begin{cases} 1 - \alpha U_p(-x) & \text{if } x < -\tilde{x}_p \\ \frac{\alpha}{2} & \text{if } -\tilde{x}_p \leq x < 0 \end{cases} \tag{38}$$

which is weakly lower than $L_{\alpha,p}(x)$.

Let $\hat{p} = \alpha p \geq 1$ and construct a new symmetric posterior which is equal to

$$L_{1,\hat{p}}(x) = \begin{cases} 1 - U_{\hat{p}}(-x) & \text{if } x < -\tilde{x}_{\hat{p}} \\ \frac{1}{2} & \text{if } -\tilde{x}_{\hat{p}} \leq x < 0 \end{cases} \tag{39}$$

in the range of negative x. Note that this is the lower bound of posteriors which support a full-coverage equilibrium with price $\hat{p} \geq 1$. In the following, we show that $L_{1,\hat{p}}$ is a 'strict' MPC of G in the sense of $\int_{-\Delta}^u L_{1,\hat{p}}(x)dx < \int_{-\Delta}^u G(x)dx$ for any $u \in (-\Delta, 0]$. (Then a similar posterior associated with a price slightly above $\hat{p}$ must be an MPC of G.) Since $L_{\alpha,p}^- \leq G$, it suffices to show $L_{1,\hat{p}}$ is a 'strict' MPC of $L_{\alpha,p}^-$. One can check that $L_{1,\hat{p}}$ crosses $L_{\alpha,p}^-$ only once and from below in the range of negative x. Therefore, it suffices to show

$$\int_{-\Delta}^0 L_{1,\hat{p}}(x)dx < \int_{-\Delta}^0 L_{\alpha,p}^-(x)dx \ .$$

Using (38) and (39), one can rewrite this condition as

$$1 - \tfrac{1}{2}\alpha p \times \left(1 + \log\left(\frac{4}{\alpha p} - 2\right)\right) < (2 - \alpha)(1 - \tfrac{1}{2}p) - \tfrac{1}{2}\alpha p \times \log\left(\frac{4}{\alpha p} - \frac{2}{\alpha}\right) \ ,$$

which further simplifies to

$$\alpha p \times \log\frac{2 - p}{2 - \alpha p} < 2(\alpha - 1)(p - 1) \ .$$

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Given $\log x \leq x - 1$, a sufficient condition for the above inequality is

$$\frac{\alpha p}{2 - \alpha p} > \frac{2(p-1)}{p} \ .$$

Since $\alpha p \geq 1$, we have $\frac{\alpha p}{2 - \alpha p} \geq 1$. Therefore, the above condition holds if $1 > \frac{2(p-1)}{p}$ or $p < 2$. This must be true given $p < 1 + \frac{\Delta}{2}$ and $\Delta \leq 2$. This completes the proof. ■

Omitted details of the consumer-optimal solution in Section 5. The consumer-optimal policy is less affected by the presence of the outside option. As mentioned in the main text, the consumer-optimal price is no greater than δ. When the prior density is log-concave, we have $\delta \leq \frac{\Delta}{4}$, and so if $\Delta \leq \frac{4}{3}$ the consumer-optimal price is no greater than $1 - \frac{\Delta}{2}$ (which is the minimum valuation for a product in our setup), in which case Proposition 3 continues to apply. For larger Δ, we have $\delta \leq \frac{1}{2}$ given $\Delta \leq 2$, so we can focus on price $p \leq \frac{1}{2}$. For such a low price, the relevant posterior upper bound in the range $x \in [-\Delta, 0]$ is

$$\tilde{U}_p(x) = \max\{U_p^M(x), U_p(x)\} \ ,$$

where U_p^M is introduced in (35). Notice that $\tilde{U}_p$ is log-convex, and F is log-concave in the range of $x \leq 0$ if its density is log-concave, and since $p < \delta \leq \frac{1}{2}p_F$ the upper bound $\tilde{U}_p$ crosses F only once and from above. Therefore, the same analysis of the consumer problem in section 4.2 applies here, after replacing the upper bound U_p there by $\tilde{U}_p$.

