Behavioral sciences and auto-transformations. Introduction

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The goal of the present article is to define transformations (named here as auto-transformations) of probability density functions of random variables into similar functions having smaller sizes of their domains. In particular, auto-transformations from infinite to finite sizes of domains will be analyzed. The goal is aroused from the well-known problems of behavioral sciences.

Content

1. Introduction

1.1. Preliminaries

The present article is devoted to modifications of the probability density functions (PDFs) of continuous random variables (r.v.s).

The next stage of this work will be a mathematical description of influence of noise (and other causes that can cause dispersion of data) on measurement data near the boundaries of the measurement intervals.

The article develops works [15]-[21], where biases of the expectations of random variables are revealed and an applied mathematical method (approach) and model are created that qualitatively explained some well-known generic problems of behavioral economics, decision theories, and the social sciences. The examples of such problems are the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, risk premium, etc.

1.2. Main goal of the article

The main goal of the article is to determine transformations of probability density functions of continuous r.v.s from some larger (especially from infinite) domains into some smaller ones and to consider general properties of these transformations (named here as auto-transformations or ATs).

The transformations can be performed from infinite into half-infinite, from infinite and half-infinite into finite, and also from longer finite into shorter finite domains (intervals). These transformations of probability density functions are the first example of such transformations and can be extended in future to the general case of the real-valued random variables.

Suppose PDFs whose domains are infinite. The initial point, that has motivated this goal, is a question how could such or similar PDFs be modified if their domains were half-infinite or finite. The modifications near the boundaries of the intervals (domains) are of particular importance.

1.3. Review of the literature

Bounds for moments, means, and functions are considered in many works, see, e.g., [7], [9], [23], [24]. Situations considered in [5], [11], [26], [27], [30], [29] are, in the mathematical aspects, the most similar to those analyzed here.

Mathematical aspects of behaviour sciences, utility and prospect theories are considered in a number of works, see, e.g., [6], [10], [25], [32]. Similar aspects are kept in mind in the present article as well. Works [1] and [31] constitute one of starting points for situations considered in [21] and here.

Qualitative inuences of noise are considered in some works. For example, stabilization and synchronization by noise are considered in a number of works, see, e.g., [2], [3], [4], [8], [12], [13]. A noise as a possible cause of some periodic behavior is considered in, e.g., [14], [28]. The forbidden zones that will be considered in the next section can be treated as some qualitative influence of noise and other sources of dispersion upon the expectations of data.

2. The problematic that motivated this article

2.1. Problems of behavioral sciences

A man as an individual actor is a key subject of economics and some other sciences. There are some basic problems concerned with the mathematical description of the behavior of a man. They are the most actual in behavioral economics, especially in utility and prospect theories, and also in decision theories, the social sciences and psychology, see, e.g., [22].

Examples of the problems are the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, risk premium, the four-fold pattern paradox, etc.

The essence of the problems consists in biases of preferences and decisions of a man in comparison with predictions of the probability theory. These biases are maximal near the boundaries of the probability scale.

2.2. Existence theorem. Applied mathematical method and model

Bounds on the expectations of r.v.s (or bounding inequalities or, in other words, forbidden zones for the expectations of r.v.s) that take on values in a finite interval are considered in, e.g., [19], [21].

Suppose a set $\{X_i\}$, $i = 1, \dots, n$, of random variables X_i whose values lie within an interval $[a, b]$. If $0 < (b\text{-}a) < \infty$ holds for $[a, b]$, and if the condition $\sigma^2_i \geq \sigma^2_{\min} > 0$ of the minimal variance $\sigma^2_{\min}$ holds for their variances σ^2_i, then their expectations μ_i are separated from the boundaries a and b of the interval $[a, b]$ by forbidden zones of non-zero width,

$$a < \left(a + \frac{\sigma^2{}_{\min}}{b-a} \right) \leq \mu_i \leq \left(b - \frac{\sigma^2{}_{\min}}{b-a} \right) < b\,. \qquad (1)$$

That is, the theorem proves the possibility of the existence of forbidden zones of non-zero width for the expectations of measurement data under the condition that the variance of the data is not less than the minimal variance of some non-zero value. Such non-zero minimal variance of measurement data can be caused, e.g., by noise, imprecision, errors, incompleteness, various types of uncertainty. Noise can be one of usual and important causes.

