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Abstract

In this paper we introduce minimum cost spanning tree problems with multiple sources. This new setting is an extension of the classical model where there is a single source. We extend several definitions of the folk rule, the most prominent rule in the classical model, to this new context: first as the Shapley value of the irreducible game; second as an obligation rule; third as a partition rule; and finally through a cone-wise decomposition. We prove that all the definitions provide the same cost allocation and present two axiomatic characterizations.

Keywords: minimum cost spanning tree problems, multiple sources, folk rule, axiomatic characterizations.

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1 Introduction

A group of agents is interested in a service provided by a supplier with multiple service stations, also called sources. Agents will be served through costly connections. They do not care whether they are connected directly or indirectly to the sources, but they want to be connected to all of them. This may occur for safety reasons. Agents will have greater assurances of the service in the sense that they can still enjoy the service even if one or more sources cease to provide it. There could also be a situation where several suppliers offer different services by using the same network (Internet, cable TV, etc.) and agents are interested in all of them. These situations generalize classical minimum cost spanning tree problems with a single source by allowing the possibility of multiple sources.

Given a cost spanning tree problem with multiple sources, the least costly way of connecting all agents to all sources (a minimum cost spanning tree) must be sought. This tree can be obtained, in polynomial time, by using the same algorithms as in the classical minimum cost spanning tree problem, for instance that of Prim (1956) or that of Kruskal (1957). Nevertheless, some variants of this problem are not so easy from a computational point of view: the fixed cost spanning forest problem studied in Granot and Granot (1992), where there are potential sites to construct sources at a fixed construction cost; the multi-source spanning tree problem studied in Farley et al. (2000), where the objective is to compute the spanning tree that minimizes the sum of the distances from each source to every other node; and the hop constrained Steiner trees with multiple root nodes studied in Gouveia et al. (2014).

Once it is known how to connect all the agents to all sources at the minimum cost, another major issue that usually arises is how to allocate that cost to the agents. Our paper studies this issue in minimum cost spanning tree problems with multiple sources. Even though many papers in the literature on Operations Research or Economics study how to allocate the minimum cost to agents in the classical setting with a single source, there are only a few devoted to this issue in the setting of multiple sources. Two of them are mentioned below.

Rosenthal (1987) introduces the minimum cost spanning forest game where there are multiple sources that offer the same service and agents want to be connected to at least one source. He associates a cooperative game with this problem and shows that its core is non-empty. Kuipers (1997) studies a problem where there are multiple sources, each of them offering a different service, and each agent specifies the set of sources that she wants to be connected to. He associates a cooperative game with this problem and seeks to determine the conditions under which the core is non-empty.

Our approach is different because we want all agents to be connected to all sources. From this perspective our problem can be seen as a particular case of Kuipers (1997) where all agents demand to be connected to all sources. Nevertheless, the cooperative game that we set up to study this problem is different. In both the papers mentioned above the cost of a coalition S is the minimum cost of connecting all members in S to some sources under the assumption that S is allowed to use nodes outside S. We follow the standard approach (as in the classical minimum cost spanning tree problem) and assume that agents in S can not use the locations of agents outside S.

In classical minimum cost spanning tree problems the most popular rule is the so called "folk rule", which is studied in many papers. The folk rule has been proved to satisfy very appealing properties. It provides allocations in the core and is monotonic in the population and in the cost matrix. It is also additive in the cost matrix, which makes it easy to compute. Our first aim is to extend the definition of the folk rule to our setting by using the following four approaches:

1. as the Shapley value of the irreducible game (Bergantiños and Vidal-Puga 2007),

2. as an obligation rule (Tijs et al. 2006; Bergantiños and Kar 2010),

3. as a partition rule (Bergantiños et al. 2010 and 2011),

4. through a cone-wise decomposition (Branzei et al. 2004; Bergantiños and Vidal-Puga 2009).

We show that all four approaches make the same recommendation: the folk rule. We also provide two axiomatic characterizations of this rule. In both characterizations we use five axioms. Four of them are common to both results and the last one is different. The four common axioms are: independence of irrelevant trees (the cost allocation should depend only on the arcs that belong to the minimal tree), cone-wise additivity (the cost allocation should be additive on cones), symmetry (agents with the same connection costs should pay the same), and equal treatment of source costs (if the connection cost between two sources increases, all agents should be affected by the same amount). Independence of irrelevant trees, cone-wise additivity, and symmetry are defined as in classical minimum cost spanning tree problems. But equal treatment of source costs, introduced in this paper, is specifically designed for the multiple source setting since it says nothing in classical minimum cost spanning tree problems. The other two axioms are core selection (the allocation should belong to the core of the problem) and separability (two subsets of agents can connect to all sources separately or jointly, if there are no savings when they connect jointly, agents must pay the same in both circumstances). Both axioms are also defined as in classical minimum cost spanning tree problems.

Bergantiños and Navarro-Ramos (2019a) prove that the folk rule can also be obtained through a painting procedure in which, following a fixed protocol, agents paint the arcs on the paths connecting them to the sources. Bergantiños and Navarro-Ramos (2019b) provide another characterization of the folk rule in a paper written after the present one. They use the properties cone-wise additivity, symmetry, and equal treatment of source costs combined with two other different properties: cost monotonicity and isolated agents.

Bergantiños and Lorenzo (2019) also follow the axiomatic approach in our setting. They characterize several families of rules satisfying some of the following properties:

cone-wise additivity, independence of irrelevant trees, core selection, and equal treatment of source costs.

The paper is structured as follows. Section 2 introduces minimum cost spanning tree problems with multiple sources. Section 3 extends the four definitions of the folk rule to our setting and show that they coincide. Section 4 presents its axiomatic characterizations. All the proofs are relegated to the Appendix.

2 The model

Let $N = \{1, \ldots, |N|\}$ be a set of agents and $M = \{s_1, ..., s_{|M|}\}$ be a set of sources. We are interested in networks whose nodes are elements of $N \cup M$. We denote by $|N|$ and $|M|$ the cardinals of N and M, respectively. For each N and M, a *cost matrix* $C = (c_{ij})_{i,j \in N \cup M}$ represents the cost of a direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each i, $j \in N \cup M$ and $c_{ii} = 0$ for each $i \in N \cup M$. Since $c_{ij} = c_{ji}$ for each i, $j \in N \cup M$, we will work with undirected arcs $\{i, j\}$. We denote the set of all cost matrices over $N \cup M$ as $\mathcal{C}^{N \cup M}$. Given C, $C' \in \mathcal{C}^{N \cup M}$, $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all i, $j \in N \cup M$. Similarly, given x, $y \in \mathbb{R}^N$, $x \leq y$ if $x_i \leq y_i$ for each $i \in N$.

A *minimum cost spanning tree problem with multiple sources* (a *problem* for short) is characterized by a triple (N, M, C) where N is the set of agents, M is the set of sources, and C is the cost matrix in $\mathcal{C}^{N \cup M}$. Given a subset $S \subset N$, we denote by (S, M, C) the restriction of the problem to the subset of agents S. The classical minimum cost spanning tree problem (*classical problem* for short) corresponds to the case where M has a single element, which is denoted by 0.

For each network g and each pair of distinct nodes i and $j \in N \cup M$, a *path from i to j in g* is a sequence of distinct arcs $g_{ij} = \{\{i_{s-1}, i_s\}\}_{s=1}^{p}$ such that $\{i_{s-1}, i_s\} \in g$ for each $s \in \{1, 2, \ldots, p\}$, $i = i_0$, and $j = i_p$. A *cycle* is a path from i to i. For each i, $j \in N \cup M$, *i and j are connected in g* if there is a path from i to j. A *tree* is a connected network that has no cycles.

For each network g, $S \subset N \cup M$ is a *connected component* if (1) for each i, $j \in S$, i and j are connected in g and (2) S is maximal, i.e., for each $T \subset N \cup M$ with $S \subsetneq T$, there are $i, j \in T$, $i \neq j$, such that i and j are not connected in g. Let $P(g) = \{S_k(g)\}_{k=1}^{n(g)}$ be the partition of $N \cup M$ into *connected components* induced by g. For each network g, let $S(P(g), i)$ be the element of $P(g)$ to which i belongs. Let $P(N \cup M)$ denote the set of all partitions of $N \cup M$ and $P = \{S_1, \ldots, S_{|P|}\}$ be a generic element of $P(N \cup M)$. For each $P, P' \in P(N \cup M)$, P is said to be *finer* than P' if for each $S \in P$ there is $T \in P'$ such that $S \subset T$. Given a finite set S, $\Delta(S) = \{x \in \mathbb{R}$ such that $x_i \in [0, 1]$ for each $i \in S$ and $\sum\limits_{i \in S} x_i = 1\}$ is the simplex over S.

For each problem (N, M, C) and each network g, the cost associated with g is defined as $c(N, M, C, g) = \sum\limits_{\{i,j\} \in g} c_{ij}$. When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N, M, C, g)$. Our first objective is to minimize the cost of connecting all agents to the sources. This is achieved by a network of links that has no cycles, which is called a *minimal tree*. Formally, a tree t is a minimal tree if $c(t) = \min\{c(g) : g \text{ is a tree}\}$. A minimal tree always exists but it does not necessarily have to be unique. Kruskal's algorithm (1956) computes a minimal tree. The idea behind this algorithm is to construct a minimal tree by sequentially adding the cheapest arc avoiding cycles.

Formally, let $A^0(C) = \{\{i,j\} : i,j \in N \cup M \text{ and } i \neq j\}$ and $g^0(C) = \emptyset$.

Step 1: Take an arc $\{i,j\} \in A^0(C)$ such that $c_{ij} = \min_{\{k,l\} \in A^0(C)} \{c_{kl}\}$. If there are several arcs satisfying this condition, select one of them. Let $\{i^1(C), j^1(C)\} = \{i,j\}$, $A^1(C) = A^0(C) \setminus \{i,j\}$ and $g^1(C) = \{i^1(C), j^1(C)\}$.

Step $p+1$ ($p = 1, \ldots, |N| + |M| - 2$): Take an arc $\{i,j\} \in A^p(C)$ such that $c_{ij} = \min_{\{k,l\} \in A^p(C)} \{c_{kl}\}$. If there are several arcs satisfying this condition, select one as before. Two cases are possible:

1. If $g^p(C) \cup \{i,j\}$ has a cycle, then go to the beginning of Step $p+1$ with $A^p(C)$ obtained from $A^p(C)$ by deleting $\{i,j\}$, that is, $A^p(C) = A^p(C) \setminus \{i,j\}$, and $g^p(C)$ the same.

2. If $g^p(C) \cup \{i,j\}$ has no cycles, then take $\{i^{p+1}(C), j^{p+1}(C)\} = \{i,j\}$, $A^{p+1}(C) = A^p(C) \setminus \{i,j\}$, $g^{p+1}(C) = g^p(C) \cup \{i^{p+1}(C), j^{p+1}(C)\}$, and go to Step $p+2$.

This process is completed in $|N| + |M| - 1$ steps, exactly the minimum number of arcs that are needed in order to connect all agents with all sources. $g^{|N|+|M|-1}(C)$ is a tree obtained from the Kruskal algorithm (the algorithm leads to a tree which is not always unique). When there is no ambiguity, we write A^p, g^p, and $\{i^p, j^p\}$ instead of $A^p(C)$, $g^p(C)$, and $\{i^p(C), j^p(C)\}$ respectively. We denote by $m(N,M,C)$ the cost of a minimal tree in (N,M,C).

Once a minimal tree is obtained, an interesting issue is how to divide its cost among the agents. A *cost allocation rule,* or a *rule,* is a map f that associates with each problem (N,M,C) a vector of cost shares $f(N,M,C) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(N,M,C) = m(N,M,C)$.

Example 1 *Let* (N,M,C) *be such that* $N = \{1,2,3\}$, $M = \{a,b\}$, $c_{1a} = 7$, $c_{12} = 8$, $c_{3b} = 9$, $c_{1b} = 10$, *and* $c_{ij} = 20$ *otherwise. The unique minimal tree is* $\{\{1,a\},\{1,2\},\{1,b\},\{3,b\}\}$ *and* $m(N,M,C) = 34$.

