# Conditional moments of noncausal alpha-stable processes and the prediction of bubble crash odds

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**Abstract**

Noncausal, or anticipative, $\alpha$-stable processes generate trajectories featuring locally explosive episodes akin to speculative bubbles in financial time series data. For $(X_t)$ a two-sided infinite $\alpha$-stable moving average (MA), conditional moments up to integer order four are shown to exist provided $(X_t)$ is anticipative enough. The functional forms of these moments at any forecast horizon under any admissible parameterisation are obtained by adding to the literature on arbitrary, not necessarily symmetric bivariate $\alpha$-stable random vectors the functional forms of the third and fourth order conditional moments, as well as the second order moment in the case $\alpha = 1$ with skewed spectral measure. The dynamics of noncausal processes simplifies during explosive episodes and allows to express ex ante crash odds at any horizon in terms of the MA coefficients and of the tail index $\alpha$. The results are illustrated in a synthetic portfolio allocation framework and an application to the Nasdaq and S&P500 series is provided.

*Keywords:* Noncausal processes, Multivariate stable distributions, Conditional dependence, Extremal dependence Explosive bubbles, Prediction, Crash odds, Portfolio allocation

*MSC classes:* 60G52, 60E07, 60G25

## 1 Introduction

Dynamic models often admit solution processes for which the current value of the variable is a function of future values of an independent error process. Such solutions, called *anticipative* or *noncausal*, have attracted increasing attention in the financial and econometric literatures. In particular, noncausal processes have been found convenient for modelling locally explosive phenomena in financial time series such as speculative bubbles [Bec et al. (2019), Cavaliere et al. (2017), Fries and Zakoian (2019), Gouriéroux and Zakoian (2017), Hecq and Sun (2019), Hecq et al. (2016), Hecq et al. (2017a), Hecq et al. (2017b), Hencic and Gouriéroux (2015)] (see also Andrews et al. (2009), Chen et al. (2017), Gouriéroux et al. (2016), Lanne et al. (2012b),

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Lanne and Saikkonen (2011), Lanne and Saikkonen (2013)). Noncausal time series models may offer a possibility to forecast the future trajectories of bubbles and to infer the odds of crashes at future horizons, enabling for instance portfolio managers to build exit strategies, risk managers to accurately assess large downside risks during prolonged bull markets, and the regulator to adjust requirements and restrictions in order to ensure resilience of the financial system. However, lack of knowledge about the predictive distribution of noncausal processes is impeding the ability to forecast them, thus limiting their use in practical applications. Numerical procedures have been proposed to empirically approximate the conditional distribution of noncausal processes [Gouriéroux and Jasiak (2016), Lanne et al. (2012a)]. These however become computationally unaffordable beyond the simpler noncausal models and one or two-step ahead prediction horizons, face accuracy limitations when it comes to capturing the dynamics during extreme events [Gouriéroux et al. (2019), Voisin and Hecq (2019)], and provide limited theoretical guarantees regarding the quality of the approximation. Partial results have been obtained by Gouriéroux and Zakoian (2017) for the noncausal autoregression of order 1 (AR(1)) driven by independent and identically distributed (i.i.d.) stable errors. This process is defined as the stationary solution of

$$X_t = \rho X_{t+1} + \varepsilon_t, \qquad \varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, \beta, \sigma, 0), \tag{1.1}$$

where $0 < |\rho| < 1$, and $\mathcal{S}(\alpha, \beta, \sigma, 0)$ denotes the univariate $\alpha$-stable distribution with tail parameter $\alpha \in (0, 2)$, asymmetry $\beta \in [-1, 1]$ and scale $\sigma > 0$. Figure 1 depicts a typical simulated path of a noncausal stable AR(1) featuring multiple bubbles. Despite being an infinite variance process, condi-

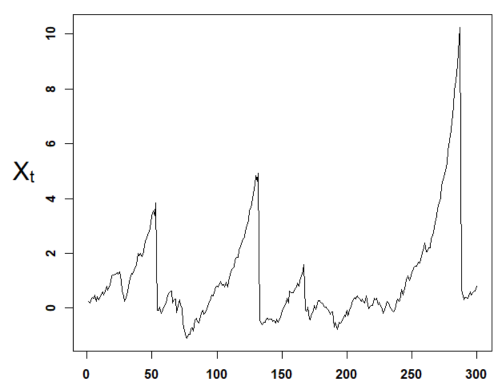


Figure 1: Sample path of the solution of (1.1) with $\varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(1.7, 0.8, 0.1, 0)$ and $\rho = 0.95$.

tional moments of $X_{t+h}$ given $X_t$ can be shown to exist up to integer order four for any horizon $h$, and Gouriéroux and Zakoian (2017) obtained expressions of the conditional expectation and variance in special cases - symmetric stable errors ($\beta = 0$) and Cauchy errors ($\alpha = 1$, $\beta = 0$) respectively. Provided the expressions of the conditional moments are derived, this suggests that point forecasts of noncausal processes based on their conditional expectation, variance, skewness and kurtosis could be formulated -as opposed to other predictors specifically introduced to circumvent the infinite variance of $\alpha$-stable processes, such as minimum $L^\alpha$-dispersion or maximum covariation (Karcher et al. (2013) and the references therein). This paper extends and exploits the literature on the conditional moments of arbitrary bivariate $\alpha$-stable random vectors [Cioczek-Georges and Taqqu (1995a), Cioczek-Georges and Taqqu (1995b), Cioczek-Georges and Taqqu (1998), Hardin et al. (1991), Samorodnitsky and Taqqu (1994) (ST94 hereafter)] to propose a complete characterisation of the first four moments of $X_{t+h}|X_t$, for $(X_t)$ an infinite *two-sided* moving average process driven by $\alpha$-stable errors

$$X_t = \sum_{k\in\mathbb{Z}} a_k \varepsilon_{t+k}, \qquad\qquad \varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(\alpha,\beta,\sigma,0), \tag{1.2}$$

where $(a_k)$ is a non-random coefficients sequence satisfying mild conditions for $(X_t)$ to be well defined. AR and ARMA models -whether causal, noncausal, invertible or non-invertible- are encompassed as a special case of our framework. While the causality or noncausality of the process is not presumed beforehand,[1] it is surprisingly found that noncausality is crucial for the existence of conditional moments higher than order $\alpha$. The functional forms of the conditional moments are derived, and we furthermore show that the characterisation non-trivially extends to aggregated stable processes defined as linear combinations of processes of the form (1.2), which were suggested by Gouriéroux and Zakoian (2017) to allow for bubbles with a variety of growth rates to appear on a single trajectory. We show that the conditional distribution of $X_{t+h}$ given $X_t = x$ displays dramatic simplifications when $x \to \pm\infty$, providing illuminating interpretations on the behaviour of noncausal processes during explosive episodes and allowing to quantify the crash odds of bubble models.

Section 2 starts by recalling characterisations and properties of multivariate stable distributions, and provides our results on the conditional moments up to order four of arbitrary bivariate $\alpha$-stable vectors. Section 3 proposes a sufficient condition on the coefficients $(a_k)$ for the existence of conditional moments, characterises their functional forms when they exist, and derives their asymptotic behaviour and the collapse odds of explosive episodes. Our results suggest that bubbles of the AR(1) feature a non-aging, or memory-less, property. We illustrate through an example how our results extend to continuous time processes. Section 4 provides the extension to aggregated stable processes. Section 5 provides an illustration of our results in a synthetic portfolio selection framework where investors optimise on the quantities of a speculative asset as well as on the holding horizon, and proposes an application of the crash odds evaluation on the Nasdaq and S&P500 series. Proofs and complementary results are collected in a Supplementary file.

[1] A moving average process (1.2) is said to be purely causal if $a_k = 0$ for $k > 0$ and purely noncausal if $a_k = 0$ for $k < 0$.

## 2 Conditional moments of bivariate $\alpha$-stable vectors

We begin by recalling some characterisations of multivariate stable distributions and then propose new functional forms of higher-order conditional moments in the bivariate case. Letting $\alpha \in (0, 2)$, a random vector $\boldsymbol{X} = (X_1, \dots, X_d)$ is said to be an $\alpha$-stable random vector in $\mathbb{R}^d$ (see Theorem 2.3.1 in ST94) if there exists a unique pair $(\Gamma, \boldsymbol{\mu}^0)$, where $\Gamma$ is a finite measure on the Euclidean unit sphere $S_d$ and $\boldsymbol{\mu}^0$ a vector in $\mathbb{R}^d$, such that, for any $\boldsymbol{u} \in \mathbb{R}^d$, the characteristic function of $\boldsymbol{X}$ writes

$$\mathbb{E}\Big[e^{i\langle \boldsymbol{u}, \boldsymbol{X}\rangle}\Big] = \exp\Big\{ - \int_{S_d} |\langle \boldsymbol{u}, \boldsymbol{s}\rangle|^{\alpha}\Big(1 - i\,\mathrm{sign}(\langle \boldsymbol{u}, \boldsymbol{s}\rangle) w(\alpha, \langle \boldsymbol{u}, \boldsymbol{s}\rangle)\Big)\Gamma(d\boldsymbol{s}) + i\,\langle \boldsymbol{u}, \boldsymbol{\mu}^0\rangle \Big\}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is the canonical scalar product, $w(\alpha, s) = \mathrm{tg}\left(\frac{\pi\alpha}{2}\right)$, if $\alpha \neq 1$, and $w(1, s) = -\frac{2}{\pi} \ln |s|$ otherwise, for $s \in \mathbb{R}$. The measure $\Gamma$ and the vector $\boldsymbol{\mu}^0$ are respectively called the *spectral measure* and the *shift vector* of $\boldsymbol{X}$. The pair $(\Gamma, \boldsymbol{\mu}^0)$ is said to be the *spectral representation* of $\boldsymbol{X}$. In the univariate case, (2.1) boils down to $\mathbb{E}[e^{iuX}] = \exp\Big\{ - \sigma^{\alpha}|u|^{\alpha}\Big(1 - i\beta\,\mathrm{sign}(u) w(\alpha, u)\Big) + iu\mu\Big\}$, for some $\sigma > 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$. Stable distributions are known to have very little moments. However, the distribution of one component conditionally on the others can have more moments according to the degree of dependence between them. In the bivariate case, if $\boldsymbol{X} = (X_1, X_2)$ is an $\alpha$-stable random vector with spectral measure $\Gamma$, satisfying

$$\int_{S_2} |s_1|^{-\nu}\Gamma(ds) < +\infty, \quad \text{for some} \quad \nu \geq 0, \tag{2.2}$$

then, $\mathbb{E}\big[|X_2|^{\gamma}|X_1 = x\big] < +\infty$ for almost every $x$ if $0 \leq \gamma < \min(\alpha + \nu, 2\alpha + 1) < 5$ (see Theorem 5.1.3 in ST94 for details).

We give formulae for the conditional moments up to order four of arbitrary (not necessarily symmetric) $\alpha$-stable bivariate vectors $(X_1, X_2)$, that is, up to the maximum admissible integer order under the most favourable dispositions of the above sufficient condition for the existence of the conditional moments. The conditional moments of bivariate $\alpha$-stable vectors were studied in a series of papers in the 90s [Cioczek-Georges and Taqqu (1994), Cioszek-Georges and Taqqu (1995a,b), Cioczek-Georges and Taqqu (1998), Hardin et al. (1991), Samorodnitsky and Taqqu (1991), ST94, Wu and Cambanis (1991)] (see also Cambanis and Fotopoulos (1995), Cambanis et al. (1992), Fotopoulos (1998), Miller (1978)) but only the functional forms of the first and second order moments received attention in the literature. The conditional expectation of arbitrary $\alpha$-stable bivariate vectors is the most comprehensively understood (see for instance Hardin et al. (1991), Samorodnitsky and Taqqu (1991)). The conditional variance was also studied but most exclusively in the Symmetric $\alpha$-Stable (S$\alpha$S) case (see Cambanis and Fotopoulos (1995), Fotopoulos (1998), Wu and Cambanis (1991)). One notable exception is Theorem 3.1 in Cioczek-Georges and Taqqu (1995a) which states without proof a functional form of the conditional variance for an arbitrary, skewed bivariate $\alpha$-stable vector for $\alpha \neq 1$. We therefore provide a proof for the second moment as well and fill the gap for $\alpha = 1$. In the rest of this section, we assume without loss of generality that the shift vector $\boldsymbol{\mu}^0 = (\mu_1^0, \mu_2^0)$ is zero.[2] We first state our results in the case $\alpha \neq 1$ and include the conditional expectation provided in Theorem 5.2.2 by Samorodnistky and Taqqu for comprehensiveness.

**Theorem 2.1** *Let $(X_1, X_2)$ be an $\alpha$-stable random vector with spectral representation $(\Gamma, \mathbf{0})$.*

*For $\alpha \in (0,2) \setminus \{1\}$, and letting $\Gamma$ satisfy (2.2) with $\nu > 1-\alpha$ if $\alpha \in (0,1)$,*

$$\mathbb{E}\Big[X_2 \Big| X_1 = x\Big] = \kappa_1 x + \frac{a(\lambda_1 - \beta_1\kappa_1)}{1 + a^2\beta_1^2}\left[a\beta_1 x + \frac{1 - xH(x)}{\pi f_{X_1}(x)}\right]. \tag{2.3}$$

*For $\alpha \in (1/2,2) \setminus \{1\}$ and $\Gamma$ satisfying (2.2) with $\nu > 2-\alpha$,*

$$\begin{aligned}\mathbb{E}\Big[X_2^2 \Big| X_1 = x\Big] = \kappa_2 x^2 + \frac{ax(\lambda_2 - \beta_1\kappa_2)}{1 + (a\beta_1)^2}&\left[a\beta_1 x + \frac{1 - xH(x)}{\pi f_{X_1}(x)}\right] \\ &- \frac{\alpha^2\sigma_1^{2\alpha}}{\pi f_{X_1}(x)}\mathcal{H}\Big(2, \boldsymbol{\theta}_1; x\Big).\end{aligned} \tag{2.4}$$

*For $\alpha \in (1,2)$ and $\Gamma$ satisfying (2.2) with $\nu > 3-\alpha$,*

$$\begin{aligned}\mathbb{E}\Big[X_2^3 \Big| X_1 = x\Big] = \kappa_3 x^3 + \frac{ax^2(\lambda_3 - \beta_1\kappa_3)}{1 + (a\beta_1)^2}&\left[a\beta_1 x + \frac{1 - xH(x)}{\pi f_{X_1}(x)}\right] \\ &- \frac{\alpha^2\sigma_1^{2\alpha}}{2\pi f_{X_1}(x)}\left[x\mathcal{H}\Big(2, \boldsymbol{\theta}_2; x\Big) + \alpha\sigma_1^\alpha\mathcal{H}\Big(3, \boldsymbol{\theta}_3; x\Big)\right].\end{aligned} \tag{2.5}$$

*For $\alpha \in (3/2,2)$ and $\Gamma$ satisfying (2.2) with $\nu > 4-\alpha$,*

$$\begin{aligned}\mathbb{E}\Big[X_2^4 \Big| X_1 = x\Big] = \kappa_4 x^4 + \frac{ax^3(\lambda_4 - \beta_1\kappa_4)}{1 + (a\beta_1)^2}&\left[a\beta_1 x + \frac{1 - xH(x)}{\pi f_{X_1}(x)}\right] \\ &- \frac{\alpha^2\sigma_1^{2\alpha}}{\pi f_{X_1}(x)}\left[\frac{x^2}{2}\mathcal{H}\Big(2, \boldsymbol{\theta}_4; x\Big) + \frac{\alpha x\sigma_1^\alpha}{6}\mathcal{H}\Big(3, \boldsymbol{\theta}_5; x\Big) + \frac{\alpha^2\sigma_1^{2\alpha}}{3}\mathcal{H}\Big(4, \boldsymbol{\theta}_6; x\Big)\right].\end{aligned} \tag{2.6}$$

*Here, $a = tg(\pi\alpha/2)$, and for $p \in \{1,2,3,4\}$, when they exist,*

$$\begin{aligned}\sigma_1^\alpha &= \int_{S_2} |s_1|^\alpha \Gamma(d\boldsymbol{s}), & \beta_1 &= \frac{\int_{S_2} s_1^{<\alpha>}\Gamma(d\boldsymbol{s})}{\sigma_1^\alpha}, \\ \kappa_p &= \frac{\int_{S_2}(s_2/s_1)^p |s_1|^\alpha \Gamma(d\boldsymbol{s})}{\sigma_1^\alpha}, & \lambda_p &= \frac{\int_{S_2}(s_2/s_1)^p s_1^{<\alpha>}\Gamma(d\boldsymbol{s})}{\sigma_1^\alpha},\end{aligned} \tag{2.7}$$

*where $y^{<r>} = sign(y)|y|^r$ for any $y, r \in \mathbb{R}$. For any $n \in \mathbb{N}$, $\boldsymbol{\theta}_i = (\theta_{i1}, \theta_{i2}) \in \mathbb{R}^2$, $x \in \mathbb{R}$, $\mathcal{H}$ is defined by*

$$\mathcal{H}(n, \boldsymbol{\theta}_i; x) = \int_0^{+\infty} e^{-\sigma_1^\alpha u^\alpha} u^{n(\alpha-1)}\Big(\theta_{i1}\cos(ux - a\beta_1\sigma_1^\alpha u^\alpha) + \theta_{i2}\sin(ux - a\beta_1\sigma_1^\alpha u^\alpha)\Big)du, \tag{2.8}$$

[2] This can be done without loss of generality because, assuming the conditional moment of order $p$ exists, $\mathbb{E}\big[X_2^p \big| X_1 = x\big] = \mathbb{E}\big[(X_2 - \mu_2^0 + \mu_2^0)^p \big| X_1 - \mu_1^0 = x - \mu_1^0\big] = \sum_{j=0}^p \mathrm{C}_p^j (\mu_2^0)^{p-j}\mathbb{E}\big[\tilde{X}_2^j \big| \tilde{X}_1 = \tilde{x}\big]$ where $\tilde{x} = x - \mu_1^0$, and $(\tilde{X}_1, \tilde{X}_2) = (X_1 - \mu_1^0, X_2 - \mu_2^0)$ has the same spectral measure as $(X_1, X_2)$ and zero shift parameter.

*and we denote* $H(\cdot) := \mathcal{H}(0,(0,1);\cdot)$, *and* $f_{X_1}(\cdot) := \frac{1}{\pi}\mathcal{H}(0,(1,0);\cdot)$.[3] *Finally,* $\boldsymbol{\theta}_1 = (\theta_{11}, \theta_{12})$ *in* (2.4) *is given by*

$$\theta_{11} = \kappa_1^2 - a^2\lambda_1^2 + a^2\beta_1\lambda_2 - \kappa_2, \qquad \theta_{12} = a(\lambda_2 + \beta_1\kappa_2) - 2a\lambda_1\kappa_1, \tag{2.9}$$

*and the remaining* $\boldsymbol{\theta}_i$*'s in* (2.5)-(2.6)*, which depend only on* $\alpha$, $\beta_1$*, and the* $\kappa_p$*'s and* $\lambda_p$*'s above, are given in* (D.1)-(D.10) *in the Supplementary file. If* $\alpha < 1$ *and* $\beta_1 = 1$ *(resp.* $\beta_1 = -1$*), Relations* (2.3) *and* (2.4) *are well defined only for* $x \geq 0$ *(resp.* $x \leq 0$*).*

We now give the formulae for the second conditional moment when $\alpha = 1$.[4] As for the conditional expectation when $(X_1, X_2)$ is *not S1S*, two different results hold according to whether the marginal distribution of $X_1$ is skewed or symmetric.

**Theorem 2.2** *Let* $(X_1, X_2)$ *be* $\alpha$*-stable, with* $\alpha = 1$ *and spectral representation* $(\Gamma, \mathbf{0})$*, where* $\Gamma$ *satisfies* (2.2) *with* $\nu > 1$*. Then, for almost every* $x$,

$$\begin{aligned}
\mathbb{E}\Big[X_2^2\Big|X_1 = x\Big] &= \sigma_1^2(a^2q_0^2 - \kappa_1^2) + \frac{2\sigma_1\lambda_1}{\beta_1}\Big(\sigma_1\kappa_1 - aq_0(x-\mu_1)\Big) + \frac{\lambda_2}{\beta_1}\Big((x-\mu_1)^2 - \sigma_1^2\Big) \\
&\quad + \Big(a\sigma_1 q_0(\lambda_1 - \beta_1\kappa_1) + (\kappa_1\lambda_1 - \lambda_2)(x-\mu_1)\Big)\frac{2\sigma_1 U(x)}{\beta_1\pi f_{X_1}(x)} \\
&\quad + \Big(\lambda_2 + \beta_1\kappa_2 - 2\kappa_1\lambda_1 + a^2\sigma_1\beta_1(\lambda_1^2 - \beta_1\lambda_2)W(x)\Big)\frac{\sigma_1}{\beta_1\pi f_{X_1}(x)},
\end{aligned}$$

*if* $\beta_1 \neq 0$*, and*

$$\begin{aligned}
\mathbb{E}\Big[X_2^2\Big|X_1 = x\Big] &= \sigma_1^2(\kappa_2 + a^2q_0^2 - \kappa_1^2) - 2a\sigma_1\kappa_1 q_0(x-\mu_1) + \kappa_2(x-\mu_1)^2 \\
&\quad + a\sigma_1(\lambda_2 - 2\lambda_1\kappa_1)\frac{F_{X_1}(x) - 1/2}{f_{X_1}(x)} + \frac{a\sigma_1\lambda_1}{\pi f_{X_1}(x)}\Big[2\Big(a\sigma_1 q_0 - \kappa_1(x-\mu_1)\Big)V(x) + a\sigma_1\lambda_1 W(x)\Big],
\end{aligned}$$

*if* $\beta_1 = 0$*. Here,* $a = 2/\pi$, $\sigma_1$, $\beta_1$*, the* $\kappa_p$*'s and the* $\lambda_p$*'s are as in* (2.7)*, and*

$$\begin{aligned}
U(x) &= \int_0^{+\infty} e^{-\sigma_1 t}\sin\Big(t(x-\mu_1) + a\sigma_1\beta_1 t\ln t\Big)dt, \\
V(x) &= \int_0^{+\infty} e^{-\sigma_1 t}(1+\ln t)\cos\Big(t(x-\mu_1) + a\sigma_1\beta_1 t\ln t\Big)dt, \\
W(x) &= \int_0^{+\infty} e^{-\sigma_1 t}(1+\ln t)^2\cos\Big(t(x-\mu_1) + a\sigma_1\beta_1 t\ln t\Big)dt,
\end{aligned}$$

$$q_0 = \frac{1}{\sigma_1}\int_{S_2} s_2\ln|s_1|\Gamma(d\boldsymbol{s}), \qquad \mu_1 = -a\int_{S_2} s_1\ln|s_1|\Gamma(d\boldsymbol{s}).$$

The previous expressions of the conditional moments simplify when one considers the asymptotics with respect to the conditioning variable, as $X_1 = x$ becomes large.

[3] Notice that $f_{X_1}$ is the density of $X_1 \sim \mathcal{S}(\alpha, \beta_1, \sigma_1, 0)$ when $\alpha \neq 1$.

[4] See Theorem 5.2.3 in ST94 for the functional form of the conditional expectation in the case $\alpha = 1$.

**Proposition 2.1** *Let $p \in \{1,2,3,4\}$ and let $(X_1, X_2)$ be $\alpha$-stable with $\alpha \in (0,2)$, and spectral representation $(\Gamma, \mathbf{0})$ such that the conditional moment of order $p$ exists. If $|\beta_1| \neq 1$, then*

$$x^{-p}\,\mathbb{E}\Big[X_2^p\Big|X_1 = x\Big] \underset{x\to+\infty}{\longrightarrow} \frac{\kappa_p + \lambda_p}{1+\beta_1}, \qquad\qquad x^{-p}\,\mathbb{E}\Big[X_2^p\Big|X_1 = x\Big] \underset{x\to-\infty}{\longrightarrow} \frac{\kappa_p - \lambda_p}{1-\beta_1},$$

*and if $|\beta_1| = 1$ and $\beta_1 x \to +\infty$, then,*

$$x^{-p}\,\mathbb{E}\Big[X_2^p\Big|X_1 = x\Big]\longrightarrow\kappa_p.$$

**Remark 2.1** When $|\beta_1| \neq 1$, both the left and right tail of the density of $X_1$ display power law decay as $O(|x|^{-\alpha-1})$. However, when $\beta_1 = -1$ for instance, the distribution of $X_1$ is said to be *totally skewed to the left*: the left tail still decays as $O(|x|^{-\alpha-1})$, but the right tail decays much faster and another asymptotics holds (see Theorem 5.2.2 in Zolotarev (1986) for details).

# 3 Conditional moments of noncausal $\alpha$-stable processes

Operating the arsenal of properties of multivariate $\alpha$-stable distributions we provide in the previous section, we study the existence and functional forms of the conditional moments of noncausal $\alpha$-stable infinite moving average processes, before focusing on the dynamics during extreme events and discussing the implications for the prediction of bubble crash odds. An example at the end of the section illustrates how the results extend to continuous time.

## 3.1 Existence and functional forms of conditional moments

Let us consider $(X_t)$ a *two-sided* MA($\infty$) process as in (1.2) with coefficients $(a_k)$ satisfying

$$\sum_{k\in\mathbb{Z}} |a_k|^s < +\infty, \text{ for some } s \in (0,\alpha)\cap[0,1], \tag{3.1}$$

$$\text{and in addition for } \alpha = 1, \beta \neq 0, \quad \sum_{k\in\mathbb{Z}} |a_k|\Big|\ln|a_k|\Big| < +\infty. \tag{3.2}$$

Conditions (3.1)-(3.2) ensure that $\sum_{k\in\mathbb{Z}} a_k\varepsilon_{t+k}$ converges absolutely almost surely so that $(X_t)$ is well defined. Because the error sequence $(\varepsilon_t)$ is $\alpha$-stable distributed, the bivariate vector $(X_t, X_{t+h})$, for any horizon $h$, is itself $\alpha$-stable and the results from the previous section apply. This is a consequence of the following lemma, which provides the spectral representation of more general, discrete time vectors of linear moving averages driven by $\alpha$-stable i.i.d. errors.

**Lemma 3.1** *Let $0 < \alpha < 2$. For $\varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, \beta, \sigma, \mu)$ and real deterministic sequences $(a_{k,i})_k$, $i = 1, \ldots, m$, $m \geq 2$, each satisfying (3.1)-(3.2), let $\boldsymbol{X_t} = (X_{1,t}, \ldots, X_{m,t})$, with $X_{i,t} = \sum_{k\in\mathbb{Z}} a_{k,i}\varepsilon_{t+k}$, and* *denote* $\boldsymbol{a}_k = (a_{k,1}, \ldots, a_{k,m})$ *for* $k \in \mathbb{Z}$. *Then,* $\boldsymbol{X_t}$ *is an* $\alpha$*-stable random vector in* $\mathbb{R}^m$*, with spectral representation* $(\Gamma, \boldsymbol{\mu}^0)$ *given by*

$$\Gamma = \sigma^\alpha \sum_{s=\pm 1} \sum_{k\in\mathbb{Z}} \frac{1+s\beta}{2} \|\boldsymbol{a}_k\|^\alpha \delta_{\left\{\frac{s\boldsymbol{a}_k}{\|\boldsymbol{a}_k\|}\right\}}, \qquad \boldsymbol{\mu}^0 = \sum_{k\in\mathbb{Z}} \boldsymbol{a}_k \mu - \mathbb{1}_{\{\alpha=1\}} \frac{2}{\pi}\sigma\beta \sum_{k\in\mathbb{Z}} \boldsymbol{a}_k \ln \|\boldsymbol{a}_k\|, \tag{3.3}$$

*where* $\delta_{\{\boldsymbol{x}\}}$ *is the Dirac measure at point* $\boldsymbol{x} \in \mathbb{R}^m$*,* $\|\cdot\|$ *stands for the Euclidean norm, and by convention, if for some* $k \in \mathbb{Z}$*,* $\boldsymbol{a}_k = \boldsymbol{0}$*, i.e.* $\|\boldsymbol{a}_k\| = 0$*, then the* $k^{th}$ *term vanishes from the sums.*

The results on bivariate stable vectors thus immediately apply to $\boldsymbol{X}_t = (X_t, X_{t+h})$ with $\boldsymbol{a}_k = (a_k, a_{k-h})$. A sufficient condition for the existence of conditional moments is given in the following proposition as well as their functional forms.

**Proposition 3.1** *Let* $(X_t)$ *be an* $\alpha$*-stable two-sided MA(*$\infty$*) process,* $0 < \alpha < 2$*,* $\beta \in [-1, 1]$*,* $\sigma > 0$*, satisfying* (1.2)*,* (3.1)*-*(3.2) *and let* $h \geq 1$*.*

$\iota$*) Assume there is* $\nu > 0$ *such that*

$$\sum_{k\in\mathbb{Z}} \left(a_k^2 + a_{k-h}^2\right)^{\frac{\alpha+\nu}{2}} |a_k|^{-\nu} < \infty. \tag{3.4}$$

*Then* $\mathbb{E}[|X_{t+h}|^\gamma | X_t] < \infty$ *for* $0 \leq \gamma < \min(\alpha + \nu, 2\alpha + 1)$*.*

$\iota\iota$*) For* $\alpha \neq 1$*, the moments* $\mathbb{E}[X_{t+h}^p | X_t]$*,* $p = 1, 2, 3, 4$*, when they exist, are given by Theorem 2.1 with*

$$\sigma_1^\alpha = \sigma^\alpha \sum_{k\in\mathbb{Z}} |a_k|^\alpha, \quad \beta_1 = \beta \frac{\sum_{k\in\mathbb{Z}} a_k^{<\alpha>}}{\sum_{k\in\mathbb{Z}} |a_k|^\alpha}, \quad \kappa_p = \frac{\sum_{k\in\mathbb{Z}} |a_k|^\alpha \left(\frac{a_{k-h}}{a_k}\right)^p}{\sum_{k\in\mathbb{Z}} |a_k|^\alpha}, \quad \lambda_p = \beta \frac{\sum_{k\in\mathbb{Z}} a_k^{<\alpha>} \left(\frac{a_{k-h}}{a_k}\right)^p}{\sum_{k\in\mathbb{Z}} |a_k|^\alpha}.$$

$\iota\iota\iota$*) For* $\alpha = 1$*, let* $(\tilde{X}_t, \tilde{X}_{t+h}) := (X_t, X_{t+h}) - \boldsymbol{\mu}^0$ *where* $\boldsymbol{\mu}^0$ *is the shift vector as in Lemma 3.1. Then, the second order moment of* $\tilde{X}_{t+h} | \tilde{X}_t$ *is given in Theorem 2.2 with the* $\kappa_p$*'s,* $\lambda_p$*'s,* $\sigma_1$*,* $\beta_1$ *as in* $\iota\iota$*) and*

$$q_0 = \beta \sum_{k\in\mathbb{Z}} a_{k-h} \ln\left(\frac{|a_k|}{a_k^2 + a_{k-h}^2}\right) \Big/ \sum_{k\in\mathbb{Z}} |a_k|, \qquad \mu_1 = -\frac{2\sigma\beta}{\pi} \sum_{k\in\mathbb{Z}} a_k \ln\left(\frac{|a_k|}{a_k^2 + a_{k-h}^2}\right).$$

*By convention, in all the points above, if* $(a_k, a_{k-h}) = (0, 0)$*, then the* $k^{th}$ *term vanishes from the sums.*

Note that the left-hand side of (3.4) is an increasing function of $\nu$. Thus, if (3.4) holds for some $\nu_0 > 0$, it then holds for any $0 \leq \nu \leq \nu_0$, and if it fails for $\nu_0$, it then fails for all $\nu \geq \nu_0$. Causal processes, say of the form $\sum_{k\leq 0} a_k \varepsilon_{t+k}$ with $a_0 = 1$, automatically fail condition (3.4) for all $\nu > 0$, as $(a_h, a_0) = (0, 1)$ and the $h^{\text{th}}$ term of the sum is finite only if $\nu = 0$.[5] Conversely, (3.4) may hold for some $\nu > 0$ for noncausal processes provided the coefficients $(a_k)$ do not decay too fast as $k \to +\infty$. In fact, the slower

[5] In the case of symmetric errors ($\beta = 0$), Theorem 1.1 by Cioczek-Georges and Taqqu (1995b) allows to conclude that causal processes hence do not have finite conditional moments for orders higher than $\alpha$.

the decay of $(a_k)$ as $k \to +\infty$, the higher the values of $\nu$ for which (3.4) will hold. It is easy to show that (3.4) holds for any $\nu \geq 0$ as soon as $(a_k)$ decays geometrically or hyperbolically, guaranteeing the existence of conditional moments up to order $2\alpha+1$ at all prediction horizons for most noncausal ARMA and fractionally integrated processes.[6]

From a computational perspective, the conditional moments of $X_{t+h}$ given $X_t = x$ can be inexpensively calculated for various horizons $h$ and conditioning values $x$. Indeed, the functions $\mathcal{H}(n, \boldsymbol{\theta}; x)$, $n = 2, 3, 4$, appearing in Theorem 2.1 can be decomposed into $a_h u_n(x) + b_h v_n(x)$, where $a_h$ and $b_h$ are constants depending only on $h$ and fixed parameters of the process, while $u_n(x) = \mathcal{H}(n, (0, 1); x)$ and $v_n = \mathcal{H}(n, (1, 0); x)$ are simple integrals which need only to be computed once for a given conditioning value $x$. Figure 2 shows the match between theoretical and empirical conditional moments of an ARMA process with causal, noncausal, invertible and noninvertible roots for different horizons as a function of the conditioning value. The empirical conditional moments were computed using Nadaraya-Watson estimator across 2000 simulated trajectories of $10^7$ observations each. The 0.05-0.95 interquantile interval across simulations are also displayed and show that even with $10^7$ observations, the uncertainty around the estimate can be large.