This theorem has led to an applied mathematical method (approach) of the biases of the expectations. The method supposes that the subjects (peoples) make their choices (at least to a considerable degree) as if there were some biases of the expectations for games.

The first stage of the approach is a concept of qualitative mathematical models. The models are concentrated on cases when the signs of the presupposed differences (for the choices of subjects), that are required to obtain the observed data, are not equal to the signs of the real differences between the expectations for the uncertain and sure games.

These general models enable formal solutions of the considered paradoxes, but some problems were aroused. In particular, the limits of their applicability need additional research (see the next subsection).

The first step of realization of the concept is a special qualitative mathematical model (see, e.g., [16]) consigned for the cases when the expectations for the sure and uncertain games are equal to each other. Such a choice allows to avoid the above limits of the applicability, that has led to the fact that the model was successfully and uniformly applied in more than one domain of experimental data.

Nevertheless, the above general problems of the limits of applicability of the models still stand over.

2.3 Need for further research

One of main questions concerning these theorem of existence of forbidden zones and method of the biases of the expectations is to analyze the widths of the forbidden zones for various PDFs. In particular, the width r_μ of the forbidden zones for the expectations can be determined from (1) as

$$r_\mu \equiv \frac{\sigma^2{}_{\min}}{b-a}. \qquad (2)$$

We see that this width can evidently be neglected with respect to the minimal standard deviation $\sigma^2{}_{\min}$ when $\sigma^2{}_{min} \to 0$.

Suppose the level of the noise (or other source of the variance of the data) tends to zero. Hence, the variance of the data also tends to zero. Then the width of the forbidden zones tends to zero as well. One can estimate also the ratio of this width to the standard deviation. This ratio represents, in particular, the decreasing of the width with respect to the decreasing of the standard deviation (when the noise level decreases).

Therefore expression (2) leads to an important consequence. This consequence states: "When (the minimum of) the standard deviation tends to zero then the ratio of the width of the forbidden zones to (the minimum of) the standard deviation tends to zero as well." It means (approximately) that the widths of the forbidden zones decrease much faster than the standard deviation. This means that, at least in some cases, the widths of such forbidden zones can be neglected at low levels of the noise (and of other sources).

So, the practical problem is whether the considered forbidden zones and biases for results of experiments can be neglected at low levels of their causes (such as noise). This leads to the future goals of the present research.

There is an additional problem as well. The above theorem has been proven for finite intervals. However, a number of important PDFs are defined on infinite and half-infinite intervals. So there is a need for a tool for estimations or at least hypotheses and assumptions about these or similar important PDFs. This leads to the main goal of the present article.

3. Auto-transformations. Main definitions

3.1. Auto-transformations as a tool for modifications and hypotheses

The domains of many important probability density functions are infinite. The normal distribution can be mentioned as one of the most important examples. Questions can arise about how such or similar probability density functions could be modified if their domains were half-infinite or finite.

Generally, questions can arise about how probability density functions can be modified when their domains are modified from larger to smaller sizes of the domains. These questions can be actual in particular in connection with possible expansions and generalizations of the results of, e.g., [19] and [21] obtained for finite intervals.

Such questions can be too hard to be solved immediately and exactly. Therefore a tool can be proposed here to modify probability density functions, and to put forward hypotheses and make assumptions about such modified functions. It will modify mainframe probability density functions into transformed ones that will be, depending on parameters of the transformations, similar to the mainframe functions to a greater or lesser degree.

This tool can be named as auto-transformations of probability density functions.