To illustrate, consider the uniform example with support $[-\Delta, \Delta]$. One can check that $F(\tilde{x}_p) \leq U_p(\tilde{x}_p)$ if $p \leq 2 - \Delta$, where recall $\tilde{x}_p = 2(p-1)$ is where U_p^M and U_p intersect. In this price range, U_p^M becomes irrelevant and the intercept point of F and $\tilde{U}_p$ in the range of negative x is the same as in section 4.2. Following the analysis there, the consumer-optimal price is $p^* = \gamma\Delta$, where recall $\gamma \approx 0.05$ is the solution to $\gamma - \log\gamma = 3$. This is indeed less than $2 - \Delta$ if $\Delta \leq \frac{2}{1+\gamma} \approx 1.9$. When $p > 2 - \Delta$, the intercept point of F and $\tilde{U}_p$ solves $F(x) = U_p^M(x)$, from which it follows that $x_p = \frac{1}{2}[2 - \Delta - \sqrt{(2-\Delta)^2 - 8\Delta(p-1)}]$. In this case, (18) becomes

$$\int_{x_p}^0 \frac{\partial \tilde{U}_p(x)}{\partial p} dx - 1 = \log\frac{2 - x_p}{2 - x_p} + \frac{1}{2}\left(\frac{p}{p - x_p} - \log\frac{p}{p - x_p} - 3\right) \ .$$

This is positive at $p = 2 - \Delta$ if $\Delta > \frac{2}{1+\gamma}$ and must be negative at $p = 1$. For instance, when $\Delta = 2$, the consumer-optimal price is $p^* \approx 0.105$, but this is almost the same as

$\gamma\Delta$. In other words, even in this case with large Δ the outside option has a negligible effect on the consumer-optimal policy.

Proof of Proposition 5. The lower bound for $x \in [-\Delta, 0]$ across the three cases depicted in Figure 10 can be succinctly defined as

$$\tilde{L}_p(x) = \begin{cases} L_p(x) & \text{if } x < \min\{0, -\tilde{x}_p\} \\ \frac{1}{2} & \text{if } \min\{0, -\tilde{x}_p\} \leq x < 0 \end{cases} ,$$

and it increases with p. When the prior density is log-concave the prior CDF F is convex with $F(-\Delta) = 0$. Therefore, $\tilde{L}_p$ crosses F at most once and from below in the range of negative x. This implies that the optimal posterior must take the form of the lower bound, and the optimal price p^* solves

$$\int_{-\Delta}^0 \tilde{L}_p(x)dx = \delta . \tag{40}$$

(This implies there is no mismatch with the firm-optimal signal structure.) We then have: (i) if $\delta \leq \int_{-\Delta}^0 \tilde{L}_1(x)dx = \frac{1}{2}(1 - \log 2)$, (40) has a unique solution $p^* = \frac{2\delta}{1-\log 2} \leq 1$ and $\tilde{L}_{p^*}$ takes the form in Figure 10a; (ii) if $\frac{1}{2}(1 - \log 2) < \delta < \int_{-\Delta}^0 \tilde{L}_{4/3}(x)dx = \frac{1}{3}$, (40) has a unique solution $p^* \in (1, \frac{4}{3})$ which solves

$$\int_{-\Delta}^0 \tilde{L}_p(x)dx = 1 - \frac{p}{2}[1 + \log(\frac{4}{p} - 2)] = \delta ,$$

and $\tilde{L}_{p^*}$ takes the form in Figure 10b; (iii) if $\delta \geq \frac{1}{3}$, which implies (4), (40) has a unique solution $p^* = 1 + \delta$ and $\tilde{L}_{p^*}$ takes the form in Figure 10c. ∎

Proof of Proposition 6. Let $\hat{x}_p$ denote the smaller of the two crossing points given p illustrated on Figure 9 (i.e. the smaller solution to $\frac{p}{p-2x} = F(x)$). As in section 4.2, two necessary conditions for G are that it satisfy the MPC constraint (7) at the intercept point $\hat{x}_p$, and that G lies below $\hat{U}_p$ for $x \in [\hat{x}_p, 0]$. The bold curve on Figure 9 shows a convenient way to do this. Note that unlike with Figure 4b above, we have not shown that this G is an MPC of F, as $\hat{U}_p$ is above F for x close to zero. Therefore, the resulting $\hat{\tau}_p$ is an upper bound on the feasible match efficiency when the MPC constraint is fully considered.