3 The folk rule in minimum cost spanning tree problems with multiple sources

In this section, we extend four definitions of the folk rule to our setting and prove that they make the same recommendation. The first one is defined as the Shapley value of the irreducible game, the second is as an obligation rule, the third is as a partition rule, and the fourth is through a cone-wise decomposition in simple problems.

3.1 The Shapley value of the irreducible game

In the classical problem, Bergantiños and Vidal-Puga (2007) define the folk rule as the Shapley value of the irreducible game. We now extend this definition to the case of multiple sources. Let (N, M, C) be a problem and t a minimal tree in (N, M, C). We define the *minimal network* (N, M, C^t) associated with t where $c^t_{ij} = \max_{\{k,\ell\} \in g_{ij}} \{c_{k\ell}\}$ and g_{ij} denotes the unique path in t from i to j. It is well known that C^t does not depend on the choice of the minimal tree. Following Bird (1976), the *irreducible problem* (N, M, C^*) of (N, M, C) can thus be defined as the minimal network (N, M, C^t) associated with any minimal tree t. C^* is referred to as the *irreducible matrix*.

A *game with transferable utility*, briefly a *game*, is a pair (N, v), where v is a real-valued function defined on all coalitions $S \subseteq N$ satisfying that $v(\emptyset) = 0$. The *irreducible game* is a pair (N, v_{C^*}) such that for each $S \subset N$, $v_{C^*}(S) = m(S, M, C^*)$, which means that the value of a coalition is the minimum cost (in C^*) of connecting the agents in S to every source using only the locations of the members in S.

Let Π_N be the set of all permutations over the finite set N. For each $\pi \in \Pi_N$, let $Pre(i, \pi)$ be the set of agents of N preceding i in the order π, i.e., $Pre(i, \pi) = \{j \in N$ such that $\pi(j) < \pi(i)\}$. For each $i \in N$, the Shapley value of a game (N, v) (Shapley 1953) is the average of her marginal contributions:

$$Sh_i(N, v) = \frac{1}{|N|!} \sum_{\pi \in \Pi_N} \big(v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi))\big).$$

Definition 1 *For each problem (N, M, C), the rule f^{Sh} is defined as the Shapley value of the irreducible game associated with (N, M, C). Namely, $f^{Sh}(N, M, C) = Sh(N, v_{C^*})$.*

We now compute f^{Sh} in Example 1. Since the unique minimal tree is $\{\{1, a\}, \{1, 2\}, \{1, b\}, \{3, b\}\}$, $c^*_{1a} = 7$, $c^*_{12} = 8$, $c^*_{1b} = 10$, and $c^*_{3b} = 9$. Besides, $c^*_{2a} = 8$, and $c^*_{ij} = 10$ otherwise. The irreducible game is as follows:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_{C^*}(S)$	17	18	19	25	26	27	34

Thus,

$$f^{Sh}(N, M, C) = \left(\frac{62}{6}, \frac{68}{6}, \frac{74}{6}\right) = (10.33, 11.33, 12.33).$$

3.2 Obligation rules

Tijs et al. (2006) define the family of obligation rules for the classical problem. These rules are defined through obligation functions. Let $N_0 = N \cup \{0\}$ be a set of nodes where 0 denotes the source in the classical problem. An obligation function is a map o that assigns to each $S \in 2^{N_0} \setminus \{\emptyset\}$ a vector $o(S)$ meeting the requirements that $o(S) \in \Delta(S)$ if $0 \notin S$, $o_i(S) = 0$ for each $i \in S$ if $0 \in S$, and for each $S, T \in 2^{N_0} \setminus \{\emptyset\}$ such that

$S \subset T$ and $i \in S$, $o_i(S) \geq o_i(T)$. An obligation function can be interpreted as follows: Assume that agents in S are connected with one another. Now, they need to construct an arc from any agent in S to the source so that they are all connected. Thus, $o_i(S)$ represents the proportion of the cost of the arc that each agent $i \in S$ must pay. If the agents in S are already connected to the source, then they do not need to construct any additional arc and so their obligation is zero, $o_i(S) = 0$ for each $i \in S$.

The obligation rule associated with an obligation function o, which is denoted by f^o, is defined through the Kruskal algorithm as follows. The cost of each arc that is constructed at each step of the Kruskal algorithm is divided among the agents who benefit from its construction. Each agent pays the difference between her obligation to the component to which she belongs before the arc is added and the one afterwards. Tijs et al. (2006) prove that f^o is well-defined, namely, it is independent of the choice of the minimal tree by the Kruskal algorithm. The folk rule corresponds to the obligation function where for each $S \subset N$ and each $i \in S$, $o_i^*(S) = \dfrac{1}{|S|}$.

We now extend this definition to our setting. Let $P = \left\{S_1, ..., S_{|P|}\right\} \in P(N \cup M)$. Note that in the classical problem, if $i \in S_k$, then the obligation of agent i depends only on S_k (the element of the partition to which i belongs). However, in our problem, it depends on the whole structure of the partition in connected components. We assume that for each $S_k \in P$, agents in S_k are connected with one another. The obligation of each $i \in N$ in P, $o_i(P)$, is defined as follows.

(1) A link that joins two components of P with sources: Since all agents in N are interested in such a link, all agents have an equal obligation over that link.

(2) A link that joins a component S_k without sources ($S_k \cap M = \emptyset$) to a component $S_{k'}$ with sources ($S_{k'} \cap M \neq \emptyset$): Since only agents in S_k are interested in such a link, only agents in S_k have obligations over it.

Formally, for each $i \in S_k \cap N$, the obligation function o^* is defined as

$$o_i^*(P) = \begin{cases} \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} & \text{if } S_k \cap M \neq \emptyset, \\ \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \dfrac{1}{|S_k|} & \text{if } S_k \cap M = \emptyset. \end{cases} \tag{1}$$

It is straightforward to see that when there is a single source ($|M| = 1$), o^* coincides with the obligation function associated with the folk rule in the classical problem.

The obligation rule f^{o^*} associated with the obligation function o^* is defined in a similar way as in the classical problem.

Definition 2 *For each problem (N, M, C) and each $i \in N$, the rule f^{o^*} is defined as*

$$f_i^{o^*}(N, M, C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p}[o_i^*(P(g^{p-1})) - o_i^*(P(g^p))].$$

In Proposition 1 we prove that f^{o^*} is well-defined, namely, for each (N, M, C), f^{o^*} divides $m(N, M, C)$ among the agents and is independent of the minimal tree selected by the Kruskal algorithm.

We now compute f^{o^*} in Example 1.

Arc	$P(g)$	$o_1^*(P(g))$	$o_2^*(P(g))$	$o_3^*(P(g))$
$\emptyset$	$\{1,2,3,a,b\}$	$\frac{2-1}{3}+\frac{1}{1}=1+\frac{1}{3}$	$\frac{2-1}{3}+\frac{1}{1}=1+\frac{1}{3}$	$\frac{2-1}{3}+\frac{1}{1}=1+\frac{1}{3}$
$\{1,a\}$	$\{1a,2,3,b\}$	$\frac{2-1}{3}=\frac{1}{3}$	$\frac{2-1}{3}+1=1+\frac{1}{3}$	$\frac{2-1}{3}+1=1+\frac{1}{3}$
$\{1,2\}$	$\{12a,3,b\}$	$\frac{2-1}{3}=\frac{1}{3}$	$\frac{2-1}{3}=\frac{1}{3}$	$\frac{2-1}{3}+1=1+\frac{1}{3}$
$\{3,b\}$	$\{12a,3b\}$	$\frac{2-1}{3}=\frac{1}{3}$	$\frac{2-1}{3}=\frac{1}{3}$	$\frac{2-1}{3}=\frac{1}{3}$
$\{1,b\}$	$\{123ab\}$	0	0	0

Thus,

$$
\begin{aligned}
f_1^{o^*}(N, M, C) &= c_{1a} + \frac{1}{3}c_{1b} = 7 + \frac{10}{3} = 10.33, \\
f_2^{o^*}(N, M, C) &= c_{12} + \frac{1}{3}c_{1b} = 8 + \frac{10}{3} = 11.33, \\
f_3^{o^*}(N, M, C) &= c_{3b} + \frac{1}{3}c_{1b} = 9 + \frac{10}{3} = 12.33.
\end{aligned}
$$

3.3 Partition rules

Bergantiños et al. (2010, 2011) introduce a family of rules using the Kruskal algorithm. At each step of the algorithm, the cost of the selected arc is divided among the agents by using a *sharing function*. A sharing function ϱ is a map that specifies the part of the cost paid by each agent at each step of the Kruskal algorithm.

We now explain the sharing function inducing the folk rule in the classical problem. Assume that when an arc is added, components S_k and S_l are joined. The sharing function is defined through the following principles.

1. When a component without the source is joined to one with the source, only agents in the component without the source obtain benefits. Thus, the full cost of the arc is paid equally by the agents in the component without the source.

2. When two components without the source are joined, agents in both components benefit. We assume that the total amount paid by one component is proportional to the number of agents in the other. We further assume that all agents in the same component pay the same amount.

For each $i \in S_k$, the proportion of the arc paid by agent i is:

$$\varrho_i(P, P') = \begin{cases} 0 & \text{if } 0 \in S_k, \\ \dfrac{1}{|S_k|} & \text{if } 0 \in S_\ell, \\ \dfrac{|S_\ell|}{|S_k \cup S_\ell||S_k|} & \text{if } 0 \notin S_k \cup S_\ell. \end{cases}$$

Next we extend the definition of this sharing function to our problem. Let $P = \{S_1, ..., S_{|P|}\} \in P(N \cup M)$. We assume that for each $S_k \in P$, agents in S_k are connected to one another. Let P' be a partition obtained from P after components S_k and S_ℓ are joined. We define the sharing function ϱ as follows: Cases 1 and 2 are similar to the ones in the classical problem, but Case 3 is new.

1. When we join a component without sources to one with sources, only agents in the component without sources benefit. Thus, the full cost of the arc is paid equally by the agents in the component without sources.

2. When we join two components without sources, agents of both components benefit. We assume that the total amount paid by one component is proportional to the number of agents in the other. We further assume that all agents in the same component pay the same amount.

3. When we join two components with sources, all agents in the problem benefit. Thus, the cost of that arc is divided equally among all agents in the problem.

Formally, for each $i \in N$, the sharing function ϱ^* is defined as

$$\varrho_i^*(P, P') = \begin{cases} \dfrac{1}{|N|} & \text{if } S_k \cap M \neq \emptyset, S_\ell \cap M \neq \emptyset, \\ \dfrac{1}{|S_k|} & \text{if } S_k \subseteq N,\ S_\ell \cap M \neq \emptyset, \text{ and } i \in S_k, \\ \dfrac{|S_\ell|}{|S_k \cup S_\ell||S_k|} & \text{if } S_k \cup S_\ell \subseteq N \text{ and } i \in S_k, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\varrho^*(P, P') \in \Delta(N)$.

Definition 3 *For each problem (N, M, C) and each $i \in N$, the rule f^{ϱ^*} is defined as*

$$f_i^{\varrho^*}(N, M, C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p}[\varrho_i^*(P(g^{p-1}), P(g^p))].$$

In Proposition 1 we prove that f^{ϱ^*} is well-defined, namely, it does not depend on the choice of the minimal tree by the Kruskal algorithm.

We now compute f^{ϱ^*} in Example 1.