3.2. Main definitions, assumptions, and abbreviations

For the purposes of the present article and research, following definitions, assumptions, and abbreviations are used here.

Random variables are abbreviated to **r.v.s**.

For the sake of simplicity and to not increase excessively the volume of the present first article devoted to this item, only continuous and piecewise continuous probability density functions are considered here as a rule.

The standard deviation is abbreviated to **SD**. Probability density functions are abbreviated to **PDFs**. The Heaviside step function is referred to as $\boldsymbol{\theta(x)}$: $\theta(x|x < 0) = 0$, $\theta(x|x \geq 0) = 1$.

For the conciseness, in the scope of this article, PDFs with bounded or compact supports are referred to as **compact** PDFs. The PDFs with not bounded supports are referred to as **noncompact** ones.

For the purposes of this article, a **finite boundary** is defined as the boundary which coordinate is finite and an **infinite boundary** is defined as the boundary which coordinate is infinite.

Consider the domain of the probability density function of a random variable. Suppose this domain is the infinite or a half-infinite (or a finite) interval.

Further, this interval is referred to as a mainframe interval (**MF-interval** or **MFI**). The corresponding probability density function of this variable is referred to as a mainframe function (**MF-function** or **MF PDF** or **MFF**).

Auto-transformations are abbreviated to **AT**s.

A half-infinite or finite part of the infinite MF-interval and a finite part of a half-infinite (or finite) MF-interval are defined as an **interval of auto-transformation** or an **auto-transformation interval** (**AT-interval** or **ATI**) under three following determining conditions:

1. The AT-interval contains (at least at its boundary) at least one of the main features of the MF PDF such as the expectation, median or mode.
2. The part of the MF-function that is situated in the ATI is identically mapped into the ATI (or, in other words, is unchanged).
3. The part or parts of the MF-function that lie outside the ATI are mapped into ATI. Types or modes of this mapping can be chosen in accordance with the goals and conditions of the mapping.

So the summarized transformation of the MF-function is fully enclosed in the AT-interval and consists of following two parts:

1) the part that is identically mapped (or, in other words, is unchanged);

2) part(s) that is (are) mapped from outside of ATI into inside of ATI (usually they are denoted as **out-ATI part(s)**).

This resultant transformed MF-function is referred to as an auto-transformed function (**AT-function** or **AT PDF** or **ATF**).

The boundaries of mainframe intervals are usually denoted with a subscript "MF" as e.g., a_{MF} or b_{MF}. They can be either infinite or finite.

The boundaries of AT-intervals denoted either without any subscript as e.g., a or b, or with the subscript "AT" as e.g., a_{AT} or b_{AT}.

Finite boundaries of the AT-intervals those are the nearest to the expectation of the MF-function are often marked by a subscript "boundary" and referred to as $a_{boundary}$ and/or $b_{boundary}$. Such left finite boundaries are often determined as $a_{boundary} = 0$. So the minimal distance between the expectation of the variable and the nearest left boundary $a_{boundary}$ of the interval is referred to as $min(|E(X)-a_{boundary}|)$ or simply $|E(X)-a_{boundary}|$.

Finite AT-intervals are usually referred to as $[a, b]$, where $-\infty < a < b < +\infty$ (while a_{MF} and b_{MF} usually can be supposed as infinite).

The medians of MF-functions are referred to as m_X or m.

Normal-like distributions are defined as symmetric probability distributions with non-increasing sides. In other words, for the PDF f we have:

$$f(E(X)+x_c) = f(E(X)-x_d)$$

and if $|x_c - E(X)| \le |x_d - E(X)|$, then $f(x_c) \ge f(x_d)$.

A contiguous situation is defined as the situation when a finite boundary of a MF-interval touches a finite boundary of the AT-interval.