**Example 3.1 (Noncausal $\alpha$-stable AR(1))** Let $(X_t)$ be the noncausal $\alpha$-stable AR(1) solution of (1.1) with $\alpha \neq 1$ (for simplicity), $\beta \in [-1, 1]$ and $0 < |\rho| < 1$. Then $\mathbb{E}[|X_{t+h}|^\gamma | X_t] < +\infty$ for $0 \leq \gamma < 2\alpha + 1$ and any $h \geq 1$, and the first four conditional moments, when they exist, are given by Proposition 3.1 with

$$\sigma_1^\alpha = \frac{\sigma^\alpha}{1 - |\rho|^\alpha}, \qquad \beta_1 = \beta \frac{1 - |\rho|^\alpha}{1 - \rho^{<\alpha>}}, \qquad \kappa_p = |\rho|^{\alpha h} \rho^{-hp}, \qquad \lambda_p = \beta_1 \big(\rho^{<\alpha>}\big)^h \rho^{-hp},$$

for $p \in \{1, 2, 3, 4\}$. For $\rho > 0$, a clear interpretation of the distribution $X_{t+h} | X_t = x$ appears during explosive/bubble episodes, that is, as $x$ becomes large relative to the central values of process $(X_t)$. Denoting by $\mu(x, h)$, $\sigma^2(x, h)$, $\gamma_1(x, h)$ and $\gamma_2(x, h)$ the conditional expectation, variance, skewness and excess kurtosis of $X_{t+h}$ given $X_t = x$ respectively, when they exist, we have

$$\mu(x, h) \sim (\rho^{-h} x) \rho^{\alpha h}, \qquad \gamma_1(x, h) \longrightarrow s \frac{1 - 2\rho^{\alpha h}}{\sqrt{\rho^{\alpha h}(1 - \rho^{\alpha h})}},$$

$$\sigma^2(x, h) \sim (\rho^{-h} x)^2 \rho^{\alpha h} (1 - \rho^{\alpha h}), \qquad \gamma_2(x, h) \longrightarrow \frac{1}{\rho^{\alpha h}} + \frac{1}{1 - \rho^{\alpha h}} - 6,$$

as $\beta_1 x \to +\infty$ if $|\beta_1| = 1$, $x \to \pm\infty$ if $|\beta_1| \neq 1$, and $s = 1$ ($s = -1$) if $x \to +\infty$ ($x \to -\infty$).

[6] It is possible to find noncausal processes for which conditional moments are finite up to order $\gamma$ strictly within $(\alpha, 2\alpha+1)$, with $\gamma$ moreover depending on the prediction horizon. See the Supplementary file for an example.

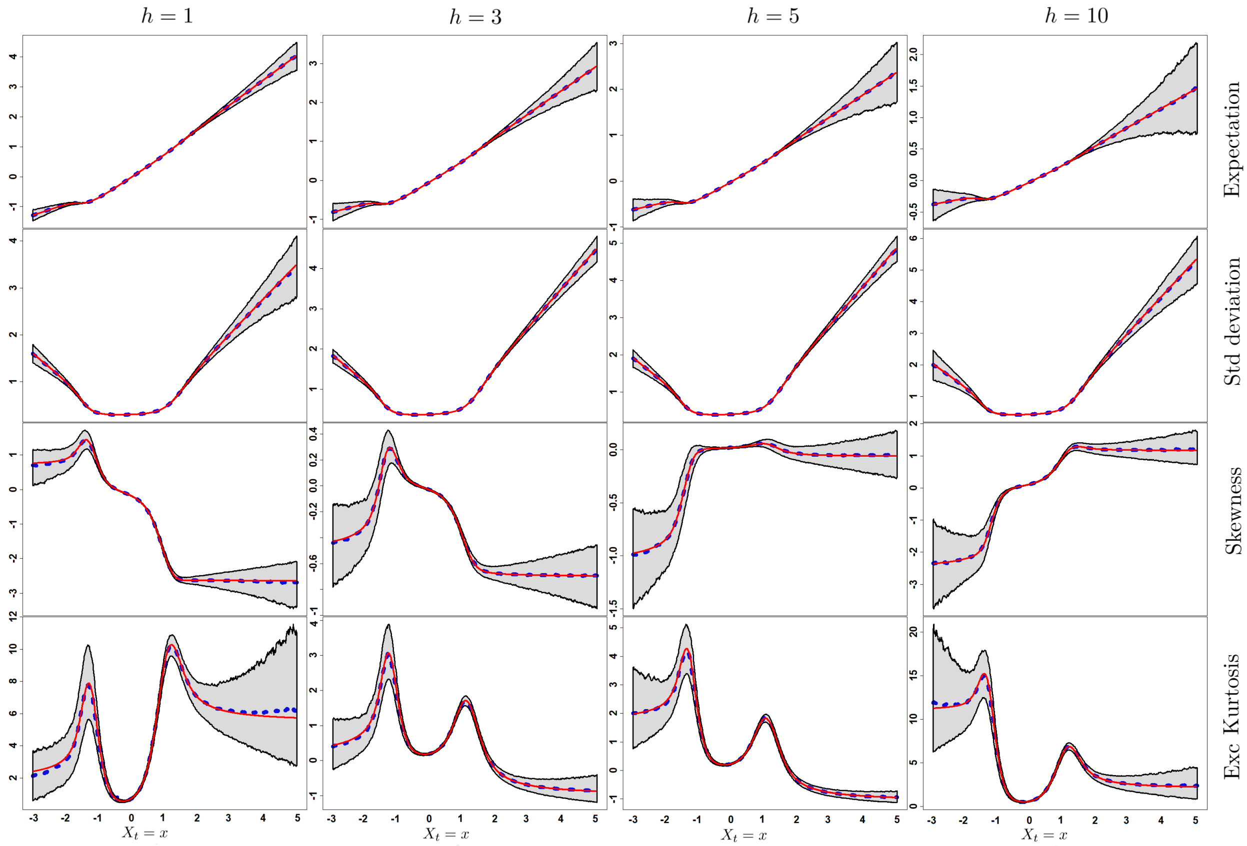


Figure 2: Conditional expectation, standard deviation, skewness and excess kurtosis (in rows) at horizons $h = 1, 3, 5, 10$ (in columns) of the ARMA process $(1-0.9F)(1-0.5B)X_t = (1+0.2F)(1-0.3B)\varepsilon_t$, $\varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(1.9, 0.8, 0.2, 0)$ for conditional values $x \in (-3, 5)$ (x-axis of each plot, the bounds -3 and 5 corresponding respectively to the 0.0003 and 0.9996 quantiles of the marginal distribution of $X_t$). Red solid lines: theoretical moments ; Blue dotted lines: average of Nadaraya-Watson estimators (bandwidth=0.1) across 2000 simulated trajectories of $10^7$ observations each ; Grey shaded areas: 0.05-0.95 interquantile interval across simulations. $F$ and $B$ denote respectively the forward and backward shift operators.

## 3.2 Extreme events and applications to crash odds for bubbles

### 3.2.1 Crash odds for bubbles of the noncausal AR(1): a memory-less property

The strikingly simplistic forms of the conditional moments during bubble episodes given above are characteristic of a weighted Bernoulli distribution charging probability $\rho^{\alpha h}$ to the value $\rho^{-h}x$ and probability $1-\rho^{\alpha h}$ to 0. It is thus natural to interpret $\rho^{\alpha h}$ as the probability that the bubble survives at least $h$ more time steps, conditionally on having reached the level $X_t = x$.[7] This interpretation implies that

[7] The interpretation of $\rho^{\alpha h}$ as a survival probability of bubbles is also reached using point processes (see the Supplementary file). The convergence in distribution of $X_{t+h}/X_t$ during extreme events towards this behaviour can furthermore be formally proven [Fries (2018)].

the survival probability does not depend on the current scale of the bubble. Surprisingly, given that the noncausal AR(1) is a Markov process, it would further imply that the survival probability of bubbles does not depend at all on the past history. In fact, the bubbles generated by the stable noncausal AR(1) appear to display a *memory-less* property characterised by an exponential survival probability exactly similar, e.g., to that of radioisotopes.[8] It can be fully characterised by its so-called *half-life*: the duration $h_{1/2}$ such that the survival probability at horizon $h_{1/2}$ is 1/2. For a noncausal AR(1) with parameters $\rho$ and $\alpha$, the half-life of bubbles is given by

$$h_{1/2} = -\frac{\ln 2}{\alpha \ln \rho}. \tag{3.5}$$

This property could be appealing from a financial and economic perspective as it implies that the crash date cannot be known with certainty by traders, hence ensuring a form of no-arbitrage condition.[9] At the same time, it would imply that no sophisticated method could allow a forecaster to say anything more regarding the future of AR(1) bubbles than «growth or crash» with the probabilities above. In the case of non-Markov noncausal processes or if the extreme errors driving bubbles are assumed to be endogenous rather than i.i.d. (as in Blasques et al. (2018)), past history would however play a more central role for prediction. We suggest lower and upper bounds of the quantity (3.5) and of crash odds for the ongoing growth episodes of the Nasdaq and S&P500 indexes in Section 5.

### 3.2.2 Dynamics of noncausal stable MA($\infty$) during extreme events

An apparent simplification of the dynamics during extreme events can also be found to hold for more general MA($\infty$). The following Corollary is an immediate consequence of Propositions 2.1 and 3.1.

**Corollary 3.1** *Let* $(X_t)$ *satisfying* (1.2), (3.1)-(3.2) *with a non-negative coefficients sequence* $(a_k)$ *satisfying* (3.4) *for some* $\nu > 0$. *For* $h \geq 1$, *let the almost surely finite random variable* $A_h$ *such that* $\mathbb{P}\Big(A_h = \frac{a_{k-h}}{a_k}\Big) = \frac{|a_k|^\alpha}{\sum_{l\in\mathbb{Z}} |a_l|^\alpha}$, *for all* $k \in \mathbb{Z}$. *Then, for* $p = 1, 2, 3, 4$, *if the moments exist,*

$$\mathbb{E}\Big[\Big(\frac{X_{t+h}}{X_t}\Big)^p \Big| X_t = x\Big] \longrightarrow \mathbb{E}[(A_h)^p],$$

*as* $\beta_1 x \to +\infty$ *if* $|\beta_1| = 1$ *and* $x \to \pm\infty$ *if* $|\beta_1| \neq 1$.

[8] Beside the fact that the survival probabilities indeed both belong to the exponential family, we use this analogy here to stress the unpredictable character of the crash occurrence. While it is possible to accurately predict the average decay of large amount of a certain radioisotope with time, predicting the disintegration of a single nucleus is more of a gamble.

[9] The scale invariance is a typical property of power-law distributed extreme events, which stems from $\alpha$-stable errors in our framework. It is thus possible that a similar memory-less property of bubbles still holds for other distributions with power-law tails such as the t-student which is commonly invoked for bubble modelling.

Although only a result about the convergence of moments, Corollary 3.1 seems to suggest that the conditional distribution of $X_{t+h}/X_t$ becomes close to that of $A_h$ during extreme events. This intuition can actually be formalised and results such as the following can be shown to hold:

$$\text{For all } k \in \mathbb{Z},\ \delta > 0, \qquad \mathbb{P}\Big(\Big|\frac{X_{t+h}}{X_t} - \frac{a_{k-h}}{a_k}\Big| < \delta \Big| X_t > x\Big) \underset{x\to\infty}{\longrightarrow} \frac{\sum\limits_{\ell\in\mathcal{J}_k} |a_\ell|^\alpha}{\sum\limits_{l\in\mathbb{Z}} |a_l|^\alpha}, \tag{3.6}$$

where $\mathcal{J}_k := \Big\{\ell \in \mathbb{Z} : \Big|\frac{a_{\ell-h}}{a_\ell} - \frac{a_{k-h}}{a_k}\Big| < \delta\Big\}$. The demonstration of such results is outside the scope of the current paper and is considered elsewhere [Fries (2018)].

### 3.2.3 Crash odds for bubbles of noncausal stable processes

This has important implications in the context of speculative bubble modelling for the evaluation of crash odds. Assume for instance that a noncausal process of the form $X_t = \sum_{k\geq 0} a_k \varepsilon_{t+k}$, $a_k > 0$ and $a_k/a_{k+1} \geq c > 0$ for all $k \geq 0$,[10] is considered to model a certain type of bubble. If (3.6) holds, the crash probability at horizon $h$ of $X_t$, observed extreme at date $t$, can then be expressed by

$$\mathbb{P}\Big(\Big|\frac{X_{t+h}}{X_t}\Big| < \delta \Big| X_t > x\Big) \underset{x\to\infty}{\longrightarrow} \frac{\sum\limits_{k=0}^{h-1} |a_k|^\alpha}{\sum\limits_{k\geq 0} |a_k|^\alpha} := p_{\infty,h}, \tag{3.7}$$

for $\delta > 0$ small enough. Similarly to the interpretation of the noncausal AR(1), one can notice that the crash probability of bubbles does not depend on their current scale. Contrary to the noncausal AR(1) however, the survival probabilities could in general be different if the past history of the bubble was accounted for in the conditioning.[11] We illustrate here through simulations that the probability on the left-hand side indeed converges towards the right-hand side limit as the conditioning value $x$ grows larger. We simulated a trajectory of $N = 10^8$ observations of a noncausal AR(3) process and computed the following estimator of the probability (3.7):

$$\hat{p}_{q,h} := \Big(\sum_{t=1}^{N-h} \mathbb{1}_{\{|X_{t+h}/X_t|<\delta\}\cap\{X_t>q\}}\Big) \Big/ \sum_{t=1}^{N-h} \mathbb{1}_{\{X_t>q\}}, \tag{3.8}$$

for several horizons $h$ and several quantiles $q$ of the marginal distribution of $X_t$. Table 1 gathers the results of this exercise and one can notice that the empirical probabilities become very close to the claimed theoretical ones as $q$ reaches the 0.99-quantile of $X_t$ and beyond. To evaluate such probability in practice, only the knowledge of the coefficients $(a_k)$ and of the tail parameter $\alpha$ is required, the asymmetry does

[10] This assumption ensures that $(a_k)$ does not display wild variations after $k = 0$ which could be mistaken with the crash.

[11] To investigate this question, one has to characterise the conditional distribution of $X_{t+h}$ given more past information, e.g., $X_t, X_{t-1}$... This problem is also out of the scope of the current paper and is addressed in Fries (2018).

| | $h$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 15 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $q_{0.8}$ | 22.9 | 34.8 | 44.8 | 52.3 | 58.2 | 62.8 | 66.4 | 69.2 | 71.5 | 73.2 | 77.9 | 79.3 |
| | $q_{0.9}$ | 25.2 | 39.4 | 51.3 | 60.3 | 67.3 | 72.5 | 76.5 | 79.6 | 82.0 | 83.4 | 88.2 | 89.4 |
| $\hat{p}_{q,h}$ | $q_{0.99}$ | 23.0 | 40.1 | 56.4 | 68.2 | 76.9 | 83.2 | 87.8 | 91.0 | 93.3 | 94.9 | 98.2 | 98.8 |
| | $q_{0.999}$ | 22.0 | 40.3 | 56.9 | 69.4 | 78.5 | 85.0 | 89.5 | 92.7 | 94.8 | 96.4 | 99.3 | 99.8 |
| | $q_{0.9999}$ | 21.6 | 40.0 | 56.8 | 69.2 | 78.3 | 84.9 | 89.5 | 92.7 | 94.8 | 96.3 | 99.3 | 99.9 |
| $p_{\infty,h}$ | | 21.6 | 40.0 | 56.7 | 69.1 | 78.2 | 84.6 | 89.2 | 92.5 | 94.7 | 96.3 | 99.4 | 99.9 |

Table 1: Theoretical and empirical crash probabilities (3.7) and (3.8) (in percentages) at several horizons $h$ of the noncausal AR(3) $(1-0.8F)(1-0.4F)(1+0.3F)X_t = \varepsilon_t, \varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(1.6, 1, 0.25, 0)$. The empirical probabilities were computed on a trajectory of $N = 10^8$ observations, with $\delta = 0.01$ and for $q = q_a$ several $a$-quantiles of the marginal distribution of $X_t$.

not intervene if the coefficients $(a_k)$ are non-negative, and the location and scale play no role.

It is worth emphasising that the asymptotics in (3.7) is with respect to the level $x$ of the trajectory and not to a sample size: in principle, the limiting probability can accurately quantify the crash odds of an extreme episode even if no data or no previous episode was observed on the trajectory before. In practice, if one estimates the coefficients $a_k$'s (for which a low-dimensional parametric form could be assumed), estimation uncertainty depending on the sample size might enter (3.7). However, even if no extreme episode has been observed before, one could still resort to theoretical considerations and priors to propose likely dynamics and bubble shapes that may occur in the future to study different scenarii -as typically done with stress tests in macroprudential analysis [Hanson et al. (2011)].

### 3.3 Continuous time: an example of power-law bubbles with long memory

With the following example, we illustrate that our results can be extended without difficulty to continuous time, and that noncausal linear processes can encompass local dynamics which are considered to be typically nonlinear or even non-stationary. The process chosen here is inspired from the Johansen-Ledoit-Sornette (JLS) bubble literature (see for instance Johansen et al. (1999), Sornette (2003), Sornette (2017), Sornette and Johansen (2001)) and is characterised by trajectories («prices») featuring bubbles with power-law growth close to the peak[12] while exhibiting long memory in the returns at the same time. We define $X_t$ for all $t \in \mathbb{R}$ as

$$X_t = \int_t^{\infty} f(x-t)M(dx), \qquad \text{with} \quad f(x) = \frac{1}{a_1 x^{d_1} + a_2 x^{d_2}} \mathbb{1}_{\{x>0\}}, \tag{3.9}$$

[12] The power-law growth here is a property of the *shape* of the trajectory close to the bubble peak which JLS derive using a physical approach. It should not be confused with the power-law distribution of the extreme events that we mentioned earlier, which is a property of the *scale* of the trajectory due to the $\alpha$-stable errors. Both coexist in this example.

where $a_1, a_2, d_1, d_2$ are positive constants, $d_1 < d_2$, and $M$ is an $\alpha$-stable random measure with constant skewness intensity equal to $\beta$ and Lebesgue control measure (see Chapter 3 Definition 3.3.1 in ST94 for details). Similarly to the baseline path interpretation in Fries and Zakoian (2019), when a realisation of the random measure $M$ attributes an extreme mass in the vicinity of a certain date $t_c$, the trajectory of $X_t$ can be locally approximated up to a multiplicative constant by $X_t \approx f(t_c - t)$. Close to the bubble peak, the trajectory is thus dominated by the term with smaller exponent and explodes at the same speed as $x^{-d_1}/a_1$, before suddenly collapsing. Further in the tail of the bubble, the trajectory is dominated by the term with greater exponent and decays as $x^{-d_2}/a_2$, inducing long memory. In contrast with the JLS framework which focuses on a single financial bubble viewed as non-stationary phenomenon resulting from a nonlinear physical system, the example process (3.9) is strictly stationary and can generate multiple bubbles whose dynamics are mimicking that of JLS bubbles close to the peak.[13] This process is well-defined and stationary if $\int_{\mathbb{R}} |f(x)|^\alpha dx < +\infty$ which is equivalent to

$$\frac{1}{d_2} < \alpha < \frac{1}{d_1}. \tag{3.10}$$

One can show that $(X_t, X_{t+h})$ is bivariate $\alpha$-stable, obtain its spectral representation, and apply the properties of Section 2. In particular, the conditional moments $\mathbb{E}[|X_{t+h}|^\gamma | X_t]$ are finite at least up to order $\gamma < \min(\alpha + \nu, 2\alpha + 1)$ for any $\nu \geq 0$ such that $\int_{\mathbb{R}_+} |f(x-h)|^{\alpha+\nu}/|f(x)|^\nu dx < +\infty$, i.e.,

$$\nu < \frac{1}{d_1} - \alpha. \tag{3.11}$$

For $\alpha \in (3/2, 2)$, the fourth order conditional moment is finite provided $0 < d_1 < 1/4$. Theorem 2.1 then provides the functional forms of the moments with,

$$\sigma_1^\alpha = \int_{\mathbb{R}} |f(x)|^\alpha dx, \qquad \beta_1 = \beta, \qquad \kappa_p = \frac{1}{\sigma_1^\alpha} \int_{\mathbb{R}_+} \left(\frac{f(x-h)}{f(x)}\right)^p |f(x)|^\alpha dx, \qquad \lambda_p = \beta \kappa_p,$$

for $p = 1, 2, 3, 4$. Similarly to the noncausal processes in discrete time, it can be shown that the conditional moments simplify by Proposition 2.1 during extreme events:

$$\mathbb{E}\left[\left(\frac{X_{t+h}}{X_t}\right)^p \middle| X_t = x\right] \underset{x \to \pm\infty}{\longrightarrow} \mathbb{E}\left[\left(\frac{f(U-h)}{f(U)}\right)^p\right],$$

where $U$ is a random variable with density $g(u) = \dfrac{|f(u)|^\alpha}{\int_{\mathbb{R}} |f(s)|^\alpha ds}$ for $u \in \mathbb{R}$. Again, from this convergence of the moments, we may suspect that the conditional distribution $X_{t+h}/X_t$ becomes close to that of $f(U-h)/f(U)$ during extreme events. If we admit this, then the crash probability can be obtained as

$$\mathbb{P}\left(\left|\frac{X_{t+h}}{X_t}\right| < \delta \middle| X_t > x\right) \underset{x \to \infty}{\longrightarrow} \int_0^h g(u) du, \qquad\qquad \text{for } \delta \text{ small enough.}$$

[13] To be fully consistent with JLS, one should also include a log-periodic oscillating component in $f$. This poses no difficulty but makes the presentation cumbersome so we omit it.

# 4 Aggregated noncausal $\alpha$-stable processes

In order to encompass trajectories featuring bubbles of different growth rates, Gouriéroux and Zakoian (2017) introduced an *aggregated* process defined as the linear combination of multiple AR(1):

$$X_t = \sum_{j=1}^{J} \pi_j X_{j,t}, \quad X_{j,t} = \rho_j X_{j,t+1} + \varepsilon_{j,t}, \quad 0 < |\rho_j| < 1, \quad j = 1, \dots, m, \tag{4.1}$$

where $\pi_j \in \mathbb{R}$ for $j = 1, \dots, J$ and $(\varepsilon_{j,t})_{t\in\mathbb{Z}} \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, \beta_j, \sigma_j, 0)$ are mutually independent sequences of i.i.d. errors. Sample trajectories of $(X_t)$ feature bubble episodes with various rates of increase $1/\rho_j$, $j = 1, \dots, J$. Unlike for the latent $(X_{j,t})$'s, nothing is known about the predictive distribution of $X_{t+h}$ given its past, even in this simpler case of an aggregation of AR(1) processes. We give results regarding the conditional distribution of $X_{t+h}$ given $X_t$ in the framework where the $(X_{j,t})$'s involved in the aggregation are two-sided MA($\infty$) processes.

**Definition 4.1** Let $(X_{1,t}), \dots, (X_{J,t})$ be $J \geq 1$ stable moving averages, each satisfying (1.2), (3.1)-(3.2), for some coefficients sequences $(a_{j,k})_k$ and mutually independent error sequences $\varepsilon_{j,t} \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, \beta_j, \sigma_j, 0)$, $j = 1, \dots, J$. Let also $(\pi_j)_{j=1,\dots,J}$ be scalars and define $(X_t)$ as

$$X_t = \sum_{j=1}^{J} \pi_j X_{j,t}, \qquad \text{for} \quad t \in \mathbb{Z}.$$

We will call such process $(X_t)$ an *$\alpha$-stable aggregated moving average*, an *aggregated process*, or simply, a *stable aggregate*, and call $(X_{j,t})$, $j = 1, \dots, J$ the *latent* moving averages of $(X_t)$.

The following proposition is a consequence of the fact that the vector $(X_t, X_{t+h}) = \sum_{j=1}^{J} \pi_j (X_{j,t}, X_{j,t+h})$ is itself $\alpha$-stable and its spectral measure $\Gamma_h$ is actually a mixture of the spectral measures $\Gamma_{j,h}$ of each vector $(X_{j,t}, X_{j,t+h})$ as: $\Gamma_h = \sum_{j=1}^{J} |\pi_j|^\alpha \Gamma_{j,h}$ (see Lemma H.1 for details).

**Proposition 4.1** *Let $(X_t)$ be an $\alpha$-stable aggregate with latent moving averages $(X_{1,t}), \dots, (X_{J,t})$ as in Definition 4.1 with $0 < \alpha < 2$. Let $h \geq 1$.*

*$\iota$) Assume there is $\nu > 0$ such that*

$$\text{for all } j = 1, \dots, J, \quad \sum_{k\in\mathbb{Z}} \left(a_{j,k}^2 + a_{j,k-h}^2\right)^{\frac{\alpha+\nu}{2}} |a_{j,k}|^{-\nu} < \infty. \tag{4.2}$$

*Then $\mathbb{E}[|X_{t+h}|^\gamma | X_t] < \infty$ for $0 \leq \gamma < \min(\alpha + \nu, 2\alpha + 1)$.*

*$\iota\iota$) For $\alpha \neq 1$, the first four conditional moments of $X_{t+h}|X_t$, when they exist, are given by Theorem 2.1 with*

$$\sigma_1^\alpha = \sum_{j=1}^{J} |\pi_j|^\alpha \sigma_{1,j}^\alpha, \qquad \beta_1 = \mathbb{E}(B), \qquad \kappa_p = \mathbb{E}(K_p), \qquad \lambda_p = \mathbb{E}(L_p), \qquad \text{for} \quad p \in \{1, 2, 3, 4\},$$

*where $B$, $K_p$ and $L_p$ are discrete random variables such that $\mathbb{P}((B, K_p, L_p) = (\beta_{1,j}, \kappa_{p,j}, \lambda_{p,j})) = w_j$, $w_j = |\pi_j|^\alpha \sigma_{1,j}^\alpha / \sum_{i=1}^J |\pi_i|^\alpha \sigma_{1,i}^\alpha$ for $j = 1, \ldots, J$, and where $\sigma_{1,j}$, $\beta_{1,j}$, $\kappa_{p,j}$ and $\lambda_{p,j}$ denote the quantities defined in Proposition 3.1 where $(a_k)_k$, $\sigma$ and $\beta$ are replaced by $(a_{j,k})_k$, $\sigma_j$ and $\beta_j$.*

*ιιι) For $\alpha = 1$, let $(\tilde{X}_t, \tilde{X}_{t+h}) := (X_t, X_{t+h}) - \boldsymbol{\mu}^0$ where $\boldsymbol{\mu}^0 = (\mu_1^0, \mu_2^0)$ where $\boldsymbol{\mu}^0$ is as in Lemma H.1. Then, the second order moment of $\tilde{X}_{t+h}|\tilde{X}_t$ is given by Theorem 2.2 with the $\kappa_p$'s, $\lambda_p$'s, $\sigma_1$ and $\beta_1$ as above and*

$$q_0 = \mathbb{E}(Q_0), \qquad \mu_1 = \sum_{j=1}^J \pi_j \mu_{1,j},$$

*where $Q_0$ is a discrete random variable such that, for $p \in \{1, 2\}$, $\mathbb{P}((B, K_p, L_p, Q_0) = (\beta_{1,j}, \kappa_{p,j}, \lambda_{p,j}, q_{0,j})) = w_j$, for $j = 1, \ldots, J$, and $q_{0,j}$, $\mu_{1,j}$ denote the quantities defined in Proposition 3.1 with $(a_k)_k$, $\sigma$ and $\beta$ replaced by $(a_{j,k})_k$, $\sigma_j$ and $\beta_j$.*

The above proposition straightforwardly applies to the aggregated noncausal stable AR(1) defined in (4.1). Notice that for the non-aggregated noncausal AR(1), $\rho > 0$ is sufficient to guarantee the linearity of the conditional expectation, but merely assuming $\rho_j > 0$ for $j = 1, \ldots, J$ for the aggregated process $(X_t)$ does not guarantee linearity in general. Linearity of the conditional expectation (2.3) is achieved if and only if $\lambda_1 - \beta_1 \kappa_1 = 0$, which is equivalent to $\text{Cov}(B, K_1) = 0$ if $\rho_j > 0$ for $j = 1, \ldots, J$. Based on this, it is easy to construct examples for which $x \longmapsto \mathbb{E}[X_{j,t+h}|X_{j,t} = x]$ are all linear in $x$ for any $j$ and $h$, and yet such that $y \longmapsto \mathbb{E}[X_{t+h}|X_t = y]$ is a non-linear function of $y$.

From a statistical perspective, a strategy to estimate agnostically the coefficients sequences $(a_{j,k})_k$, $j = 1, \ldots, J$ could exploit 1) the fact that exceedances above high thresholds of a stable MA process behave as a marked point process [Rootzen (1978)], the marks being normalised sample paths of, say, $(X_{j,t})$, and are asymptotically of the same shape as $(\ldots, a_{j,-1}, a_{j,0}, a_{j,1}, \ldots)$. Practical procedures to identify these marks and provide estimates of (at least some) $a_{j,k}$'s could leverage declustering schemes such as in Ferro and Segers (2003). An estimation strategy could also exploit 2) the fact that the spectral measure of, say, $(X_t, \ldots, X_{t+n})$ is a mixture of the spectral measures of the latent $(X_{j,t}, \ldots, X_{j,t+n})$. The extremal dependence of sample paths of $(X_t)$ could thus be analysed by adapting Boldi and Davison (2007) to the case of mixtures of spectral measures of sum-stable vectors.

# 5 Applications

This section presents two applications of our results. The first one uses the conditional moments up to order four in a synthetic portfolio allocation framework. The second one illustrates how one can evaluate crash odds of real series by fitting noncausal models.