Arc	$P(g^{p-1}), P(g^p)$	$\varrho_1^*(P(g^{p-1}), P(g^p))$	$\varrho_2^*(P(g^{p-1}), P(g^p))$	$\varrho_3^*(P(g^{p-1}), P(g^p))$
$\{1,a\}$	$\{1,a,2,3,b\}$ $\{1a,2,3,b\}$	1	0	0
$\{1,2\}$	$\{1a,2,3,b\}$ $\{12a,3,b\}$	0	1	0
$\{3,b\}$	$\{12a,3,b\}$ $\{12a,3b\}$	0	0	1
$\{1,b\}$	$\{12a,3b\}$ $\{123ab\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Thus,

$$
\begin{aligned}
f_1^{\varrho^*}(N,M,C) &= c_{1a} + \frac{1}{3}c_{1b} = 7 + \frac{10}{3} = 10.33,\\
f_2^{\varrho^*}(N,M,C) &= c_{12} + \frac{1}{3}c_{1b} = 8 + \frac{10}{3} = 11.33,\\
f_3^{\varrho^*}(N,M,C) &= c_{3b} + \frac{1}{3}c_{1b} = 9 + \frac{10}{3} = 12.33.
\end{aligned}
$$

3.4 The cone-wise decomposition

Norde et al. (2004) prove that each classical problem can be written as a non-negative linear combination of classical *simple problems* where the costs of the arcs are either 0 or 1. Branzei et al. (2004) define the folk rule first in classical simple problems as follows. Agents connected to the source through a 0 cost path pay nothing. Agents connected with one another through a 0 cost path pay the cost of connecting to the source equally. Then they extend this definition to the general classical problem in a linear way following the result by Norde et al. (2004).

We first introduce the folk rule in classical simple problems following Branzei et al. (2004). For each simple problem (N_0, C) and each $S \subset N$, two agents i, $j \in N$, $i \neq j$, are (C,S)-*connected* if there exists a path g_{ij} from i to j satisfying that for all $\{k,\ell\} \in g_{ij}$, $c_{k\ell} = 0$ and $\{k,\ell\} \subset S$. Also, $S \subset N$ is a C-*component* if two conditions hold: First, for all i, $j \in S$, i and j are (C,S)-*connected*. Second, S is maximal, i.e., if $S \subsetneq T$, then there exist i, $j \in T$, $i \neq j$, such that i and j are not (C,T)-*connected*. It is obvious that the set of C-components is a partition of N.

For each simple problem (N_0, C), the folk rule is defined as follows. For each $i \in N$, let S_i be the C-component to which i belongs. Then,

$$
f_i(N_0, C) = \begin{cases} \dfrac{1}{|S_i|} & \text{if } c_{0j} = 1 \text{ for each } j \in S_i, \\ 0 & \text{otherwise.} \end{cases}
$$

Namely, agents in a C-component who are connected to the source at 0 cost pay nothing, whereas agents in a C-component who are connected to the source at 1 cost divide this cost equally among the members.

Next lemma adapts the results of Norde et al. (2004) to our setting.

Lemma 1 *For each problem (N, M, C), there exist a positive number $m(C) \in \mathbb{N}$, a sequence $\{C^q\}_{q=1}^{m(C)}$ of cost matrices, and a sequence $\{x^q\}_{q=1}^{m(C)}$ of non-negative real numbers satisfying three conditions:*

(1) $C = \sum_{q=1}^{m(C)} x^q C^q$.

(2) *For each $q \in \{1, \ldots, m(C)\}$, there exists a network g^q such that $c_{ij}^q = 1$ if $\{i, j\} \in g^q$ and $c_{ij}^q = 0$ otherwise.*

(3) *For each $q \in \{1, \ldots, m(C)\}$ and each $\{i, j, k, \ell\} \subset N_0$, if $c_{ij} \leq c_{k\ell}$, then $c_{ij}^q \leq c_{k\ell}^q$.*

Branzei et al. (2004) extend the definition of the folk rule to a classical problem (N_0, C) using Lemma 1, so that the folk rule is defined as

$$\sum_{q=1}^{m(C)} x^q f\left(N_0, C^q\right)$$

where $f\left(N_0, C^q\right)$ denotes the folk rule in the simple problem (N_0, C^q).

We now apply this approach to our problem. Since we have multiple sources, we need to adapt the procedure. First, we need to modify the definition of C-component. Instead of considering each component as a subset of N, we now consider a C-component as a subset of $N \cup M$.

Let (N, M, C) be a simple problem. Denote by $P = \{S_1, \ldots, S_{|P|}\}$ the set of C-components. The rule f^{CW} for simple problems is defined as follows. We first connect each component without sources to a component with sources and divide the cost equally among the agents in the component. Then we connect the components with sources with one another and divide the cost equally among all agents. Formally, for each $i \in N$, let $S(P, i)$ be the C-component to which i belongs. Then,

$$f_i^{CW}(N, M, C) = \begin{cases} \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} & \text{if } S(P, i) \cap M \neq \emptyset, \\ \dfrac{1}{|S(P, i)|} + \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} & \text{if } S(P, i) \cap M = \emptyset. \end{cases}$$

Definition 4 *For each problem (N, M, C) and each $i \in N$, the rule f^{CW} is defined as*

$$f_i^{CW}(N, M, C) = \sum_{q=1}^{m(C)} x^q f_i^{CW}(N, M, C^q).$$

We now compute f^{CW} in Example 1. Note that $C = \sum_{q=1}^{5} x^q C^q$ where $x^1 = 7$, $x^2 = x^3 = x^4 = 1$, $x^5 = 10$, and

Arcs	C^1	C^2	C^3	C^4	C^5
$\{a,1\}$	1	0	0	0	0
$\{1,2\}$	1	1	0	0	0
$\{b,3\}$	1	1	1	0	0
$\{b,1\}$	1	1	1	1	0
$\{a,b\}$	1	1	1	1	1
$\{a,2\}$	1	1	1	1	1
$\{a,3\}$	1	1	1	1	1
$\{b,2\}$	1	1	1	1	1
$\{1,3\}$	1	1	1	1	1
$\{2,3\}$	1	1	1	1	1

We compute $f^{CW}(N, M, C^q)$ for each $q = 1, ..., 5$.

1. C^1-components are $\{1, 2, 3, a, b\}$.

$$f^{CW}(N, M, C^1) = \left(1 + \frac{1}{3}, 1 + \frac{1}{3}, 1 + \frac{1}{3}\right).$$

2. C^2-components are $\{a1, 2, 3, b\}$.

$$f^{CW}(N, M, C^2) = \left(\frac{1}{3}, 1 + \frac{1}{3}, 1 + \frac{1}{3}\right).$$

3. C^3-components are $\{a12, 3, b\}$.

$$f^{CW}(N, M, C^3) = \left(\frac{1}{3}, \frac{1}{3}, 1 + \frac{1}{3}\right).$$

4. C^4-components are $\{a12, b3\}$.

$$f^{CW}(N, M, C^4) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

5. C^5-components are $\{ab123\}$.

$$f^{CW}(N, M, C^5) = (0, 0, 0).$$

Then,

$$
\begin{aligned}
f^{CW}(N,M,C) &= \sum_{q=1}^{5} x^q f^{CW}(N,M,C^q) \\
&= 7\left(1+\frac{1}{3},1+\frac{1}{3},1+\frac{1}{3}\right)+\left(\frac{1}{3},1+\frac{1}{3},1+\frac{1}{3}\right) \\
&\quad +\left(\frac{1}{3},\frac{1}{3},1+\frac{1}{3}\right)+\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)+10\,(0,0,0) \\
&= (10.33, 11.33, 12.33)\,.
\end{aligned}
$$

3.5 Equivalence of the four approaches

In Proposition 1 we prove that the obligation rule f^{o^*} and the Kruskal sharing rule f^{ϱ^*} are well-defined. In Theorem 1, we prove that all four approaches make the same recommendation. The proofs of Proposition 1 and Theorem 1 are in the Appendix.

Proposition 1 *f^{o^*} and f^{ϱ^*} are well-defined.*

Theorem 1 *For each problem (N,M,C),*

$$f^{Sh}(N,M,C)=f^{o^*}(N,M,C)=f^{\varrho^*}(N,M,C)=f^{CW}(N,M,C).$$

4 Axiomatic characterizations of the folk rule

In this section we provide two axiomatic characterizations of the folk rule in the multiple source setting. We begin with an extension of several axioms discussed in the classical problem. The first axiom, *independence of irrelevant trees*, requires that the cost allocation chosen by a rule should depend only on the arcs that belong to a minimal tree. This axiom is introduced in Bergantiños and Vidal-Puga (2007) and also used in Bogomolnaia and Moulin (2010) under the name of reductionism.

Independence of irrelevant trees (IIT). For each (N,M,C) and (N,M,C'), if they have a common minimal tree t such that $c_{ij}=c'_{ij}$ for each $\{i,j\}\in t$, then $f(N,M,C)=f(N,M,C')$. Equivalently, IIT can be stated as for each (N,M,C), $f(N,M,C)=f(N,M,C^*)$, where C^* is the irreducible matrix associated with (N,M,C).

Cost monotonicity requires that if some cost increases, no agent ends up better off. This axiom has been widely discussed in the literature: Dutta and Kar (2004); Tijs et al. (2006); Bergantiños and Vidal-Puga (2007); Lorenzo and Lorenzo-Freire (2009); and Bergantiños and Kar (2010).

Cost monotonicity (CM). For each (N,M,C) and (N,M,C'), if $C\le C'$, then $f(N,M,C)\le f(N,M,C')$. It is easy to check that CM implies IIT.

Additivity requires that a cost allocation should be additive in the cost matrix, that is, for each (N, M, C) and (N, M, C'), $f(N, M, C + C') = f(N, M, C) + f(N, M, C')$. However, there is no rule satisfying additivity in the classical problem and so there is no rule satisfying this property in the multiple source setting either. Therefore, as in the classical problem, we formulate a weaker version of additivity, *cone-wise additivity* (Norde et al. 2004; Bergantiños and Kar 2010; Bogomolnaia and Moulin 2010) which requires the additivity property to hold only for a pair of problems where the orders of all arcs (in increasing cost) coincide for both problems.

Cone-wise additivity (CA). Let (N, M, C) and (N, M, C') be two problems satisfying that there exists an order $\sigma : \{\{i, j\}\}_{i,j \in N \cup M, i<j} \to \left\{1, 2, ..., \frac{|N \cup M|(|N \cup M|+1)}{2}\right\}$, such that for each $i, j, k, \ell \in N \cup M$ satisfying that $\sigma\{i, j\} \leq \sigma\{k, \ell\}$, then $c_{ij} \leq c_{k\ell}$ and $c'_{ij} \leq c'_{k\ell}$. Thus, $f(N, M, C + C') = f(N, M, C) + f(N, M, C')$.

We now introduce a monotonicity property concerned with the changes in the set of agents. *Population monotonicity* requires that if new agents join the problem, then no agent in the initial problem should be worse off. This property has been widely discussed in the literature: Dutta and Kar (2004); Tijs et al. (2006); Bergantiños and Vidal-Puga (2007, 2008); Lorenzo and Lorenzo-Freire (2009); Bergantiños and Kar (2010); and Bogomolnaia and Moulin (2010).

Population monotonicity (PM). For each (N, M, C), each $S \subset T \subseteq N$, and each $i \in S$, $f_i(S, M, C) \geq f_i(T, M, C)$.

Core selection requires that no coalition of agents has an incentive to deviate from the grand coalition to build their own minimal tree.

Core selection (CS). For each (N, M, C) and each $S \subset N$, $\sum_{i \in S} f_i(N, M, C) \leq m(S, M, C)$.

It is straightforward to show that PM implies CS. For each $S \subset N$ and each $i \in S$, PM implies that $f_i(N, M, C) \leq f_i(S, M, C)$, so that $\sum\limits_{i \in S} f_i(N, M, C) \leq \sum\limits_{i \in S} f_i(S, M, C)$. Since $\sum\limits_{i \in S} f_i(S, M, C) = m(S, M, C)$, PM implies CS.

Suppose that two subsets, S and $N \setminus S$, can connect to all sources separately or jointly. *Separability* (Bergantiños and Vidal-Puga 2007 and 2009; Bergantiños et al. 2011) requires that if there are no savings when they connect jointly, the agents must pay the same in both circumstances.