A hypothetical adhesion situation is modified from the hypothetical reflection situation. The reflected part of the MF PDF is "adhered" to the boundary $d_{refl} = a_{boundary} = 0$. The MF PDF is transformed to the function of the mixed type, such that its discrete part is equal to the integral of the reflected part of the MF PDF. In other words, the reflected part of the mainframe PDF is adhered to the point $x = 0$. A statement will be proven that adhesion ATs provide the minimal distance from the boundary of ATIs to the expectation of AT PDFs.

A hypothetical reflection situation is defined as the situation when the probability density function f of a MFF is modified to the hypothetical PDF f_{refl} of an ATF, such that the left part of f is reflected with respect to some point (dot) $d_{refl} \equiv d_{reflection}$ of reflection. Often this point is the median of MF-function $d_{refl} = m_X$. Usually, m_X and d_{refl} are supposed to be $d_{refl} = m_X = 0$ and

$$f_{refl}(x) = \theta(x)[f(x) + f(-x)].$$

The hypothetical reflection situation is, in a sense similar to the reflection of a wave of light from a mirror.

If the auto-transformation interval is finite, especially if the mainframe interval is infinite or at least semi-infinite, then the reflection can be multiple. This can occur also in similar cases but in the absence of reflection. In all these cases the AT can be referred to as the **repeated auto-transformation** or **multiple AT** or **many-fold AT** or **two-mirror AT**. Otherwise (and as a rule in this article), ATs can be referred to as **one-fold AT**s.

Auto-transformations that transform the mainframe PDFs of out-ATI parts into linear parts of auto-transformed PDFs are referred to as **linear AT**s.

Auto-transformations that transform the mainframe PDFs of out-ATI parts into constant parts of auto-transformed PDFs are referred to as **constant auto-transformations** (that can be many-fold).

Auto-transformations that transform the mainframe PDFs of out-ATI parts into parts of auto-transformed PDFs those values are equal at the corresponding boundaries to the values of the MF PDFs, are referred to as **boundary continuous AT**s. When those values are equal to the means of the MF PDFs), then those ATs are referred to as **mean-matched AT**s.

For the purposes of the present research, one can denote a vanishing auto-transformation as the auto-transformation such that the additional part Δf of AT PDF starts at the corresponding boundary of ATI with a non-zero value and ends with the zero value.

Vanishing ATs can be of convex, triangle, concave, and mixed types. Stepwise one-step ATs can be considered in a sense as the upper (unattainable) limit cases for convex vanishing ATs.

In all situations of auto-transformed functions, the standard deviations of the mainframe functions are used.

Usually, $\boldsymbol{\hbar}$ denotes the value (***h***eight) of the probability density function (***h*** will be used in future for the probability mass function) and $\boldsymbol{l}$ denotes the length. Usually, the index “1” denotes the center or top of a function, that is $\boldsymbol{\hbar_1} \equiv \hbar_{centre}$ and $\boldsymbol{l_1} \equiv l_{centre}$. The index “2” denotes the side or tail or bottom of a function, that is $\boldsymbol{\hbar_2} \equiv \hbar_{side} \equiv \hbar_{tail}$ and $\boldsymbol{l_2} \equiv l_{side} \equiv l_{tail}$.

Auto-transformations can be used as a basis for approximation, modelling etc. Approaches and first approximation models that use auto-transformations can be referred to as AT-approaches and AT-models.

4. Auto-transformations. General considerations

4.1. Modifications of out-ATI parts

Let us denote the norms of f on the out-ATI parts as

$$\delta_{out.left} \equiv \int_{a_{MF}}^{a_{AT}} f(x)dx \equiv \int_{a_{MF}}^{a} f(x)dx$$

and

$$\delta_{out.right} \equiv \int_{b_{AT}}^{b_{MF}} f(x)dx \equiv \int_{b}^{b_{MF}} f(x)dx$$

and denote

$$\delta_{out} \equiv \delta_{out.left} + \delta_{out.right} .$$

Let us consider some possible cases of the modifications of the out-ATI parts of the MF-functions into the AT-functions.