## 5.1 Portfolio selection

It has been recently found that the incorporation of higher order moments for portfolio optimisation can lead to substantial improvements of the assets allocation strategies [Harvey et al. (2010), Holly et al. (2011), Jondeau and Rockinger (2006), Lai (1991), Lai et al. (2006)], and efforts are deployed to efficiently capture time-varying higher moments into the allocation program [Bernardi and Catania (2018), Boudt et al. (2015), González-Pedraz et al. (2015), Harvey and Siddique(1999), Jondeau and Rockinger (2012)]. Two approaches to account for higher order moments in the choice of the optimal portfolio are polynomial goal programming (PGP) and the maximisation of the Taylor expansion of a utility function, a common one being the constant relative risk aversion (CRRA) utility. For speculative assets typically, asymmetry and heavy-tails in returns can be expected to be of crucial importance for the (non-)investment decision. We illustrate in the framework of noncausal stable processes how the functional forms of the conditional moments in Theorem 2.1 can be used to perform portfolio selection. We consider a simple framework where an investor endowed with an initial wealth $W_t$ at present date $t$ has the choice between a speculative asset $X_t$ and a safer asset $S_t$. The investor has an investment horizon $H$: at date $t$, she will decide of the share $\omega$ (resp. $1-\omega$) to invest in the speculative asset (resp. safer asset), and of the intermediate horizon $h \leq H$ at which she commits to liquidate its holding of speculative asset and to invest the proceedings in the safer asset until $t+H$. This leads to an optimisation problem of the terminal wealth $W_{t+H}$ (or overall return $R_{t+H} = (W_{t+H} - W_t)/W_t)$) in both the allocation $\omega$ and the intermediate horizon $h$. We will consider time to be continuous and that $X_t$ follows a continuous time noncausal stable AR(1) as in (1.1) with a non-zero location parameter,[14] and that the safer asset follows a geometric Brownian motion (GBM) dynamics with drift $r$ and volatility $\sigma$. The processes $(X_t)$ and $(S_t)$ will be assumed independent. For a given strategy $(\omega, h)$, the terminal wealth can be expressed as

$$W_{t+H} = W_t \frac{S_{t+H}}{S_{t+h}} \left( \omega \frac{X_{t+h}}{X_t} + (1-\omega) \frac{S_{t+h}}{S_t} \right).$$

The CRRA utility maximisation program of the terminal wealth and its fourth order Taylor approximation around the expected terminal wealth read [Jondeau and Rockinger (2006)]

$$\max_{(\omega,h)} \ \mathbb{E}[U(W_{t+H})|X_t, S_t] \approx \sum_{k=0}^{4} \frac{U^{(k)}(\overline{W}_{t+H})}{k!} \mathbb{E}\Big[ (W_{t+H} - \overline{W}_{t+H})^k \Big| X_t, S_t \Big], \tag{5.1}$$

[14] I.e., a noncausal stable Ornstein-Uhlenbeck (OU) process. There is a one-to-one correspondence between a given stable AR(1) and its OU analogue and one can show that it is valid to use the results of Example 3.1 as if $h$ was real instead of integer. We therefore define $(X_t)$ as in (1.1) to avoid introducing additional notations.

where $U(c) = c^{1-\gamma}/(1-\gamma)$, for a risk aversion parameter $\gamma > 0$, and $\overline{W}_{t+H} = \mathbb{E}[W_{t+H}|X_t, S_t]$. A PGP program can be specified as (inspired from Aksaraylı and Pala (2018), Lai (1991), Lai et al. (2006))

$$\begin{aligned} &\min_{(\omega,h)} \ \Big(1 + |d_1 - R^*|\Big)^{\gamma_1} + \Big(1 + |d_2 - V^*|\Big)^{\gamma_2} + \Big(1 + |d_3 - S^*|\Big)^{\gamma_3} + \Big(1 + |d_4 - K^*|\Big)^{\gamma_4}, \\ &\text{s.t.} \quad R_{\omega,h} + d_1 = R^*, \quad V_{\omega,h} - d_2 = V^*, \quad S_{\omega,h} + d_3 = S^*, \quad K_{\omega,h} - d_4 = K^*, \quad d_i \geq 0, \end{aligned} \tag{5.2}$$

where $R_{\omega,h}$, $V_{\omega,h}$, $S_{\omega,h}$, $K_{\omega,h}$ denote respectively the conditional expectation, variance, skewness and excess kurtosis of the returns $R_{t+H}$ for a given strategy $(\omega, h)$ ; $R^*$, $V^*$, $S^*$, $K^*$ denote the optima of the subprograms $\max_{(\omega,h)} R_{\omega,h}$, $\min_{(\omega,h)} V_{\omega,h}$, $\max_{(\omega,h)} S_{\omega,h}$, $\min_{(\omega,h)} K_{\omega,h}$ ; and the $\gamma_i$'s are non-negative parameters weighting the preference of the investor to pursue optimality of one moment over the others. In both approaches, it is just a matter of algebra using the independence between $(X_t)$ and $(S_t)$ to express the objective functions in terms the moments of $X_{t+h}|X_t$ and the parameters.

As an experiment, we numerically solve the above programs for the following parameterisations. For the process $X_t$, we set $\rho = 0.7$, with errors $\varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(1.7, 1, 2, 3)$. One unit of time can be thought as a year and we take $H = 2$ ; the bubbles of $X_t$ hence grow roughly at an annual rate of $1/\rho \approx 43\%$, and have a half-life of $-\ln 2/\alpha \ln \rho \approx 13.7$ months. For the safer asset, we set both the annual return $r$ and volatility $\sigma$ to 2%. We consider a CRRA investor with $\gamma = 5$ and a more risk averse one with $\gamma = 10$, as well as two PGP investors with equal weighting $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (1, 1, 1, 1)$ and more kurtosis sensitive weighting $(1, 1, 1, 4)$. While the starting value of $S_t$ does not matter, the starting value of $X_t$ deeply modifies the investment landscape. We thus set several starting values for $X_t = x$ corresponding to quantiles of the marginal distribution of $X_t$, from central to extreme. We assume unit initial wealth and search for optima $(\omega^*, h^*)$ in the set $[-1, 1] \times [0, 2]$, thus allowing short strategies. Table 2 reports the results. Given that the programs are likely non-convex, there is in general no unique optimum. All attained solutions achieving comparable (global) optimality are reported. We rounded $\omega^*$ to the closest percentage point, $h^*$ to the closest month, and by convention, if $\omega^* = 0$, we report $h^* = 0$ as well.

One can notice that for initial values of the speculative asset close to the stationary baseline, the optimal strategies are rather passive. The price $X_t$ is more likely to follow a noisy trajectory around its central level, while the safer asset offers a higher and surer reward. Higher initial values of the speculative asset give evidence that the coming months or years will be dominated by the explosive regime: there is a possibility of gaining immense returns compared to the safer asset, but with great risk of loosing the bet. The optimal strategies are much more active in this case, both in quantities and in holding horizons. The CRRA investors almost exclusively bet on a crash occurring at some point before the terminal horizon, and will opt to short the speculative asset. The more risk averse will halve its bet in terms of quantities compared to the less risk averse one. The PGP investors may choose between two types of equally

| $X_t = x$ | $q_{0.5}$ | $q_{0.6}$ | $q_{0.7}$ | $q_{0.8}$ | $q_{0.9}$ | $q_{0.95}$ | $q_{0.99}$ | $q_{0.999}$ | $q_{0.9999}$ |
|---|---|---|---|---|---|---|---|---|---|
| CRRA | (7,1) | (4,1) | (-12,24) | (-18,24) | (-23,24) | (-23,24) | (-15,24) | (-10,24) | (-10,24) |
| $\gamma = 5$ | (2,10) | (-5,24) | | | | | | | |
| | (1,20) | | | | | | | | |
| | (7,1) | (5,1) | (3,1) | (1,1) | (-11,24) | (-11,24) | (-7,24) | (-5,24) | (-5,24) |
| $\gamma = 10$ | (2,6) | (-2,24) | (-6,24) | (-9,24) | | | | | |
| | (1,15) | | | | | | | | |
| PGP | (0,0) | (0,0) | (8,21) | (22,18) | (30,18) | (33,18) | (30,19) | (24,19) | (23,19) |
| (1,1,1,1) | | | (0,0) | (-59,10) | (-82,10) | (-89,10) | (-71,10) | (-53,10) | (-48,10) |
| | | | (-27,9) | | | | | | |
| (1,1,1,4) | (0,0) | (-1,7) | (-2,12) | (28,15) | (44,15) | (50,15) | (47,15) | (38,15) | (36,15) |
| | | | | (-62,11) | (-98,11) | (-100,11) | (-95,11) | (-69,11) | (-62,11) |

Table 2: Optimal investment strategies $(\omega^*, h^*)$, where $\omega^*$ is reported in percent of the portfolio and $h^*$ in months, of programs (5.1) and (5.2). The speculative asset $X_t$ is assumed noncausal AR(1) as in (1.1) with $\rho = 0.7$, $\varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(1.7, 1, 2, 3)$, while the safer follows a GBM with drift $r = 0.02$ and volatility $\sigma = 0.02$. Initial price of the speculative asset is set to $x = q_a$, for several $a$-quantiles of the marginal distribution. Reading example: for a PGP investor with weights $(1, 1, 1, 4)$, and for an initial value of the speculative asset $x = q_{0.8}$, two distinct strategies achieve comparable global optimality: 1) a strategy long by 28% of the speculative asset with holding horizon of 15 months, and 2) a strategy short by -62% with holding horizon of 11 months.

optimal -according to their criterion- strategies: long or short. The long strategies are characterised by lower (absolute) quantities but longer holding horizons compared to the short strategies. If the more kurtosis-sensitive investor chooses the long strategy, she will bet significantly higher quantities compared to the less kurtosis-sensitive investor, but with holding horizons down by several months. If she opts for the short strategy, she will bet more aggressively on the collapse of the bubble both in quantities and horizons. Unlike the CRRA investors, the PGP investors will not short these aggressive quantities beyond a year. The risk would be to reach the terminal horizon with the bubble still ongoing and hence endure heavy losses.

### 5.2 Evaluating the odds of crashes of real series

In this section, we consider two series commonly studied in the speculative bubble literature: the Nasdaq and S&P500 indexes (see e.g. Phillips et al. (2015), Phillips et al. (2011)). We will focus on the almost uninterrupted growth episodes since the aftermath of the 2008 crisis up to 2019 and suggest an ex ante analysis. At the cost of assuming that these explosive episodes in the data can be modelled as ongoing realisations of AR(1) bubbles climbing towards exogenous power-law-scaled peaks, we will be

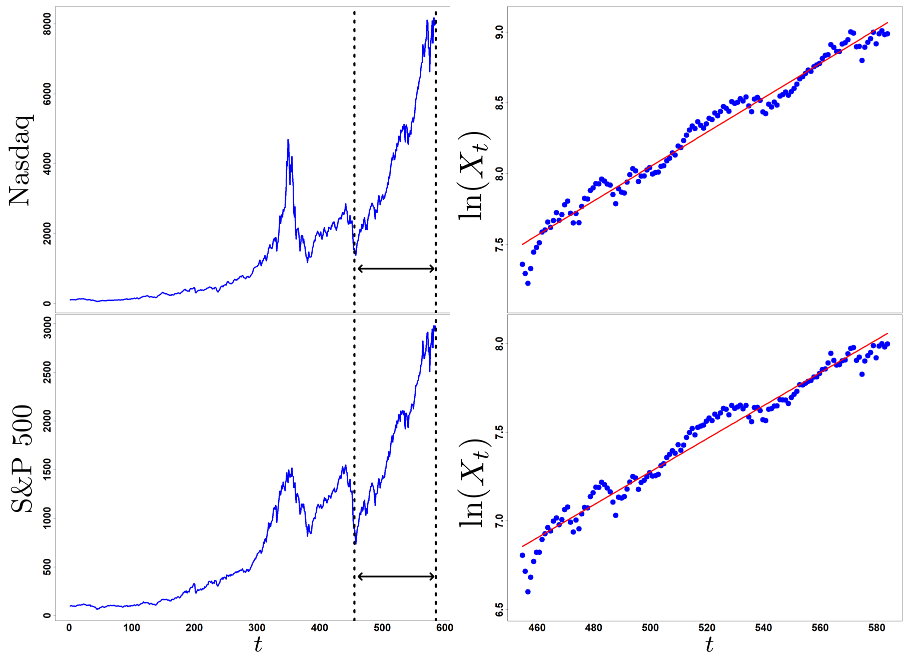


Figure 3: Monthly Nasdaq and S&P500 indexes, non-adjusted for inflation (upper and lower left respectively), from 02/1971 to 09/2019. The arrows and vertical dotted lines indicate the period of analysis, from 12/2008 to 09/2019. Right panels: regressions of log prices against time (data in points, fit in red solid lines).

in the position to propose an evaluation of the crash odds based on the half-lives $h_{0.5}$ given in (3.5). This requires to provide values for the AR coefficient $\rho$ and the tail parameter $\alpha$. Under the AR(1) assumption, bubbles should have an exponential shape $t \mapsto \rho^{-t}$ up to a multiplicative constant, and the parameter $\rho$ could thus be estimated locally by fitting an exponential trend on the explosive episode - or conveniently, by fitting a linear regression on the logarithm of the data. Fitting the regression $\ln(X_t) = at + b$ on the monthly Nasdaq and S&P500 series from December 2008 to September 2019, we obtain estimates $\hat{a}$ of $a$, from which we deduce $\hat{\rho} = \exp(-\hat{a})$. Turning to the literature regarding the tail parameter $\alpha$, studies mostly report values ranging from slightly below one to four for financial series (Ibragimov and Prokhorov (2016) and the references therein).[15] The widest range of plausible values compatible with our framework would thus be $\alpha \in [0.5, 2]$ (we include 2 as the limit for an $\alpha$-stable index arbitrarily close to 2). Assuming a uniform prior for $\alpha$ on $[0.5, 2]$ and neglecting the (small) estimation uncertainty around $\hat{a}$, this suggests the range $\frac{\ln 2}{2\hat{a}} \le h_{0.5} \le \frac{2\ln 2}{\hat{a}}$ for the half-lives of corresponding AR(1) bubbles. Furthermore, from a half-life $h_{0.5}$, one can compute the likelihood of collapse at any desired horizon $h$ as $1-(1/2)^{h/h_{0.5}}$. We provide the corresponding ranges for the odds of a crash occurring within

[15] We further note that reported values above two are not necessarily evidence against the infinite variance $\alpha$-stable hypothesis [McCulloch (1997)].

| | | Growth rate $\hat{a}$ | Annualised AR coef. $\hat{\rho}$ | Plausible range for $\alpha$ | Half-life range in years | Odds of crash within one year |
|---|---|---|---|---|---|---|
| Nasdaq | Nominal | $1.2 \cdot 10^{-2}$ | 0.86 | [0.5 – 2] | [2.4 – 9.5] | [7.0% – 25%] |
| | Infl. adj. | $1.1 \cdot 10^{-2}$ | 0.88 | [0.5 – 2] | [2.7 – 11] | [6.2% – 23%] |
| S&P500 | Nominal | $9.3 \cdot 10^{-3}$ | 0.89 | [0.5 – 2] | [3.0 – 13] | [5.3% – 21%] |
| | Infl. adj. | $8.0 \cdot 10^{-3}$ | 0.91 | [0.5 – 2] | [3.5 – 15] | [4.5% – 18%] |

Table 3: Estimated growth rates $\hat{a}$ of exponential trends fitted on the nominal and real Nasdaq and S&P500 indexes (monthly data from 12/2008 to 09/2019) ; Corresponding annualised AR(1) coefficients $\hat{\rho} = \exp(-12\hat{a})$ ; Ranges of the half-lives $\hat{h}_{0.5} = \ln 2/12\hat{a}\alpha$ (in years) with uniform prior on $\alpha \in [0.5, 2]$ ; Corresponding ranges for crash odds within one year $1 - (1/2)^{1/\hat{h}_{0.5}}$.

the next year. Figure 3 displays the series and the fits, and Table 3 gathers the estimates. To remain agnostic as to whether we should consider nominal or real prices, depending on what is more relevant with respect to the behaviours and motives of economic agents sustaining the growth, we include estimates for the inflation-adjusted indexes.[16] This analysis suggests relatively important crash odds within one year ranging from 4.5 to 25%. Tighter ranges could be obtained by estimating the tail parameter $\alpha$. Recent approaches robust to unavailable extreme values such as developed in Zou et al. (2017) could be promising in that respect, as one could typically consider the crash date to be missing from the dataset.

## 6 Concluding remarks

We provided functional forms for the conditional moments up to order four of arbitrary bivariate $\alpha$-stable random vectors $(X_1, X_2)$ as well as their asymptotic behaviours when the conditioning variable takes extreme values. Embedding two-sided MA($\infty$) processes into this framework, we could describe in detail the conditional dependence of $X_{t+h}$ on $X_t$. We have shown that noncausality plays a crucial role in the finiteness of conditional moments, and provided functional forms for the latter up to the fourth order, when they exist. We furthermore obtained unique insights into the extremal dependence of $(X_t, X_{t+h})$, which is a topic of interest on its own [Ledford and Tawn (2003), Wadsworth et al. (2017)], but especially in the context of bubble modelling: during the extreme «bubble» episodes that such processes generate, we have shown that the dynamics simplifies and can be easily interpreted, revealing for instance a memory-less or non-aging property of AR(1) bubbles. We demonstrated how crash odds can be evaluated ex ante in the framework of these models, even on local bubble events of real data. We

[16] We use a seasonally adjusted Consumer Price Index provided by the Federal Bank of Saint Louis. fred.stlouisfed.org/series/CPIAUCSL

illustrated through examples the ability of noncausal processes to encompass local dynamics which are considered typically non-linear or even non-stationary, and how they can be applied in practice for horizon selection in portfolio problems with speculative assets. Statistical methods for agnostically estimating the coefficients $(a_k)$ of the MA representation, e.g. under low dimensionial restrictions, and for robustly estimating the tail index $\alpha$ in locally non-stationary events could enable more refined evaluation of the crash odds. We also have shown how the main results extend to aggregated processes, including the existence and the form of the conditional moments. Thorough investigation of their a priori much richer dynamics and of the statistical aspects is left for further research.

## References


Andrews, B., Calder, M., and R., Davis. 2009. Maximum likelihood estimation for $\alpha$-stable autoregressive process. *Annals of Statistics*, 37, 1946-1982.

Aksaraylı, M., and O., Pala. 2018. A polynomial goal programming model for portfolio optimization based on entropy and higher moments. *Expert Systems with Applications*, 94, 185-192.

Bec, F., Nielsen, H. B., and S., Saïdi. 2019. Mixed Causal-Noncausal Autoregressions: Bimodality Issues in Estimation and Unit Root Testing. Université de Cergy-Pontoise. No. hal-02175760. 2019.

Bernardi, M., and L., Catania. 2018. Portfolio optimisation under flexible dynamic dependence modelling. *Journal of Empirical Finance*, 48, 1-18.

Blasques, F., Koopman, S. J., and M., Nientker. 2018. A Time-Varying Parameter Model for Local Explosions. Tinbergen Institute Discussion Paper, No. TI 2018-088/III.

Boldi, M. O., and A. C., Davison. 2007. A mixture model for multivariate extremes. *Journal of the Royal Statistical Society: Series B*, 69, 217-229.

Boudt, K., Lu, W., and B., Peeters. 2015. Higher order comoments of multifactor models and asset allocation. *Finance Research Letters*, 13, 225-233.

Cambanis, S., and S., Fotopoulos. 1995. Conditional variance for stable random vectors. *Probability and Mathematical Statistics*, Wrocław University, 15, 195-214.

Cambanis, S., and W., Wu. 1992. Multiple regression on stable vectors. *Journal of multivariate analysis*, 41, 243-272.

Cavaliere, G., Nielsen, H.B, and A. Rahbek. 2017. Bootstrapping non-causal autoregressions: with applications to explosive bubble modelling. Forthcoming in *Journal of Business and Economic Statistics*.

Chen, B., Choi, J., and J. C., Escanciano. 2017. Testing for fundamental vector moving average representations. *Quantitative Economics*, 8, 149-180.

Cioczek-Georges, R., and M. S., Taqqu. 1994. How do conditional moments of stable vectors depend on the spectral measure? *Stochastic processes and their applications*, 54, 95-111.

Cioczek-Georges, R., and M. S., Taqqu. 1995. Form of the conditional variance for stable random variables. *Statistica Sinica*, 351-361.

Cioczek-Georges, R., and M. S., Taqqu. 1995. Necessary conditions for the existence of conditional moments of stable random variables. *Stochastic processes and their applications*, 56, 233-246.

Cioczek-Georges, R., and M. S., Taqqu. 1998. Sufficient conditions for the existence of conditional moments of stable random variables. *Stochastic Processes and Related Topics*, Birkhäuser Boston, 35-67.

Ferro, C. A., and J., Segers. 2003. Inference for clusters of extreme values. *Journal of the Royal Statistical Society: Series B*, 65, 545-556.

Fotopoulos, S. B. 1998. On the conditional variance-covariance of stable random vectors, II. *Stochastic Processes and Related Topics*, Birkhäuser Boston, 123-140.

Fries, S. 2018. Path prediction of aggregated $\alpha$-stable moving averages using semi-norm representations. ArXiv preprint arXiv:1809.03631.

Fries, S., and J.-M., Zakoian. 2019. Mixed causal-noncausal AR processes and the modelling of explosive bubbles. *Econometric Theory*, 35, 1234-1270.

González-Pedraz, C., Moreno, M., and J. I., Peña. 2015. Portfolio selection with commodities under conditional copulas and skew preferences. *Quantitative Finance*, 15, 151-170.

Gouriéroux, C., Hencic, A., and J., Jasiak. 2019. Forecast Performance and Bubble Analysis in Noncausal MAR (1,1) Processes. Toulouse School of Economics. 10.13140/RG.2.2.25135.38568.

Gouriéroux, C., and J., Jasiak. 2016. Filtering, prediction and simulation methods for noncausal processes. *Journal of Time Series Analysis*, 37, 405-430.

Gouriéroux, C., Jasiak, J. and A., Monfort. 2016. Stationary bubble equilibria in rational expectation models. Working Papers from Centre de Recherche en Economie et Statistique (CREST), No 2016-31.

Gouriéroux, C. and J.-M., Zakoian. 2017. Local explosion modelling by non-causal process. *Journal of the Royal Statistical Society: Series B*. 79, 737-756.

Hanson, S. G., Kashyap, A. K., and J. C., Stein. 2011. A macroprudential approach to financial regulation. *Journal of economic Perspectives*, 25, 3-28.

Hardin Jr, C. D., Samorodnitsky, G., and M. S., Taqqu. 1991. Nonlinear regression of stable random variables. *The Annals of Applied Probability*, 582-612.

Harvey, C. R., Liechty, J. C., Liechty, M. W., and P., Müller. 2010. Portfolio selection with higher moments. *Quantitative Finance*, 10, 469-485.

Harvey, C. R., and A., Siddique. 1999. Autoregressive conditional skewness. *Journal of financial and quantitative analysis*, 34, 465-487.

Hecq, A., and L., Sun. 2019. Identification of Noncausal Models by Quantile Autoregressions. arXiv preprint arXiv:1904.05952.

Hecq, A., Lieb, L., and S. M., Telg. 2016. Identification of mixed causal-noncausal models in finite samples. *Annals of Economics and Statistics*, 123/124, 307-331.

Hecq, A., Telg, S., and L., Lieb. 2017. Do seasonal adjustments induce noncausal dynamics in inflation rates? *Econometrics*, 5, 48.

Hecq, A., Telg, S., and L, Lieb. 2017. Simulation, estimation and selection of mixed causal-noncausal autoregressive models: The MARX Package. Available at SSRN: https://ssrn.com/abstract=3015797.

Hencic, A., and C., Gouriéroux. 2015. Noncausal autoregressive model in application to Bitcoin/USD exchange rates. *Econometrics of Risk*, Springer International Publishing, 17-40.

Holly, A., Monfort, A., and M., Rockinger. 2011. Fourth order pseudo maximum likelihood methods. *Journal of Econometrics*, 162, 278-293.

Ibragimov, R., and A., Prokhorov. 2016. Heavy tails and copulas: Limits of diversification revisited. *Economics Letters*, 149, 102-107.

Johansen, A., Sornette, D., and O., Ledoit. 1999. Predicting financial crashes using discrete scale invariance. arXiv preprint cond-mat/9903321.

Jondeau, E., and M., Rockinger. 2006. Optimal portfolio allocation under higher moments. *European Financial Management*, 12, 29-55.

Jondeau, E., and M., Rockinger. 2012. On the importance of time variability in higher moments for asset allocation. *Journal of Financial Econometrics*, 10, 84-123.

Karcher, W., Shmileva, E., and E., Spodarev. 2013. Extrapolation of stable random fields. *Journal of Multivariate Analysis*, 115, 516-536.

Lai, T. Y. 1991. Portfolio selection with skewness: a multiple-objective approach. *Review of Quantitative Finance and Accounting*, 1, 293.

Lai, K. K., Yu, L., and S., Wang. 2006. Mean-variance-skewness-kurtosis-based portfolio optimization. In First International Multi-Symposiums on Computer and Computational Sciences (IMSCCS'06). 2, 292-297. IEEE.

Lanne, M., Luoto, J., and P, Saikkonen. 2012. Optimal forecasting of noncausal autoregressive time series. *International Journal of Forecasting*, 28, 623-631.

Lanne, M., Nyberg, H., and E., Saarinen. 2012. Does noncausality help in forecasting economic time series? *Economics Bulletin*, 32, 2849-2859.

Lanne, M., and P., Saikkonen. 2011. Noncausal autogressions for economic time series. *Journal of Time Series Econometrics*, 3.

Lanne, M., and P., Saikkonen. 2013. Noncausal vector autoregression. *Econometric Theory*, 29, 447-481.

Ledford, A. W., and J. A., Tawn. 2003. Diagnostics for dependence within time series extremes. *Journal of the Royal Statistical Society: Series B*, 65, 521-543.

McCulloch, J. H. 1997. Measuring tail thickness to estimate the stable index $\alpha$: A critique. *Journal of Business & Economic Statistics*, 15, 74-81.

Miller, G. 1978. Properties of certain symmetric stable distributions. *Journal of Multivariate Analysis*, 8, 346–360.

Phillips, P. C., Shi, S., and J., Yu. 2015. Testing for multiple bubbles: Historical episodes of exuberance and collapse in the S&P 500. *International economic review*, 56, 1043-1078.

Phillips, P. C., Wu, Y., and J., Yu. 2011. Explosive behavior in the 1990s Nasdaq: When did exuberance escalate asset values?. *International economic review*, 52, 201-226.

Rootzen, H. 1978. Extremes of moving averages of stable processes. *The Annals of Probability*, 6, 847-869.

Samorodnitsky, G., and M. S., Taqqu. 1991. Conditional moments and linear regression for stable random variables. *Stochastic processes and their applications*, 39, 183-199.

Samorodnitsky, G., and M. S., Taqqu. 1994. *Stable non-Gaussian random processes*, Chapman & Hall, London, 516-536.

Sornette, D. 2003. Critical market crashes. *Physics Reports*, 378, 1-98.

Sornette, D. 2017. *Why stock markets crash: critical events in complex financial systems* Vol. 49. Princeton University Press.

Sornette, D., and A., Johansen. 2001. Significance of log-periodic precursors to financial crashes. *Quantitative Finance*, 1, 452-471.

Voisin, E., and A., Hecq. 2019. Forecasting bubbles with mixed causal-noncausal autoregressive models. Maastricht University. https://mpra.ub.uni-muenchen.de/92734/

Wadsworth, J. L., Tawn, J. A., Davison, A. C., and D. M., Elton. 2017. Modelling across extremal dependence classes. *Journal of the Royal Statistical Society: Series B*, 79, 149-175.

Wu, W., and S., Cambanis. 1991. Conditional variance of symmetric stable variables. *Stable Processes and Related Topics* (pp. 85-99). Birkhäuser Boston.

Zolotarev, V. M. 1986. *One-dimensional stable distributions*. Vol. 65, American Mathematical Society.

Zou, J., Davis, R. A., and G., Samorodnitsky. 2017. Extreme value analysis without the largest values: what can be done?. *Probability in the Engineering and Informational Sciences*, 1-21.

# Supplementary file
# [For Online Publication only]

Conditional moments of noncausal alpha-stable processes and the prediction of bubble crash odds

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# A Complementary results

## A.1 Existence of moments and superexponential decay of $(a_k)$: a boundary case

As pointed after Proposition 3.1, noncausal ARMA and fractionally integrated processes whose MA coefficients decay at geometric and hyperbolic speed satisfy condition (3.4) for all $\nu > 0$.[17] Such processes hence admit finite conditional moments at least up to order $2\alpha + 1$. Theorem 5.1.3 by Samorodnitsky and Taqqu, Theorems 1.1, 1.2 in Cioczek-Georges and Taqqu (1995b) however point to the fact that intermediate cases may arise where moments are finite at most up to order $\alpha+\nu$ for some value of $\nu$ such that $\alpha < \alpha+\nu < 2\alpha+1$. We propose here a noncausal MA($\infty$) process with super-exponentially decaying MA coefficients which can reach any intermediate value of the boundary. Consider the noncausal process defined for all $t \in \mathbb{Z}$ by $X_t = \sum_{k=0}^{+\infty} a_k \varepsilon_{t+k}$ with $a_k = \exp\{1 - e^{ak}\}$, $a > 0$, for all $k \geq 0$, and let $(\varepsilon_t)$ be an i.i.d. symmetrically distributed $\alpha$-stable error sequence. Letting $\nu \geq 0$, the general term of the series in (3.4) reads for all $k \geq h$

$$
\begin{aligned}
(a_k^2 + a_{k-h}^2)^{\frac{\alpha+\nu}{2}} |a_k|^{-\nu} &= \left(1 + (a_{k-h}/a_k)^2\right)^{\frac{\alpha+\nu}{2}} |a_k|^{\alpha} \\
&= \Big(1 + \exp\{2e^{ak}(1 - e^{-ah})\}\Big)^{\frac{\alpha+\nu}{2}} \exp\{-\alpha(1 - e^{ak})\} \\
&\underset{k\to+\infty}{\sim} \exp\Big\{e^{ak}\left[(1 - e^{-ah})(\alpha+\nu) - \alpha\right] + \alpha\Big\},
\end{aligned}
$$

which is the term of an absolutely convergent series if and only if $(1 - e^{-ah})(\alpha + \nu) - \alpha < 0$, hence if and only if

$$
\nu < \alpha \left( \frac{1}{1 - e^{-ah}} - 1 \right). \tag{A.1}
$$

Because we assume $(\varepsilon_t)$ to be symmetrically distributed, Theorems 1.1 and 1.2 in Cioczek-Georges and Taqqu (1995b) allow to consider (3.4) and (A.1) as sufficient and necessary

[17] Provided there are no index $k$ such that $a_{k-h} \neq 0$ and $a_k = 0$.

conditions for the finiteness of $\mathbb{E}[|X_{t+h}|^\gamma|X_t]$, $0 \leq \gamma < \min(\alpha+\nu, 2\alpha+1)$, in most configurations of $\alpha$ and $\nu$ (see within Cioczek-Georges and Taqqu (1995b) for details). In particular, one can see that for a fixed prediction horizon $h \geq 1$, the upper bound (A.1) on $\nu$ can lie anywhere between 0 and $+\infty$ according to the parameter $a$. The smaller $a > 0$, i.e., the slower the decay, the higher the bound on $\nu$, and conversely, the greater $a$ (faster decay), the smaller the upper bound on $\nu$ for the existence of conditional moments.

Furthermore, contrary to the case where $(a_k)$ decays at geometric or hyperbolic speeds, the finiteness of $\mathbb{E}[|X_{t+h}|^\gamma|X_t]$ also depends on the prediction horizon $h$. Most notably, for any fixed decay speed $a$, on can see that the bound (A.1) tends to 0 as $h \to +\infty$. For a decay parameter $a$ small enough, the moments $\mathbb{E}[|X_{t+h}|^\gamma|X_t]$ may thus be finite up to order $2\alpha+1$ for short-term prediction horizons while being finite only up to order $\alpha$ for longer-term prediction horizons.

## A.2 Interpreting $\rho^{\alpha h}$ using point processes

The quantity $\rho^{\alpha h}$ appearing in Example 3.1 and subsequent comments has the intuitive interpretation of a survival probability at horizon $h$ of a bubble generated by (1.1). This conclusion can also be reached using point processes under the less restrictive assumption that the errors of (1.1) belong to the domain of attraction of an $\alpha$-stable distribution. Consider $n$ observations $X_1, \ldots, X_n$ of (1.1) where now $(\varepsilon_t)$ is an i.i.d. sequence of random variables such that:

$$\mathbb{P}(|\varepsilon_0| > x) = x^{-\alpha}L(x), \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\mathbb{P}(\varepsilon_0 > x)}{\mathbb{P}(|\varepsilon_0| > x)} \to c \in [0,1],$$

with $L$ a slowly varying function at infinity. Let $a_n = \inf\{u : \mathbb{P}(|\varepsilon_0| > u) \leq n^{-1}\}$. Then, adapting Section 3.D in Davis and (1985), we can study the time indexes $k \in \{1, \ldots, n\}$ for which $a_n^{-1}X_k$ falls outside the interval $(-x, x)$, for $x > 0$, that is, the time indexes for which $(X_t)$ undergoes extreme events. The corresponding point process converges as the number of observations $n$ grows to infinity:

$$\sum_{k=1}^{n} \delta_{\left(k/n, a_n^{-1}X_k\right)}\Big( \ \cdot \ \cap B_x\Big) \xrightarrow{d} \sum_{k=1}^{+\infty} \xi_k \delta_{\Upsilon_k},$$

where $\delta$ is the Dirac measure, $B_x = (0, +\infty) \times \Big((-\infty, -x) \cup (x, +\infty)\Big)$, $\{\Upsilon_k, k \geq 1\}$ are the points of a homogeneous Poisson Random Measure (PRM) on $(0, +\infty)$ with rate $x^{-\alpha}$,[18] and $\xi_k = \text{Card}\{i \in \mathbb{Z} : J_k|\rho^i| > 1\}$ where $\{J_k, k \geq 1\}$ are i.i.d. on $(1, +\infty)$, independent of $\{\Upsilon_k\}$, with common density:

$$f(z) = \alpha z^{-\alpha-1}\mathbb{1}_{(1,+\infty)}(z). \tag{A.2}$$

[18] See Daley and Vere-Jones (2007): $\{\Upsilon_k, k \geq 1\}$ are the points of a homogeneous PRM on $(0, +\infty)$ with rate $x^{-\alpha}$ if and only if, for any $\ell \geq 1$, nonnegative integers $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$ such that $a_i < b_i \leq a_{i+1}$, $i = 1, \ldots, \ell$, and any

The sequences $\{\Upsilon_k\}$ and $\{\xi_k\}$ are interpreted (see Leadbetter and Nandagopalan (1989)) as describing respectively the occurrence dates of clusters of extreme events and the size of these clusters (i.e. the number of co-occurring extreme events, which here corresponds to the duration of bubble episodes). Since $\xi_k = \text{Card}\{i \in \mathbb{Z} : J_k|\rho^i| > 1\} = \arg\max_{i \geq 1}\{J_k > |\rho|^{-i}\}$, we can obtain explicitly the distribution of the bubble duration using (A.2). For any $h \geq 1$,

$$\mathbb{P}\Big(\xi_k \geq h\Big) = \mathbb{P}\Big(J_k > |\rho|^{-h}\Big) = |\rho|^{\alpha h},$$

which as announced, is precisely the probability parameter of the Bernoulli variable intervening in the suggested interpretation in Example 3.1.