Separability (SEP). For each (N, M, C) and each $S \subset N$, if $m(N, M, C) = m(S, M, C) + m(N \setminus S, M, C)$, then

$$f_i(N, M, C) = \begin{cases} f_i(S, M, C) & \text{if } i \in S, \\ f_i(N \setminus S, M, C) & \text{if } i \in N \setminus S. \end{cases}$$

Note that PM also implies SEP. By PM, for each $i \in S$, $f_i(N, M, C) \leq f_i(S, M, C)$ and for each $i \in N \setminus S$, $f_i(N, M, C) \leq f_i(N \setminus S, M, C)$. If $m(N, M, C) = m(S, M, C) + m(N \setminus S, M, C)$, then from the definition of a rule, we have the desired conclusion.

Symmetry requires that if two agents are symmetric in the sense that they have the same connection costs to the rest of the agents and the sources, then they should pay the same.

Symmetry (SYM). For each (N, M, C) and each $i, j \in N$, if $c_{ik} = c_{jk}$ for each $k \in N \cup M \setminus \{i, j\}$, then $f_i(N, M, C) = f_j(N, M, C)$.

We now introduce a property specifically designed for our problem, which requires that if the cost between two sources increases, then all agents should be affected by the same amount.

Equal treatment of source costs (ETSC). For each (N, M, C) and (N, M, C') and each $a, b \in M$, if for each $k, l \in M \cup N$ such that $\{k, l\} \neq \{a, b\}$, $c_{kl} = c'_{kl}$, then for each i, $j \in N$, $f_i(N, M, C') - f_i(N, M, C) = f_j(N, M, C') - f_j(N, M, C)$.

In the classical problem, this axiom is related to *constant share of extra costs* (Bergantiños and Kar 2010), which requires that if all agents have the same connection cost to the source and this cost is greater that any other cost in the network, if this cost increases then agents should share this extra cost in the same way in both problems. However, *constant share of extra costs* is concerned with a cost change in the arcs between the agents and the source, and ETSC is concerned with a cost change in an arc between two sources.

Next we present the axiomatic characterizations of the folk rule. First, we prove that the folk rule satisfies all the axioms introduced above.

Proposition 2 *The folk rule satisfies IIT, CM, CA, PM, CS, SEP, SYM, and ETSC.*

The proof is in the Appendix.

Theorem 2 (*a*) *A rule satisfies IIT, CA, CS, SYM, and ETSC if and only if it is the folk rule.*

(*b*) *A rule satisfies IIT, CA ,SEP, SYM, and ETSC if and only if it is the folk rule.*

The proof is in the Appendix. Also, in the Appendix, we show that all the axioms in Theorem 2 are independent.

We end this section by discussing our characterizations with other results in the literature.

In the classical problem, Bergantiños et al. (2011) characterize the folk rule by imposing the axioms of CM, CA, CS (or SEP), and SYM. Since CM implies IIT and the folk rule satisfies CM, the folk rule can alternatively be characterized by imposing CM instead of IIT. By adding ETSC to the list, we obtain characterizations of the folk rule in our problem. This axiom is important since we need to specify how a rule should respond to cost changes between sources differently from the classical problem.

Bergantiños and Navarro-Ramos (2019b) characterize the folk rule with the following properties: CA, SYM, ETSC, CM and isolated agents (this property is not used in this paper). This characterization is unrelated with our characterizations in the sense that there is no implication relationship between the properties. Besides, the proofs of the characterizations are quite different. See Bergantiños and Navarro-Ramos (2019b) for a more detailed discussion on this issue.

Bergantiños and Lorenzo (2019) also follow the axiomatic approach. They characterize several families of rules. First they characterize the family of rules satisfying CA and IIT. They prove that this family can be obtained through Kruskal's algorithm. The also characterize the family of rules satisfying the previous two properties and CS. Finally, they characterize the family of rules satisfying the previous three properties and ETSC.

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Appendix

Proof of Proposition 1. We need to prove two statements. First, f^{o^*} and f^{ϱ^*} divide the cost of the minimal tree $m(N,M,C)$ among the agents. Second, the definition of f^{o^*} and f^{ϱ^*} does not depend on the choice of the minimal tree by the Kruskal algorithm.

We start with f^{o^*}. In order to prove that f^{o^*} divides $m(N,M,C)$ among the agents, it suffices to prove that for each $p=1,\ldots,|N|+|M|-1$, the cost of arc $\{i^p,j^p\}$ is allocated in full among the agents in N.

Given $P=\left\{S_1,...,S_{|P|}\right\}\in P(N\cup M)$ it is trivial to see that $\sum\limits_{i\in N}o_i^*\left(P\right)=|P|-1$. Then,

$$\begin{aligned}\sum_{i\in N}[o_i^*(P(g^{p-1}))-o_i^*(P(g^p))] &= \sum_{i\in N}o_i^*(P(g^{p-1}))-\sum_{i\in N}o_i^*(P(g^p))\\ &= |P(g^{p-1})|-1-(|P(g^p)|-1)\\ &= |P(g^{p-1})|-|P(g^p)|\\ &= 1\end{aligned}$$

Next we prove that f^{o^*} does not depend on the choice of the minimal tree by the Kruskal algorithm. Given a tree $t=\{\{i^p,j^p\}\}_{p=1}^{|N|+|M|-1}$ obtained by the Kruskal algorithm, we define the following:

- $B^0(t)=\emptyset$, $c^0(t)=c^0=0$.
- $c^1(t)=\min\limits_{\{k,\ell\}\in t\setminus B^0(t)}\{c_{k\ell}\}$, $c^1=\min\limits_{\{k,\ell\}\subset N\cup M, c_{k\ell}>c^0}\{c_{k\ell}\}$, and

  $B^1(t)=\{\{i,j\}\in t: c_{ij}=c^1(t)\}$.
- In general, $c^q(t)=\min\limits_{\{k,\ell\}\in t\setminus\cup_{r=0}^{q-1}B^r(t)}\{c_{k\ell}\}$, $c^q=\min\limits_{\{k,\ell\}\subset N\cup M, c_{k\ell}>c^{q-1}}\{c_{k\ell}\}$, and $B^q(t)=\{\{i,j\}\in t: c_{ij}=c^q(t)\}$.

This process ends when we find $m(t)\le|N|+|M|-1$ such that $\cup_{r=0}^{m(t)-1}B^r(t)\subsetneq t=\cup_{r=0}^{m(t)}B^r(t)$. Note that $m(t)$ denotes the number of arcs in t with different costs.

By the Kruskal algorithm, for all $q=1,...,m\left(t\right)$, $c^q(t)=c^q$. Next, we prove that $P(B^1(t))=P(\{\{i,j\}:\ c_{ij}\le c^1\})$. Since $B^1(t)\subset\{\{i,j\}:\ c_{ij}\le c^1\}$, $P(B^1(t))$ is finer than $P(\{\{i,j\}:\ c_{ij}\le c^1\})$. Suppose that $P(B^1(t))\ne P(\{\{i,j\}:\ c_{ij}\le c^1\})$. Then, there exist S, $S'\in P(B^1(t))$, $S\ne S'$, $k\in S$, and $\ell\in S'$ such that $c_{k\ell}\le c^1$. Thus, $B^1(t)\cup\{\{k,\ell\}\}$ has no cycles and $\{k,\ell\}\notin t$, which contradicts the construction of t by the Kruskal algorithm. Then, $P(B^1(t))=P(\{\{i,j\}:\ c_{ij}\le c^1\})$.

Suppose now that for all $q<q_0$,

$$P\left(\cup_{r=0}^{q}B^r(t)\right)=P(\{\{k,\ell\}:\ c_{k\ell}\le c^q\}).$$

Using arguments similar to those used in the case $q=1$, we can prove that

$$P(\cup_{r=0}^{q_0} B^r(t)) = P(\{\{i,j\}:\ c_{ij} \leq c^{q_0}\}).$$

Since $t = \cup_{r=1}^{m(t)} B^r(t)$ and $c_{ij} = c^r$ for all $\{i,j\} \in B^r(t)$ and all $r = 0, \ldots, m(t)$,

$$\begin{aligned}
f_i^{o^*}(N,M,C) &= \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \left[o_i^*(P(g^{p-1})) - o_i^*(P(g^p))\right] \\
&= \sum_{q=1}^{m(t)} \left(\sum_{p=|\cup_{r=0}^{q-1} B^r(t)|+1}^{|\cup_{r=0}^{q} B^r(t)|} c_{i^p j^p} \left[o_i^*(P(g^{p-1})) - o_i^*(P(g^p))\right] \right) \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(g^{|\cup_{r=0}^{q-1} B^r(t)|})) - o_i^*(P(g^{|\cup_{r=0}^{q} B^r(t)|}))\right] \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(\cup_{r=0}^{q-1} B^r(t))) - o_i^*(P(\cup_{r=0}^{q} B^r(t)))\right] \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(\{\{i,j\}: c_{ij} \leq c^{q-1}\})) - o_i^*(P(\{\{i,j\}: c_{ij} \leq c^{q}\}))\right]. \quad (2)
\end{aligned}$$

Thus, f^{o^*} does not depend on the minimal tree t.

To prove that f^{ϱ^*} is well-defined, it is enough to show that at each step p of the Kruskal algorithm and for each $i \in N$,

$$\varrho_i^*(P(g^{p-1}), P(g^p)) = o_i^*(P(g^{p-1})) - o_i^*(P(g^p)).$$

Assume without loss of generality that $g^p = g^{p-1} \cup \{k,\ell\}$, $P(g^{p-1}) = \{S_1, \ldots, S_r\}$, $k \in S_1$, $\ell \in S_2$, and $P(g^p) = \{S_2', \ldots, S_r'\}$ where $S_2' = S_1 \cup S_2$ and $S_j' = S_j$ for each $j = 3, \ldots, r$. We consider four cases:

Case 1. $S_1 \cup S_2 \subset N$:

Subcase 1.a. $i \notin S_2'$. Since $S_i' = S_i$, it is trivial to see that

$$o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) = 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)).$$

Subcase 1.b. $i \in S_2'$. Assume that $i \in S_1$ (since the other case is similar, we omit it). Then,

$$\begin{aligned}
o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{1}{|S_1|} - \frac{1}{|S_1 \cup S_2|} = \frac{|S_2|}{|S_1 \cup S_2||S_1|} \\
&= \varrho_i^*(P\left(g^{p-1}\right), P(g^p)).
\end{aligned}$$

Case 2. $S_1 \cap M \neq \emptyset$ and $S_2 \cap M \neq \emptyset$:

Subcase 2.a. $i \notin S_2'$ and $S_i \subset N$.

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \frac{1}{|S_i|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} - \frac{1}{|S_i'|} \\ &= \frac{1}{|N|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

Subcase 2.b. $i \notin S_2'$ and $S_i \cap M \neq \emptyset$.

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= \frac{1}{|N|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

Subcase 2.c. $i \in S_2'$. Suppose that $i \in S_1$ (since the other case is analogous, we omit it). Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= \frac{1}{|N|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

Case 3. $S_1 \subset N$ and $S_2 \cap M \neq \emptyset$ (since the case $S_1 \cap M \neq \emptyset$ and $S_2 \subset N$ is similar, we omit it):

Subcase 3.a. $i \notin S_2'$ and $S_i \subset N$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{1}{|S_i|} - \frac{1}{|S_i'|} \\ &= 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

Subcase 3.b. $i \notin S_2'$ and $S_i \cap M \neq \emptyset$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

Subcase 3.c. $i \in S_2' \cap S_1$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1}) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \frac{1}{|S_1|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= \frac{1}{|S_1|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

Subcase 3.d. $i \in S_2' \cap S_2$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1}) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \qquad \blacksquare \end{aligned}$$

Proof of Theorem 1. From the proof of Proposition 1 we have that $f^{o^*} = f^{\varrho^*}$. We now prove that $f^{Sh} = f^{CW}$ and $f^{\varrho^*} = f^{CW}$.