As with expression (17), an upper bound on consumer surplus given p is therefore

$$\mu + \int_{-\Delta}^{\hat{x}_p} F(x)dx + \int_{\hat{x}_p}^0 \hat{U}_p(x)dx - p . \tag{41}$$

The derivative of this expression with respect to p is

$$\int_{\hat{x}_p}^{-\frac{1}{2}p} \frac{\partial \hat{U}_p(x)}{\partial p} dx - 1 = \frac{1}{2} \left(\frac{p}{p - 2\hat{x}_p} - \log \frac{2p}{p - 2\hat{x}_p} - \frac{5}{2} \right) .$$

It equals $\frac{1}{2}(F(\hat{x}_p) - \log(2F(\hat{x}_p)) - \frac{5}{2})$ by using $\frac{p}{p-2\hat{x}_p} = F(\hat{x}_p)$. Note that $\hat{x}_p$ increases with p given that $\hat{U}_p$ crosses F from above at the smaller of the two crossing points. This derivative therefore decreases with p, and so the point $\hat{x}^*$ which maximizes the upper bound (41) satisfies $F(\hat{x}^*) = \hat{\gamma}$, where $\hat{\gamma} \approx 0.043$ is the root of $\gamma - \log(2\gamma) = \frac{5}{2}$. Evaluating the upper bound (41) at this crossing point $\hat{x}^*$ shows the maximum consumer surplus upper bound to be

$$\mu + \int_{-\Delta}^{\hat{x}^*} F(x)dx - \hat{x}^* F(\hat{x}^*) = \mu - \int_{-\Delta}^{\hat{x}^*} xdF(x) , \qquad (42)$$

which is the same expression as (30) in the consumer-optimal problem in section 4.2 but using $\hat{x}^* > x^*$ rather than x^*.

Finally, we show that consumer surplus with pure strategies comes close to reaching this upper bound. The ratio of consumer surplus with pure strategies to this upper bound is

$$\frac{\mu - \int_{-\Delta}^{x^*} xdF(x)}{\mu - \int_{-\Delta}^{\hat{x}^*} xdF(x)} = \frac{\mu - x^* F(x^*) + \int_{-\Delta}^{x^*} F(x)dx}{\mu - \hat{x}^* F(\hat{x}^*) + \int_{-\Delta}^{\hat{x}^*} F(x)dx} > \frac{\mu - x^* F(x^*) + \int_{-\Delta}^{x^*} F(x)dx}{\mu - x^* F(\hat{x}^*) + \int_{-\Delta}^{x^*} F(x)dx}$$

$$> \frac{\mu - x^* F(x^*)}{\mu - x^* F(\hat{x}^*)} > \frac{\mu + \Delta F(x^*)}{\mu + \Delta F(\hat{x}^*)} \geq \frac{\Delta + \Delta F(x^*)}{\Delta + \Delta F(\hat{x}^*)} = \frac{1 + F(x^*)}{1 + F(\hat{x}^*)} = \frac{1 + \frac{1}{2}\gamma}{1 + \hat{\gamma}} \approx 0.984 .$$

Here, the first inequality uses $\int_{x^*}^{\hat{x}^*} F(x)dx < (\hat{x}^* - x^*)F(\hat{x}^*)$, the third inequality uses $-\Delta < x^* < 0$, and the final inequality uses the fact that $V \geq \Delta$ implies $\mu \geq \Delta$. Thus, when the prior has log-concave density the maximum consumer surplus attainable with pure strategies attains at least 98.4% of the consumer surplus which could be available when mixed pricing strategies were permitted. ■