# B Preliminary elements for the proof of the main results

## B.1 Notations for the proofs of Theorem 2.1 and Proposition 2.1

The proof of Theorem 2.1 is quite involved and relies on techniques used in [Cioczek-Georges and Taqqu (1994), Cioczek-Georges and Taqqu (1998)]. It consists in differentiating the conditional characteristic function of $X_2|X_1$ up to the fourth derivation order and evaluating the derivatives at 0 to obtain the conditional moments. Formal computation of the derivatives yields divergent terms for the third and fourth order derivatives, as well as for the second order derivative when $1/2 < \alpha < 1$ and special manipulations are needed (in particular the «appropriate integration by parts» in Cioczek-Georges and Taqqu (1994) (p.106) as well as an additional manipulation to obtain the fourth derivative). We first introduce some notations to make the presentation of the proof as compact as possible, then provide the derivatives in Lemma B.1 and finally show Theorem 2.1 by obtaining the functional forms of the conditional moments.

Let $\boldsymbol{X} = (X_1, X_2)$ be an $\alpha$-stable vector, with $0 < \alpha < 2$, $\alpha \neq 1$, and spectral representation $(\Gamma, \mathbf{0})$. Its characteristic function will be denoted $\varphi_{\boldsymbol{X}}(t, r)$ for any $(t, r) \in \mathbb{R}^2$, and reads

$$\varphi_{\boldsymbol{X}}(t, r) = \exp\left\{-\int_{S_2} g_1(ts_1 + rs_2)\Gamma(ds)\right\}, \tag{B.1}$$

where $g_1(z) = |z|^\alpha - iaz^{<\alpha>}$ for $z \in \mathbb{R}$, and $a = \text{tg}(\pi\alpha/2)$. As we assume $\sigma_1 > 0$ so that $X_1$ is not degenerate, the conditional characteristic function of $X_2$ given $X_1 = x$, denoted $\phi_{X_2|x}(r)$ for $r \in \mathbb{R}$,

nonnegative integers $n_1, \ldots, n_\ell$:

$$\mathbb{P}\Big(N(a_i, b_i] = n_i, i = 1, \ldots, \ell\Big) = \prod_{i=1}^{\ell} \frac{[x^{-\alpha}(b_i - a_i)]^{n_i}}{n_i!} \exp\left\{-x^{-\alpha}(b_i - a_i)\right\},$$

where $N(a_i, b_i]$ denotes the number of terms of $\{\Upsilon_k, k \geq 1\}$ falling in the half-open interval $(a_i, b_i]$, $i = 1, \ldots, \ell$.

equals

$$\phi_{X_2|x}(r) := 1 + \frac{1}{2\pi f_{X_1}(x)} \int_{\mathbb{R}} e^{-itx}\Big(\varphi_{\boldsymbol{X}}(t,r) - \varphi_{\boldsymbol{X}}(t,0)\Big)dt. \tag{B.2}$$

where $f_{X_1}$ denotes the density of $X_1 \sim \mathcal{S}(\alpha, \beta_1, \sigma_1, 0)$. The following notation of the $\mathcal{H}$ family function will be more handy than that in (2.8): for any $y > -1$ and $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$, define the function $\mathcal{H}(y, \boldsymbol{\theta}; \cdot)$ for $x \in \mathbb{R}$ as

$$\mathcal{H}(y,\boldsymbol{\theta};x) = \int_0^{+\infty} e^{-\sigma_1^\alpha u^\alpha} u^y \Big(\theta_1 \cos(ux - a\beta_1\sigma_1^\alpha u^\alpha) + \theta_2 \sin(ux - a\beta_1\sigma_1^\alpha u^\alpha)\Big)du, \tag{B.3}$$

For $z \in \mathbb{R}$, denote also,

$$g_2(z) = z^{<\alpha-1>} - ia|z|^{\alpha-1}, \tag{B.4}$$

$$g_3(z) = |z|^{\alpha-2} - iaz^{<\alpha-2>}. \tag{B.5}$$

Often, we shall invoke functions of the form

$$r \longmapsto \int_{\mathbb{R}} e^{-itx}\varphi_{\boldsymbol{X}}(t,r) f_1^{p_1}(t,r)\dots f_m^{p_m}(t,r)dt, \tag{B.6}$$

where $m \leq 3$ and the $f_i$'s will be functions of the type $f_i(t,r) = \int_{S_2} g_{j_i}(ts_1 + rs_2)s_1^{k_i}s_2^{\ell_i}\Gamma(d\boldsymbol{s})$, for $j_i = 2, 3$, $k_i, \ell_i \in \mathbb{Z}$ for which $f_i$ is well defined and positive integer exponents $p_i$'s. As a shorthand when no ambiguity is possible, we shall denote functions like (B.6) by

$$\Lambda\Big(\int_{S_2} g_{j_1} s_1^{k_1} s_2^{\ell_1}\Big)^{p_1}\Big(\int_{S_2} g_{j_2} s_1^{k_2} s_2^{\ell_2}\Big)^{p_2}\dots$$

up to the $m^{\text{th}}$ term.

## B.2 Lemma B.1 for the proof of Theorem 2.1

**Lemma B.1** *Let $(X_1, X_2)$ be an $\alpha$-stable vector, $0 < \alpha < 2, \alpha \neq 1$, with conditional characteristic function $\phi_{X_2|x}$ as given in* (B.2)*. Let $r \in \mathbb{R}$. If $1 < \alpha < 2$, or if $0 < \alpha < 1$ and* (2.2) *holds with $\nu > 1-\alpha$, the first derivative of $\phi_{X_2|x}$ is given by*

$$\phi^{(1)}_{X_2|x}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)}\Lambda\Big(\int_{S_2} g_2 s_2\Big). \tag{B.7}$$

*If $1/2 < \alpha < 2$ and* (2.2) *holds with $\nu > 2-\alpha$, the second derivative is given by*

$$\phi^{(2)}_{X_2|x}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)}\left[ix\Lambda\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big) + \alpha\left\{\Lambda\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big)\Big(\int_{S_2} g_2 s_1\Big) - \Lambda\Big(\int_{S_2} g_2 s_2^2\Big)^2\right\}\right], \tag{B.8}$$

*If $1 < \alpha < 2$ and* (2.2) *holds with $\nu > 3-\alpha$, the third derivative is given by*

$$\phi^{(3)}_{X_2|x}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)}\Big(ix\Big((\alpha-1)I_1 - \alpha I_2\Big) + \alpha^2(I_3 - I_4) + \alpha(\alpha-1)(I_5 + I_6 - 2I_7)\Big), \tag{B.9}$$

*with*

$$I_1 = \Lambda\Big(\int_{S_2} g_3 s_2^3 s_1^{-1}\Big), \qquad I_5 = \Lambda\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big)\Big(\int_{S_2} g_3 s_2 s_1\Big),$$

$$I_2 = \Lambda\Big(\int_{S_2} g_2 s_2\Big)\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big), \qquad I_6 = \Lambda\Big(\int_{S_2} g_2 s_1\Big)\Big(\int_{S_2} g_3 s_2^3 s_1^{-1}\Big),$$

$$I_3 = \Lambda\Big(\int_{S_2} g_2 s_2\Big)^3, \qquad I_7 = \Lambda\Big(\int_{S_2} g_2 s_2\Big)\Big(\int_{S_2} g_3 s_2^2\Big),$$

$$I_4 = \Lambda\Big(\int_{S_2} g_2 s_1\Big)\Big(\int_{S_2} g_2 s_2\Big)\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big).$$

*If* $3/2 < \alpha < 2$ *and* (2.2) *holds with* $\nu > 4 - \alpha$*, the fourth derivative is given by*

$$\begin{aligned}\phi^{(4)}_{X_2|x}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)}\Bigg[\; & i\alpha x\Big(\alpha\Big(3J_1 - 2J_2\Big) + (\alpha-1)\Big(2J_3 - 3J_4 + J_5\Big)\Big) + \alpha x^2 J_6 - (\alpha-1)x^2 J_7 \\ & + \alpha^2(\alpha-1)\Big(J_8 + J_9 + J_{10} - 3\Big(2J_{11} + J_{12} - J_{13}\Big)\Big) \\ & + \alpha(\alpha-1)^2\Big(4J_{14} - 3J_{15} - J_{16}\Big) + \alpha^3\Big(3J_{17} - J_{18} - J_{19}\Big)\Bigg],\end{aligned} \tag{B.10}$$

*with*

$$J_1 = \Lambda\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big)\Big(\int_{S_2} g_2 s_2\Big)^2, \qquad J_{11} = \Lambda\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big)\Big(\int_{S_2} g_3 s_2 s_1\Big)\Big(\int_{S_2} g_2 s_2\Big),$$

$$J_2 = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big)\Big(\int_{S_2} g_2 s_1\Big)\Big(\int_{S_2} g_2 s_2\Big), \qquad J_{12} = \Lambda\Big(\int_{S_2} g_3 s_2^3 s_1^{-1}\Big)\Big(\int_{S_2} g_2 s_1\Big)\Big(\int_{S_2} g_2 s_2\Big),$$

$$J_3 = \Lambda\Big(\int_{S_2} g_3 s_2^4 s_1^{-2}\Big)\Big(\int_{S_2} g_2 s_1\Big), \qquad J_{13} = \Lambda\Big(\int_{S_2} g_3 s_2^2\Big)\Big(\int_{S_2} g_2 s_2\Big)^2,$$

$$J_4 = \Lambda\Big(\int_{S_2} g_3 s_2^3 s_1^{-1}\Big)\Big(\int_{S_2} g_2 s_2\Big), \qquad J_{14} = \Lambda\Big(\int_{S_2} g_3 s_2^3 s_1^{-1}\Big)\Big(\int_{S_2} g_3 s_2 s_1\Big),$$

$$J_5 = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big)\Big(\int_{S_2} g_3 s_2 s_1\Big), \qquad J_{15} = \Lambda\Big(\int_{S_2} g_3 s_2^2\Big)^2,$$

$$J_6 = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big)\Big(\int_{S_2} g_2 s_2\Big), \qquad J_{16} = \Lambda\Big(\int_{S_2} g_3 s_2^4 s_1^{-2}\Big)\Big(\int_{S_2} g_3 s_1^2\Big),$$

$$J_7 = \Lambda\Big(\int_{S_2} g_3 s_2^4 s_1^{-2}\Big), \qquad J_{17} = \Lambda\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big)\Big(\int_{S_2} g_2 s_1\Big)\Big(\int_{S_2} g_2 s_2\Big)^2,$$

$$J_8 = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big)\Big(\int_{S_2} g_3 s_1^2\Big)\Big(\int_{S_2} g_2 s_2\Big), \qquad J_{18} = \Lambda\Big(\int_{S_2} g_2 s_2\Big)^4,$$

$$J_9 = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big)\Big(\int_{S_2} g_3 s_2 s_1\Big)\Big(\int_{S_2} g_2 s_1\Big), \qquad J_{19} = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big)\Big(\int_{S_2} g_2 s_1\Big)^2\Big(\int_{S_2} g_2 s_2\Big),$$

$$J_{10} = \Lambda\Big(\int_{S_2} g_3 s_2^4 s_1^{-2}\Big)\Big(\int_{S_2} g_2 s_1\Big)^2.$$

# C Proof of Lemma B.1

For each of the derivatives, the proof involves two main steps: 1) justifying inversion of integral and derivation signs 2) computation of the derivative.

## C.1 Justifying inversion of integral and derivation signs

### C.1.1 Justifying inversion: First derivative

#### Case $\alpha \in (0,1)$

Assume $\alpha \in (0,1)$. We begin with the first derivative of the imaginary part of $\phi_{X_2|x}$.

$$
\begin{aligned}
&\frac{d}{dr}\Big(\mathrm{Im}\phi_{X_2|x}(r)\Big)\\
&\quad= \frac{-1}{2\pi f_{X_1}(x)} \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\Bigg[e^{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(ds)}\sin\Big(tx - a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(ds)\Big)\\
&\qquad\qquad - e^{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(ds)}\sin\Big(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\Big)\Bigg]dt\\
&\quad= \frac{-1}{2\pi f_{X_1}(x)} \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\Bigg[\sin\Big(tx - a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(ds)\Big)\\
&\qquad\qquad - \sin\Big(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\Big)\Bigg]\\
&\qquad\qquad \times \exp\Big\{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(ds)\Big\}dt\\
&\qquad - \frac{1}{2\pi f_{X_1}(x)} \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\Bigg[\exp\Big\{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(ds)\Big\}\\
&\qquad\qquad - \exp\Big\{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(ds)\Big\}\Bigg]\\
&\qquad\qquad \times \sin\Big(tx - a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(ds)\Big)\,dt\\
&\quad:= I_1 + I_2. \qquad\qquad (\mathrm{C.1})
\end{aligned}
$$

The integrand of $I_1$ converges to

$$
-\alpha a\cos\Big(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\Big) \times \int_{S_2}|ts_1+rs_2|^{\alpha-1}s_2\Gamma(ds) \times \exp\Big\{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(ds)\Big\}
$$

Using the mean value theorem, the triangle inequality and the inequality $-|x+y|^\alpha \le -|x|^\alpha + |y|^\alpha$ when $0<\alpha<1$, the integrand of $I_1$ can be bounded for any $h$, $|h|<|r|$, by

$$
\begin{aligned}
&\Big|cos(y)\Big|\Big(\Big|\frac{a}{h}\Big|\int_{S_2}\Big|(ts_1+(r+h)s_2)^{<\alpha>} - (ts_1+rs_2)^{<\alpha>}\Big|\Gamma(ds)\Big)\exp\Big\{\int_{S_2}-|ts_1|^\alpha + |rs_2|^\alpha\Gamma(ds)\Big\}\\
&\le 2|a|e^{|r|^\alpha\sigma_2^\alpha}e^{-\sigma_1^\alpha|t|^\alpha}\int_{S_2}|ts_1+rs_2|^{\alpha-1}\Gamma(ds), \qquad\qquad (\mathrm{C.2})
\end{aligned}
$$

where $\sigma_2 = \Big(\int_{S_2}|s_2|^\alpha\Gamma(d\boldsymbol{s})\Big)^{1/\alpha}$, $y\in\mathbb{R}$, and we used the bound

$$
\left|\frac{(ts_1+(r+h)s_2)^{<\alpha>} - (ts_1+rs_2)^{<\alpha>}}{h}\right| \le 2|ts_1+rs_2|^{\alpha-1}|s_2|, \qquad (\mathrm{C.3})
$$

for $ts_1+rs_2\neq 0$, which is a consequence of $||1+z|^{<\alpha>}-1|\leq 2|z|$, for $z\in\mathbb{R}$ (see Lemma C.3 ($\iota\iota$) below). Bound (C.2) does not depend on $h$ and is integrable with respect to $t$. Indeed, invoking Lemma C.5 with $\eta=\alpha-1$, $b=p=0$, and (2.2) with $\nu>2-\alpha>1-\alpha$

$$\begin{aligned}
&\left|\int_{\mathbb{R}} e^{-\sigma_1^\alpha|t|^\alpha}\int_{S_2}\left|t+r\frac{s_2}{s_1}\right|^{\alpha-1}|s_1|^{\alpha-1}\Gamma(ds)dt-\int_{\mathbb{R}}\int_{S_2} e^{-\sigma_1^\alpha|t|^\alpha}|t|^{\alpha-1}|s_1|^{\alpha-1}\Gamma(d\boldsymbol{s})dt\right|\\
&\qquad\leq\int_{S_2}|s_1|^{\alpha-1}\int_{\mathbb{R}} e^{-\sigma_1^\alpha|t|^\alpha}\left|\left|t+r\frac{s_2}{s_1}\right|^{\alpha-1}-|t|^{\alpha-1}\right|dt\Gamma(ds)\\
&\qquad\leq\text{const}\int_{S_2}|s_1|^{\alpha-1+\nu}|s_1|^{-\nu}\Gamma(ds)\\
&\qquad\leq\text{const}\int_{S_2}|s_1|^{-\nu}\Gamma(ds)\\
&\qquad<+\infty,
\end{aligned}\tag{C.4}$$

and the integrability with respect to $t$ follows from the fact that $\int_{\mathbb{R}} e^{-\sigma_1^\alpha|t|^\alpha}|t|^{\alpha-1}dt<+\infty$. Hence the Lebesgue dominated convergence theorem applies to $I_1$ and we can invert integration and derivation. Focusing on $I_2$, its integrand tends to

$$-\alpha\int_{S_2}(ts_1+rs_2)^{<\alpha-1>}s_2\Gamma(ds)\exp\left\{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(ds)\right\}\sin\left(tx-a\int_{S_2}|ts_1+rs_2|^{<\alpha>}\Gamma(ds)\right).$$

Using the inequality

$$\left|\frac{(ts_1+(r+h)s_2)^\alpha-(ts_1+rs_2)^\alpha}{h}\right|\leq|ts_1+rs_2|^{\alpha-1}|s_2|,$$

for $ts_1+rs_2\neq 0$, which is a consequence of $||1+z|^\alpha-1|\leq|z|$, for $z\in\mathbb{R}$ (Lemma C.3 ($\iota$) below) and the inequality $|e^{-x}-e^{-y}|\leq e^{-y}e^{|x-y|}|x-y|$, for $x,y\in\mathbb{R}$, we can bound the integrand of $I_2$ for any $|h|<|r|$ by

$$\begin{aligned}
&\exp\left\{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(ds)\right\}\exp\left\{\left|\int_{S_2}|ts_1+(r+h)s_2|^\alpha-|ts_1+rs_2|^\alpha\Gamma(ds)\right|\right\}\\
&\qquad\qquad\qquad\qquad\times\left|\frac{1}{h}\int_{S_2}|ts_1+(r+h)s_2|^\alpha-|ts_1+rs_2|^\alpha\Gamma(ds)\right|\\
&\leq e^{2|r|^\alpha\sigma_2^\alpha}e^{-\sigma_1^\alpha|t|^\alpha}\int_{S_2}\left|t+r\frac{s_2}{s_1}\right|^{\alpha-1}|s_1|^{\alpha-1}\Gamma(ds).
\end{aligned}$$

The integrability with respect to $t$ is deduced as for (C.4) using Lemma C.5 with $\eta=\alpha-1$, $b=p=0$. Thus, the Lebesgue-dominated convergence theorem applies to $I_2$ and we can invert integration and derivation. The real part of $\phi_{X_2|x}(r)$ can be treated in a similar way, allowing us to derivate under the integral.

**Case $\alpha \in (1,2)$**

Assume $\alpha \in (1,2)$. Just as for the case $\alpha \in (0,1)$, the imaginary part of $\phi_{X_2|x}$ is given by (C.1)

$$\frac{d}{dr}\Big(\mathrm{Im}\phi_{X_2|x}(r)\Big) = I_1 + I_2.$$

The integrands of $I_1$ and $I_2$ still converges to the same limits, however a different argument is needed to bound them. For $|h| < |r|$, the mean value theorem, the triangle inequality and the inequality of Lemma C.4, yield the following bound for the integrand of $I_1$

$$\left(\Big|\frac{a}{h}\Big| \int_{S_2} \Big|(ts_1 + (r+h)s_2)^{<\alpha>} - (ts_1 + rs_2)^{<\alpha>}\Big|\Gamma(ds)\right) e^{|r|^\alpha \sigma_2^\alpha} e^{-2^{1-\alpha}\sigma_1^\alpha |t|^\alpha}, \tag{C.5}$$

where $y \in \mathbb{R}$. By the triangle inequality and the mean value theorem, we have for some $u \in \Big(\min\Big(ts_1 + (r+h)s_2, ts_1 + rs_2\Big), \max\Big(ts_1 + (r+h)s_2, ts_1 + rs_2\Big)\Big)$

$$\begin{aligned}
\Big|\int_{S_2} (ts_1 + (r+h)s_2)^{<\alpha>} - (ts_1 + rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big| &= \Big|\int_{S_2} \alpha h s_2 |u|^{\alpha-1}\Gamma(d\boldsymbol{s})\Big| \\
&\leq \alpha|h|\Big|\int_{S_2} |t|^{\alpha-1} + 2|r|^{\alpha-1}\Gamma(d\boldsymbol{s}) \\
&\leq \alpha|h|\Gamma(S_2)(|t|^{\alpha-1} + 2|r|^{\alpha-1})
\end{aligned} \tag{C.6}$$

Thus, (C.5) can be bounded by

$$\alpha|a|\Gamma(S_2)e^{|r|^\alpha\sigma_2^\alpha} e^{-2^{1-\alpha}\sigma_1^\alpha|t|^\alpha}(|t|^{\alpha-1} + 2|r|^{\alpha-1}),$$

which is certainly integrable with respect to $t$ on $\mathbb{R}$ for $\alpha > 1$. Let us now turn to $I_2$. We have again by the mean value theorem,

$$\Big|\frac{|ts_1 + (r+h)s_2|^\alpha - |ts_1 + rs_2|^\alpha}{h}\Big| \leq \alpha(|t|^{\alpha-1} + 2|r|^{\alpha-1}),$$

if $|h| < |r|$, and thus

$$\begin{aligned}
&\Big|\frac{e^{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(d\boldsymbol{s})} - e^{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(d\boldsymbol{s})}}{h}\Big| \\
&\qquad\leq \max\Big(e^{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(d\boldsymbol{s})}, e^{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(d\boldsymbol{s})}\Big) \\
&\qquad\quad\times \int_{S_2}\Big|\frac{|ts_1 + (r+h)s_2|^\alpha - |ts_1 + rs_2|^\alpha}{h}\Big|\Gamma(d\boldsymbol{s}) \\
&\qquad\leq \Gamma(S_2)e^{|2r|^\alpha\sigma_2^\alpha} e^{-2^{1-\alpha}\sigma_1^\alpha|t|^\alpha}\alpha(|t|^{\alpha-1} + 2|r|^{\alpha-1}),
\end{aligned} \tag{C.7}$$

by Lemma C.1 (C.18) and Lemma C.4. The latter bound is again integrable with respect to $t$ on $\mathbb{R}$. Hence the dominated convergence theorem applies to $I_1$, $I_2$ and therefore to $\frac{d}{dr}\Big(\mathrm{Im}\phi_{X_2|x}(r)\Big)$ and we can invert the integration and derivation signs. Similar arguments show the dominated convergence theorem applies to the real part of the conditional characteristic function as well.

### C.1.2 Justifying inversion: Second derivative

**Case $\boldsymbol{\alpha \in (1/2, 1)}$**

In an expanded fashion, $\phi^{(1)}_{X_2|x}(r)$ can be written,

$$\phi^{(1)}_{X_2|x}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)}\Big[J_1 - aJ_2 - i(J_3 + aJ_4)\Big], \tag{C.8}$$

with,

$$\begin{aligned}
J_1(r) &= \int_{\mathbb{R}} e^{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(ds)} \cos\left(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\right)\int_{S_2}(ts_1+rs_2)^{<\alpha-1>}s_2\Gamma(ds)dt,\\
J_2(r) &= \int_{\mathbb{R}} e^{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(ds)} \sin\left(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\right)\int_{S_2}|ts_1+rs_2|^{\alpha-1}s_2\Gamma(ds)dt,\\
J_3(r) &= \int_{\mathbb{R}} e^{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(ds)} \sin\left(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\right)\int_{S_2}(ts_1+rs_2)^{<\alpha-1>}s_2\Gamma(ds)dt,\\
J_4(r) &= \int_{\mathbb{R}} e^{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(ds)} \cos\left(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\right)\int_{S_2}|ts_1+rs_2|^{\alpha-1}s_2\Gamma(ds)dt.
\end{aligned}$$

To obtain $\phi^{(2)}_{X_2|x}(r)$, we will show that the dominated convergence theorem applies to $J_1'$. Let us consider,

$$\begin{aligned}
J_1'(r) = \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\Bigg[&\exp\Big\{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(d\boldsymbol{s})\Big\}\cos\Big(tx - a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(ds)\Big)\\
&\times\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha-1>}s_2\Gamma(ds)\\
&-\exp\Big\{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(d\boldsymbol{s})\Big\}\cos\Big(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\Big)\\
&\times\int_{S_2}(ts_1+rs_2)^{<\alpha-1>}s_2\Gamma(ds)\Bigg]dt\\
= \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\Bigg[&\exp\Big\{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(d\boldsymbol{s})\Big\} - \exp\Big\{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(d\boldsymbol{s})\Big\}\Bigg]\\
&\times\cos\Big(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\Big)\int_{S_2}(ts_1+rs_2)^{<\alpha-1>}s_2\Gamma(ds)dt\\
+ \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}&\exp\Big\{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(d\boldsymbol{s})\Big\}\\
&\times\Bigg[\cos\Big(tx - a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(ds)\Big)\\
&-\cos\Big(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\Big)\Bigg]
\end{aligned} \tag{C.9}$$

$$
\begin{aligned}
&\qquad\qquad\qquad \times \int_{S_2} (ts_1 + rs_2)^{<\alpha-1>} s_2 \Gamma(d\boldsymbol{s}) dt \\
&+ \lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}} \exp\Big\{ - \int_{S_2} |ts_1 + (r+h)s_2|^{\alpha} \Gamma(d\boldsymbol{s}) \Big\} \cos\Big( tx - a \int_{S_2} (ts_1 + (r+h)s_2)^{<\alpha>} \Gamma(ds) \Big) \\
&\qquad \times \Bigg[ \int_{S_2} (ts_1 + (r+h)s_2)^{<\alpha-1>} s_2 \Gamma(ds) - \int_{S_2} (ts_1 + rs_2)^{<\alpha-1>} s_2 \Gamma(ds) \Bigg] dt \\
&:= K_1 + K_2 + K_3. \qquad\qquad (C.10)
\end{aligned}
$$

It can be shown that the dominated convergence theorem applies to $K_1$ following the proof in Cioczek-Georges and Taqqu (1994) (p.105) for $I_1$. Consider $K_2$. The integrand converges to

$$
\begin{aligned}
\alpha a \bigg( \int_{S_2} |ts_1 + rs_2|^{\alpha-1} s_2 \Gamma(ds) \bigg) \bigg( \int_{S_2} (ts_1 + rs_2)^{<\alpha-1>} s_2 \Gamma(ds) \bigg) \\
\times \sin\Big( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \Big) \exp\Big\{ - \int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds) \Big\}.
\end{aligned}
$$

Using the mean value theorem, (C.3) and the triangle inequality, we can bound the integrand for any $|h| < |r|$ by

$$
\begin{aligned}
&\bigg| \frac{1}{h} \int_{S_2} (ts_1 + (r+h)s_2)^{<\alpha>} - (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \bigg| \\
&\qquad \times |\sin(y)| e^{2|r|^{\alpha}\sigma_2^{\alpha}} e^{-|t|^{\alpha}\sigma_1^{\alpha}} \int_{S_2} \Big| t + r\frac{s_2}{s_1} \Big|^{\alpha-1} |s_2| |s_1|^{\alpha-1} \Gamma(ds) \\
&\le 2 e^{2|r|^{\alpha}\sigma_2^{\alpha}} \bigg( \int_{S_2} \Big| t + r\frac{s_2}{s_1} \Big|^{\alpha-1} |s_1|^{\alpha-1} \Gamma(ds) \bigg)^2 e^{-|t|^{\alpha}\sigma_1^{\alpha}} \qquad (C.11)
\end{aligned}
$$

where $y \in \mathbb{R}$. The bound (C.11) does not depend on $h$ and is integrable with respect to $t$: invoking (2.9) Lemma 2.2 in Cioczek-Georges and Taqqu (1994),

$$
\begin{aligned}
&\bigg| \int_{\mathbb{R}} \int_{S_2} \int_{S_2} e^{-\sigma_1^{\alpha}|t|^{\alpha}} \Big| t + r\frac{s_2}{s_1} \Big|^{\alpha-1} \Big| t + r\frac{s_2'}{s_1'} \Big|^{\alpha-1} |s_1'|^{\alpha-1} |s_1|^{\alpha-1} \Gamma(ds)\Gamma(ds')dt \qquad (C.12) \\
&\qquad\qquad - \int_{\mathbb{R}} \int_{S_2} \int_{S_2} e^{-\sigma_1^{\alpha}|t|^{\alpha}} |t|^{2\alpha-2} dt \Gamma(d\boldsymbol{s}) \Gamma(d\boldsymbol{s}') \bigg| \\
&= \bigg| \int_{S_2} \int_{S_2} |s_1'|^{\alpha-1} |s_1|^{\alpha-1} \int_{\mathbb{R}} e^{-\sigma_1^{\alpha}|t|^{\alpha}} \bigg[ \Big| t + r\frac{s_2}{s_1} \Big|^{\alpha-1} \Big| t + r\frac{s_2'}{s_1'} \Big|^{\alpha-1} - \Big| t + r\frac{s_2}{s_1} \Big|^{\alpha-1} \Big| t \Big|^{\alpha-1} \\
&\qquad\qquad + \Big| t + r\frac{s_2}{s_1} \Big|^{\alpha-1} \Big| t \Big|^{\alpha-1} - |t|^{2\alpha-2} \bigg] dt \Gamma(ds) \Gamma(ds') \bigg| \\
&\le \int_{S_2} \int_{S_2} |s_1'|^{\alpha-1} |s_1|^{\alpha-1} \int_{\mathbb{R}} e^{-\sigma_1^{\alpha}|t|^{\alpha}} \bigg[ \bigg| \Big| t + r\frac{s_2'}{s_1'} \Big|^{\alpha-1} - |t|^{\alpha-1} \bigg| \Big| t + r\frac{s_2}{s_1} \Big|^{\alpha-1} \\
&\qquad\qquad + \bigg| \Big| t + r\frac{s_2}{s_1} \Big|^{\alpha-1} - |t|^{\alpha-1} \bigg| |t|^{\alpha-1} \bigg] dt \Gamma(ds) \Gamma(ds')
\end{aligned}
$$

$$
\begin{aligned}
&\leq \text{const}\bigg(\int_{S_2} |s_1|^{\alpha-1}\Gamma(ds)\bigg)^2 \\
&< +\infty, \qquad\qquad (C.13)
\end{aligned}
$$

where const is a constant depending only on $\alpha$ and $\sigma_1^\alpha$. The integrability of (C.11) follows from (C.13), the fact that $\int_{\mathbb{R}} e^{-\sigma_1^\alpha |t|^\alpha}|t|^{2\alpha-2}dt < +\infty$ and (2.2) with $\nu > 2-\alpha > 1-\alpha$. Hence the dominated convergence theorem applies to $K_2$. Let us now turn to $K_3$: «this is [a] case when appropriate "integration by parts" is needed» (Cioczek-Georges and Taqqu (1994)). With the change of variable $t' = t + \dfrac{hs_2'}{s_1'}$,

$$
\begin{aligned}
K_3 = \lim_{h\to 0}\frac{1}{h}\bigg[&\int_{\mathbb{R}} \exp\Big\{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(d\boldsymbol{s})\Big\}\cos\Big(tx - a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(ds)\Big)\\
&\qquad\times\int_{S_2}(t+\frac{hs_2'}{s_1'}+\frac{rs_2'}{s_1'})^{<\alpha-1>}s_2's_1'^{<\alpha-1>}\Gamma(ds')dt\\
&-\int_{\mathbb{R}} \exp\Big\{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(d\boldsymbol{s})\Big\}\cos\Big(tx - a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(ds)\Big)\\
&\qquad\times\int_{S_2}(t+\frac{rs_2'}{s_1'})^{<\alpha-1>}s_2's_1'^{<\alpha-1>}\Gamma(ds')dt\bigg]\\
= \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\bigg[&\exp\Big\{-\int_{S_2}\Big|\Big(t-\frac{hs_2'}{s_1'}\Big)s_1+(r+h)s_2\Big|^\alpha\Gamma(d\boldsymbol{s})\Big\}\\
&\times\cos\bigg(\Big(t-\frac{hs_2'}{s_1'}\Big)x - a\int_{S_2}\Big(\Big(t-\frac{hs_2'}{s_1'}\Big)s_1+(r+h)s_2\Big)^{<\alpha>}\Gamma(ds)\bigg)\\
&-\exp\Big\{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(d\boldsymbol{s})\Big\}\cos\Big(tx - a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(ds)\Big)\bigg]\\
&\times\Big(t+r\frac{s_2'}{s_1'}\Big)^{<\alpha-1>}s_2's_1'^{<\alpha-1>}\Gamma(ds')dt\\
= \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\frac{1}{\frac{hs_2'}{s_1'}}\bigg[&\cos\bigg(\Big(t-\frac{hs_2'}{s_1'}\Big)x - a\int_{S_2}\Big(\Big(t-\frac{hs_2'}{s_1'}\Big)s_1+(r+h)s_2\Big)^{<\alpha>}\Gamma(ds)\bigg)\\
&-\cos\Big(tx - a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(ds)\Big)\bigg]\\
&\times\exp\Big\{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(d\boldsymbol{s})\Big\}\Big(t+r\frac{s_2'}{s_1'}\Big)^{<\alpha-1>}s_2'^2|s_1'|^{\alpha-2}\Gamma(ds')dt\\
+\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\frac{1}{\frac{hs_2'}{s_1'}}\bigg[&\exp\Big\{-\int_{S_2}\Big|\Big(t-\frac{hs_2'}{s_1'}\Big)s_1+(r+h)s_2\Big|^\alpha\Gamma(d\boldsymbol{s})\Big\}-\exp\Big\{-\int_{S_2}|ts_1+(r+h)s_2|^\alpha\Gamma(d\boldsymbol{s})\Big\}\bigg]\\
&\times\cos\bigg(\Big(t-\frac{hs_2'}{s_1'}\Big)x - a\int_{S_2}\Big(\Big(t-\frac{hs_2'}{s_1'}\Big)s_1+(r+h)s_2\Big)^{<\alpha>}\Gamma(ds)\bigg)\\
&\times\Big(t+r\frac{s_2'}{s_1'}\Big)^{<\alpha-1>}s_2'^2|s_1'|^{\alpha-2}\Gamma(ds')dt
\end{aligned}
$$

$$= K_{31} + K_{32}.$$

The case of $K_{32}$ is similar to that of $I_{22}$ in Cioczek-Georges and Taqqu (1994) (p.106-108), the dominated convergence theorem applies. We focus on $K_{31}$. Its integrand converges to

$$\sin\Big(tx - a\int_{S_2}(ts_1 + rs_2)^{<\alpha>}\Gamma(ds)\Big)\exp\Big\{-\int_{S_2}|ts_1 + rs_2|^{\alpha}\Gamma(ds)\Big\}$$
$$\times\Big(x - \alpha a\int_{S_2}|ts_1 + rs_2|^{\alpha-1}s_1\Gamma(ds)\Big)\Big(\int_{S_2}(ts_1' + rs_2')^{<\alpha-1>}{s_2'}^2{s_1'}^{-1}\Gamma(ds')\Big).$$