We first prove that f^{CW} and f^{Sh} coincide in simple problems. Let (N, M, C) be a simple problem. Let $P = \{S_1, \ldots, S_{|P|}\}$ be the set of C-components. For each $i \in N \cup M$, let $S(P, i)$ be the C-component to which i belongs. Assume that t is a minimal tree. It is easy to prove that all the elements inside a component are connected at zero cost in t, while the components connect to one another through arcs of cost 1. Note that in the irreducible problem (N, M, C^*) we have that $c_{ij}^* = 0$ when $S(P, i) = S(P, j)$ while $c_{ij}^* = 1$ when $S(P, i) \neq S(P, j)$. Thus, the set of C-components and C^*-components coincide. Recall that for each $i \in N$,

$$\begin{aligned} f_i^{CW}(N, M, C) &= \begin{cases} \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} & \text{if } S(P, i) \cap M \neq \emptyset, \\ \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \dfrac{1}{|S(P, i)|} & \text{otherwise.} \end{cases} \\ f_i^{Sh}(N, M, C) &= Sh_i(N, v_{C^*}) = \frac{1}{|N|!} \sum_{\pi \in \Pi} (v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi))). \end{aligned}$$

We consider two cases:

Case 1. $S(P, i) \cap M \neq \emptyset$. For each order $\pi \in \Pi$, if $\pi(i) = 1$, agent i has to pay the cost of connecting her component to all sources. Thus, $v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi)) = |\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1$. If $\pi(i) > 1$, this means that when this agent arrives all the components with sources are already connected. Thus, $v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = 0$. Therefore,

$$\begin{aligned} f_i^{Sh}(N,M,C) &= \frac{1}{|N|!}\sum_{\pi \in \Pi}(v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi))) \\ &= \frac{1}{|N|!}\sum_{\pi \in \Pi : \pi(i)=1}(|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1) \\ &= \frac{1}{|N|!}(|N|-1)!\,(|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1) \\ &= \frac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &= f_i^{CW}(N,M,C). \end{aligned}$$

Case 2. $S(P,i) \cap M = \emptyset$. For each order $\pi \in \Pi$, we compute $v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi))$ distinguishing several cases.

Subcase 2.a. $Pre(i,\pi) \cap S(P,i) \neq \emptyset$. Thus, $v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = 0$.

Subcase 2.b. $Pre(i,\pi) \cap S(P,i) = \emptyset = Pre(i,\pi)$. Then $\pi(i) = 1$. Thus, $v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = |\{S_j \in P : S_j \cap M \neq \emptyset\}|$.

Subcase 2.c. $Pre(i,\pi) \cap S(P,i) = \emptyset \neq Pre(i,\pi)$. In this case, $\pi(i) > 1$. Thus, $v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = 1$.

Let $\Pi^* = \{\pi \in \Pi : Pre(i,\pi) \cap S(P,i) = \emptyset \text{ and } \pi(i) > 1\}$. Taking into account the computations above, we have that

$$f_i^{Sh}(N,M,C) = \frac{1}{|N|}|\{S_j \in P : S_j \cap M \neq \emptyset\}| + \frac{1}{|N|!}|\Pi^*|.$$

Note that

$$\frac{1}{|N|!}|\Pi^*| = \frac{1}{|N|!}\sum_{k=1}^{|N|-|S(P,i)|}\frac{(|N|-|S(P,i)|)!}{(|N|-|S(P,i)|-k)!}(|N|-k-1)!.$$

We consider $|S(P,i)| = m+1$. Then,

$$\begin{aligned} \frac{1}{|N|!}|\Pi^*| &= \sum_{k=1}^{|N|-m-1}\frac{(|N|-m-1)!(|N|-k-1)!}{(|N|-m-k-1)!|N|!} \\ &= \frac{(|N|-m-1)!m!}{|N|!}\sum_{k=1}^{|N|-m-1}\binom{|N|-k-1}{m}. \end{aligned}$$

Since

$$\begin{aligned}\binom{x+1}{y+1}-\binom{x}{y+1} &= \frac{(x+1)!}{(y+1)!\,(x-y)!}-\frac{x!}{(y+1)!\,(x-y-1)!} \\ &= \frac{[(x+1)-(x-y)]\,x!}{(y+1)!\,(x-y)!} \\ &= \frac{x!}{y!\,(x-y)!} \\ &= \binom{x}{y}\end{aligned}$$

we have that

$$\begin{aligned}\sum_{k=1}^{|N|-m-1}\binom{|N|-k-1}{m} &= \sum_{k=1}^{|N|-m-2}\binom{|N|-k-1}{m}+\binom{m}{m} \\ &= \sum_{k=1}^{|N|-m-2}\left[\binom{|N|-k}{m+1}-\binom{|N|-k-1}{m+1}\right]+\binom{m}{m} \\ &= \binom{|N|-1}{m+1}-\binom{m+1}{m+1}+\binom{m}{m} \\ &= \binom{|N|-1}{m+1}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{|N|!}|\Pi^*| &= \frac{(|N|-m-1)!m!}{|N|!}\binom{|N|-1}{m+1} \\ &= \frac{(|N|-m-1)!m!}{|N|!}\frac{(|N|-1)!}{(m+1)!\,(|N|-m-2)!} \\ &= \frac{|N|-m-1}{|N|(m+1)} \\ &= \frac{1}{m+1}-\frac{1}{|N|} \\ &= \frac{1}{|S(P,i)|}-\frac{1}{|N|}.\end{aligned}$$

Therefore,

$$\begin{aligned}f_i^{Sh}(N,M,C) &= \frac{|\{S_j\in P : S_j\cap M\neq\emptyset\}|}{|N|}+\frac{1}{|S(P,i)|}-\frac{1}{|N|} \\ &= \frac{|\{S_j\in P : S_j\cap M\neq\emptyset\}|-1}{|N|}+\frac{1}{|S(P,i)|} \\ &= f_i^{CW}(N,M,C).\end{aligned}$$

Now we consider a general problem (N,M,C) and $i\in N$. Thus,

$$f_i^{CW}(N,M,C)=\sum_{q=1}^{m(C)}x^q f_i^{CW}(N,M,C^q)=\sum_{q=1}^{m(C)}x^q Sh_i(N,v_{(C^q)^*}).$$

Since the Shapley value satisfies additivity on v,

$$\sum_{q=1}^{m(C)}x^q Sh_i(N,v_{(C^q)^*})=Sh_i\left(N,v_{\sum_{q=1}^{m(C)}x^q(C^q)^*}\right).$$

It only remains to prove that $C^*=\sum\limits_{q=1}^{m(C)}x^q(C^q)^*$. Let t be a minimal tree and g_{ij} the unique path in t from i to j. We know that $c_{ij}^*=\max\limits_{\{k,\ell\}\in g_{ij}}\{c_{k\ell}\}=c_{i'j'}$. By Lemma 1, we know that the order of the arcs according to its cost is preserved in each C^q. So t is also a minimal tree for each simple problem C^q. Thus, $c_{ij}^{q*}=\max\limits_{\{k,\ell\}\in g_{ij}}\{c_{k\ell}^q\}=c_{i'j'}^q$ and hence

$$c_{ij}^*=c_{i'j'}=\sum_{q=1}^{m(C)}x^q c_{i'j'}^q=\sum_{q=1}^{m(C)}x^q c_{ij}^{q*}.$$

We now prove that f^{o^*} coincides with f^{CW}. Let (N,M,C) be a problem and t, $m(t)$, and c^k $(k=1,...,m(t))$ be as in the proof of Proposition 1 when we proved that f^{o^*} does not depend on the minimal tree chosen by the Kruskal algorithm. By Lemma 1, $C=\sum\limits_{q=1}^{m(C)}x^qC^q$. Besides, by Norde et al. (2004), we have that $c^1=\min\{c_{ij}:c_{ij}>0\}$ and

$$c_{ij}^1=\begin{cases}0 & \text{when } c_{ij}<c^1,\\ 1 & \text{when } c_{ij}\geq c^1.\end{cases}$$

In general, for each $q=2,\ldots,m(C)$,

$$c^q=\min\{c_{ij}:c_{ij}>c^{q-1}\},$$

$$c_{ij}^q=\begin{cases}0 & \text{when } c_{ij}<c^q,\\ 1 & \text{when } c_{ij}\geq c^q,\end{cases}$$

and

$$x^q=\begin{cases}c^1 & \text{when } q=1,\\ c^q-c^{q-1} & \text{when } q>1.\end{cases}$$

For each $q=1,...,m(C)$, the set of C^q-components coincides with $P(\{\{i,j\}:c_{ij}\leq c^{q-1}\})$. Obviously, $m(t)\leq m(C)$ and t is a minimal tree in C^q for each $q=1,...,m\,(C)$. Besides, for each $q>m(t)$ and each $\{i,j\}\in t$, $c_{ij}^q=0$. By definition of f^{o^*}, for each $i\in N$ and each $q=m(t)+1,...,m(C)$, $f_i^{CW}(N,M,C^q)=0$. Then,

$$f^{CW}(N,M,C)=\sum_{q=1}^{m(C)}x^q f^{CW}(N,M,C^q)=\sum_{q=1}^{m(t)}x^q f^{CW}(N,M,C^q).$$

By definition of o^* and f^{CW}, for each $i \in N$ and each $q = 1, ..., m(t)$,

$$f_i^{CW}(N, M, C^q) = o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{q-1}\})),$$

where we denote $c^0 = 0$.

Therefore,

$$\begin{aligned}
f_i^{CW}(N, M, C) &= \sum_{q=1}^{m(t)} x^q f_i^{CW}(N, M, C^q) \\
&= \sum_{q=1}^{m(t)} x^q o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{q-1}\})) \\
&= c^1 o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^0\})) \\
&\quad + \sum_{q=2}^{m(t)} (c^q - c^{q-1}) o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{q-1}\})) \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{q-1}\})) - o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^q\})) \right] \\
&\quad + c^{m(t)} o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{m(t)}\})).
\end{aligned}$$

Since $P(\{\{i,j\} : c_{ij} \leq c^{m(t)}\}) = \{N \cup M\}$, for each $i \in N$, $o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{m(t)}\})) = 0$. Therefore,

$$f_i^{CW}(N, M, C) = \sum_{q=1}^{m(C)} c^q \left[o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{q-1}\})) - o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^q\})) \right].$$

By (2), we deduce that $f_i^{CW}(N, M, C) = f_i^{o^*}(N, M, C)$. ■

Proof of Proposition 2.

(1) *The folk rule satisfies IIT*: By Theorem 1 the folk rule can be defined as the Shapley value of the irreducible game. Thus, the folk rule satisfies IIT.

(2) *The folk rule satisfies CM*: Let (N, M, C) and (N, M, C') be such that $C \leq C'$. We will prove that $f^{o^*}(N, M, C) \leq f^{o^*}(N, M, C')$. It is enough to prove it when there exists $a, b \in N \cup M$ such that $c_{ab} < c'_{ab}$ and $c_{ij} = c'_{ij}$ when $\{i, j\} \neq \{a, b\}$.

Suppose that there is a minimal tree t in (N, M, C) such that $\{a, b\} \notin t$. This means that t is also a minimal tree in the problem (N, M, C') with exactly the same costs. Since the folk rule satisfies IIT, $f^{o^*}(N, M, C) = f^{o^*}(N, M, C')$. Now suppose that $\{a, b\} \in t$ for each minimal tree t in (N, M, C). Let T be the set of trees in (N, M, C) that do not contain the arc $\{a, b\}$ and $x = \min_{t \in T} c(N, M, C, t) - m(N, M, C)$.

We distinguish several cases:

Case 1. $c'_{ab} - c_{ab} \leq x$. Given a minimal tree t in (N, M, C), we have that t is also a minimal tree in (N, M, C'). Consider the set

$$A = \{\{i, j\} \in t : c_{ab} < c_{ij} < c'_{ab}\}.$$

We have two subcases:
Subcase 1.a. $A = \emptyset$. We can apply the Kruskal algorithm to problems (N, M, C) and (N, M, C') in such a way that we select the arcs of t in the same order. Therefore, for each $i \in N$,

$$f_i^{o^*}(N, M, C') - f_i^{o^*}(N, M, C) = (c'_{ab} - c_{ab}) \left(o_i^*(P) - o_i^*(P^{ab})\right)$$

where P is the partition in connected components before arc $\{a, b\}$ is selected by the Kruskal algorithm and P^{ab} is the partition obtained after arc $\{a, b\}$ is selected. Note that $P^{ab} = P \setminus \{S(P, a), S(P, b)\} \cup (S(P, a) \cup S(P, b))$. Let $i \in N$.