4.1.1. Non-increasing auto-transformations

Auto-transformations that transform the MF PDFs of out-ATI parts into parts of AT functions (that are summarized with the MF PDFs of AT interval to constitute the resulting AT PDFs) that do not increase in the direction from the corresponding boundary to the middle of AT-intervals are referred to as n**on-increasing auto-transformations**.

Non-increasing auto-transformations correspond to intuitive condition that the auto-transformed out-ATI part should contribute near the boundary that is the nearest to this out-ATI part, at least, not less than near the opposite boundary. Constant ATs belong evidently to non-increasing ATs. Non-increasing ATs are the main type of ATs considered here.

4.1.2. Full-ATI auto-transformations. Uniform auto-transformations

Auto-transformations that transform MF PDFs of out-ATI parts into the total AT-interval are referred to as **full-ATI auto-transformations**.

Suppose that out-ATI parts are mapped into the ATI, e.g., by means of some uniform rising coefficient or of some uniform addition part.

Uniform multiplication. The MF PDF that is situated in the AT-interval is multiplied by the uniform rising coefficient

$$f_{AT}(x \mid x \in [a,b]) = \frac{f_{MF}(x \mid x \in [a,b])}{1-\delta_{out}} .$$

Uniform addition. The out-ATI parts are uniformly added to the AT-interval part of the MF-function and the transformed PDF is

$$f_{AT}(x \mid x \in [a,b]) = f_{MF}(x \mid x \in [a,b]) + \frac{\delta_{out}}{b-a} .$$

4.2 About the distances from the expectations to the boundaries

A simple but necessary proof that the distances from the expectations of AT-functions to the nearest boundaries of AT-intervals are minimal for the adhesion auto-transformations will be given in next articles.

For the reflected auto-transformations it will be proven that the distances from the expectations of AT-functions to the nearest boundaries of AT-intervals are minimal when the points (dots) of reflection coincide with the medians of the MF-functions.

4.3 Necessary auto-transformations

An auto-transformation of a PDF is referred to as the necessary AT or Norm-necessary AT or N-necessary AT if

$$\int_a^b f(x)dx \equiv \int_{a_{AT}}^{b_{AT}} f(x)dx \geq \frac{1}{2}\int_{a_{MF}}^{b_{MF}} f(x)dx = \frac{1}{2},$$

where a_{MF} and b_{MF} can be supposed as both finite or infinite; or, equivalently,

$$\int_a^b f(x)dx \equiv \int_{a_{MF}}^{a_{AT}} f(x)dx + \int_{b_{AT}}^{b_{MF}} f(x)dx \leq \frac{1}{2}.$$

That is the norm of the unchanged part of the ATF is not less than 1/2. That is the difference between the norms calculated for the MFF and the unchanged part of the ATF is not more than 1/2.

4.4 Sufficient auto-transformations

An auto-transformation of a PDF is referred to as the sufficient auto-transformation or Norm-sufficient AT or N-sufficient AT if

$$\int_{a_{MF}}^{a} f(x)dx + \int_{b}^{b_{MF}} f(x)dx << \int_{a_{MF}}^{b_{MF}} f(x)dx = 1,$$

that is the difference between the norms calculated for the MFF and ATF is negligibly small in comparison with the norm calculated for the MFF. That is the difference is much less than unit.

Sufficient ATs evidently belong to necessary ATs.

For the normal distribution, the auto-transformation interval that corresponds to the “three-sigma rule” can be used as a sufficient AT-interval.

5. Conclusions

The present article proposes a new tool to modify or to transform the probability density functions of random variables into similar functions having smaller sizes of their domains. This tool can be treated as transformations of distributions of r.v.s or simply as transformations of r.v.s.

The tool is intended to put forward hypotheses and make assumptions about such modified PDFs. The tool is named as the auto-transformations (ATs) of PDFs. It can be treated as as a first step to general ATs of real-valued random variables.

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