Using the mean value theorem and Lemma C.3 $(\iota\iota)$, we can bound the integrand of $K_{31}$ for any $|h| < |r|$ by

$$|\sin(y)|e^{2|r|^{\alpha}\sigma_2^{\alpha}}e^{-|t|^{\alpha}\sigma_1^{\alpha}}\int_{S_2}\Big|t + r\frac{s_2'}{s_1'}\Big|^{\alpha-1}{s_2'}^2|s_1'|^{\alpha-2}$$
$$\times\left|\frac{1}{\frac{hs_2'}{s_1'}}\right|\left|-\frac{hs_2'}{s_1'}x - a\int_{S_2}\Big(\Big(t - \frac{hs_2'}{s_1'}\Big)s_1 + (r+h)s_2\Big)^{<\alpha>} - (ts_1 + (r+h)s_2)^{<\alpha>}\Gamma(ds)\Big)\right|\Gamma(ds')$$
$$\leq e^{2|r|^{\alpha}\sigma_2^{\alpha}}e^{-|t|^{\alpha}\sigma_1^{\alpha}}\int_{S_2}\Big|t + r\frac{s_2'}{s_1'}\Big|^{\alpha-1}{s_2'}^2|s_1'|^{\alpha-2}\Big(|x| + 2a\int_{S_2}\Big|t + (r+h)\frac{s_2}{s_1}\Big|^{\alpha-1}|s_1|\Gamma(ds)\Big)\Gamma(ds')$$
$$\leq |x|e^{2|r|^{\alpha}\sigma_2^{\alpha}}e^{-|t|^{\alpha}\sigma_1^{\alpha}}\int_{S_2}\Big|t + r\frac{s_2'}{s_1'}\Big|^{\alpha-1}{s_2'}^2|s_1'|^{\alpha-2}\Gamma(ds')$$
$$+ 2ae^{2|r|^{\alpha}\sigma_2^{\alpha}}e^{-|t|^{\alpha}\sigma_1^{\alpha}}\int_{S_2}\int_{S_2}\Big|t + r\frac{s_2'}{s_1'}\Big|^{\alpha-1}\Big|t + (r+h)\frac{s_2}{s_1}\Big|^{\alpha-1}|s_1|{s_2'}^2|s_1'|^{\alpha-2}\Gamma(ds)\Gamma(ds').$$

The integrability with respect to $t$ of the first (resp. second) term is obtained in the same way as for (C.4) (resp. (C.13)) and concluding using (2.2) with $\nu > 2-\alpha$. Thus, the dominated convergence theorem applies to $K_{31}$, which finally shows that the dominated convergence theorem applies to $J_1'$. The other $J$'s can be treated in a similar fashion.

**Case $\alpha \in (1,2)$**

After derivation, $\phi^{(1)}_{X_2|x}(r)$ is given by (C.8) with functions $J$'s of the form

$$\int_{\mathbb{R}} e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(ds)}\mathrm{trig}\Big(tx - a\int_{S_2}|ts_1 + rs_2|^{<\alpha>}\Gamma(ds)\Big)\int_{S_2}(ts_1 + rs_2)^{<\alpha-1>\text{ or }\alpha-1}s_2\Gamma(ds)dt,$$

which are similar to deal with. Consider for instance $J_1(r)$. It's derivative can be written as in (C.10)

$$J_1'(r) = K_1 + K_2 + K_3.$$

For the integrand of $K_1$, we can use (C.7) and the triangle inequality to bound it by

$$\Gamma(S_2)e^{|2r|^\alpha\sigma_2^\alpha}e^{-2^{1-\alpha}\sigma_1^\alpha|t|^\alpha}\alpha(|t|^{\alpha-1}+2|r|^{\alpha-1})\int_{S_2}|ts_1+rs_2|^{\alpha-1}|s_2|\Gamma(ds).$$

Since $0<\alpha-1<1$, we can further bound it by

$$\Gamma(S_2)e^{|2r|^\alpha\sigma_2^\alpha}e^{-2^{1-\alpha}\sigma_1^\alpha|t|^\alpha}\alpha(|t|^{\alpha-1}+2|r|^{\alpha-1})^2,$$

which is integrable with respect to $t$. The same bound can be obtained for the integrand of $K_2$ using the mean value theorem, (C.6) and Lemma C.4. As for $K_3$, there is no need to perform "appropriate integration by parts" since $0<\alpha-1<1$. Its integrand converges to

$$(\alpha-1)\exp\Big\{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(d\boldsymbol{s})\Big\}\cos\Big(tx-a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(ds)\Big)\int_{S_2}|ts_1+rs_2|^{\alpha-2}s_2^2\Gamma(ds).$$

Using Lemmas C.4 and C.3 ($\iota\iota$), it can be bounded for any $|h|<|r|$ by

$$\begin{aligned}&\frac{2}{|h|}\Gamma(S_2)e^{|2r|^\alpha\sigma_2^\alpha}e^{-2^{1-\alpha}\sigma_1^\alpha|t|^\alpha}\int_{S_2}|ts_1+rs_2|^{\alpha-2}|hs_2|\Gamma(ds),\\ &\le\Gamma(S_2)e^{|2r|^\alpha\sigma_2^\alpha}e^{-2^{1-\alpha}\sigma_1^\alpha|t|^\alpha}\int_{S_2}\Big|t+\frac{rs_2}{s_1}\Big|^{\alpha-2}|s_1|^{\alpha-2}\Gamma(ds).\end{aligned}$$

We can show that this bound is integrable with respect to $t$ using Lemma C.5 with $\eta=\alpha-2$, $b=0$ and $p=0$, the fact that $\int_{\mathbb{R}}e^{-2^{1-\alpha}\sigma_1^\alpha|t|^\alpha}|t|^{\alpha-2}dt<+\infty$ for $\alpha\in(1,2)$ and (2.2) with $\nu>2-\alpha$. The dominated convergence theorem thus applies and we get

$$\begin{aligned}\phi^{(2)}_{X_2|x}(r)=\frac{-\alpha}{2\pi f_{X_1}(x)}\Bigg[&-\alpha\int_{\mathbb{R}}e^{-itx}\varphi_{\boldsymbol{X}}(t,r)\Big(\int_{S_2}g_2(ts_1+rs_2)s_2\Gamma(ds)\Big)^2dt\\ &+(\alpha-1)\int_{\mathbb{R}}e^{-itx}\varphi_{\boldsymbol{X}}(t,r)\Big(\int_{S_2}g_3(ts_1+rs_2)s_2^2\Gamma(ds)\Big)dt\Bigg],\end{aligned}\tag{C.14}$$

with $g_3(z)=|z|^{\alpha-2}-iaz^{<\alpha-2>}$ for $z\in\mathbb{R}$. Integrating by parts the terms $|ts_1+rs_2|^{<\alpha-2>\text{ or }\alpha-2}$ involved in the expression $\int_{\mathbb{R}}e^{-itx}\varphi_{\boldsymbol{X}}(t,r)\Big(\int_{S_2}g_3(ts_1+rs_2)s_2^2\Gamma(ds)\Big)dt$ yields the expression (B.8) obtained in the case $\alpha\in(1/2,1)$. Hence, the same functional form for the second order conditional moment (2.4) in Theorem 2.1 holds when $\alpha>1$.

### C.1.3 Justifying inversion: Third derivative

Let $\alpha\in(1,2)$ and let (2.2) hold with $\nu>3-\alpha$. Starting from the second derivative of $\phi^{(2)}_{X_2|x}(r)$ given at (B.8), with obvious notations

$$\phi^{(2)}_{X_2|x}(r)=\frac{-\alpha}{2\pi f_{X_1}(x)}\Big[ixI_1(r)+\alpha(I_3(r)-I_2(r))\Big]$$

On the one hand, it can be shown that the dominated convergence theorem applies to $I_1'$ using the usual arguments the fact that (2.2) holds with $\nu > 3-\alpha$. On the other hand, after some elementary manipulations, we get that

$$
\begin{aligned}
I_3 - I_2 = \int_{\mathbb{R}} & e^{-itx+ia\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})} e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})} \\
& \times \int_{S_2}\int_{S_2}\Big\{(ts_1+rs_2)^{<\alpha-1>}(ts_1'+rs_2')^{<\alpha-1>} - a^2|ts_1+rs_2|^{\alpha-1}|ts_1'+rs_2'|^{\alpha-1} \\
& \qquad - ia\Big(|ts_1+rs_2|^{\alpha-1}(ts_1'+rs_2')^{<\alpha-1>} + (ts_1+rs_2)^{<\alpha-1>}|ts_1'+rs_2'|^{\alpha-1}\Big)\Big\} \\
& \qquad\qquad \times \Big[s_2^2 s_1^{-1}s_1' - s_2 s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt
\end{aligned}
$$

The previous expression can be decomposed into terms of the form

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{S_2}\int_{S_2} & \operatorname{trig}\Big(-tx + a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big) \\
& \times e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})} \\
& \times |ts_1+rs_2|^{<\alpha-1> \text{ or } \alpha-1} \times |ts_1'+rs_2'|^{<\alpha-1> \text{ or } \alpha-1} \\
& \times \Big[s_2^2 s_1^{-1}s_1' - s_2 s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt,
\end{aligned}
$$

where «trig» is to be replaced by a sine or cosine function. Each of these terms can be treated in a similar way to show that the dominated convergence theorem applies. We will consider

$$
\begin{aligned}
J(r) = \int_{\mathbb{R}}\int_{S_2}\int_{S_2} & \cos\Big(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big) e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})} \\
& \times |ts_1+rs_2|^{\alpha-1}(ts_1'+rs_2')^{<\alpha-1>}\Big[s_2^2 s_1^{-1}s_1' - s_2 s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt.
\end{aligned}
$$

We have

$$
\begin{aligned}
J'(r) = \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\int_{S_2} & \Big[\cos\Big(tx - a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big) \\
& \qquad - \cos\Big(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)\Big] \\
& \times e^{-\int_{S_2}|ts_1+(r+h)s_2|^{\alpha}\Gamma(d\boldsymbol{s})}|ts_1+(r+h)s_2|^{\alpha-1}(ts_1'+(r+h)s_2')^{<\alpha-1>} \\
& \times \Big[s_2^2 s_1^{-1}s_1' - s_2 s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt \\
+ \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\int_{S_2} & \cos\Big(tx - a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big) \\
& \times \Big[e^{-\int_{S_2}|ts_1+(r+h)s_2|^{\alpha}\Gamma(d\boldsymbol{s})} - e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad \times |ts_1+(r+h)s_2|^{\alpha-1}(ts_1'+(r+h)s_2')^{<\alpha-1>}\Big[s_2^2s_1^{-1}s_1'-s_2s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt\\
&+\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\int_{S_2}\cos\Big(tx-a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})}\\
&\qquad\qquad \times\Big[|ts_1+(r+h)s_2|^{\alpha-1}-|ts_1+rs_2|^{\alpha-1}\Big]\\
&\qquad\qquad \times (ts_1'+(r+h)s_2')^{<\alpha-1>}\Big[s_2^2s_1^{-1}s_1'-s_2s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt\\
&+\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\int_{S_2}\cos\Big(tx-a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})}\\
&\qquad\qquad \times\Big[(ts_1'+(r+h)s_2')^{<\alpha-1>}-(ts_1'+rs_2')^{<\alpha-1>}\Big]\\
&\qquad\qquad \times |ts_1+rs_2|^{\alpha-1}\Big[s_2^2s_1^{-1}s_1'-s_2s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt\\
&:= K_1+K_2+K_3+K_4.
\end{aligned}
$$

We will show that we can apply the dominated convergence theorem to the $K_i$'s. Let us begin with $K_1$. Its integrand converges to

$$
\begin{aligned}
&\alpha a\int_{S_2\times S_2\times S_2}\sin\Big(tx-a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})}\\
&\qquad \times|ts_1+rs_2|^{\alpha-1}(ts_1'+rs_2')^{<\alpha-1>}|ts_1''+rs_2''|^{\alpha-1}s_2''\Big[s_2^2s_1^{-1}s_1'-s_2s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')\Gamma(d\boldsymbol{s}'').
\end{aligned}
$$

For any $h$, $|h|<|r|$, the integrand of $K_1$ can be bounded using the mean value theorem on the cosine and Lemma C.4 by

$$
\begin{aligned}
&\frac{|a|}{|h|}\Big|\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}-(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big|e^{2^{\alpha}|r|^{\alpha}\sigma_2^{\alpha}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}\\
&\quad\times\Big|\int_{S_2}\int_{S_2}|ts_1+(r+h)s_2|^{\alpha-1}(ts_1'+(r+h)s_2')^{<\alpha-1>}\Big[s_2^2s_1^{-1}s_1'-s_2s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')\Big|. \qquad \text{(C.15)}
\end{aligned}
$$

Hence, by inequality (C.6) and given that $0<\alpha-1<1$, the quantity (C.15) can be bounded by

$$
\begin{aligned}
&\alpha|a|\Gamma(S_2)e^{2^{\alpha}|r|^{\alpha}\sigma_2^{\alpha}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}(|t|^{\alpha-1}+2|r|^{\alpha-1})\\
&\quad\times\Big|\int_{S_2}\int_{S_2}|ts_1+(r+h)s_2|^{\alpha-1}(ts_1'+(r+h)s_2')^{<\alpha-1>}\Big[s_2^2s_1^{-1}s_1'-s_2s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')\Big|\\
&\le\alpha|a|\Gamma(S_2)e^{2^{\alpha}|r|^{\alpha}\sigma_2^{\alpha}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}(|t|^{\alpha-1}+2|r|^{\alpha-1})^3\Big(\Gamma(S_2)+\int_{S_2}|s_1|^{-1}\Gamma(d\boldsymbol{s})\Big)\\
&\le \text{const }e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}(|t|^{\alpha-1}+2|r|^{\alpha-1})^3,
\end{aligned}
$$

where const is a finite nonnegative constant because of (2.2) with $\nu>3-\alpha>1$ and the fact that $\Gamma$ is a finite measure. This last bound, independent of $h$, is integrable with respect to $t$ on $\mathbb{R}$. The dominated convergence theorem applies to $K_1$. Consider now $K_2$. Its integrand converges to

$$\alpha\int_{S_2\times S_2\times S_2}\cos\left(tx-a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\right)e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})} \tag{C.16}$$
$$\times|ts_1+rs_2|^{\alpha-1}(ts_1'+rs_2')^{<\alpha-1>}(ts_1''+rs_2'')^{<\alpha-1>}s_2''\Big[s_2^2s_1^{-1}s_1'-s_2s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')\Gamma(d\boldsymbol{s}'')$$

By (C.7), the integrand of $K_2$ can be bounded by

$$\Gamma(S_2)e^{|2r|^{\alpha}\sigma_2^{\alpha}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}\alpha(|t|^{\alpha-1}+2|r|^{\alpha-1})$$
$$\left|\int_{S_2}\int_{S_2}|ts_1+(r+h)s_2|^{\alpha-1}(ts_1'+(r+h)s_2')^{<\alpha-1>}\Big[s_2^2s_1^{-1}s_1'-s_2s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')\right|$$

Which can be further bounded by an integrable function of $t$ in a similar way as for the integrand of $K_1$. The dominated convergence theorem applies to $K_2$. Consider now $K_3$. Its integrand converges to

$$(\alpha-1)\int_{S_2}\int_{S_2}\cos\left(tx-a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\right)e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})}$$
$$\times(ts_1+rs_2)^{<\alpha-2>}(ts_1'+(r+h)s_2')^{<\alpha-1>}s_2\Big[s_2^2s_1^{-1}s_1'-s_2s_2'\Big]\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')$$

Using Lemmas C.4, C.3 ($\iota$) and the triangle inequality, the integrand of $K_3$ can be bounded by

$$\frac{1}{|h|}e^{|r|^{\alpha}\sigma_2^{\alpha}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}\int_{S_2}\int_{S_2}|hs_2||ts_1+rs_2|^{\alpha-2}|ts_1'+(r+h)s_2'|^{\alpha-1}\Big|s_2^2s_1^{-1}s_1'-s_2s_2'\Big|\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')$$
$$\leq e^{|r|^{\alpha}\sigma_2^{\alpha}}\Gamma(S_2)\int_{S_2}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}|ts_1+rs_2|^{\alpha-2}(|t|^{\alpha-1}+2|r|^{\alpha-1})\Big|1+|s_1|^{-1}\Big|\Gamma(d\boldsymbol{s})$$

To show the integrability with respect to $t$ of the last bound we make use of Lemma C.5 with $\eta=\alpha-2$, $b=0,\alpha-1$ and $p=0$ and the fact that with $1<\alpha<2$, $\int_{\mathbb{R}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}|t|^{\alpha-2}dt<+\infty$ and $\int_{\mathbb{R}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}|t|^{2\alpha-3}dt<+\infty$

$$e^{|r|^{\alpha}\sigma_2^{\alpha}}\Gamma(S_2)\int_{S_2}\Big|1+|s_1|^{-1}\Big|\int_{\mathbb{R}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}|s_1|^{\alpha-2}\Big|t+r\frac{s_2}{s_1}\Big|^{\alpha-2}(|t|^{\alpha-1}+2|r|^{\alpha-1})dt\Gamma(d\boldsymbol{s})$$
$$\leq e^{|r|^{\alpha}\sigma_2^{\alpha}}\Gamma(S_2)\int_{S_2}\Big|1+|s_1|^{-1}\Big||s_1|^{\alpha-2}\Bigg[\int_{\mathbb{R}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}\Bigg|\Big|t+r\frac{s_2}{s_1}\Big|^{\alpha-2}-|t|^{\alpha-2}+|t|^{\alpha-2}\Bigg||t|^{\alpha-1}dt$$
$$+2|r|^{\alpha-1}\int_{\mathbb{R}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}\Bigg|\Big|t+r\frac{s_2}{s_1}\Big|^{\alpha-2}-|t|^{\alpha-2}+|t|^{\alpha-2}\Bigg|dt\Bigg]\Gamma(d\boldsymbol{s})$$
$$\leq e^{|r|^{\alpha}\sigma_2^{\alpha}}\Gamma(S_2)\int_{S_2}\Big|1+|s_1|^{-1}\Big||s_1|^{\alpha-2}\Bigg[\int_{\mathbb{R}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}\Bigg|\Big|t+r\frac{s_2}{s_1}\Big|^{\alpha-2}-|t|^{\alpha-2}\Bigg||t|^{\alpha-1}dt$$
$$+2|r|^{\alpha-1}\int_{\mathbb{R}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}\Bigg|\Big|t+r\frac{s_2}{s_1}\Big|^{\alpha-2}-|t|^{\alpha-2}\Bigg|dt$$
$$+\int_{\mathbb{R}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}|t|^{2\alpha-3}dt$$
$$+2|r|^{\alpha-1}\int_{\mathbb{R}}e^{-2^{1-\alpha}\sigma_1^{\alpha}|t|^{\alpha}}|t|^{\alpha-2}dt\Bigg]\Gamma(d\boldsymbol{s})$$

$$\le \text{const } \int_{S_2} \Big|1 + |s_1|^{-1}\Big| |s_1|^{\alpha-2}\Gamma(d\boldsymbol{s})$$
$$\le \text{const } \Big(\int_{S_2} |s_1|^{\alpha-2}\Gamma(d\boldsymbol{s}) + \int_{S_2} |s_1|^{\alpha-3}\Gamma(d\boldsymbol{s})\Big),$$

which is finite because of (2.2) with $\nu > 3-\alpha$. Hence, the dominated convergence theorem applies to $K_3$. The case of $K_4$ is similar, using Lemma C.3 ($\iota\iota$) instead of ($\iota$) to bound the term $\Big|(ts_1' + (r+h)s_2')^{<\alpha-2>} - (ts_1' + rs_2')^{<\alpha-2>}\Big|$. The dominated convergence theorem applies to all the $K_i$'s and we can invert the integration and derivation signs in $J'$.

### C.1.4 A special manipulation to obtain the fourth derivative

Before derivating $\phi^{(3)}_{X_2|x}$, we follow the advice stated in Cioczek-Georges and Taqqu (1998) (p.48) and integrate by parts the terms containing $\int_{S_2} g_3(ts_1 + rs_2)s_2^3 s_1^{-1}\Gamma(d\boldsymbol{s})$ and $\int_{S_2} g_3(ts_1 + rs_2)s_2^2\Gamma(d\boldsymbol{s})$, namely $I_1$, $I_6$ and $I_7$. This is done in order to guarantee the validity of the representation of the fourth derivative when (2.2) holds for any $\nu > 4-\alpha$. If we did not do this step first, the obtained fourth derivative would be valid only when (2.2) holds with $\nu > 5-\alpha$. We obtain

$$\phi^{(3)}_{X_2|x}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)}\Big[i\alpha x\Big(I_{11} - I_2 + I_{62} - 2I_{72}\Big) - x^2 I_{12} + \alpha^2\Big(I_3 - I_4 - 2I_{71} + I_{61}\Big) + \alpha(\alpha-1)\Big(I_5 - I_{63} + 2I_{73}\Big)\Big], \tag{C.17}$$

where, in addition to $I_2$, $I_3$, $I_4$ and $I_5$ defined in the Lemma,

$$I_{11} = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big)\Big(\int_{S_2} g_2 s_1\Big), \qquad I_{12} = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big),$$
$$I_{61} = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big)\Big(\int_{S_2} g_2 s_1\Big)^2, \qquad I_{71} = \Lambda\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big)\Big(\int_{S_2} g_2 s_1\Big)\Big(\int_{S_2} g_2 s_2\Big),$$
$$I_{62} = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big)\Big(\int_{S_2} g_2 s_1\Big), \qquad I_{72} = \Lambda\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big)\Big(\int_{S_2} g_2 s_2\Big),$$
$$I_{63} = \Lambda\Big(\int_{S_2} g_2 s_2^3 s_1^{-2}\Big)\Big(\int_{S_2} g_3 s_1^2\Big), \qquad I_{73} = \Lambda\Big(\int_{S_2} g_2 s_2^2 s_1^{-1}\Big)\Big(\int_{S_2} g_3 s_2 s_1\Big).$$

Both justification and computation of the fourth derivative are obtained by starting from the above representation of the third derivative.

### C.1.5 Justifying inversion: Fourth derivative

Showing that the dominated convergence theorem holds when differentiating (C.17) is the most delicate for the terms: $I_5$, $I_{63}$ and $I_{73}$ -the terms involving the function $g_3$, that is, $|ts_1 + rs_2|$ to the power $\alpha - 2$. Arguments and bounds that have already been encountered can be used for the other ones.

Let us show the dominated convergence theorem applies to $I_5$. The cases of $I_{63}$ and $I_{73}$ are similar. We decompose $I_5$ into terms of the form

$$\int_{\mathbb{R}}\int_{S_2}\int_{S_2}\operatorname{trig}\Big(-tx+a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})}$$
$$\times|ts_1+rs_2|^{\alpha-1 \text{ or } <\alpha-1>}|ts_1'+rs_2'|^{\alpha-2 \text{ or } <\alpha-2>}s_2^2s_1^{-1}s_2's_1'\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt.$$

Consider for instance

$$J(r):=\int_{\mathbb{R}}\int_{S_2}\int_{S_2}\cos\Big(-tx+a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})}$$
$$\times|ts_1+rs_2|^{\alpha-1}|ts_1'+rs_2'|^{\alpha-2}s_2^2s_1^{-1}s_2's_1'\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt.$$

We have

$$\begin{aligned}
J'(r)=\lim_{h\to0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\int_{S_2}&\Big[|ts_1'+(r+h)s_2'|^{\alpha-2}-|ts_1'+rs_2'|^{\alpha-2}\Big]|ts_1+(r+h)s_2|^{\alpha-1}\\
&\times\cos\Big(-tx+a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)\\
&\times e^{-\int_{S_2}|ts_1+(r+h)s_2|^{\alpha}\Gamma(d\boldsymbol{s})}s_2^2s_1^{-1}s_2's_1'\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt\\
+\lim_{h\to0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\int_{S_2}&|ts_1'+rs_2'|^{\alpha-2}\Big[|ts_1+(r+h)s_2|^{\alpha-1}-|ts_1+rs_2|^{\alpha-1}\Big]\\
&\times\cos\Big(-tx+a\int_{S_2}(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)\\
&\times e^{-\int_{S_2}|ts_1+(r+h)s_2|^{\alpha}\Gamma(d\boldsymbol{s})}s_2^2s_1^{-1}s_2's_1'\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt\\
+\lim_{h\to0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\int_{S_2}&|ts_1'+rs_2'|^{\alpha-2}|ts_1+rs_2|^{\alpha-1}\\
\times\Big[\cos\Big(-tx+a\int_{S_2}&(ts_1+(r+h)s_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)-\cos\Big(-tx+a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)\Big]\\
&\times e^{-\int_{S_2}|ts_1+(r+h)s_2|^{\alpha}\Gamma(d\boldsymbol{s})}s_2^2s_1^{-1}s_2's_1'\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt\\
+\lim_{h\to0}\frac{1}{h}\int_{\mathbb{R}}\int_{S_2}\int_{S_2}&|ts_1'+rs_2'|^{\alpha-2}|ts_1+rs_2|^{\alpha-1}\\
&\times\cos\Big(-tx+a\int_{S_2}(ts_1+rs_2)^{<\alpha>}\Gamma(d\boldsymbol{s})\Big)\\
&\times\Big[e^{-\int_{S_2}|ts_1+(r+h)s_2|^{\alpha}\Gamma(d\boldsymbol{s})}-e^{-\int_{S_2}|ts_1+rs_2|^{\alpha}\Gamma(d\boldsymbol{s})}\Big]s_2^2s_1^{-1}s_2's_1'\Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}')dt\\
:=K_1+K_2+K_3+K_4&
\end{aligned}$$

The integrand of $K_4$ can be bounded using inequality (C.16), (C.7) and invoking Lemma C.5 and (2.2) with $\nu>4-\alpha$. The integrand of $K_3$ can be bounded using (C.6) Lemma C.4, and concluding with

Lemma C.5 and (2.2) with $\nu > 4-\alpha$. Focus now on $K_2$. Using Lemmas C.4 and C.3 ($\iota$), its integrand can be bounded by

$$e^{|2r|^\alpha \sigma_2^\alpha} e^{-2^{1-\alpha}\sigma_1^\alpha |t|^\alpha} \left|t + \frac{rs_2'}{s_1'}\right|^{\alpha-2} \left|t + \frac{rs_2}{s_1}\right|^{\alpha-2} s_2^3 |s_1|^{\alpha-3} |s_1'|^{\alpha-1} |s_2'|.$$

The later bound does not depend on $h$ and can be shown to be integrable with respect to $t$ using (2.2) with $\nu > 4-\alpha$, Lemma C.6 with $\eta = \alpha - 2$, $z_2 = z_4 = 0$, $p = 0$ and the fact that $\int_{\mathbb{R}} e^{-c|t|^\alpha} |t|^{2(\alpha-2)} < +\infty$ for $\alpha \in (3/2, 2)$. Let us now turn to the term $K_1$ which is more intricate. Appropriate «integration by parts» is required. With the change of variable $t = t + \frac{hs_2'}{s_1'}$,

$$\begin{aligned}
K_1 = \lim_{h\to 0} \frac{1}{h} \int_{S_2}\int_{S_2}\int_{\mathbb{R}} &\left[ e^{-\int_{S_2} \left| \left(t - \frac{hs_2'}{s_1'}\right)s_1 + (r+h)s_2 \right|^\alpha \Gamma(d\boldsymbol{s})} - e^{-\int_{S_2} |ts_1 + (r+h)s_2|^\alpha \Gamma(d\boldsymbol{s})} \right] \\
&\times \cos\left( \left(t - \frac{hs_2'}{s_1'}\right)x - a \int_{S_2} \left( \left(t - \frac{hs_2'}{s_1'}\right)s_1 + (r+h)s_2 \right)^{<\alpha>} \Gamma(d\boldsymbol{s}) \right) \\
&\times \left| \left(t - \frac{hs_2'}{s_1'}\right)s_1 + (r+h)s_2 \right|^{\alpha-1} |ts_1' + rs_2'|^{\alpha-2} s_2^2 s_1^{-1} s_2' s_1' dt \Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}') \\
+ \lim_{h\to 0} \frac{1}{h} \int_{S_2}\int_{S_2}\int_{\mathbb{R}} & e^{-\int_{S_2} |ts_1 + (r+h)s_2|^\alpha \Gamma(d\boldsymbol{s})} \\
&\times \cos\left( \left(t - \frac{hs_2'}{s_1'}\right)x - a \int_{S_2} \left( \left(t - \frac{hs_2'}{s_1'}\right)s_1 + (r+h)s_2 \right)^{<\alpha>} \Gamma(d\boldsymbol{s}) \right) \\
&\times \left[ \left| \left(t - \frac{hs_2'}{s_1'}\right)s_1 + (r+h)s_2 \right|^{\alpha-1} - \left| ts_1 + (r+h)s_2 \right|^{\alpha-1} \right] \\
&\times |ts_1' + rs_2'|^{\alpha-2} s_2^2 s_1^{-1} s_2' s_1' dt \Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}') \\
+ \lim_{h\to 0} \frac{1}{h} \int_{S_2}\int_{S_2}\int_{\mathbb{R}} & e^{-\int_{S_2} |ts_1 + (r+h)s_2|^\alpha \Gamma(d\boldsymbol{s})} \\
&\times \left[ \cos\left( \left(t - \frac{hs_2'}{s_1'}\right)x - a \int_{S_2} \left( \left(t - \frac{hs_2'}{s_1'}\right)s_1 + (r+h)s_2 \right)^{<\alpha>} \Gamma(d\boldsymbol{s}) \right) \right. \\
&\qquad \left. - \cos\left( tx - a \int_{S_2} \left( ts_1 + (r+h)s_2 \right)^{<\alpha>} \Gamma(d\boldsymbol{s}) \right) \right] \\
&\times |ts_1 + (r+h)s_2|^{\alpha-1} |ts_1' + rs_2'|^{\alpha-2} s_2^2 s_1^{-1} s_2' s_1' dt \Gamma(d\boldsymbol{s})\Gamma(d\boldsymbol{s}') \\
:= K_{11} + K_{12} + K_{13}.
\end{aligned}$$

It can be shown that the generalised Lebesgue convergence theorem applies to the terms $K_{11}$ and $K_{12}$ following the proof in Cioczek-Georges and Taqqu (1998) (p.50-52). Regarding the integrand of $K_{13}$, using the mean value theorem on the cosine, Lemma C.4 and (C.6), we get for $|h| < |r|$

$$\frac{1}{\left|\frac{hs_2'}{s_1'}\right|} e^{|2r|^\alpha \sigma_2^\alpha} e^{-2^{1-\alpha}\sigma_1^\alpha |t|^\alpha} |ts_1 + (r+h)s_2|^{\alpha-1} |ts_1' + rs_2'|^{\alpha-2} s_2^2 |s_1|^{-1} |s_2'|^2$$

$$
\begin{aligned}
&\times \left|\frac{hs_2'}{s_1'}x + a\int_{S_2}\Big(\Big(t-\frac{hs_2'}{s_1'}\Big)s_1+(r+h)s_2\Big)^{<\alpha>} - \Big(ts_1+(r+h)s_2\Big)^{<\alpha>}\Gamma(d\boldsymbol{s})\right| \\
\le\ & \frac{1}{\left|\frac{hs_2'}{s_1'}\right|} e^{|2r|^\alpha \sigma_2^\alpha} e^{-2^{1-\alpha}\sigma_1^\alpha |t|^\alpha} \big|ts_1+(r+h)s_2\big|^{\alpha-1} |ts_1'+rs_2'|^{\alpha-2} s_2^2 |s_1|^{-1}|s_2'|^2 \\
&\times \left[\Big|\frac{hs_2'}{s_1'}x\Big| + \Big|a\frac{hs_2'}{s_1'}\Big|\int_{S_2}|s_1||ts_1+(r+h)s_2|^{\alpha-1}\Gamma(d\boldsymbol{s})\right] \\
\le\ & e^{|2r|^\alpha \sigma_2^\alpha} e^{-2^{1-\alpha}\sigma_1^\alpha |t|^\alpha} \Big|t+\frac{rs_2'}{s_1'}\Big|^{\alpha-2} s_2^2|s_1|^{-1}{s_2'}^2|s_1'|^{\alpha-2} \\
&\times \left(|t|^{\alpha-1}+|2r|^{\alpha-1}\right)\Big[|x| + |a|\Gamma(S_2)(|t|^{\alpha-1}+|2r|^{\alpha-1})\Big].
\end{aligned}
$$

The last bound can be shown to be integrable with respect to $t$ using Lemma C.7 with $\eta = \alpha - 2$, $b = 0, \alpha-1, 2(\alpha-1)$, $p = 0$ and (2.2) with $\nu > 4 - \alpha$. We established that we can invert the derivation and integration signs in all the $K_i$'s, hence in $J'$.