Subcase 1.a.i. $S(P, a) \cap M \neq \emptyset$ and $S(P, b) \cap M \neq \emptyset$. Then,

$$\begin{aligned}
& (c'_{ab} - c_{ab})(o_i^*(P) - o_i^*(P^{ab})) \\
= \; & (c'_{ab} - c_{ab}) \left(\frac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|} - \frac{|S_k \in P : S_k \cap M \neq \emptyset| - 2}{|N|} \right) \\
= \; & \frac{c'_{ab} - c_{ab}}{|N|} \\
\geq \; & 0.
\end{aligned}$$

Subcase 1.a.ii. $S(P, a) \cap M \neq \emptyset$ and $S(P, b) \cap M = \emptyset$. Since the case $S(P, a) \cap M = \emptyset$ and $S(P, b) \cap M \neq \emptyset$ is similar, we omit it.
(1) If $i \notin S(P, a) \cup S(P, b)$, then

$$(c'_{ab} - c_{ab}) \left(o_i^*(P) - o_i^*(P^{ab})\right) = 0.$$

(2) If $i \in S(P, a)$, then

$$(c'_{ab} - c_{ab}) \left(o_i^*(P) - o_i^*(P^{ab})\right) = 0.$$

(3) If $i \in S(P, b)$, then

$$(c'_{ab} - c_{ab}) \left(o_i^*(P) - o_i^*(P^{ab})\right) = \frac{(c'_{ab} - c_{ab})}{|S(P, b)|} \geq 0.$$

Subcase 1.a.iii. $S(P, a) \cap M = \emptyset$ and $S(P, b) \cap M = \emptyset$.
(1) If $i \notin S(P, a) \cup S(P, b)$, then

$$(c'_{ab} - c_{ab}) \left(o_i^*(P) - o_i^*(P^{ab})\right) = 0.$$

(2) If $i \in S(P,a)$ (since the case $i \in S(P,b)$ is similar, we omit it), then

$$(c'_{ab} - c_{ab}) \left(o_i^*(P) - o_i^*(P^{ab})\right) = (c'_{ab} - c_{ab}) \left(\frac{1}{|S(P,a)|} - \frac{1}{|S(P,a) \cup S(P,b)|}\right) \geq 0.$$

Subcase 1.b. $A \neq \emptyset$. When we apply the Kruskal algorithm to problems (N, M, C) and (N, M, C'), the arc $\{a, b\}$ is selected later in (N, M, C'). Let

$$c^0 = c_{ab} \text{ and } c_{ij}^0 = \begin{cases} c^0 & \text{if } \{i,j\} = \{a,b\}, \\ c_{ij} & \text{otherwise.} \end{cases}$$

For each $k \geq 1$, let

$$\begin{aligned} c^k &= \min\{c_{ij} : \{i,j\} \in A, c_{ij} > c^{k-1}\} \text{ and} \\ c_{ij}^k &= \begin{cases} c^k & \text{if } \{i,j\} = \{a,b\}, \\ c_{ij} & \text{otherwise.} \end{cases} \end{aligned}$$

We apply this procedure until we find r such that $c_{ab}^r = \max\{c_{ij} : \{i,j\} \in A\}$. By setting $C^{r+1} = C'$, we have a sequence of problems $\{(N, M, C^k)\}_{k \in \{0,\dots,r+1\}}$ such that $C^0 = C$ and $C^{r+1} = C'$. Note that t is a minimal tree in each of those problems. Besides, for each pair of problems (N, M, C^k) and (N, M, C^{k+1}) we can select the arcs of t in the same order following the Kruskal algorithm.

Thus, using arguments similar to those used in subcase 1.a, for each $k = 0, ..., r$ and each $i \in N$,

$$f_i^{o^*}(N, M, C^{r+1-k}) - f_i^{o^*}(N, M, C^{r-k}) \geq 0.$$

Then, for each $i \in N$,

$$f_i^{o^*}(N, M, C') - f_i^{o^*}(N, M, C) = \sum_{k=0}^{r} \left[f_i^{o^*}(N, M, C^{r+1-k}) - f_i^{o^*}(N, M, C^{r-k})\right] \geq 0.$$

Case 2. $c'_{ab} - c_{ab} > x$. Let the problem (N, M, C'') be such that $c''_{ab} = c_{ab} + x$ and $c''_{ij} = c_{ij}$ otherwise. Let t' be a minimal tree in (N, M, C'). Obviously $\{a, b\} \notin t'$ and t' is also a minimal tree in (N, M, C''). Since the folk rule f^{o^*} satisfies IIT, for each $i \in N$,

$$f_i^{o^*}(N, M, C') - f_i^{o^*}(N, M, C) = f_i^{o^*}(N, M, C'') - f_i^{o^*}(N, M, C).$$

Since (N, M, C'') satisfies the condition of Case 1, for each $i \in N$,

$$f_i^{o^*}(N, M, C'') - f_i^{o^*}(N, M, C) \geq 0.$$

(3) *The folk rule satisfies CA*: By Theorem 1 the folk rule can be defined as f^{CW}, the cone-wise decomposition. Thus, it is obvious that it satisfies CA.

(4) *The folk rule satisfies PM*: It is enough to show that for each $k \in N$ and each $i \in N \setminus \{k\}$, $f_i^{o^*}(N,M,C) \leq f_i^{o^*}(N \setminus \{k\}, M, C)$. Without loss of generality, let $k = |N| = n$.

First, we claim that if $c_{ns} = \alpha$ for each $s \in M$, $c_{ni} = \beta$ for each $i \in N \setminus \{n\}$, and $\beta > \alpha > \max\limits_{i,j \in N \cup M \setminus \{n\}} \{c_{ij}\}$, then for each $i \in N \setminus \{n\}$, $f_i^{o^*}(N,M,C) \leq f_i^{o^*}(N \setminus \{n\}, M, C)$.

Let $t = \{\{i^p(N,M,C), j^p(N,M,C)\}\}_{p=1}^{|N|+|M|-1}$ be a minimal tree chosen by the Kruskal algorithm. Then, (i) $\{i^{|N|+|M|-1}(N,M,C), j^{|N|+|M|-1}(N,M,C)\} = \{n,s\}$ for some $s \in M$, (ii) $\{n,s\}$ is the only arc that agent n is linked in the tree t, and (iii) $N \setminus \{n\}$ and M are already connected under $g^{|N|+|M|-2}(N,M,C)$. Also, the subtree $\{\{i^p(N,M,C), j^p(N,M,C)\}\}_{p=1}^{|N|+|M|-2}$ is a minimal tree in $(N \setminus \{n\}, M, C)$ and for each $p = 1, \ldots, |N|+|M|-2$, $\{i^p(N,M,C), j^p(N,M,C)\} = \{i^p(N \setminus \{n\}, M, C), j^p(N \setminus \{n\}, M, C)\}$. Then, for each $i \in N \setminus \{n\}$,

$$\begin{aligned} f_i^{o^*}(N,M,C) &= \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \left[o_i^* \left(P \left(g^{p-1}(N,M,C) \right) \right) - o_i^* \left(P \left(g^p(N,M,C) \right) \right) \right] \\ &= \sum_{p=1}^{|N|+|M|-2} c_{i^p j^p} \left[o_i^* \left(P \left(g^{p-1}(N,M,C) \right) \right) - o_i^* \left(P \left(g^p(N,M,C) \right) \right) \right] \end{aligned}$$

where the last equality comes from the fact that for each $i \in N \setminus \{n\}$,

$$o_i^* \left(P \left(g^{|N|+|M|-2}(N,M,C) \right) \right) = o_i^* \left(P \left(g^{|N|+|M|-1}(N,M,C) \right) \right) = 0.$$

Note that for each $p = 1, \ldots, |N|+|M|-2$, $P\left(g^p(N,M,C)\right) \setminus \{n\} = P\left(g^p(N \setminus \{n\}, M, C)\right)$, for each $i \in N \setminus \{n\}$, $S\left(P\left(g^p(N,M,C)\right), i\right) = S\left(P\left(g^p(N \setminus \{n\}, M, C)\right), i\right)$, and $\{S_j \in P\left(g^p(N,M,C)\right) : S_j \cap M \neq \emptyset\} = \{S_j \in P\left(g^p(N \setminus \{n\}, M, C)\right) : S_j \cap M \neq \emptyset\}$.

Let $i \in N \setminus \{n\}$. For each $p = 1, \ldots, |N|+|M|-2$, let

$$q^p = |\{S_j \in P\left(g^p(N,M,C)\right) : S_j \cap M \neq \emptyset\}| = |\{S_j \in P\left(g^p(N \setminus \{n\}, M, C)\right) : S_j \cap M \neq \emptyset\}|$$

and

$$s^p = |S(P(g^p(N,M,C)), i)| = |S(P(g^p(N \setminus \{n\}, M, C)), i)|.$$

We consider several cases:

Case 1. $S\left(P\left(g^{p-1}(N,M,C)\right), i\right) \cap M \neq \emptyset$. Then, $S\left(P\left(g^p(N,M,C)\right), i\right) \cap M \neq \emptyset$. Now

$$\begin{aligned} & o_i^* \left(P \left(g^{p-1}(N,M,C) \right) \right) - o_i^* \left(P \left(g^p(N,M,C) \right) \right) \\ = \; & \frac{q^{p-1}-1}{|N|} - \frac{q^p - 1}{|N|} \leq \frac{q^{p-1}-1}{|N \setminus \{n\}|} - \frac{q^p - 1}{|N \setminus \{n\}|} \\ = \; & o_i^* \left(P \left(g^{p-1}(N \setminus \{n\}, M, C) \right) \right) - o_i^* \left(P \left(g^p(N \setminus \{n\}, M, C) \right) \right). \end{aligned}$$

Case 2. $S\left(P\left(g^{p-1}(N,M,C)\right),i\right)\cap M=\emptyset$ and $S\left(P\left(g^{p}(N,M,C)\right),i\right)\cap M\neq\emptyset$. Now,

$$\begin{aligned}
& o_i^*\left(P\left(g^{p-1}(N,M,C)\right)\right)-o_i^*\left(P\left(g^{p}(N,M,C)\right)\right)\\
= & \frac{q^{p-1}-1}{|N|}+\frac{1}{s^{p-1}}-\frac{q^{p}-1}{|N|}\leq\frac{q^{p-1}-1}{|N\setminus\{n\}|}+\frac{1}{s^{p-1}}-\frac{q^{p}-1}{|N\setminus\{n\}|}\\
= & o_i^*\left(P\left(g^{p-1}(N\setminus\{n\},M,C)\right)\right)-o_i^*\left(P\left(g^{p}(N\setminus\{n\},M,C)\right)\right).
\end{aligned}$$

Case 3. $S\left(P\left(g^{p-1}(N,M,C)\right),i\right)\cap M=\emptyset$ and $S\left(P\left(g^{p}(N,M,C)\right),i\right)\cap M=\emptyset$. Now,

$$\begin{aligned}
& o_i^*\left(P\left(g^{p-1}(N,M,C)\right)\right)-o_i^*\left(P\left(g^{p}(N,M,C)\right)\right)\\
= & \frac{q^{p-1}-1}{|N|}+\frac{1}{s^{p-1}}-\frac{q^{p}-1}{|N|}-\frac{1}{s^{p}}\leq\frac{q^{p-1}-1}{|N\setminus\{n\}|}+\frac{1}{s^{p-1}}-\frac{q^{p}-1}{|N\setminus\{n\}|}-\frac{1}{s^{p}}\\
= & o_i^*\left(P\left(g^{p-1}(N\setminus\{n\},M,C)\right)\right)-o_i^*\left(P\left(g^{p}(N\setminus\{n\},M,C)\right)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_i^{o^*}(N,M,C) &= \sum_{p=1}^{|N|+|M|-1} c_{i^pj^p}\left[o_i^*\left(P\left(g^{p-1}(N,M,C)\right)\right)-o_i^*\left(P\left(g^{p}(N,M,C)\right)\right)\right]\\
&\leq \sum_{p=1}^{|N|+|M|-2} c_{i^pj^p}\left[o_i^*\left(P\left(g^{p-1}(N\setminus\{n\},M,C)\right)\right)-o_i^*\left(P\left(g^{p}(N\setminus\{n\},M,C)\right)\right)\right]\\
&= f_i^{o^*}(N\setminus\{n\},M,C), \qquad (3)
\end{aligned}$$

as desired.

