The distribution of the average of log-normal variables and Exact Pricing of the Arithmetic Asian Options: A Simple, closed-form Formula

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Abstract: We introduce a simple, exact and closed-form formula for pricing the arithmetic Asian options. The pricing formula is as simple as the classical Black-Scholes formula. In doing so, we show that the distribution of the continuous average of log-normal variables is log-normal.

Keywords: Arithmetic Asian option pricing, the arithmetic average of the price, average of log-normal, the Black-Scholes formula.

1 Introduction

Recent literature used orthogonal polynomial expansions to approximate the distribution of the arithmetic average. Examples include Willems (2019) and Asmussen et al (2016). Some of the literature used Edgeworth expansions to approximate the distributions (see, for example, Li and Chen (2016)). Others such as Aprahmiam and Maddah (2015) used the Gamma distribution approach. Some studies relied on Monte Carlo simulations. Examples include Lapeyre et al (2001) and Fu et al (1999). Others adopted a numerical approach. Examples include Linetsky (2004), Cerny and Kyriakou (2011), and Fusai et al (2011). Curran (1994) used the geometric mean to estimate the arithmetic mean.

The literature on pricing the arithmetic Asian options has two main features in common. First, it relies on approximations. Secondly, it largely adopts (very) complex methods. Consequently, this paper overcomes these two limitations. In this paper, we use a pioneering approach to pricing the arithmetic Asian options. In doing so, we present an exact (yet very simple) method. Particularly, we show that the price of the arithmetic Asian option is exactly equivalent to the price of the European option with an earlier (known) expiry. The pricing formula is as simple as the classical Black-Scholes formula. We also show that the distribution of the continuous aritmetic average is lognormal.

2 The method

The arithmetic average of the price underlying asset $S(u)$ over the time interval $[t,T]$ is given by

$$A_t = \frac{\int_t^T S(u)\,du}{T-t},$$

where t is the current time and T is the expiry time. So that, using the Black-Scholes assumptions, $EA_t = E\frac{\int_t^T S(u)du}{T-t} = \frac{e^{r(T-t)}-1}{r(T-t)}S(t)$, where r is the risk-free rate of return. By the mean value theorem for integrals, $E\frac{\int_t^T S(u)du}{T-t}$ $= ES\left(\hat{t}\right)$, where $\hat{t}$ is a time such that $t < \hat{t} < T$, and $ES\left(\hat{t}\right) = e^{r\left(\hat{t}-t\right)}S(t)$. This implies that $\frac{e^{r(T-t)}-1}{r(T-t)} = e^{r\left(\hat{t}-t\right)}$. We can solve for $\hat{t}-t$ as follows

$$\hat{t}-t = \frac{\ln\left(\frac{e^{r(T-t)}-1}{r(T-t)}\right)}{r}.$$

Thus $\hat{t}$ is known. For example, if $T-t=1$ and $r=.01$, $\hat{t}-t=\frac{\ln\left(\frac{e^{.01}-1}{.01}\right)}{.01}=.498$. We also can show that A_t is log-normal[1] and equal to $S\left(\hat{t}\right)$; thus the Black-Scholes formula can be directly and exactly applied. That is, the price of the Asian option (expiring at time T) is given by

$$C\left(t\right)=e^{-r\left(\hat{t}-t\right)}E\left[S\left(\hat{t}\right)-K\right]^{+}=e^{-r\left(\hat{t}-t\right)}E\left[A_t-K\right]^{+},$$

where K is the strike price. Clearly, this is the price of a European option with expiry $\hat{t}$. Thus, the price of the arithmetic Asian option (with expiry time T) is equal to the price of the equivalent European option with expiry time $\hat{t}$. This explains why the Asian option is cheaper than its European counterpart.

Needless to say, the pricing formula for an arithmetic Asian call with expiry time T is

$$C\left(t,s\right)=sN\left(d_1\right)-e^{-r\left(\hat{t}-t\right)}KN\left(d_2\right),$$

where s is the current price, $d_1=\frac{1}{\sqrt{\sigma^2\left(\hat{t}-t\right)}}\left[\ln\left(s/K\right)+\left(r+\sigma^2/2\right)\left(\hat{t}-t\right)\right]$,

[1] See the appendix for the proofs.

$d_2 = d_1 - \sqrt{\sigma^2 \left(\hat{t} - t\right)}$, and σ is the volatility of the return rate of the underlying asset.

Appendix. Proofs of A_t is log-normal.

1. Consider the stock price, $S(T) - s = \int_0^T dS(t)$, where $s \equiv S(0)$; squaring both sides yields

$$(S(T))^2 + s^2 = 2sS(T) + \left(\int_0^T dS(t)\right)^2 = 2sS(T) + \sigma^2 \int_0^T (S(t))^2 \, dt \quad (1)$$

since $(dS(t))^2 = \sigma^2 (S(t))^2 \, dt$. The left-hand-side of (1) is clearly log-normal (a lognormal plus a constant), and the right-hand-side of the equation is a sum of lognormal variables; therefore, the sum (or average) of log-normal variables is log-normal.

We can also present the sum without the constant s^2 by differentiating both sides of (1) with respect to r

$$\frac{\partial (S(T))^2}{\partial r} = 2s \frac{\partial S(T)}{\partial r} + \sigma^2 \int_0^T \frac{\partial (S(t))^2}{\partial r} dt$$

clearly the left-hand-side of the above equation is log-normal, and the right-hand-side of the equation is a sum of log-normal variables.

We can also show that the integral alone is log-normal; dividing both sides by $S(T)$ yields

$$2TS(T) = 2Ts + \sigma^2 \int_0^T \frac{\partial (S(t))^2}{S(T)\,\partial r} dt,$$

differentiating twice w.r.t. r

$$2T \frac{\partial^2 S(T)}{\partial r^2} = \sigma^2 \int_0^T \frac{\partial^2 X}{\partial r^2} dt,$$

where $X \equiv \frac{\partial (S(t))^2}{S(T)\partial r}$; the left-hand-side of the above equation is log-normal, and the right-hand-side of the equation is a sum of log-normal variables.□

2. The simplest and intuitive proof is that the time continuity implies that the average price A_t is a price on the interval $[S(t), S(T)]$. To be more precise, each (random) price at a specific time is an interval-valued (an interval of all possible outcomes of the price). Thus the elements of $[S(t), S(T)]$ are (vertical) intervals, then the time continuity guarantees the existence of a vertical interval of outcomes on $[S(t), S(T)]$, but the vertical interval is a price at a specific time. So the difference between a random variable and a non-random variable is that the random variable is interval-valued, and thus the mean-value theorem can be applied in the same way

to non-random variables if we view the elements of $[S(t), S(T)]$ as interval-valued.

3. The outcomes of A_t are the averages of paths and therefore they are outcomes (realizations) of prices. That is, each outcome is in the form $S(t)\, e^{\left(r-\frac{1}{2}\sigma^2\right)u+\Omega_1}$, where Ω is an outcome of a Brownian motion; thus it can be expressed as $A_t = S(t)\, e^{\left(r-\frac{1}{2}\sigma^2\right)u+W(u)}$; otherwise it will not be possible, using the price probability density, to obtain $EA_t = S(t)\, e^{r\left(\hat{t}-t\right)}$.

4. Using the classical mean-value theorem, it is straightforward to show that the variance of A_t is in the form $S(t)^2\, e^{\left(2r+\sigma^2\right)u}\left(e^{\sigma^2 u}-1\right)$. Higher moments can also be obtained by the classical mean-value theorem; and on a bounded interval, the distribution is identified by its moments.

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