#### C.1.6 Lemmas for justifying the inversions in the proof of Lemma B.1

The following elementary lemmas, stated without proof, are used to establish Lemma B.1.

**Lemma C.1** *For $x, y \in \mathbb{R}$,*

$$|e^{-x} - e^{-y}| \le e^{-\min(x,y)}|x-y|, \tag{C.18}$$

$$|e^{-x} - e^{-y}| \le e^{-y}e^{|x-y|}|x-y|. \tag{C.19}$$

**Lemma C.2** *For $\alpha > 1$ and $x, y \in \mathbb{R}$,*

$$\max\Big(2^{1-\alpha}|x|^\alpha - |y|^\alpha, 2^{1-\alpha}|y|^\alpha - |x|^\alpha\Big) \le |x+y|^\alpha \le 2^{\alpha-1}\Big(|x|^\alpha + |y|^\alpha\Big).$$

**Lemma C.3** *For $z \in \mathbb{R}$ and $0 < b \le 1$,*

$$
\begin{aligned}
&(\iota) \qquad \Big||1+z|^b - 1\Big| \le |z|, \\
&(\iota\iota) \qquad \Big||1+z|^{<b>} - 1\Big| \le 2|z|.
\end{aligned}
$$

**Lemma C.4 (Lemma 3.3, Cioszek-Georges and Taqqu (1998))** *For $\alpha > 1$ and $t, r \in \mathbb{R}$,*

$$\exp\Big\{-\int_{S_2}|ts_1+rs_2|^\alpha\Gamma(d\boldsymbol{s})\Big\} \le \exp\{|r|^\alpha\sigma_2^\alpha\}\exp\{-2^{1-\alpha}\sigma_1^\alpha|t|^\alpha\}.$$

**Lemma C.5 (Lemma 3.1, Cioszek-Georges and Taqqu (1998))** *The following inequality holds for $c > 0$, $0 < \alpha < 2$, $-1 < \eta < 0$ and $-1-\eta < b$:*

$$\int_{\mathbb{R}}\exp(-c|t|^\alpha)\Big||t+z|^\eta - |t|^\eta\Big||t|^b dt \le const.\,|z|^p$$

*with*

$$0 \leq p < b + \eta + 1 \quad for \quad -1 - \eta < b < 0,$$

*and*

$$0 \leq p < \eta + 1 \quad or \quad b \leq p < b + \eta + \eta + 1, p \leq 1 \quad for \quad 0 \leq b.$$

*const. depends only on* $c$, $\alpha$, $\eta$, $b$ *and* $p$.

**Lemma C.6 (Corollary 3.1, Cioszek-Georges and Taqqu (1998))** *The following inequality holds for* $c > 0$, $0 < \alpha < 2$, $-1/2 < \eta < 0$ *and* $0 \leq p < 2\eta + 1$*:*

$$\int_{\mathbb{R}} \exp(-c|t|^{\alpha}) \Big| |t + z_1|^{\eta} |t + z_3|^{\eta} - |t + z_2|^{\eta} |t + z_4|^{\eta} \Big| dt \leq const. \left(|z_1 - z_2|^p + |z_3 - z_4|^p\right),$$

*where const depends only on* $c$, $\alpha$, $\eta$ *and* $p$.

**Lemma C.7 (Lemma 3.12, Cioszek-Georges and Taqqu (1998))** *The following inequality holds for* $c > 0$, $0 < \alpha < 2$, $-1 < \eta < 0$, $b \geq 0$ *and* $0 \leq p < \eta + 1$*:*

$$\int_{\mathbb{R}} \exp(-c|t|^{\alpha}) \Big| |t + z_1|^{\eta} - |t + z_2|^{\eta} \Big| |t|^b dt \leq const. \, |z_1 - z_2|^p,$$

*where const depends only on* $c$, $\alpha$, $\eta$, $b$ *and* $p$.

## C.2 Computation of the derivatives

We detail the computation of the second order derivative highlighting where appropriate integration by parts intervenes. The computations are similar for the third and fourth order derivatives.

Note that if $f(x) = |x|^b$, for $x, b \in \mathbb{R}$, $b \neq 0$, then for $x \neq 0$, $f'(x) = b x^{<b-1>}$ and if $f : x \longmapsto x^{<b>}$, then $f'(x) = b|x|^{b-1}$. This can be shown by distinguishing the cases $x > 0$ and $x < 0$.

$$\begin{aligned}
\phi^{(2)}_{X_2|x}(r) &= \frac{\partial}{\partial r} \phi^{(1)}_{X_2|x}(r) \\
&= \frac{-\alpha}{2\pi f_{X_1}(x)} \lim_{h \to 0} \frac{1}{h} \Bigg[ \int_{\mathbb{R}} \int_{S_2} e^{-itx} \varphi_{\boldsymbol{X}}(t, r + h) g_2(ts_1 + (r + h)s_2) s_2 \Gamma(d\boldsymbol{s}) dt \\
&\qquad\qquad\qquad\qquad - \int_{\mathbb{R}} \int_{S_2} e^{-itx} \varphi_{\boldsymbol{X}}(t, r) g_2(ts_1 + rs_2) s_2 \Gamma(d\boldsymbol{s}) dt \Bigg] \\
&= \frac{-\alpha}{2\pi f_{X_1}(x)} \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} e^{-itx} \Big[ \varphi_{\boldsymbol{X}}(t, r + h) - \varphi_{\boldsymbol{X}}(t, r) \Big] g_2(ts_1 + (r + h)s_2) s_2 \Gamma(d\boldsymbol{s}) dt \\
&\quad + \frac{-\alpha}{2\pi f_{X_1}(x)} \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} e^{-itx} \varphi_{\boldsymbol{X}}(t, r) \Big[ g_2(ts_1 + (r + h)s_2) - g_2(ts_1 + rs_2) \Big] s_2 \Gamma(d\boldsymbol{s}) dt
\end{aligned}$$

$$:= A_1 + A_2.$$

The first limit can be straightforwardly obtained:

$$\begin{aligned} A_1 &= \frac{\alpha^2}{2\pi f_{X_1}(x)} \int_{\mathbb{R}} e^{-itx} \varphi_{\boldsymbol{X}}(t,r) \Big( \int_{S_2} g_2(ts_1 + rs_2) s_2 \Gamma(d\boldsymbol{s}) \Big)^2 dt \\ &= \frac{\alpha^2}{2\pi f_{X_1}(x)} \Lambda \Big( \int_{S_2} g_2 s_2 \Big)^2. \end{aligned}$$

The second one requires appropriate integration by parts. With the change of variable $t' = t + \dfrac{hs_2}{s_1}$,

$$\begin{aligned} A_2 &= \frac{-\alpha}{2\pi f_{X_1}(x)} \lim_{h\to 0} \frac{1}{h} \Big[ \int_{S_2} \int_{\mathbb{R}} e^{-itx} \varphi_{\boldsymbol{X}}(t,r) g_2(ts_1 + (r+h)s_2) s_2 dt \Gamma(d\boldsymbol{s}) \\ &\qquad\qquad - \int_{S_2} \int_{\mathbb{R}} e^{-itx} \varphi_{\boldsymbol{X}}(t,r) g_2(ts_1 + rs_2) s_2 dt \Gamma(d\boldsymbol{s}) \Big] \\ &= \frac{-\alpha}{2\pi f_{X_1}(x)} \lim_{h\to 0} \frac{1}{h} \Big[ \int_{S_2} \int_{\mathbb{R}} e^{-i\left(t - \frac{hs_2}{s_1}\right)x} \varphi_{\boldsymbol{X}}\Big(t - \frac{hs_2}{s_1}, r\Big) g_2(ts_1 + rs_2) s_2 dt \Gamma(d\boldsymbol{s}) \\ &\qquad\qquad - \int_{S_2} \int_{\mathbb{R}} e^{-itx} \varphi_{\boldsymbol{X}}(t,r) g_2(ts_1 + rs_2) s_2 dt \Gamma(d\boldsymbol{s}) \Big] \\ &= \frac{\alpha}{2\pi f_{X_1}(x)} \int_{S_2} \int_{\mathbb{R}} s_2^2 s_1^{-1} g_2(ts_1 + rs_2) \lim_{h\to 0} \frac{1}{-\frac{hs_2}{s_1}} \Big[ e^{-i\left(t - \frac{hs_2}{s_1}\right)x} \varphi_{\boldsymbol{X}}\Big(t - \frac{hs_2}{s_1}, r\Big) \\ &\qquad\qquad - e^{-itx} \varphi_{\boldsymbol{X}}(t,r) \Big] dt \Gamma(d\boldsymbol{s}) \\ &= \frac{\alpha}{2\pi f_{X_1}(x)} \int_{S_2} \int_{\mathbb{R}} s_2^2 s_1^{-1} g_2(ts_1 + rs_2) \Big[ -ixe^{-itx} \varphi_{\boldsymbol{X}}(t,r) + e^{-itx} \frac{\partial}{\partial t} \varphi_{\boldsymbol{X}}(t,r) \Big] dt \Gamma(d\boldsymbol{s}) \\ &= \frac{-i\alpha x}{2\pi f_{X_1}(x)} \int_{\mathbb{R}} e^{-itx} \varphi_{\boldsymbol{X}}(t,r) \Big( \int_{S_2} s_2^2 s_1^{-1} g_2(ts_1 + rs_2) \Gamma(d\boldsymbol{s}) \Big) dt \\ &\quad - \frac{\alpha^2}{2\pi f_{X_1}(x)} \int_{\mathbb{R}} e^{-itx} \varphi_{\boldsymbol{X}}(t,r) \Big( \int_{S_2} s_1 g_2(ts_1 + rs_2) \Gamma(d\boldsymbol{s}) \Big) \Big( \int_{S_2} s_2^2 s_1^{-1} g_2(ts_1 + rs_2) \Gamma(d\boldsymbol{s}) \Big) dt \\ A_2 &= \frac{-i\alpha x}{2\pi f_{X_1}(x)} \Lambda \Big( \int_{S_2} g_2 s_2^2 s_1^{-1} \Big) - \frac{\alpha^2}{2\pi f_{X_1}(x)} \Lambda \Big( \int_{S_2} g_2 s_2^2 s_1^{-1} \Big) \Big( \int_{S_2} g_2 s_1 \Big) \end{aligned}$$

Combining the expressions obtained for $A_1$ and $A_2$ yields the second derivative.

# D Proof of Theorem 2.1

We here finally evaluate the derivatives of Lemma B.1 at $r = 0$ to obtain the functional forms of the conditinal moments. These proofs yield in particular the expressions of the constants $\boldsymbol{\theta}_i$, $i = 1, \ldots, 6$ which intervene in Theorem 2.1. Lemmas at the end of this section are used to regroup terms and simplify as much as possible the functional forms.

## D.1 Proof of second order conditional moment (2.4) in Theorem 2.1

The second order derivative of the characteristic function of $X_2|X_1 = x$ is given by (B.8) in Lemma B.1. Evaluating it at $r = 0$ yields

$$\begin{aligned}
&\mathbb{E}\Big[X_2^2\Big|X_1 = x\Big] \\
&\quad = -\phi^{(2)}_{X_2|x}(0) \\
&\quad = \frac{\alpha}{2\pi f_{X_1}(x)} \int_{\mathbb{R}} e^{-itx + ia\sigma_1^\alpha \beta_1 t^{<\alpha>}} e^{-\sigma_1^\alpha |t|^\alpha} \\
&\qquad\qquad \times \Big[ix\sigma_1^\alpha(\kappa_2 t^{<\alpha-1>} - ia\lambda_2|t|^{\alpha-1}) - \alpha\sigma_1^{2\alpha}(\kappa_1 t^{<\alpha-1>} - ia\lambda_1|t|^{\alpha-1})^2 \\
&\qquad\qquad\qquad + \alpha\sigma_1^{2\alpha}(\kappa_2 t^{<\alpha-1>} - ia\lambda_2|t|^{\alpha-1})(t^{<\alpha-1>} - ia\beta_1|t|^{\alpha-1})\Big] dt \\
&\quad = \frac{\alpha\sigma_1^\alpha}{2\pi f_{X_1}(x)} \int_{\mathbb{R}} e^{-itx + ia\sigma_1^\alpha \beta_1 t^{<\alpha>}} e^{-\sigma_1^\alpha |t|^\alpha} \\
&\qquad\qquad \times \Big[xa\lambda_2|t|^{\alpha-1} + \alpha\sigma_1^\alpha |t|^{2(\alpha-1)}\Big(\kappa_2 - a^2\beta_1\lambda_2 - \kappa_1^2 + a^2\lambda_1^2\Big) \\
&\qquad\qquad\qquad + ix\kappa_2 t^{<\alpha-1>} + i\alpha\sigma_1^\alpha t^{<2(\alpha-1)>}\Big(2a\lambda_1\kappa_1 - a(\lambda_2 + \beta_1\kappa_2\Big)\Big] dt \\
&\quad = \frac{\alpha\sigma_1^\alpha}{\pi f_{X_1}(x)} \Big[ax\lambda_2 C_1(x) + \kappa_2 x S_1(x) \\
&\qquad\qquad - \alpha\sigma_1^\alpha\Big(\kappa_1^2 - a^2\lambda_1^2 + a^2\beta_1\lambda_2 - \kappa_2\Big)C_2(x) - \alpha\sigma_1^\alpha\Big(a(\lambda_2 + \beta_1\kappa_2) - 2a\lambda_1\kappa_1\Big)S_2(x)\Big],
\end{aligned}$$

where the $\kappa_i$'s and $\lambda_i$'s are given in (2.7). Invoking Lemma D.1 ($\iota\iota\iota$) yields

$$\begin{aligned}
\mathbb{E}\Big[X_2^2\Big|X_1 = x\Big] &= \frac{x}{1 + (a\beta_1)^2}\left[(a^2\lambda_2\beta_1 + \kappa_2)x + a(\lambda_2 - \kappa_2\beta_1)\frac{1 - xH(x)}{\pi f_{X_1}(x)}\right] \\
&\quad - \frac{\alpha^2\sigma_1^{2\alpha}}{\pi f_{X_1}(x)}\mathcal{H}\Big(2(\alpha-1), \boldsymbol{\theta}_1; x\Big) \\
&= \kappa_2 x^2 + \frac{ax(\lambda_2 - \beta_1\kappa_2)}{1 + (a\beta_1)^2}\left[a\beta_1 x + \frac{1 - xH(x)}{\pi f_{X_1}(x)}\right] - \frac{\alpha^2\sigma_1^{2\alpha}}{\pi f_{X_1}(x)}\mathcal{H}\Big(2(\alpha-1), \boldsymbol{\theta}_1; x\Big),
\end{aligned}$$

where $\mathcal{H}$ is given in (B.3) with

$$\theta_{11} = \kappa_1^2 - a^2\lambda_1^2 + a^2\beta_1\lambda_2 - \kappa_2, \qquad \theta_{12} = a(\lambda_2 + \beta_1\kappa_2) - 2a\lambda_1\kappa_1.$$

### D.2 Proof of third order conditional moment (2.5) in Theorem 2.1

The third order derivative of the characteristic function of $X_2|X_1 = x$ is given by (B.9) in Lemma B.1. It can be shown that the $I$'s evaluated at $r = 0$ write

$$
\begin{aligned}
I_1 &= 2\sigma_1^{\alpha}\mathcal{H}\Big(\alpha-2, \boldsymbol{\theta}_1^I; x\Big), & \boldsymbol{\theta}_1^I &= \Big(\kappa_3, -a\lambda_3\Big),\\
I_2 &= 2\sigma_1^{2\alpha}\mathcal{H}\Big(2(\alpha-1), \boldsymbol{\theta}_2^I; x\Big), & \boldsymbol{\theta}_2^I &= \Big(L, -aK\Big),\\
iI_3 &= 2\sigma_1^{3\alpha}\mathcal{H}\Big(3(\alpha-1), \boldsymbol{\theta}_3^I; x\Big), & \boldsymbol{\theta}_3^I &= \Big(a\lambda_1(3\kappa_1^2 - a^2\lambda_1^2), \kappa_1^3 - 3a^2\kappa_1\lambda_1^2\Big),\\
iI_4 &= 2\sigma_1^{3\alpha}\mathcal{H}\Big(3(\alpha-1), \boldsymbol{\theta}_4^I; x\Big), & \boldsymbol{\theta}_4^I &= \Big(a\Big(K + \beta_1 L\Big), L - a^2\beta_1 K\Big),\\
iI_5 = iI_7 &= 2\sigma_1^{2\alpha}\mathcal{H}\Big(2\alpha-3, \boldsymbol{\theta}_5^I; x\Big), & \boldsymbol{\theta}_5^I &= \Big(aK, L\Big),\\
iI_6 &= 2\sigma_1^{2\alpha}\mathcal{H}\Big(2\alpha-3, \boldsymbol{\theta}_6^I; x\Big), & \boldsymbol{\theta}_6^I &= \Big(a(\lambda_3 + \beta_1\kappa_3), \kappa_3 - a^2\beta_1\lambda_3\Big),
\end{aligned}
$$

with $K = \kappa_1\lambda_2 + \lambda_1\kappa_2$ and $L = \kappa_1\kappa_2 - a^2\lambda_1\lambda_2$. Hence,

$$
\mathbb{E}\Big[X_2^3\Big|X_1 = x\Big] = -i\phi_{X_2|x}^{(3)}(0) = \frac{\alpha}{\pi f_{X_1}(x)}\Bigg[-x\Big((\alpha-1)K_1 - \alpha K_2\Big) + \alpha^2 K_3 + \alpha(\alpha-1)K_4\Bigg],
$$

with

$$
\begin{aligned}
K_1 &= \sigma_1^{\alpha}\mathcal{H}\Big(\alpha-2, \boldsymbol{\theta}_1^K; x\Big), & &\text{with} \quad \boldsymbol{\theta}_1^K = \boldsymbol{\theta}_1^I,\\
K_2 &= \sigma_1^{2\alpha}\mathcal{H}\Big(2(\alpha-1), \boldsymbol{\theta}_2^K; x\Big), & &\text{with} \quad \boldsymbol{\theta}_2^K = \boldsymbol{\theta}_2^I,\\
K_3 &= \sigma_1^{3\alpha}\mathcal{H}\Big(3(\alpha-1), \boldsymbol{\theta}_3^K; x\Big), & &\text{with} \quad \boldsymbol{\theta}_3^K = \boldsymbol{\theta}_3^I - \boldsymbol{\theta}_4^I\\
K_4 &= \sigma_1^{2\alpha}\mathcal{H}\Big(2\alpha-3, \boldsymbol{\theta}_4^K; x\Big), & &\text{with} \quad \boldsymbol{\theta}_4^K = \boldsymbol{\theta}_6^I - \boldsymbol{\theta}_5^I.
\end{aligned}
$$

Invoking Lemma D.1 ($\iota\iota$) for $n = 1, 2$ and regrouping the terms, we get

$$
\begin{aligned}
\mathbb{E}\Big[X_2^3\Big|X_1 = x\Big] = &\frac{\alpha x^2\sigma_1^{\alpha}}{\pi f_{X_1}(x)}\Big(\theta_{12}^K C_1(x) - \theta_{11}^K S_1(x)\Big)\\
&+ \frac{\alpha}{\pi f_{X_1}(x)}\Bigg[\ \frac{\alpha x\sigma_1^{2\alpha}}{2}C_2(x)\Big(-2\Big(\theta_{11}^K + a\beta_1\theta_{12}^K\Big) + 2\theta_{21}^K - \theta_{42}^K\Big)\\
&\qquad + \frac{\alpha x\sigma_1^{2\alpha}}{2}S_2(x)\Big(-2\Big(\theta_{12}^K - a\beta_1\theta_{11}^K\Big) + 2\theta_{22}^K + \theta_{41}^K\Big)\\
&\qquad + \frac{\alpha^2\sigma_1^{3\alpha}}{2}C_3(x)\Big(2\theta_{31}^K + \theta_{41}^K + a\beta_1\theta_{42}^K\Big)\\
&\qquad + \frac{\alpha^2\sigma_1^{3\alpha}}{2}S_3(x)\Big(2\theta_{32}^K + \theta_{42}^K - a\beta_1\theta_{41}^K\Big)\Bigg].
\end{aligned}
$$

Using Lemma D.1 ($\iota\iota\iota$) yields the conclusion with $\boldsymbol{\theta}_2 = (\theta_{21}, \theta_{22})$, $\boldsymbol{\theta}_3 = (\theta_{31}, \theta_{32})$ such that

$$\theta_{21} = 3(L + a^2\beta_1\lambda_3 - \kappa_3), \tag{D.1}$$

$$\theta_{22} = 3a(\lambda_3 + \beta_1\kappa_3 - K), \tag{D.2}$$

$$\theta_{31} = a\Big(\lambda_3(1 - a^2\beta_1^2) + 2\beta_1\kappa_3 + 2\lambda_1(3\kappa_1^2 - a^2\lambda_1^2) - 3(K + \beta_1 L)\Big), \tag{D.3}$$

$$\theta_{32} = \kappa_3(1 - a^2\beta_1^2) - 2a^2\beta_1\lambda_3 + 2(\kappa_1^3 - 3a^2\kappa_1\lambda_1^2) + 3(a^2\beta_1 K - L), \tag{D.4}$$

with $K = \kappa_1\lambda_2 + \kappa_2\lambda_1$, $L = \kappa_1\kappa_2 - a^2\lambda_1\lambda_2$.

## D.3 Proof of fourth order conditional moment (2.6) in Theorem 2.1

The conditional moments are obtained by evaluating the derivatives of the conditional characteristic function at $r = 0$. We provide here the proof for the fourth order, which yields the expressions of the vectors $\boldsymbol{\theta}_4$, $\boldsymbol{\theta}_5$ and $\boldsymbol{\theta}_6$ appearing in Equation (2.6) of Theorem 2.1. The fourth order derivative of the characteristic function of $X_2|X_1 = x$ is given by (B.10) in Lemma B.1. It can be shown that the $J$'s evaluated at $r = 0$ write

$$\begin{aligned}
iJ_1 &= 2\sigma_1^{3\alpha}\mathcal{H}\Big(3(\alpha-1), \boldsymbol{\theta}_1^J; x\Big), & J_{11} = J_{13} &= 2\sigma_1^{3\alpha}\mathcal{H}\Big(3\alpha-4, \boldsymbol{\theta}_{11}^J; x\Big),\\
iJ_2 &= 2\sigma_1^{3\alpha}\mathcal{H}\Big(3(\alpha-1), \boldsymbol{\theta}_2^J; x\Big), & J_{14} &= 2\sigma_1^{2\alpha}\mathcal{H}\Big(2\alpha-4, \boldsymbol{\theta}_{14}^J; x\Big),\\
iJ_3 &= 2\sigma_1^{2\alpha}\mathcal{H}\Big(2\alpha-3, \boldsymbol{\theta}_3^J; x\Big), & J_{15} &= 2\sigma_1^{2\alpha}\mathcal{H}\Big(2\alpha-4, \boldsymbol{\theta}_{15}^J; x\Big),\\
iJ_4 = iJ_5 &= 2\sigma_1^{2\alpha}\mathcal{H}\Big(2\alpha-3, \boldsymbol{\theta}_4^J; x\Big), & J_{16} &= 2\sigma_1^{2\alpha}\mathcal{H}\Big(2\alpha-4, \boldsymbol{\theta}_{16}^J; x\Big),\\
J_6 &= 2\sigma_1^{2\alpha}\mathcal{H}\Big(2(\alpha-1), \boldsymbol{\theta}_6^J; x\Big), & J_{17} &= 2\sigma_1^{4\alpha}\mathcal{H}\Big(4(\alpha-1), \boldsymbol{\theta}_{17}^J; x\Big),\\
J_7 &= 2\sigma_1^{\alpha}\mathcal{H}\Big(\alpha-2, \boldsymbol{\theta}_7^J; x\Big), & J_{18} &= 2\sigma_1^{4\alpha}\mathcal{H}\Big(4(\alpha-1), \boldsymbol{\theta}_{18}^J; x\Big),\\
J_8 = J_9 = J_{12} &= 2\sigma_1^{3\alpha}\mathcal{H}\Big(3\alpha-4, \boldsymbol{\theta}_8^J; x\Big), & J_{19} &= 2\sigma_1^{4\alpha}\mathcal{H}\Big(4(\alpha-1), \boldsymbol{\theta}_{19}^J; x\Big),\\
J_{10} &= 2\sigma_1^{3\alpha}\mathcal{H}\Big(3\alpha-4, \boldsymbol{\theta}_{10}^J; x\Big), & &
\end{aligned}$$

where $\boldsymbol{\theta}_i^J = (\theta_{i1}^J, \theta_{i2}^J)$, for $i = 1, \ldots, 19$,

$$\begin{aligned}
\theta_{11}^J &= a\Big(\lambda_2(\kappa_1^2 - a^2\lambda_1^2) + 2\kappa_1\kappa_2\lambda_1\Big), & \theta_{12}^J &= \kappa_2(\kappa_1^2 - a^2\lambda_1^2) - 2a^2\kappa_1\lambda_1\lambda_2,\\
\theta_{21}^J &= a\Big(K + \beta_1 L\Big), & \theta_{22}^J &= L - a^2\beta_1 K,\\
\theta_{31}^J &= a\Big(\beta_1\kappa_4 + \lambda_4\Big), & \theta_{32}^J &= \kappa_4 - a^2\beta_1\lambda_4,\\
\theta_{41}^J &= aK, & \theta_{42}^J &= L,\\
\theta_{61}^J &= L, & \theta_{62}^J &= -aK,
\end{aligned}$$

$$
\begin{aligned}
\theta^J_{71} &= \kappa_4, & \theta^J_{72} &= -a\lambda_4,\\
\theta^J_{81} &= L - a^2\beta_1 K, & \theta^J_{82} &= -a\Big(K + \beta_1 L\Big),\\
\theta^J_{101} &= \kappa_4(1-a^2\beta_1^2) - 2a^2\beta_1\lambda_4, & \theta^J_{102} &= -a\Big(\lambda_4(1-a^2\beta_1^2) + 2\beta_1\kappa_4\Big),\\
\theta^J_{111} &= \theta^J_{12}, & \theta^J_{112} &= -\theta^J_{11},\\
\theta^J_{141} &= L, & \theta^J_{142} &= -aK,\\
\theta^J_{151} &= \kappa_2^2 - a^2\lambda_2^2, & \theta^J_{152} &= -2a\kappa_2\lambda_2,\\
\theta^J_{161} &= \kappa_4 - a^2\beta_1\lambda_4, & \theta^J_{162} &= -a\Big(\lambda_4 + \beta_1\kappa_4\Big),\\
\theta^J_{171} &= \theta^J_{12} - a\beta_1\theta^J_{11}, & \theta^J_{172} &= -\theta^J_{11} + a\theta^J_{12},\\
\theta^J_{181} &= \kappa_1^4 - 6a^2\kappa_1^2\lambda_1^2 + a^4\lambda_1^4, & \theta^J_{182} &= -4a\kappa_1\lambda_1(\kappa_1^2 - a^2\lambda_1^2),\\
\theta^J_{191} &= L(1-a^2\beta_1^2) - 2a^2\beta_1 K, & \theta_{192} &= -a\Big(K(1-a^2\beta_1^2) + 2\beta_1 L\Big),
\end{aligned}
$$

and $K = \kappa_1\lambda_3 + \lambda_1\kappa_3$, $L = \kappa_1\kappa_3 - a^2\lambda_1\lambda_3$. Hence,

$$
\begin{aligned}
&\mathbb{E}\Big[X_2^4\Big|X_1 = x\Big] = \phi^{(4)}_{X_2|x}(0)\\
&\quad = \frac{-\alpha}{\pi f_{X_1}(x)}\Bigg[\alpha x\Big(\alpha K_1 + (\alpha-1)K_2\Big) + \alpha x^2 K_6 - (\alpha-1)x^2 K_7 + \alpha^2(\alpha-1)K_3 + \alpha(\alpha-1)^2 K_4 + \alpha^3 K_5\Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
K_1 &= \sigma_1^{3\alpha}\mathcal{H}\Big(3(\alpha-1), \boldsymbol{\theta}^K_1; x\Big), &&\text{with}\quad \boldsymbol{\theta}^K_1 = 3\boldsymbol{\theta}^J_1 - 2\boldsymbol{\theta}^J_2,\\
K_2 &= \sigma_1^{2\alpha}\mathcal{H}\Big(2\alpha-3, \boldsymbol{\theta}^K_2; x\Big), &&\text{with}\quad \boldsymbol{\theta}^K_2 = 2(\boldsymbol{\theta}^J_3 - \boldsymbol{\theta}^J_4),\\
K_3 &= \sigma_1^{3\alpha}\mathcal{H}\Big(3\alpha-4, \boldsymbol{\theta}^K_3; x\Big), &&\text{with}\quad \boldsymbol{\theta}^K_3 = \boldsymbol{\theta}^J_{10} - 3\boldsymbol{\theta}^J_{11} - \boldsymbol{\theta}^J_8,\\
K_4 &= \sigma_1^{2\alpha}\mathcal{H}\Big(2\alpha-4, \boldsymbol{\theta}^K_4; x\Big), &&\text{with}\quad \boldsymbol{\theta}^K_4 = 4\boldsymbol{\theta}^J_{14} - 3\boldsymbol{\theta}^J_{15} - \boldsymbol{\theta}^J_{16},\\
K_5 &= \sigma_1^{4\alpha}\mathcal{H}\Big(4(\alpha-1), \boldsymbol{\theta}^K_5; x\Big), &&\text{with}\quad \boldsymbol{\theta}^K_5 = 3\boldsymbol{\theta}^J_{17} - \boldsymbol{\theta}^J_{18} - \boldsymbol{\theta}^J_{19},\\
K_6 &= \sigma_1^{2\alpha}\mathcal{H}\Big(2(\alpha-1), \boldsymbol{\theta}^K_6; x\Big), &&\text{with}\quad \boldsymbol{\theta}^K_6 = \boldsymbol{\theta}^J_6,\\
K_7 &= \sigma_1^{\alpha}\mathcal{H}\Big(\alpha-2, \boldsymbol{\theta}^K_7; x\Big), &&\text{with}\quad \boldsymbol{\theta}^K_7 = \boldsymbol{\theta}^J_7.
\end{aligned}
$$