Let $\alpha=\max\limits_{i,j\in N\cup M}\{c_{ij}\}+1$ and $\beta=\alpha+1$. Let $C^0\in\mathcal{C}^{N\cup M}$ be such that $c^0_{ns}=\alpha$ for each $s\in M$ and $c^0_{ij}=c_{ij}$ otherwise. For each $r=1,...,|N|-1$, let $C^r\in\mathcal{C}^{N\cup M}$ be such that $c^r_{nr}=\beta$ and for each $\{i,j\}\neq\{n,r\}$, $c^r_{ij}=c^{r-1}_{ij}$. Let $i\in N\setminus\{n\}$. Since f^{o^*} satisfies CM,

$$f_i^{o^*}(N,M,C)\leq f_i^{o^*}(N,M,C^0)\leq f_i^{o^*}(N,M,C^1)\leq\cdots\leq f_i^{o^*}(N,M,C^{|N|-1}).$$

Applying (3) to $C^{|N|-1}$,

$$f_i^{o^*}(N,M,C^{|N|-1})\leq f_i^{o^*}(N\setminus\{n\},M,C^{|N|-1}).$$

Since $c^{|N|-1}_{ij}=c_{ij}$ for each $i,j\in N\cup M\setminus\{n\}$,

$$f_i^{o^*}(N\setminus\{n\},M,C^{|N|-1})=f_i^{o^*}(N\setminus\{n\},M,C)$$

and so f^{o^*} satisfies PM.

(5) *The folk rule satisfies CS and SEP*: Since PM implies CS and SEP, the result holds.

(6) *The folk rule satisfies SYM*: By Theorem 1 the folk rule can be obtained as the Shapley value of the game associated with the irreducible problem. It is trivial to prove that if two agents are symmetric in the problem (N, M, C), then they will also be symmetric in the irreducible problem (N, M, C^*) and hence, in the game associated with the irreducible problem. Since the Shapley value satisfies SYM, the folk rule also does.

(7) *The folk rule satisfies ETSC*: Let (N, M, C) and (N, M, C') be two problems satisfying the conditions in the statement of ETSC. Suppose that there is a minimal tree in (N, M, C) such that $\{a, b\} \notin t$. Thus, t is also a minimal tree in (N, M, C') with the same costs. Since the folk rule satisfies IIT, we have that $f^{o^*}(N, M, C) = f^{o^*}(N, M, C')$. Assume that $\{a, b\} \in t$ for each minimal tree t in (N, M, C). Let T be the set of all trees in (N, M, C) that do not contain $\{a, b\}$. Let $x = \min_{t \in T} c(N, M, C, t) - m(N, M, C)$. We consider several cases.
Case 1. $c'_{ab} - c_{ab} \leq x$. Note that a minimal tree t in (N, M, C) is also a minimal tree in (N, M, C'). Now consider the set $A = \{\{i, j\} \in t : c_{ab} < c_{ij} < c'_{ab}\}$. The proof is divided into two subcases:

Subcase 1.a. $A = \emptyset$. We can apply the Kruskal algorithm to (N, M, C) and (N, M, C') in such a way that the arcs of t are selected in the same order. Then, for each $i \in N$,

$$f_i^{o^*}(N, M, C') - f_i^{o^*}(N, M, C) = (c'_{ab} - c_{ab})(o_i^*(P) - o_i^*(P^{ab}))$$

where P is the partition in connected components before arc $\{a, b\}$ is selected by the Kruskal algorithm and P^{ab} is the partition obtained after arc $\{a, b\}$ is selected. Note that $P^{ab} = P \setminus \{S(P, a), S(P, b)\} \cup \{S(P, a) \cup S(P, b)\}$. By the definition of o^*, for each $i \in N$,

$$\begin{aligned}
&(c'_{ab} - c_{ab})(o_i^*(P) - o_i^*(P^{ab})) \\
&= (c'_{ab} - c_{ab})\Big(\frac{|\{S_k \in P : S_k \cap M \neq \emptyset\}| - 1}{|N|} - \frac{|\{S_k \in P : S_k \cap M \neq \emptyset\}| - 2}{|N|}\Big) \\
&= \frac{c'_{ab} - c_{ab}}{|N|}.
\end{aligned}$$

Subcase 1.b. $A \neq \emptyset$. When we apply the Kruskal algorithm to (N, M, C) and (N, M, C'), the arc $\{a, b\}$ is selected later in (N, M, C') than in (N, M, C). Let

$$c^0 = c_{ab} \text{ and } c_{ij}^0 = \begin{cases} c^0 & \text{if } \{i, j\} = \{a, b\}, \\ c_{ij} & \text{otherwise.} \end{cases}$$

For each $r \geq 1$, let $c^r = \min\{c_{ij} : \{i, j\} \in A, c_{ij} > c^{r-1}\}$ and

$$c_{ij}^r = \begin{cases} c^r & \text{if } \{i, j\} = \{a, b\}, \\ c_{ij} & \text{otherwise.} \end{cases}$$

We apply this procedure until we find $\bar{r}$ such that $c_{ab}^{\bar{r}} = \max\{c_{ij} : \{i,j\} \in A\}$. By setting $C^{\bar{r}+1} = C'$, we have a sequence of problems $\{(N,M,C^r)\}_{r\in\{0,\ldots,\bar{r}+1\}}$ such that $C^0 = C$ and $C^{\bar{r}+1} = C'$. Note that t is a minimal tree in each of those problems. In addition, for each pair of consecutive problems (N,M,C^r) and (N,M,C^{r+1}), $r \in \{0,...,\bar{r}\}$, we can select the arcs of t in the same order by following the Kruskal algorithm. Therefore, by using arguments similar to those used in Subcase 1.a,

$$\begin{aligned} f_i^{o^*}(N,M,C') - f_i^{o^*}(N,M,C) &= \sum_{r=0}^{\bar{r}} \left[f_i^{o^*}(N,M,C^{\bar{r}+1-r}) - f_i^{o^*}(N,M,C^{\bar{r}-r}) \right] \\ &= \sum_{r=0}^{\bar{r}} \frac{c^{\bar{r}+1-r} - c^{\bar{r}-r}}{|N|} = \frac{c^{\bar{r}+1} - c^0}{|N|} = \frac{c'_{ab} - c_{ab}}{|N|}. \end{aligned}$$

Case 2. $c'_{ab} - c_{ab} > x$. Let (N,M,C'') be such that $c''_{ab} = c_{ab} + x$ and $c''_{ij} = c_{ij}$ otherwise. Let t' be a minimal tree in (N,M,C'). Obviously, $\{a,b\} \notin t'$ and t' is also a minimal tree in (N,M,C''). Since f^{o^*} satisfies IIT, for each $i \in N$,

$$f_i^{o^*}(N,M,C') - f_i^{o^*}(N,M,C) = f_i^{o^*}(N,M,C'') - f_i^{o^*}(N,M,C).$$

Since (N,M,C'') satisfies the condition of Case 1, for each $i \in N$,

$$f_i^{o^*}(N,M,C'') - f_i^{o^*}(N,M,C) = \frac{c'_{ab} - c_{ab}}{|N|} = \frac{x}{|N|}$$

as desired. ■

Proof of Theorem 2.
(a) By Proposition 2, the folk rule satisfies the five axioms. Conversely, let f be a rule satisfying the five axioms. For each partition $P = \{S_1, S_2, ..., S_{|P|}\} \in P(N \cup M)$, we define the function $o(P) = f(N,M,C^P)$ where $c_{ij}^P = 0$ if $i,j \in S_k$ for some $k \in \{1,...,|P|\}$ and $c_{ij}^P = 1$ otherwise. Note that

$$\sum_{i\in N} o_i(P) = \sum_{i\in N} f_i(N,M,C^P) = m(N,M,C^P) = |P| - 1.$$

We claim that $f = f^o$ where for each (N,M,C) and each $i \in N$,

$$f_i^o(N,M,C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \Big[o_i\big(P(g^{p-1})\big) - o_i\big(P(g^p)\big) \Big].$$

Since f and f^o satisfy CA, by Lemma 1, $f(N,M,C) = \sum_{q=1}^{m(C)} f(N,M,x^q C^q)$ and $f^o(N,M,C) = \sum_{q=1}^{m(C)} f^o(N,M,x^q C^q)$. Therefore, it is enough to prove that f coincides with f^o in problems (N, M, C) where there exists a network g such that $c_{ij} = x$ if $\{i, j\} \in g$ and $c_{ij} = 0$ otherwise. Let $P(g) = \{S_1, ..., S_r\}$ be the partition induced by g over $N \cup M$.

When we use the Kruskal algorithm in this problem, we first connect the nodes within the same component with zero cost until step $(|N|+|M|-r)$. Then, we connect the nodes from different components with the constant cost x. Thus, for each $i \in N$,

$$\begin{aligned}
f_i^o(N, M, C) &= \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p}\Big[o_i\Big(P(g^{p-1})\Big) - o_i\Big(P(g^p)\Big)\Big] \\
&= \sum_{p=|N|+|M|-r+1}^{|N|+|M|-1} x\Big[o_i\Big(P(g^{p-1})\Big) - o_i\Big(P(g^p)\Big)\Big] \\
&= x\Big[o_i\Big(P(g^{|N|+|M|-r+1})\Big) - o_i\Big(P(g^{|N|+|M|-1})\Big)\Big] \\
&= x\Big[f_i(N, M, C^{P(g^{|N|+|M|-r+1})}) - f_i(N, M, C^{P(g^{|N|+|M|-1})})\Big].
\end{aligned}$$

Note that $P(g^{|N|+|M|-r+1}) = P(g)$ and $P(g^{|N|+|M|-1}) = N \cup M$.

Since $c_{ij}^{N\cup M} = 0$ for each i, $j \in N \cup M$ and f satisfies CA, for each $i \in N$, $f_i(N, M, C^{N\cup M}) = 0$, which implies that $f_i^o(N, M, C) = x f_i(N, M, C^{P(g)})$. Now, consider C' such that $c'_{ij} = \frac{1}{x}c_{ij}$ for each i, $j \in N \cup M$. Note that $C'^* = C^{P(g)}$. By IIT, $f(N, M, C^{P(g)}) = f(N, M, C')$. By CA, $f^o(N, M, C) = xf(N, M, C')$.

Using similar arguments as in Bergantiños et al. (2010, p.708), we can prove that $xf(N, M, C') = f(N, M, xC')$, which implies that $f^o(N, M, C) = f(N, M, xC')$. Since $xC' = C$, we conclude that $f = f^o$, as desired.