Invoking Lemmas D.1 ($\iota\iota$) for $n = 1, 2, 3$ and D.2, we get

$$
\begin{aligned}
\mathbb{E}\Big[X_2^4\Big|X_1 = x\Big] = \frac{-\alpha}{\pi f_{X_1}(x)}\Bigg[& x^3\sigma_1^{\alpha}\Big(\theta^K_{72}C_1(x) - \theta^K_{71}S_1(x)\Big)\\
&+ \frac{\alpha x^2\sigma_1^{2\alpha}}{2}C_2(x)\Big(-\theta^K_{22} + 2\theta^K_{61} - 2\Big(\theta^K_{71} + a\beta_1\theta^K_{72}\Big) - \frac{\alpha-1}{2\alpha-3}\theta^K_{41}\Big)
\end{aligned}
$$

$$
+\frac{\alpha x^2\sigma_1^{2\alpha}}{2}S_2(x)\Big(\theta_{21}^K+2\theta_{62}^K-2\Big(\theta_{72}^K-a\beta_1\theta_{71}^K\Big)-\frac{\alpha-1}{2\alpha-3}\theta_{42}^K\Big)\Big]
$$

$$
+\frac{\alpha^2 x\sigma_1^{3\alpha}}{6}C_3(x)\Big(6\theta_{11}^K+3\Big(\theta_{21}^K+a\beta_1\theta_{22}^K\Big)-2\theta_{32}^K+5\frac{\alpha-1}{2\alpha-3}\Big(a\beta_1\theta_{41}^K-\theta_{42}^K\Big)\Big)
$$

$$
+\frac{\alpha^2 x\sigma_1^{3\alpha}}{6}S_3(x)\Big(6\theta_{12}^K+3\Big(\theta_{22}^K-a\beta_1\theta_{21}^K\Big)+2\theta_{31}^K+5\frac{\alpha-1}{2\alpha-3}\Big(\theta_{41}^K+a\beta_1\theta_{42}^K\Big)\Big)
$$

$$
+\frac{\alpha^3\sigma_1^{4\alpha}}{3}C_4(x)\Big(\theta_{31}^K+a\beta_1\theta_{32}^K+\frac{\alpha-1}{2\alpha-3}\Big(\theta_{41}^K(1-a^2\beta_1^2)+2a\beta_1\theta_{42}^K\Big)+3\theta_{51}^K\Big)
$$

$$
+\frac{\alpha^3\sigma_1^{4\alpha}}{3}S_4(x)\Big(\theta_{32}^K-a\beta_1\theta_{31}^K+\frac{\alpha-1}{2\alpha-3}\Big(\theta_{42}^K(1-a^2\beta_1^2)-2a\beta_1\theta_{41}^K\Big)+3\theta_{52}^K\Big)\Big].
$$

Using Lemma D.1 ($\iota\iota\iota$) yields the conclusion. The coefficients $\boldsymbol{\theta}$'s in the expression (2.6) are deduced from the $\boldsymbol{\theta}^K$'s and $\boldsymbol{\theta}^J$'s as follows:

$$
\theta_{41}=-\theta_{22}^K+2\theta_{61}^K-2\Big(\theta_{71}^K+a\beta_1\theta_{72}^K\Big)-\frac{\alpha-1}{2\alpha-3}\theta_{41}^K, \tag{D.5}
$$

$$
\theta_{42}=\theta_{21}^K+2\theta_{62}^K-2\Big(\theta_{72}^K-a\beta_1\theta_{71}^K\Big)-\frac{\alpha-1}{2\alpha-3}\theta_{42}^K, \tag{D.6}
$$

$$
\theta_{51}=6\theta_{11}^K+3\Big(\theta_{21}^K+a\beta_1\theta_{22}^K\Big)-2\theta_{32}^K+5\frac{\alpha-1}{2\alpha-3}\Big(a\beta_1\theta_{41}^K-\theta_{42}^K\Big), \tag{D.7}
$$

$$
\theta_{52}=6\theta_{12}^K+3\Big(\theta_{22}^K-a\beta_1\theta_{21}^K\Big)+2\theta_{31}^K+5\frac{\alpha-1}{2\alpha-3}\Big(\theta_{41}^K+a\beta_1\theta_{42}^K\Big), \tag{D.8}
$$

$$
\theta_{61}=\theta_{31}^K+a\beta_1\theta_{32}^K+\frac{\alpha-1}{2\alpha-3}\Big(\theta_{41}^K(1-a^2\beta_1^2)+2a\beta_1\theta_{42}^K\Big)+3\theta_{51}^K, \tag{D.9}
$$

$$
\theta_{62}=\theta_{32}^K-a\beta_1\theta_{31}^K+\frac{\alpha-1}{2\alpha-3}\Big(\theta_{42}^K(1-a^2\beta_1^2)-2a\beta_1\theta_{41}^K\Big)+3\theta_{52}^K. \tag{D.10}
$$

## D.4 Lemmas for the proof of Theorem 2.1

The following elementary Lemmas, stated without proof, are used to establish Theorem 2.1.

**Lemma D.1** *Let $\alpha\in(1,2)$, $b>0$, $c\in\mathbb{R}$. Define for $n\geq 1$ and $x\in\mathbb{R}$*

$$
C_n(x)=\int_0^{+\infty}e^{-bt^\alpha}t^{n(\alpha-1)}\cos(tx-ct^\alpha)dt, \qquad F_n(x)=\int_0^{+\infty}e^{-bt^\alpha}t^{n(\alpha-1)-1}\cos(tx-ct^\alpha)dt,
$$

$$
S_n(x)=\int_0^{+\infty}e^{-bt^\alpha}t^{n(\alpha-1)}\sin(tx-ct^\alpha)dt, \qquad G_n(x)=\int_0^{+\infty}e^{-bt^\alpha}t^{n(\alpha-1)-1}\sin(tx-ct^\alpha)dt.
$$

$\iota$) *Then the following hold for any $n\geq 1$ and $x\in\mathbb{R}$*

$$
F_n(x)=\frac{\alpha\Big(bC_{n+1}(x)-cS_{n+1}(x)\Big)+xS_n(x)}{n(\alpha-1)}, \qquad G_n(x)=\frac{\alpha\Big(cC_{n+1}(x)+bS_{n+1}(x)\Big)-xC_n(x)}{n(\alpha-1)}.
$$

$\iota\iota$) *For any $n\geq 1$, $\theta_1,\theta_2\in\mathbb{R}$ and $x\in\mathbb{R}$:*

$$
\theta_1F_n(x)+\theta_2G_n(x)=\frac{\alpha\Big[C_{n+1}(x)\Big(b\theta_1+c\theta_2\Big)+S_{n+1}(x)\Big(b\theta_2-c\theta_1\Big)\Big]+x\Big[-\theta_2C_n(x)+\theta_1S_n(x)\Big]}{n(\alpha-1)}.
$$

*ιιι) We have for $x \in \mathbb{R}$, $b = \sigma_1^\alpha$ and $c = a\beta_1\sigma_1^\alpha$:*

$$C_1(x) = \frac{a\beta_1 x\pi f_{X_1}(x) + 1 - xH(x)}{\alpha\sigma_1^\alpha(1+(a\beta_1)^2)}, \qquad S_1(x) = \frac{x\pi f_{X_1}(x) - a\beta_1(1 - xH(x))}{\alpha\sigma_1^\alpha(1+(a\beta_1)^2)}.$$

**Lemma D.2** *Let $\alpha \in (3/2, 2)$, $b > 0$, $c \in \mathbb{R}$. Define for $x \in \mathbb{R}$*

$$h_c(x) = \int_0^{+\infty} e^{-bt^\alpha} t^{2\alpha-4}\cos(tx - ct^\alpha)dt, \qquad h_s(x) = \int_0^{+\infty} e^{-bt^\alpha} t^{2\alpha-4}\sin(tx - ct^\alpha)dt.$$

*Then for any $\theta_1, \theta_2 \in \mathbb{R}$ and $x \in \mathbb{R}$,*

$$\begin{aligned}\theta_1 h_c(x) + \theta_2 h_s(x) = {} & \frac{\alpha^2}{3(2\alpha-3)(\alpha-1)}\Big[C_4(x)\Big(\theta_1(b^2-c^2) + 2bc\theta_2\Big) + S_4(x)\Big(\theta_2(b^2-c^2) - 2bc\theta_1\Big)\Big] \\ & + \frac{5\alpha x}{6(2\alpha-3)(\alpha-1)}\Big[C_3(x)\Big(c\theta_1 - b\theta_2\Big) + S_3(x)\Big(b\theta_1 + c\theta_2\Big)\Big] \\ & - \frac{x^2}{2(2\alpha-3)(\alpha-1)}\Big[\theta_1 C_2(x) + \theta_2 S_2(x)\Big].\end{aligned}$$

# E Proof of Proposition 2.1 in the case $\alpha \neq 1$

First assume that $|\beta_1| \neq 1$. We will focus on the case $x \to +\infty$. The case $x \to -\infty$ can be obtained by considering the vector $(X_1, X_2)$, whose parameter are $\beta_1^* = -\beta_1$, $\kappa_1^* = -\kappa_1$ and $\lambda_1^* = \lambda_1$ and noticing that $\mathbb{E}\Big[X_2^p\Big|X_1 = x\Big] = \mathbb{E}\Big[X_2^p\Big| - X_1 = -x\Big]$. For $p = 1$, the result is already known (see Hardin et al. (1991)). For $p = 2, 3, 4$, we have from the proofs of (2.4)-(2.6), that

$$\mathbb{E}\Big[X_2^p\Big|X_1 = x\Big] = \frac{\alpha\sigma_1^\alpha}{\pi f_{X_1}(x)}\Big[x^{p-1}\mathcal{H}\Big(\alpha-1, (a\lambda_p, \kappa_p); x\Big) + \sum_{i=2}^{p} b_{i,p}x^{p-i}\mathcal{H}\Big(i(\alpha-1), \boldsymbol{\nu}_i; x\Big)\Big],$$

for some coefficients $b$'s. From the proof of Corollary 3.2 in Hardin et al. (1991), we deduce the following limit:

$$x^\alpha\mathcal{H}\Big(\alpha-1, (a\lambda_p, \kappa_p); x\Big) \underset{x\to+\infty}{\longrightarrow} \Big(\kappa_p + \lambda_p\Big)\sin\Big(\frac{\pi\alpha}{2}\Big)\Gamma(\alpha).$$

We also have

$$x^{\alpha+1}f_{X_1}(x) \underset{x\to+\infty}{\longrightarrow} \frac{1}{\pi}\sigma_1^\alpha(1+\beta_1)\sin\Big(\frac{\pi\alpha}{2}\Big)\Gamma(1+\alpha). \tag{E.1}$$

Hence,

$$x^{-p}\frac{\alpha\sigma_1^\alpha x^{p-1}}{\pi f_{X_1}(x)}\mathcal{H}\Big(\alpha-1, (a\lambda_p, \kappa_p); x\Big) \longrightarrow \frac{\kappa_p + \lambda_p}{1+\beta_1},$$

as $x \to +\infty$. It remains to be shown that $\dfrac{\sum_{i=2}^p b_{i,p}x^{p-i}\mathcal{H}\Big(i(\alpha-1), \boldsymbol{\nu}_i; x\Big)}{x^{p-1}\mathcal{H}\Big(\alpha-1, (a\lambda_p, \kappa_p); x\Big)} \underset{x\to+\infty}{\longrightarrow} 0$. By Theorem 127 in Titchmarsh (1948), for $i = 2, 3, 4$,

$$\mathcal{H}\Big(i(\alpha-1), \boldsymbol{\nu}_i; x\Big) \underset{x\to+\infty}{=} O\Big(x^{-i(\alpha-1)-1}\Big).$$

Hence,

$$\left|\frac{x^{p-i}\mathcal{H}\Big(i(\alpha-1),\boldsymbol{\nu}_i;x\Big)}{x^{p-1}\mathcal{H}\Big(\alpha-1,(a\lambda_p,\kappa_p);x\Big)}\right| \underset{x\to+\infty}{=} O\Big(x^{\alpha(1-i)}\Big) \longrightarrow 0.$$

Now assume that $|\beta_1| = 1$. For instance if $\beta_1 = 1$, the distribution of $X_1$ is *totally skewed to the right*. On the one hand, we have $\lambda_p = \beta_1\kappa_p$. On the other hand, the right tail of $f_{X_1}$ still decays as (E.1), yielding the conclusion.

# F Proof of Lemma 3.1

The characteristic function of $\boldsymbol{X_t}$ reads, for any $\boldsymbol{u} = (u_1,\dots,u_m) \in \mathbb{R}^m$:

$$\varphi_{\boldsymbol{X_t}}(\boldsymbol{u}) = \mathbb{E}\left(\exp\left\{i\sum_{j=1}^m u_j X_{j,t}\right\}\right) = \prod_{k\in\mathbb{Z}} \mathbb{E}\left[i\left(\sum_{j=1}^m u_j a_{k,j}\right)\varepsilon_{t+k}\right].$$

We obtain for $\alpha \neq 1$,

$$\varphi_{\boldsymbol{X_t}}(\boldsymbol{u}) = \exp\left\{-\sum_{k\in\mathbb{Z}}\sigma^\alpha\Big|\sum_{j=1}^m u_j a_{k,j}\Big|^\alpha\left(1 - i\beta\mathrm{sign}\Big(\sum_{j=1}^m u_j a_{k,j}\Big)\mathrm{tg}\frac{\pi\alpha}{2}\right) + i\sum_{j=1}^m u_j\sum_{k\in\mathbb{Z}} a_{k,j}\mu\right\}. \tag{F.1}$$

And for $\alpha = 1$,

$$\varphi_{\boldsymbol{X_t}}(\boldsymbol{u}) = \exp\left\{-\sum_{k\in\mathbb{Z}}\sigma\Big|\sum_{j=1}^m u_j a_{k,j}\Big|\left(1 + i\beta\frac{2}{\pi}\mathrm{sign}\Big(\sum_{j=1}^m u_j a_{k,j}\Big)\ln\Big|\sum_{j=1}^m u_j a_{k,j}\Big|\right) + i\sum_{j=1}^m u_j\sum_{k\in\mathbb{Z}} a_{k,j}\mu\right\}. \tag{F.2}$$

Replacing (3.3) in (2.1), we retrieve the two above formulae.

# G Proof of the asymptotic moments in Example 3.1

The results in Example 3.1 follow from Proposition 3.1 applied to $X_t = \sum_{k\in\mathbb{Z}}\rho^k\mathbb{1}_{\{k\geq 0\}}\varepsilon_{t+k}$. Regarding the asymptotic behaviours of moments, we give the proof for the excess kurtosis. The other limits and equivalents are obtained in a similar manner. Letting $\alpha \in (3/2, 2)$ ensures the existence of the fourth order moment. Since we assume $\rho > 0$, it follows that $\lambda_p = \beta_1\kappa_p$ for $p = 1,2,3,4$. Using Proposition 2.1, one can show that as $x$ tends to infinity

$$\gamma_2(x,h) \longrightarrow \frac{\kappa_4 - 4\kappa_1\kappa_3 + 6\kappa_1^2\kappa_2 - 3\kappa_1^4}{\Big(\kappa_2 - \kappa_1^2\Big)^2} - 3.$$

Substituting the $\kappa_p$'s by $\rho^{h(\alpha-p)}$ and rearranging terms yields the conclusion.

# H Proof of Proposition 4.1

We start by showing that when $(X_t)$ is an $\alpha$-stable aggregate as in Definition 4.1, the bivariate vector $(X_t, X_{t+h})$ is also $\alpha$-stable and that its spectral measure is a linear combination of the spectral measures of the $(X_{j,t}, X_{j,t+h})$. We will then be in a position to apply Theorem 2.1.

**Lemma H.1** *Let $(X_t)$ be an $\alpha$-stable aggregate, $0 < \alpha < 2$, with latent moving averages $(X_{1,t}), \dots, (X_{J,t})$ as in Definition 4.1. By Lemma 3.1, $(X_{j,t}, X_{j,t+h})$, $j = 1, \dots, J$ are all bivariate $\alpha$-stable. Denote $(\Gamma_{j,h}, \boldsymbol{\mu}_j^0)$ with $\boldsymbol{\mu}_j^0 = (\mu_{1,j}^0, \mu_{2,j}^0)$ their respective spectral representations.*
*Then, for any $h \geq 1$, $(X_t, X_{t+h})$ is a bivariate $\alpha$-stable vector and its spectral representation, denoted $(\Gamma_h, \boldsymbol{\mu}^0)$ with $\boldsymbol{\mu}^0 = (\mu_1^0, \mu_2^0)$, is given by*

$$\Gamma_h = \sum_{j=1}^{J} |\pi_j|^\alpha \Gamma_{j,h},$$

*and,*

$$\mu_1^0 = \sum_{j=1}^{J} \pi_j \Big( \mu_{1,j}^0 - \mathbb{1}_{\{\alpha=1\}} \frac{2}{\pi} \sigma_{1,j} \beta_{1,j} \ln |\pi_j| \Big), \qquad \mu_2^0 = \sum_{j=1}^{J} \pi_j \Big( \mu_{2,j}^0 - \mathbb{1}_{\{\alpha=1\}} \frac{2}{\pi} \sigma_{1,j} \lambda_{1,j} \ln |\pi_j| \Big).$$

*Proof.*

Using the independence between the $X_{j,t}$'s and denoting $\boldsymbol{X}_j = (X_{j,t}, X_{j,t+h})$,

$$\begin{aligned}
\mathbb{E}\Big[e^{iuX_t + ivX_{t+h}}\Big] &= \mathbb{E}\Bigg[\exp\Bigg\{iu \sum_{j=1}^{J} \pi_j X_{j,t} + iv \sum_{j=1}^{J} \pi_j X_{j,t+h}\Bigg\}\Bigg] = \prod_{j=1}^{J} \mathbb{E}\Bigg[\exp\Big\{i\langle \boldsymbol{u}\pi_j, \boldsymbol{X}_j\rangle\Bigg] \\
&= \prod_{j=1}^{J} \exp\Bigg\{ - \int_{S_2} |\langle \boldsymbol{u}\pi_j, \boldsymbol{s}\rangle|^\alpha \Big(1 - i \operatorname{sign}(\langle \boldsymbol{u}\pi_j, \boldsymbol{s}\rangle) w(\alpha, \langle \boldsymbol{u}\pi_j, \boldsymbol{s}\rangle)\Big) \Gamma_{j,h}(d\boldsymbol{s}) + i \langle \boldsymbol{u}\pi_j, \boldsymbol{\mu}^0\rangle \Bigg\},
\end{aligned}$$

When $\alpha \neq 1$, then $w(\alpha, \cdot) = \mathrm{tg}(\pi\alpha/2)$ and

$$\begin{aligned}
\mathbb{E}\Big[e^{iuX_t + ivX_{t+h}}\Big] &= \exp\Bigg\{ - \sum_{j=1}^{J} |\pi_j|^\alpha \int_{S_2} |\langle \boldsymbol{u}, \boldsymbol{s}\rangle|^\alpha \Big(1 - i \operatorname{sign}(\langle \boldsymbol{u}, \boldsymbol{s}\rangle) w(\alpha, \langle \boldsymbol{u}, \boldsymbol{s}\rangle)\Big) \Gamma_{j,h}(d\boldsymbol{s}) \Bigg\} \\
&= \exp\Bigg\{ - \int_{S_2} |\langle \boldsymbol{u}, \boldsymbol{s}\rangle|^\alpha \Big(1 - i \operatorname{sign}(\langle \boldsymbol{u}, \boldsymbol{s}\rangle) w(\alpha, \langle \boldsymbol{u}, \boldsymbol{s}\rangle)\Big) \Gamma_h(d\boldsymbol{s}) \Bigg\}.
\end{aligned}$$

When $\alpha = 1$, with $a = 2/\pi$,

$$\begin{aligned}
\mathbb{E}\Big[e^{iuX_t + ivX_{t+h}}\Big] &= \prod_{j=1}^{J} \exp\Bigg\{ - \int_{S_2} |\langle \boldsymbol{u}\pi_j, \boldsymbol{s}\rangle| + ia\langle \boldsymbol{u}\pi_j, \boldsymbol{s}\rangle \ln |\langle \boldsymbol{u}c\pi_j, \boldsymbol{s}\rangle| \Gamma_{j,h}(d\boldsymbol{s}) + i \langle \boldsymbol{u}\pi_j, \boldsymbol{\mu}_j^0\rangle) \Bigg\} \\
&= \exp\Bigg\{ - \int_{S_2} |\langle \boldsymbol{u}, \boldsymbol{s}\rangle| + ia\langle \boldsymbol{u}, \boldsymbol{s}\rangle \ln |\langle \boldsymbol{u}, \boldsymbol{s}\rangle| \sum_{j=1}^{J} |\pi_j| \Gamma_{j,h}(d\boldsymbol{s}) \\
&\qquad + i \sum_{j=1}^{J} \Big( \langle \boldsymbol{u}, \pi_j \boldsymbol{\mu}_j^0\rangle - a\pi_j \ln |\pi_j| \int_{S_2} \langle \boldsymbol{u}, \boldsymbol{s}\rangle \Gamma_{j,h}(d\boldsymbol{s}) \Big) \Bigg\},
\end{aligned}$$

and

$$i\sum_{j=1}^{J}\Big(\langle\boldsymbol{u},\pi_j\boldsymbol{\mu}_j^0\rangle-a\pi_j\ln|\pi_j|\int_{S_2}\langle\boldsymbol{u},\boldsymbol{s}\rangle\Gamma_{j,h}(d\boldsymbol{s})\Big)=i\langle\boldsymbol{u},\sum_{j=1}^{J}\pi_j\Big(\boldsymbol{\mu}_j^0-a\ln|\pi_j|\int_{S_2}\boldsymbol{s}\Gamma_{j,h}(d\boldsymbol{s})\Big)\rangle$$
$$=i\langle\boldsymbol{u},\sum_{j=1}^{J}\pi_j\Big(\boldsymbol{\mu}_j^0-a\sigma_{1,j}\ln|\pi_j|\begin{pmatrix}\beta_{1,j}\\ \lambda_{1,j}\end{pmatrix}\Big)\rangle.$$

□

Let us now prove Proposition 4.1.

$\iota$) By Lemma H.1, we know that the spectral measure of $(X_t,X_{t+h})$ writes $\Gamma_h=\sum_{j=1}^J|\pi_j|^\alpha\Gamma_{j,h}$, for $0<\alpha<2$, where the $\Gamma_{j,h}$'s are the spectral measures of $(X_{j,t},X_{j,t+h})$. Hence, $\int_{S_2}|s_1|^{-\nu}\Gamma_h(ds)<+\infty$ if and only if for all $j=1,\dots,J$, $\int_{S_2}|s_1|^{-\nu}\Gamma_{j,h}(ds)<+\infty$, which proves point $\iota$).

$\iota\iota$) and $\iota\iota\iota$) The forms of the conditional moments follow from Theorems 2.1 and 2.2. The parameters are obtained using Lemma H.1 by first noticing that,

$$\sigma_1^\alpha=\int_{S_2}|s_1|^\alpha\Gamma_h(d\boldsymbol{s})=\sum_{j=1}^{J}|\pi_j|^\alpha\int_{S_2}|s_1|^\alpha\Gamma_{j,h}(d\boldsymbol{s})=\sum_{j=1}^{J}|\pi_j|^\alpha\sigma_{1,j}^\alpha.$$

And thus, for instance,

$$\kappa_p=\frac{1}{\sigma_1^\alpha}\int_{S_2}(s_2/s_1)^p|s_1|^\alpha\Gamma_h(d\boldsymbol{s})=\frac{1}{\sigma_1^\alpha}\sum_{j=1}^{J}|\pi_j|^\alpha\int_{S_2}(s_2/s_1)^p|s_1|^\alpha\Gamma_{j,h}(d\boldsymbol{s})=\sum_{j=1}^{J}\frac{|\pi_j|^\alpha\sigma_{1,j}^\alpha}{\sum_{i=1}^J|\pi_i|^\alpha\sigma_{1,i}^\alpha}\kappa_{p,j}.\quad\square$$

# I Proof of Theorem 2.2

Let $\boldsymbol{X}=(X_1,X_2)$ be an $\alpha$-stable vector with $\alpha=1$ and spectral representation $(\Gamma,\mathbf{0})$. Its characteristic function, denoted $\varphi_{\boldsymbol{X}}(t,r)$ for any $(t,r)\in\mathbb{R}^2$, reads

$$\varphi_{\boldsymbol{X}}(t,r)=\exp\left\{-\int_{S_2}|ts_1+rs_2|+ia(ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(ds)\right\},\tag{I.1}$$

with $a=2/\pi$. The conditional characteristic function of $X_2$ given $X_1=x$, denoted $\phi_{X_2|x}(r)$ for $r\in\mathbb{R}$, is still given by (B.2).

**Lemma I.1** *Let $(X_1,X_2)$ be an $\alpha$-stable random vector with $\alpha=1$ and spectral representation $(\Gamma,\mathbf{0})$. If (2.2) holds with $\nu>0$, the first derivative of $\phi_{X_2|x}$ is given by*

$$\phi^{(1)}_{X_2|x}(r)=\frac{-1}{2\pi f_{X_1}(x)}\Big(A_1+iaA_2\Big),$$

*with*

$$A_1=\int_{\mathbb{R}}e^{-itx}\varphi_{\boldsymbol{X}}(t,r)\Big(\int_{S_2}s_2(ts_1+rs_2)^{<0>}\Gamma(d\boldsymbol{s})\Big)dt,\tag{I.2}$$

$$A_2 = \int_{\mathbb{R}} e^{-itx}\varphi_{\boldsymbol{X}}(t,r)\Big(\int_{S_2} s_2(1+\ln|ts_1+rs_2|)\Gamma(d\boldsymbol{s})\Big)dt \tag{I.3}$$

*If* (2.2) *holds with* $\nu > 1$, *the second derivative of* $\phi_{X_2|x}$ *is given by*

$$\phi^{(2)}_{X_2|x}(r) = \frac{-1}{2\pi f_{X_1}(x)}\Big(-B_1 + ixB_2 + B_3\Big), \tag{I.4}$$

*where,*

$$\begin{aligned}
B_1 &= \int_{\mathbb{R}} e^{-itx}\varphi_{\boldsymbol{X}}(t,r)\Big(\int_{S_2} s_2(ts_1+rs_2)^{<0>} + ias_2(1+\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\Big)^2 dt,\\
B_2 &= \int_{\mathbb{R}} e^{-itx}\varphi_{\boldsymbol{X}}(t,r)\Big(\int_{S_2} \Big((ts_1+rs_2)^{<0>} + ia(1+\ln|ts_1+rs_2|\Big)s_2^2 s_1^{-1}\Gamma(d\boldsymbol{s})\Big)dt,\\
B_3 &= \int_{\mathbb{R}} e^{-itx}\varphi_{\boldsymbol{X}}(t,r)\Big(\int_{S_2} s_1(ts_1+rs_2)^{<0>} + ias_1(1+\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\Big)\\
&\qquad\qquad \times\left(\int_{S_2}\Big((ts_1+rs_2)^{<0>} + ia(1+\ln|ts_1+rs_2|\Big)s_2^2 s_1^{-1}\Gamma(d\boldsymbol{s})\right)dt.
\end{aligned}$$

## I.1 Justifying inversion of integral and derivative signs

### First derivative

The terms depending on $r$ in the right-hand side of (I.1) are of the form (omitting the factor $1/2\pi f_{X_1}(x)$)

$$\int_{\mathbb{R}} e^{-\int_{S_2}|ts_1+rs_2|\Gamma(ds)}\mathrm{trig}\Big(-tx - a\int_{S_2}(ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\Big)dt.$$

Consider for instance the term obtained by replacing trig by the cosine function, denoted $I_1$.

$$\begin{aligned}
I_1'(r) = \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}&\left[e^{-\int_{S_2}|ts_1+(r+h)s_2|\Gamma(ds)} - e^{-\int_{S_2}|ts_1+rs_2|\Gamma(ds)}\right]\\
&\qquad\times\cos\left(tx + a\int_{S_2}(ts_1+(r+h)s_2)\ln|ts_1+(r+h)s_2|\Gamma(d\boldsymbol{s})\right)dt\\
+\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}&e^{-\int_{S_2}|ts_1+rs_2|\Gamma(ds)}\left[\cos\left(tx + a\int_{S_2}(ts_1+(r+h)s_2)\ln|ts_1+(r+h)s_2|\Gamma(d\boldsymbol{s})\right)\right.\\
&\qquad\left.-\cos\left(tx + a\int_{S_2}(ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\right)\right]dt\\
:= I_{11} + I_{12}&
\end{aligned}$$

The integrand of $I_{11}$ converges to

$$-e^{-\int_{S_2}|ts_1+rs_2|\Gamma(ds)}\cos\left(tx + a\int_{S_2}(ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\right)\int_{S_2}s_2(ts_1+rs_2)^{<0>}\Gamma(d\boldsymbol{s}).$$

Using (C.19) we can bound the integrand of $I_{11}$ by

$$\frac{1}{|h|}\left|\int_{S_2}|ts_1+(r+h)s_2|-|ts_1+rs_2|\Gamma(d\boldsymbol{s})\right|e^{-\int_{S_2}|ts_1+rs_2|\Gamma(d\boldsymbol{s})}e^{\left|\int_{S_2}|ts_1+(r+h)s_2|-|ts_1+rs_2|\Gamma(d\boldsymbol{s})\right|}.$$

By Lemma C.3 ($\iota$) and the triangle inequality, we can further bound it for $|h|<|r|$ by

$$\sigma_2 e^{\sigma_2(1+|r|)-\sigma_1|t|},$$

which does not depend on $h$ and is integrable with respect to $t$ on $\mathbb{R}$. The dominated convergence theorem applies to $I_{11}$. Turning to $I_{12}$, its integrand converges to

$$-ae^{-\int_{S_2}|ts_1+rs_2|\Gamma(d\boldsymbol{s})}\sin\left(tx+a\int_{S_2}(ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\right)\int_{S_2}s_2(1+\ln|ts_1+rs_2|)\Gamma(d\boldsymbol{s}).$$