It remains to prove that $o = o^*$. Now, let $P = \{S_1, ..., S_q, ..., S_{|P|}\}$ be a partition such that $S_k \cap M \neq \emptyset$ when $k \leq q$ and $S_k \subset N$ when $k > q$. Note that $|\{S_k \in P : S_k \cap M \neq \emptyset\}| = q$. We introduce a sequence of problems $\{(N, M, C^r)\}_{r=1,2,...,q}$ where $C^1 = C^P$ and for each $r > 1$, C^r is obtained from C^{r-1} such that $c^r_{a^{r-1}a^r} = 0$ if $a^{r-1} \in S_{r-1} \cap M$ and $a^r \in S_r \cap M$, and $c^r_{ij} = c^{r-1}_{ij}$ otherwise. By ETSC, for each $r = 2, \ldots, q$ and each i, $j \in N$,

$$f_i(N, M, C^{r-1}) - f_i(N, M, C^r) = f_j(N, M, C^{r-1}) - f_j(N, M, C^r).$$

Since

$$\begin{aligned}
\sum_{i\in N} \big[f_i(N, M, C^{r-1}) - f_i(N, M, C^r)\big] &= \sum_{i\in N} f_i(N, M, C^{r-1}) - \sum_{i\in N} f_i(N, M, C^r) \\
&= m(N, M, C^{r-1}) - m(N, M, C^r) \\
&= 1,
\end{aligned}$$

for each $i \in N$,

$$f_i(N, M, C^{r-1}) - f_i(N, M, C^r) = \frac{1}{|N|}.$$

Therefore, for each $i \in N$,

$$f_i(N,M,C^1) - f_i(N,M,C^q) = \sum_{r=2}^{q} \left[f_i(N,M,C^{r-1}) - f_i(N,M,C^r) \right]$$
$$= \frac{q-1}{|N|}.$$

Thus,

$$o(P) = f_i(N,M,C^1) = \frac{q-1}{|N|} + f_i(N,M,C^q). \tag{4}$$

By CS, for each $k = q+1, \ldots, |P|$, $\sum\limits_{i \in S_k \cap N} f_i(N,M,C^q) \leq m(S_k \cap N, M, C^q) = 1$ and for each $i \in (\cup_{k=1}^q S_k) \cap N$, $f_i(N,M,C^q) \leq m(\{i\}, M, C^q) = 0$. Since $\sum\limits_{i \in N} f_i(N,M,C^q) = m(N,M,C^q) = |P| - q$, for each $k = q+1, \ldots, |P|$, $\sum\limits_{i \in S_k \cap N} f_i(N,M,C^q) = 1$ and for each $i \in (\cup_{k=1}^q S_k) \cap N$, $f_i(N,M,C^q) = 0$. By (1) and (4), for each $i \in (\cup_{k=1}^q S_k) \cap N$,

$$o_i(P) = \frac{q-1}{|N|} = o_i^*(P)$$

For each $k = q+1, ..., |P|$ and each i, $j \in S_k$, i and j are symmetric, so that by SYM, for each $i \in S_k$ $(k > q)$, $f_i(N,M,C^q) = \frac{1}{|S_k|}$. From (1) and (4), we have that

$$o_i(P) = \frac{q-1}{|N|} + \frac{1}{|S_k|} = o_i^*(P).$$

(b) By Proposition 2, the folk rule satisfies the five axioms. Conversely, let f be a rule satisfying the five axioms. From the same argument as in (a), we obtain (4). Note that $m(N,M,C^q) = |P| - q$, for each $k = q+1, \ldots, |P|$, $m(S_k, M, C^q) = 1$, and for each $i \in (\cup_{k=1}^q S_k) \cap N$, $m(\{i\}, M, C^q) = 0$, which together imply that $m(N,M,C^q) = \sum\limits_{k=1}^{q} \big(\sum\limits_{i \in S_k \cap N} m(\{i\}, M, C^q) \big) + \sum\limits_{k=q+1}^{|P|} m(S_k, M, C^q)$. By SEP, for each $k = q+1, \ldots, |P|$, $f_i(N,M,C^q) = f_i(S_k, M, C^q)$, which implies that $\sum\limits_{i \in S_k \cap N} f_i(N,M,C^q) = \sum\limits_{i \in S_k \cap N} f_i(S_k, M, C^q) = m(S_k, M, C^q) = 1$ and for each $i \in (\cup_{k=1}^q S_k) \cap N$, $f_i(N,M,C^q) = f_i(\{i\}, M, C^q) = m(\{i\}, M, C^q) = 0$. Once again, by using the same argument as in the proof of (a), we conclude that f coincides with the folk rule. ■

Next, we show that all axioms are independent in Theorem 2.

(1) Dropping *Independence of irrelevant trees*: Let f^w be a rule defined for simple problems. For each simple problem (N,M,C), we consider $g = \{\{i,j\} \subset N \cup M : c_{ij} = 0\}$. For each $i \in N$, let

$$w_i = \begin{cases} \frac{1}{|\{\{i,j\}: j \in S(P(g),i) \ \ and c_{ij}=0\}|} & \text{if } S(P(g),i) \neq \{i\}, \\ 1 & \text{otherwise.} \end{cases}$$

For each $i \in N$, let f^w be

$$f_i^w(N,M,C) = \begin{cases} \frac{|\{S_k \in P(g): S_k \cap M \neq \emptyset\}|-1}{|N|} & \text{if } S(P(g),i) \cap M \neq \emptyset, \\ \frac{|\{S_k \in P(g): S_k \cap M \neq \emptyset\}|-1}{|N|} + \frac{w_i}{\sum\limits_{j \in S(P(g),i)} w_j} & \text{otherwise.} \end{cases}$$

This rule is extended to general problems using Lemma 1. The rule f^w satisfies CA, CS, SEP, SYM, and ETSC, but not IIT.

(2) Dropping *Cone-wise additivity*: We first introduce some notion in the classical problem following Bergantiños and Vidal-Puga (2015). For each classical problem (N_0, C) and each $S \subset N$, let

$$\delta_S = \begin{cases} \min\limits_{j \in N_0 \setminus \{i\}} c_{ij} & \text{if } S = \{i\}, \\ \min\limits_{i \in S, j \in N_0 \setminus S} c_{ij} - \max\limits_{\{i,j\} \in \tau(S)} c_{ij} & \text{if } |S| > 1, \end{cases}$$

where $N_0 = N \cup \{0\}$ and $\tau(S)$ is a minimal tree in (S, C_S) connecting all agents in S. Let $Ne(N_0, C)$ be a set of all coalitions $S \subset N$ and $|S| > 1$ such that $\delta_S > 0$. Let $\hat{o} = \{\hat{o}^x\}_{x \in \mathbb{R}_+}$ be a parametric family of functions defined as

$$\hat{o}_i^x(N) = \begin{cases} \frac{1}{|N|} & \text{if } |N| \neq 2, \\ \frac{1}{2} & \text{if } |N| = 2 \text{ and } x > 1, \\ \max\{\frac{1}{3}, \min\{\frac{c_{0i}}{c_{01}+c_{02}}, \frac{2}{3}\}\} & \text{if } |N| = 2 \text{ and } x \leq 1. \end{cases}$$

Let C^* be the irreducible cost matrix of C. For each (C^*, x) and each $i \in N$, let

$$e_i(C^*, x) = \int_0^x \hat{o}_i^t(N)dt.$$

Now, we define the rule f^e such that for each classical problem (N_0, C) and each $i \in N$,

$$f_i^e(N_0, C) = c_{0i}^* - \sum_{S \in Ne(N_0,C), i \in S} (\delta_S - e_i((S, C_S^*), \delta_S)).$$

Next, we extend this rule to our problem. For all (N, M, C), let t be a minimal tree in the irreducible problem (N, M, C^*) where all sources are connected among themselves. Let t_M be the restriction of t to M. We now consider the classical problem $(N_0, \bar{C})$ such that $\bar{t} = \{\{i,j\} \in t : i,j \in N\} \cup \{\{0,i\} : i \in N \text{ and } \{i,j\} \in t \text{ for some } j \in M\}$. It is easy to see that $\bar{t}$ is a tree that connects all agents in N to 0. Let $\bar{c_{ij}} = c_{ij}^*$ if $i,j \in N$ and $\{i,j\} \in \bar{t}$; $\bar{c_{0i}} = \max\{c_{k\ell}^* : \{k,\ell\} \in g_{ii^M}^t\}$, where i^M is the first source in the unique path connecting agent i to each source in t; and $\bar{c_{ij}} = \max\{\bar{c_{k\ell}} : \{k,\ell\} \in g_{ij}^{\bar{t}}\}$ if $i,j \in N$ and $\{i,j\} \notin t$. For each problem (N, M, C), let

$$f^e(N,M,C) = \frac{c(t_M)}{|N|} + f^e(N_0, \bar{C}).$$

The rule f^e satisfies CM (thus, IIT), PM (thus, CS and SEP), SYM and ETSC, but not CA.

(3) Dropping *Core selection* or *Separability*: The egalitarian rule f^E, defined as for each (N, M, C) and each $i \in N$, $f_i^E(N, M, C) = \frac{m(N,M,C)}{|N|}$, satisfies CA, IIT, SYM, and ETSC, but not CS or SEP.

(4) Dropping *Symmetry*: Let $\tilde{o}$ be a function such that for each $P \in P(N \cup M)$ and each $i \neq n$,

$$\tilde{o}_i(P) = \begin{cases} \frac{|\{S \in P : S \cap M \neq \emptyset\}|-1}{|N|} & \text{if } i \in S_k,\ S_k \cap M = \emptyset \text{ and } n \in S_k, \\ \frac{|\{S \in P : S \cap M \neq \emptyset\}|-1}{|N|} + \frac{1}{|S_k|} & \text{if } i \in S_k,\ S_k \cap M = \emptyset \text{ and } n \notin S_k, \\ \frac{|\{S \in P : S \cap M \neq \emptyset\}|-1}{|N|} & \text{if } i \in S_k \text{ and } S_k \cap M \neq \emptyset, \end{cases}$$

and

$$\tilde{o}_n(P) = \begin{cases} \frac{|\{S \in P : S \cap M \neq \emptyset\}|-1}{|N|} + 1 & \text{if } n \in S_k \text{ and } S_k \cap M = \emptyset, \\ \frac{|\{S \in P : S \cap M \neq \emptyset\}|-1}{|N|} & \text{if } n \in S_k \text{ and } S_k \cap M \neq \emptyset. \end{cases}$$

Let $f^{\tilde{o}}$ be a rule such that for each (N, M, C) and each $i \in N$,

$$f_i^{\tilde{o}}(N, M, C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \Big[\tilde{o}_i\big(P(g^{p-1})\big) - \tilde{o}_i\big(P(g^p)\big)\Big].$$

The rule $f^{\tilde{o}}$ satisfies CA, IIT, CS, SEP, and ETSC, but not SYM.

(5) Dropping *Equal treatment of source costs*: Let $P = \{S_1, ..., S_q, ..., S_{|P|}\}$ be a partition in $P(N \cup M)$, where $S_k \cap M \neq \emptyset$ if $k \leq q$ and $S_k \cap M = \emptyset$ if $k > q$. Let t be a number of agents in an element in P containing no source, i.e., $t = |\{i \in N : i \in S_k\ (S_k \in P)$ and $S_k \cap M = \emptyset\}|$. Let ϵ be an arbitrarily small number such that $\epsilon \in (0, \frac{1}{|N||M|})$. Let o^ϵ be a function such that for each $P \in P(N \cup M)$ and each $i \in N$, if $0 < t < |N|$,

$$o_i^\epsilon(P) = \begin{cases} \frac{1-\epsilon}{|N|}(|\{S \in P : S \cap M \neq \emptyset\}| - 1) + \frac{1}{|S_k|} & \text{if } i \in S_k \text{ and } S_k \cap M = \emptyset, \\ \frac{|N|-t(1-\epsilon)}{|N|(|N|-t)}(|\{S \in P : S \cap M \neq \emptyset\}| - 1) & \text{if } i \in S_k \text{ and } S_k \cap M \neq \emptyset, \end{cases}$$

and if $t = 0$ or $t = |N|$, $o_i^\epsilon(P) = o^*(P)$. Let f^{o^ϵ} be a rule such that for each (N, M, C) and each $i \in N$,

$$f_i^{o^\epsilon}(N, M, C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \Big[o_i^\epsilon\big(P(g^{p-1})\big) - o_i^\epsilon\big(P(g^p)\big)\Big].$$

The rule f^{o^ϵ} satisfies CA, IIT, CS, SEP, and SYM, but not ETSC.

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