Using the mean value theorem on the cosine, its integrand can be bounded by

$$\begin{aligned}
&\frac{a}{|h|}e^{-\int_{S_2}|ts_1+rs_2|\Gamma(d\boldsymbol{s})}\left|\int_{S_2}(ts_1+(r+h)s_2)\ln|ts_1+(r+h)s_2|-(ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\right|\\
&\le ae^{\sigma_2|r|-\sigma_1|t|}\frac{1}{|h|}\int_{S_2}\Big|(ts_1+(r+h)s_2)\ln|ts_1+(r+h)s_2|-(ts_1+rs_2)\ln|ts_1+rs_2|\Big|\Gamma(d\boldsymbol{s})\\
&:= ae^{\sigma_2|r|-\sigma_1|t|}\Big(Q_1+Q_2\Big), \qquad \text{(I.5)}
\end{aligned}$$

where the two terms $Q_1$ and $Q_2$ involve integrals over $S_2\cap\{\boldsymbol{s}:|ts_1+rs_2|\ge 2|h|\}$ and $S_2\cap\{\boldsymbol{s}:|ts_1+rs_2|<2|h|\}$. Focus on $Q_2$. Introduce the function $f:\mathbb{R}_+\to\mathbb{R}_+$ defined for any $z\ge 0$ by $f(z)=z|\ln z|$. It is such that $f(0)=0$ and for $z$ small enough ($0<z<e^{-1}$), $f$ is monotone increasing. Since $|ts_1+rs_2|<2|h|$, we also have $|ts_1+(r+h)s_2|<3|h|$. Thus, for $0<|h|<(3e)^{-1}$, the integrand of $Q_2$ can be bounded by

$$|h|^{-1}\Big(\Big|f(|3h|)\Big|+\Big|f(|2h|)\Big|\Big)\le 2|h|^{-1}\Big|f(|3h|)\Big|\le 6\Big|ln|3h|\Big|$$

Using Lemma J.1, we can bound the later quantity for any $v>0$ by

$$6v^{-1}\Big(2+|3h|^{v}+|3h|^{-v}\Big).$$

From $|ts_1+rs_2|/2<|h|<(3e)^{-1}$, we deduce that $|3h|^{-v}<\Big(3|ts_1+rs_2|/2\Big)^{-v}$ and

$$6v^{-1}\Big(2+|3h|^{v}+|3h|^{-v}\Big)\le 6v^{-1}\Big(2+e^{-v}+\Big(3|ts_1+rs_2|/2\Big)^{-v}\Big)\le \text{const}_1+\text{const}_2|ts_1+rs_2|^{-v},$$

for some nonnegative constants $\text{const}_1$ and $\text{const}_2$. Hence, the term involving $Q_2$ in I.5 can be further bounded for any $v>0$ by

$$ae^{\sigma_2|r|-\sigma_1|t|}\Big(\text{const}_1+\text{const}_2\int_{S_2}\Big|t+\frac{rs_2}{s_1}\Big|^{-v}|s_1|^{-v}\Gamma(d\boldsymbol{s})\Big). \qquad \text{(I.6)}$$

The term with $\text{const}_1$ is clearly integrable with respect to $t$ on $\mathbb{R}$. Letting (2.2) hold with $\nu > 0$, choose some $v \in (0, \min(\nu, 1))$. We show that the second term is bounded by an integrable function of $t$ as we did in Equation (C.4) using Lemma C.5 with $\eta = v$, $b = 0$, $p = 0$, the fact that $\int_{\mathbb{R}} e^{-\sigma_1|t|}|t|^{-v}dt < +\infty$ and (2.2) with $\nu > v > 0$. There remains to be bounded the part involving $Q_1$ in (I.5). For this term, we apply the mean value theorem to the function $z \longmapsto z \ln|z|$ and get that

$$\begin{aligned}|h|^{-1}\Big|(ts_1 + (r+h)s_2)\ln|ts_1 + (r+h)s_2| - (ts_1 + rs_2)\ln|ts_1 + rs_2|\Big|& \\ \le |h|^{-1}|hs_2|\Big|1 + \ln|u|\Big|& \\ \le 1 + \Big|\ln|u|\Big|&,\end{aligned}$$

for some $u \in [ts_1 + (r+h)s_2 \wedge ts_1 + rs_2, ts_1 + (r+h)s_2 \vee ts_1 + rs_2]$. Since $Q_1$ is an integral over $S_2 \cap \{\boldsymbol{s} : |ts_1 + rs_2| \ge 2|h|\}$, we have $|u| \in \Big[\frac{|ts_1+rs_2|}{2}, 2|ts_1 + rs_2|\Big]$, and because of the quasi-convexity of the function $z \longmapsto \Big|\ln|z|\Big|$, we can bound the above term by

$$1 + \left|\ln\left|\frac{ts_1 + rs_2}{2}\right|\right| + \Big|\ln|2(ts_1 + rs_2)|\Big| \le \text{const} + 2\Big|\ln|ts_1 + rs_2|\Big|.$$

Using Lemma J.1, we can bound this term for any $v > 0$ by

$$\text{const} + 2v^{-1}\Big(2 + |ts_1 + rs_2|^v + |ts_1 + rs_2|^{-v}\Big) \le \text{const}_1 + \text{const}_2|t|^v + \text{const}_3\Big|t + \frac{rs_2}{s_1}\Big|^{-v}|s_1|^{-v}$$

Hence, the term in (I.5) involving $Q_1$ can be bounded for any $v > 0$ by

$$ae^{\sigma_2|r| - \sigma_1|t|}\Big(\text{const}_1 + \text{const}_2|t|^v + \text{const}_3\int_{S_2}\Big|t + \frac{rs_2}{s_1}\Big|^{-v}|s_1|^{-v}\Gamma(d\boldsymbol{s})\Big). \tag{I.7}$$

which can be shown to be integrable with respect to $t$ on $\mathbb{R}$ as we did above for the term with $Q_2$. The dominated convergence theorem applies to $I_{12}$ and thus to $I_1$. We can derivate $\phi_{X_2|x}$ under the integral sign.

**Second derivative**

Let us start with $A_2$, which is the most delicate. It is composed of terms of the form

$$\begin{aligned}\int_{\mathbb{R}} e^{-\int_{S_2}|ts_1 + rs_2|\Gamma(d\boldsymbol{s})}\text{trig}\Big(-tx - a\int_{S_2}(ts_1 + rs_2)\ln|ts_1 + rs_2|\Gamma(d\boldsymbol{s})\Big)& \\ \times\Big(\int_{S_2} s_2(1 + \ln|ts_1 + rs_2|)\Gamma(d\boldsymbol{s})\Big)dt&,\end{aligned}$$

where «trig» stands for sine or cosine. Denoting the one with cosine as $K_2$, we have

$$K_2 = \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\Big[e^{-\int_{S_2}|ts_1 + (r+h)s_2|\Gamma(d\boldsymbol{s})} - e^{-\int_{S_2}|ts_1 + rs_2|\Gamma(d\boldsymbol{s})}\Big]$$

$$
\begin{aligned}
&\times \cos\Big(tx + a\int_{S_2}(ts_1+(r+h)s_2)\ln|ts_1+(r+h)s_2|\Gamma(d\boldsymbol{s})\Big)\\
&\qquad\qquad\qquad\qquad \times\Big(\int_{S_2} s_2(1+\ln|ts_1+(r+h)s_2|)\Gamma(d\boldsymbol{s})\Big)dt\\
&+\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}} e^{-\int_{S_2}|ts_1+rs_2|\Gamma(d\boldsymbol{s})}\Big[\cos\Big(tx + a\int_{S_2}(ts_1+(r+h)s_2)\ln|ts_1+(r+h)s_2|\Gamma(d\boldsymbol{s})\Big)\\
&\qquad\qquad\qquad -\cos\Big(tx + a\int_{S_2}(ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\Big)\Big]\\
&\qquad\qquad\qquad\qquad \times\Big(\int_{S_2} s_2(1+\ln|ts_1+(r+h)s_2|)\Gamma(d\boldsymbol{s})\Big)dt\\
&+\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}} e^{-\int_{S_2}|ts_1+rs_2|\Gamma(d\boldsymbol{s})}\cos\Big(tx + a\int_{S_2}(ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\Big)\\
&\qquad\qquad \times\Big[\int_{S_2} s_2\ln|ts_1+(r+h)s_2| - s_2\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\Big]dt\\
&:= K_{21}+K_{22}+K_{23}.
\end{aligned}
$$

The integrand of $K_{21}$ converges to

$$
\begin{aligned}
&-e^{-\int_{S_2}|ts_1+rs_2|\Gamma(d\boldsymbol{s})}\cos\Big(tx + a\int_{S_2}(ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\Big)\\
&\qquad\qquad \times\Big(\int_{S_2} s_2(ts_1+rs_2)^{<0>}\Gamma(d\boldsymbol{s})\Big)\Big(\int_{S_2} s_2(1+\ln|ts_1+rs_2|)\Gamma(d\boldsymbol{s})\Big).
\end{aligned}
$$

Using (C.19), the triangle inequality and (C.4), it can be bounded by

$$
\sigma_2 e^{\sigma_2(1+|r|)-\sigma_1|t|}\int_{S_2}|s_2|\Big|1+\ln|ts_1+(r+h)s_2|\Big|\Gamma(d\boldsymbol{s}). \tag{I.8}
$$

The integrand of the above expression can be bounded using Lemma J.1 for any $v>0$ by

$$
\begin{aligned}
&1+v^{-1}\Big(2+|ts_1+(r+h)s_2|^{v}+|ts_1+(r+h)s_2|^{-v}\Big)\\
&\qquad\qquad \le \mathrm{const}_1+\mathrm{const}_2|t|^{v}+\mathrm{const}_3\Big|t+\frac{(r+h)s_2}{s_1}\Big|^{-v}|s_1|^{-v},
\end{aligned}
$$

hence, (I.8) is bounded by

$$
\sigma_2 e^{\sigma_2(1+|r|)-\sigma_1|t|}\Big(\mathrm{const}_1+\mathrm{const}_2|t|^{v}+\mathrm{const}_3\int_{S_2}\Big|t+\frac{(r+h)s_2}{s_1}\Big|^{-v}|s_1|^{-v}\Gamma(d\boldsymbol{s})\Big).
$$

The terms involving $\mathrm{const}_1$ and $\mathrm{const}_2$ are clearly integrable with respect to $t$. The last term is more intricate as it still depends on $h$. We will show that the generalised Lebesgue dominated convergence theorem (Theorem 19, p.89 in Royden and Fitzpatrick (2010)) applies. Denoting

$$
T(h)=e^{-\sigma_1|t|}\Big|t+\frac{(r+h)s_2}{s_1}\Big|^{-v}|s_1|^{-v},
$$

it can be shown that $T(0)$ is integrable with respect to $t$ on $\mathbb{R}$ and $\Gamma$ on $S_2$ invoking the usual arguments. Also, choosing some $v \in (0,1)$, with have by Lemma C.7 with $\eta = -v$, $b = 0$ and $0 < p < 1 - v$,

$$\begin{aligned}
\left|\int T(h) - T(0)\right| &\leq \int_{S_2} |s_1|^{-v} \int_{\mathbb{R}} e^{-\sigma_1|t|} \left|\left|t + \frac{(r+h)s_2}{s_1}\right|^{-v} - \left|t + \frac{rs_2}{s_1}\right|^{-v}\right| dt\Gamma(d\boldsymbol{s}) \\
&\leq \text{const} \int_{S_2} |s_1|^{-v} \left|\frac{hs_2}{s_1}\right|^{p} \Gamma(d\boldsymbol{s}) \\
&\leq \text{const } |h|^p \int_{S_2} |s_1|^{-v-p} \Gamma(d\boldsymbol{s}) \underset{h\to 0}{\longrightarrow} 0,
\end{aligned}$$

because (2.2) holds with $\nu > 1$ and $v + p < v + 1 - v < 1$. Since $T(0)$ is integrable and $\lim_{h\to 0} \int T(h) = \int T(0)$, the generalised dominated convergence theorem applies to $K_{21}$. We turn to $K_{22}$. Its integrand converges to

$$\begin{aligned}
-\, ae^{-\int_{S_2} |ts_1 + rs_2|\Gamma(ds)} \sin\Big(tx + a\int_{S_2} (ts_1 + rs_2)\ln|ts_1 + rs_2|\Gamma(d\boldsymbol{s})\Big) \\
\times \Big(\int_{S_2} s_2(1 + \ln|ts_1 + rs_2|)\Gamma(d\boldsymbol{s})\Big)^2.
\end{aligned}$$

With the usual inequalities and Lemma J.1, it can be bounded for any $v > 0$ by

$$\begin{aligned}
&\frac{a}{|h|} e^{\sigma_2|r| - \sigma_1|t|} \left|\int_{S_2} (ts_1 + (r+h)s_2)\ln|ts_1 + (r+h)s_2| - (ts_1 + rs_2)\ln|ts_1 + rs_2|\Gamma(d\boldsymbol{s})\right| \\
&\qquad\qquad \times \left|\int_{S_2} s_2(1 + \ln|ts_1 + (r+h)s_2|)\Gamma(d\boldsymbol{s})\right| \\
&\leq ae^{\sigma_2|r| - \sigma_1|t|}\Big(Q_1 + Q_2\Big)\Big(\sigma_2 + \int_{S_2} \Big|\ln|ts_1 + (r+h)s_2|\Big|\Gamma(d\boldsymbol{s})\Big) \\
&\leq ae^{\sigma_2|r| - \sigma_1|t|}\Big(Q_1 + Q_2\Big)\Big(\text{const}_1 + \text{const}_2|t|^v + \text{const}_3 \int_{S_2} \left|t + \frac{(r+h)s_2}{s_1}\right|^{-v} |s_1|^{-v}\Gamma(d\boldsymbol{s})\Big),
\end{aligned}$$

where, similarly to (I.5), the two terms $Q_1$ and $Q_2$ involve integrals over $S_2 \cap \{\boldsymbol{s} : |ts_1 + rs_2| \geq 2|h|\}$ and $S_2 \cap \{\boldsymbol{s} : |ts_1 + rs_2| < 2|h|\}$. After expansion, the terms with $\text{const}_1$ and $\text{const}_2$ are readily dealt with by following the method developed for (I.5). Focus on the remaining term

$$a\int_{S_2} e^{\sigma_2|r| - \sigma_1|t|}(Q_1 + Q_2)\left|t + \frac{(r+h)s_2}{s_1}\right||s_1|^{-v}\Gamma(d\boldsymbol{s}).$$

In view of the bounds (I.6) and (I.7), the integrand can be bounded (up to a multiplicative constant) by

$$U(h) = e^{-\sigma_1|t|}\left|t + \frac{rs_2}{s_1}\right|^{-v}\left|t + \frac{(r+h)s_2'}{s_1'}\right|^{-v}|s_1|^{-v}|s_1'|^{-v}.$$

Choosing some $v \in (0, 1/2)$, we can invoke Lemma (C.6) with $\eta = -v$, $p = 0$ and the fact that $\int_{\mathbb{R}} e^{-\sigma_1|t|}|t|^{-2v}dt < +\infty$ to show that $U(0)$ is integrable on the one hand. On the other hand we can again invoke Lemma (C.6), this time with $\eta = -v$, $0 < p < 1 - 2v$, and the fact that (2.2) holds with

$\nu > 1 > v + 1 - 2v > v + p$ to show that $\int U(h) \to \int U(0)$. The generalised dominated convergence theorem applies to $K_{12}$.

We turn to $K_{23}$ for which «appropriate integration by parts» is required. After obvious manipulations,

$$
\begin{aligned}
K_{23} = \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} & s_2' \ln |ts_1' + rs_2'| \left[ e^{-\int_{S_2} \left| \left(t - \frac{hs_2'}{s_1'}\right) s_1 + rs_2 \right| \Gamma(ds)} - e^{-\int_{S_2} |ts_1 + rs_2| \Gamma(ds)} \right] \\
& \times \cos\left( \left(t - \frac{hs_2'}{s_1'}\right) x + a \int_{S_2} \left( \left(t - \frac{hs_2'}{s_1'}\right) s_1 + rs_2 \right) \ln \left| \left(t - \frac{hs_2'}{s_1'}\right) s_1 + rs_2 \right| \Gamma(d\boldsymbol{s}) \right) \Gamma(d\boldsymbol{s}') \\
+ \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} & s_2' \ln |ts_1' + rs_2'| e^{-\int_{S_2} |ts_1 + rs_2| \Gamma(d\boldsymbol{s})} \\
& \times \left[ \cos\left( \left(t - \frac{hs_2'}{s_1'}\right) x + a \int_{S_2} \left( \left(t - \frac{hs_2'}{s_1'}\right) s_1 + rs_2 \right) \ln \left| \left(t - \frac{hs_2'}{s_1'}\right) s_1 + rs_2 \right| \Gamma(d\boldsymbol{s}) \right) \right. \\
& \qquad \left. - \cos\left( tx + a \int_{S_2} (ts_1 + rs_2) \ln |ts_1 + rs_2| \Gamma(d\boldsymbol{s}) \right) \right] \Gamma(d\boldsymbol{s}') \\
:= L_1 + L_2. &
\end{aligned}
$$

Starting with $L_1$, its integrand converges to

$$
\begin{aligned}
e^{-\int_{S_2} |ts_1 + rs_2| \Gamma(d\boldsymbol{s})} & \cos\left( tx + a \int_{S_2} (ts_1 + rs_2) \ln |ts_1 + rs_2| \Gamma(d\boldsymbol{s}) \right) \\
& \times \left( \int_{S_2} s_1 (ts_1 + rs_2)^{<0>} \Gamma(d\boldsymbol{s}) \right) \left( \int_{S_2} \ln |ts_1 + rs_2| {s_2}^2 {s_1}^{-1} \Gamma(d\boldsymbol{s}) \right)
\end{aligned}
$$

It can be bounded using (C.18) and Lemma C.3 ($\iota$) by

$$
\begin{aligned}
& \left| \frac{s_2' \ln |ts_1' + rs_2'|}{h} \right| \exp\left\{ -\min\left( \int_{S_2} \left| \left(t - \frac{hs_2'}{s_1'}\right) s_1 + rs_2 \right| \Gamma(d\boldsymbol{s}), \int_{S_2} |ts_1 + rs_2| \Gamma(d\boldsymbol{s}) \right) \right\} \\
& \qquad \times \left| \int_{S_2} \left| \left(t - \frac{hs_2'}{s_1'}\right) s_1 + rs_2 \right| - |ts_1 + rs_2| \Gamma(d\boldsymbol{s}) \right| \\
& \leq e^{\sigma_2 |r|} \exp\left\{ -\sigma_1 \min\left( \left| t - \frac{hs_2'}{s_1'} \right|, |t| \right) \right\} \left| s_2' \ln |ts_1' + rs_2'| \right| \frac{1}{|h|} \int_{S_2} \left| \frac{hs_2'}{s_1'} s_1 \right| \Gamma(d\boldsymbol{s}) \\
& \leq \sigma_1 e^{\sigma_2 |r|} \exp\left\{ -\sigma_1 \min\left( \left| t - \frac{hs_2'}{s_1'} \right|, |t| \right) \right\} \left| \ln |ts_1' + rs_2'| \right| |s_2'|^2 |s_1'|^{-1} \\
& := V(h).
\end{aligned}
$$

We follow a similar procedure as the one used in Cioczek-Georges and Taqqu (1998) (p.51) to deal with the min inside the exponential. Focus on the case $\dfrac{hs_2}{s_1} > 0$ (the converse case is similar). We have

$$
\min\left( \left| t - \frac{hs_2'}{s_1'} \right|, |t| \right) = \begin{cases} \left| t - \dfrac{hs_2'}{s_1'} \right|, & \text{if} \quad t \geq hs_2'/2s_1', \\ |t|, & \text{if} \quad t < hs_2'/2s_1'. \end{cases}
$$

Thus, up to a multiplicative constant,

$$\begin{aligned}\int_{\mathbb{R}} V(h)dt &= \int_{\frac{hs_2}{2s_1}}^{+\infty} e^{-\sigma_1|t-\frac{hs_2}{s_1}|}\Big|\ln|ts_1+rs_2|\Big||s_2|^2|s_1|^{-1}dt + \int_{-\infty}^{-\frac{hs_2}{2s_1}} e^{-\sigma_1|t|}\Big|\ln|ts_1+rs_2|\Big||s_2|^2|s_1|^{-1}dt \\ &= \int_{-\frac{hs_2}{2s_1}}^{+\infty} e^{-\sigma_1|t|}\Big|\ln\Big|ts_1+rs_2+\frac{hs_2}{s_1}\Big|\Big||s_2|^2|s_1|^{-1}dt + \int_{-\infty}^{-\frac{hs_2}{2s_1}} e^{-\sigma_1|t|}\Big|\ln|ts_1+rs_2|\Big||s_2|^2|s_1|^{-1}dt \\ &= \int_{\mathbb{R}} e^{-\sigma_1|t|}\Big[\Big|\ln|ts_1+(r+h)s_2|\Big|\mathbb{1}_{\{t\geq -hs_2/2s_1\}} + \Big|\ln|ts_1+rs_2|\Big|\mathbb{1}_{\{t\leq -hs_2/2s_1\}}\Big]|s_2|^2|s_1|^{-1}dt.\end{aligned}$$

Thus, using Lemma J.1, we can bound the integrand for any $v>0$ and $|h|<|r|$ by

$$\begin{aligned}&e^{-\sigma_1|t|}\Big[\Big|\ln|ts_1+(r+h)s_2|\Big| + \Big|\ln|ts_1+rs_2|\Big|\Big]|s_2|^2|s_1|^{-1} \\ &\quad\leq v^{-1}e^{-\sigma_1|t|}\Big[\mathrm{const}_1+\mathrm{const}_2|t|^v \\ &\qquad\qquad + \mathrm{const}_3\Big|t+\frac{rs_2}{s_1}\Big|^{-v}|s_1|^{-v} + \mathrm{const}_4\Big|t+\frac{(r+h)s_2}{s_1}\Big|^{-v}|s_1|^{-v}\Big]|s_2|^2|s_1|^{-1}.\end{aligned}$$

Clearly, the terms involving $\mathrm{const}_1$ and $\mathrm{const}_2$ are integrable with respect to $t$ and $\Gamma$. Denoting the last term as $V_4(h) := e^{-\sigma_1|t|}\Big|t+\dfrac{(r+h)s_2}{s_1}\Big|^{-v}|s_2|^2|s_1|^{-1-v}$, we show that the generalised dominated convergence theorem applies. As (2.2) holds for some $\nu>1$, choose $v=\dfrac{\nu-1}{2}>0$ if $\nu<2$, and some $v\in(0,1)$ if $\nu\geq 2$. The integrability of $V_4(0)$ (and at the same time, of the term involving $\mathrm{const}_3$) is obtained from Lemma C.5 with $\eta=-v$, $b=0$, $p=0$ and the fact that $\int_{\mathbb{R}} e^{-\sigma_1|t|}|t|^{-v}dt<+\infty$. Doing so indeed yields

$$\begin{aligned}&\left|\int_{S_2}|s_2|^2|s_1|^{-1-v}\int_{\mathbb{R}} e^{-\sigma_1|t|}\Big|t+\frac{rs_2}{s_1}\Big|^{-v} - |t|^{-v}|s_2|^2|s_1|^{-1-v}dt\right|\Gamma(d\boldsymbol{s}) \\ &\qquad\leq \int_{S_2}\int_{\mathbb{R}} e^{-\sigma_1|t|}\left|\Big|t+\frac{rs_2}{s_1}\Big|^{-v} - |t|^{-v}\right|dt\Gamma(d\boldsymbol{s}) \\ &\qquad\leq \mathrm{const}\int_{S_2}|s_1|^{-\nu}|s_1|^{\nu-1-v}\Gamma(d\boldsymbol{s}) \\ &\qquad\leq \mathrm{const}\int_{S_2}|s_1|^{-\nu}\Gamma(d\boldsymbol{s}) \\ &\qquad< +\infty,\end{aligned}$$

since $\nu-1-v=\dfrac{\nu-1}{2}>0$ if $\nu\in(1,2)$ and $\nu-1-v>\nu-2>0$ if $\nu\geq 2$. The convergence $\int V_4(h)\to\int V_4(0)$ can be obtained from Lemma C.7 with $\eta=-v$, $b=0$ and $0<p<v$. The generalised dominated convergence hence applies to $L_1$.

We turn to $L_2$. Its integrand converges to

$$e^{-\int_{S_2}|ts_1+rs_2|\Gamma(d\boldsymbol{s})}\sin\left(tx+a\int_{S_2}(ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\right)$$

$$\times \left(x + a\int_{S_2} s_1(1+\ln|ts_1+rs_2|)\Gamma(d\boldsymbol{s})\right)\ln|ts_1'+rs_2'|{s_2'}^2{s_1'}^{-1}.$$

Applying the mean value theorem to the cosine function and the usual bounds, we can bound it by

$$\begin{aligned}
&e^{\sigma_2|r|-\sigma_1|t|}\Big|s_2'^2 s_1'^{-1}\ln|ts_1'+rs_2'|\Big|\\
&\quad\frac{1}{\left|\frac{hs_2'}{s_1'}\right|}\left|-\frac{hs_2'}{s_1'}x + a\int_{S_2}\Big(\big(t-\frac{hs_2'}{s_1'}\big)s_1+rs_2\Big)\ln\Big|\big(t-\frac{hs_2'}{s_1'}\big)s_1+rs_2\Big| - (ts_1+rs_2)\ln|ts_1+rs_2|\Gamma(d\boldsymbol{s})\right|\\
&\le e^{\sigma_2|r|-\sigma_1|t|}\Big|s_2'^2 s_1'^{-1}\ln|ts_1'+rs_2'|\Big|\\
&\quad\left(|x| + \frac{a}{\left|\frac{hs_2'}{s_1'}\right|}\int_{S_2}\left|\Big(\big(t-\frac{hs_2'}{s_1'}\big)s_1+rs_2\Big)\ln\Big|\big(t-\frac{hs_2'}{s_1'}\big)s_1+rs_2\Big| - (ts_1+rs_2)\ln|ts_1+rs_2|\right|\Gamma(d\boldsymbol{s})\right).
\end{aligned} \tag{I.9}$$

The term involving $|x|$ can be treated using the usual arguments. The one with the integral is of course the most delicate. Let us split this integral into two parts as:

$$\begin{aligned}
&\int_{S_2}\frac{1}{\left|\frac{hs_2'}{s_1'}\right|}\left|\Big(\big(t-\frac{hs_2'}{s_1'}\big)s_1+rs_2\Big)\ln\Big|\big(t-\frac{hs_2'}{s_1'}\big)s_1+rs_2\Big| - (ts_1+rs_2)\ln|ts_1+rs_2|\right|\Gamma(d\boldsymbol{s})\\
&\qquad := Q_1+Q_2,
\end{aligned}$$

where $Q_1$ and $Q_2$ involve integrals over $S_2\cap\{\boldsymbol{s}: |ts_1+rs_2|\ge 2|hs_2'/s_1'|\}$ and $S_2\cap\{\boldsymbol{s}: |ts_1+rs_2| < 2|hs_2'/s_1'|\}$ respectively. We will first majorise $Q_1$ and $Q_2$, and then use these bounds in inequality (I.9). Consider $Q_2$ and define the function $g$ such that for any $z>0$

$$g(z) = \begin{cases} f(z) = z|\ln z|, & \text{if } \ 0<z<e^{-1},\\ z(2+\ln z), & \text{if } \quad z\ge e^{-1}.\end{cases}$$

It is easily checked that $g$ is continuous, strictly increasing and such that for any $z>0$, $0\le f(z)\le g(z)$. The integrand of $Q_2$ can be bounded as

$$\begin{aligned}
\frac{1}{\left|\frac{hs_2'}{s_1'}\right|}\left(\Big|f\Big(\big(t-\frac{hs_2'}{s_1'}\big)s_1+rs_2\Big)\Big| + \Big|f\big(ts_1+rs_2\big)\Big|\right) &\le \frac{1}{\left|\frac{hs_2'}{s_1'}\right|}\left(\Big|g\Big(\big(t-\frac{hs_2'}{s_1'}\big)s_1+rs_2\Big)\Big| + \Big|g\big(ts_1+rs_2\big)\Big|\right)\\
&\le \frac{1}{\left|\frac{hs_2'}{s_1'}\right|}\left(\Big|g\Big(\Big|\frac{3hs_2'}{s_1'}\Big|\Big)\Big| + \Big|g\Big(\Big|\frac{2hs_2'}{s_1}\Big|\Big)\Big|\right)\\
&\le \frac{2}{\left|\frac{hs_2'}{s_1'}\right|}g\Big(\frac{3hs_2'}{s_1'}\Big).
\end{aligned}$$

By Lemma (J.1), with bound further the right-hand side for any $v>0$ by

$$\frac{2}{\left|\frac{hs_2'}{s_1'}\right|}g\Big(\frac{3hs_2'}{s_1'}\Big) \le \text{const}_1 + \text{const}_2\Big|\frac{3hs_2'}{s_1'}\Big|^{v} + \text{const}_3\Big|\frac{3hs_2'}{s_1'}\Big|^{-v}.$$

On the one hand if $\left|\dfrac{3hs_2'}{s_1'}\right| < e^{-1}$, given that $(3|ts_1+rs_2|/2)^{-v} > (3hs_2'/s_1')^{-v}$,

$$\mathrm{const}_1 + \mathrm{const}_2\left|\frac{3hs_2'}{s_1'}\right|^{v} + \mathrm{const}_3\left|\frac{3hs_2'}{s_1'}\right|^{-v} \leq \mathrm{const}_1 + \mathrm{const}_2\left|t+\frac{rs_2}{s_1}\right|^{-v}|s_1|^{-v}.$$

On the other hand if $\left|\dfrac{3hs_2'}{s_1'}\right| \geq e^{-1}$, then for $|h|<|r|$,

$$\mathrm{const}_1 + \mathrm{const}_2\left|\frac{3hs_2'}{s_1'}\right|^{v} + \mathrm{const}_3\left|\frac{3hs_2'}{s_1'}\right|^{-v} \leq \mathrm{const}_1 + \mathrm{const}_2|s_1'|^{-v}. \tag{I.10}$$

Focusing now on $Q_1$, we can use the mean value theorem to bound its integrand by

$$|s_1|\Big|1+\ln|u|\Big|,$$

for some $u \in \Big[ts_1+rs_2 - hs_2's_1/s_1' \wedge ts_1+rs_2, ts_1+rs_2-hs_2's_1/s_1' \vee ts_1+rs_2\Big]$. Given that $|ts_1+rs_2| \geq 2|hs_2'/s_1'|$, we have $|u| \in \Big[\frac{|ts_1+rs_2|}{2}, 2|ts_1+rs_2|\Big]$ and thus, we further bound the above inequality using Lemma J.1 for any $v>0$ by

$$\begin{aligned}|s_1|\Big(\mathrm{const}_1 + \mathrm{const}_2|ts_1+rs_2|^{v} + \mathrm{const}_3|ts_1+rs_2|^{-v}\Big)&\\ \leq \mathrm{const}_1 + \mathrm{const}_2|t|^{v} + \mathrm{const}_3\left|t+\frac{rs_2}{s_1}\right|^{-v}|s_1|^{1-v}.&\end{aligned} \tag{I.11}$$

Hence, using (I.10) and (I.11) in (I.9), and making use again of Lemma (J.1) to bound $\Big|\ln|ts_1'+rs_2'|\Big|$, we can bound integrand of $L_2$ for any $v>0$ by

$$\begin{aligned}&e^{-\sigma_1|t|}\left(\mathrm{const}_1 + \mathrm{const}_2|t|^{v} + \mathrm{const}_3\left|t+\frac{rs_2'}{s_1'}\right|^{-v}\right)|s_1'|^{-1-v}\\ &\qquad\times\left(|x| + \mathrm{const}_4 + \mathrm{const}_5|t|^{v} + \mathrm{const}_6|s_1'|^{-v} + \mathrm{const}_7\left|t+\frac{rs_2}{s_1}\right|^{-v}|s_1|^{1-v}\right)\end{aligned}$$

It can be shown that all the terms obtained after expansion can be bounded by functions integrable with respect to $t$ and $\Gamma$ using the usual combinations of either Lemma C.5 or Lemma C.6 with $\eta=-v$, $b=0$, $p=0$, the fact that $\int_{\mathbb{R}} e^{-\sigma_1|t|}|t|^{-v} < +\infty$, $\int_{\mathbb{R}} e^{-\sigma_1|t|}|t|^{-2v} < +\infty$ for appropriately chosen values $v>0$, and (2.2) with $\nu>1$. The detail we have to pay attention to is precisely to chose an appropriate exponent $v>0$ so that it satisfies the constraint (2.2) and ensures the finiteness of the two integrals in $t$. The later imposes us to have $v \in (0,1/2)$. Regarding the former, we identify that the most negative power of which $|s_1|$ appears in the above bound after expansion is $-1-2v$. We need $\nu-1-2v>0$. Choosing $v=(\nu-1)/4$ if $1<\nu<3$ and any $v\in(0,1/2)$ if $\nu \geq 3$ enables to satisfy both constraints, validating the use of the dominated convergence theorem for $L_2$, and finally, for $B_2$ in (I.3).

The proof is essentially similar, somewhat easier, for $B_1$ in (I.2) for which the only difficulty is to perform the «appropriate integration by parts» when it comes to differentiating the term involving $(ts_1+rs_2)^{<0>}$.

### I.2 Evaluating at $r = 0$

Since $\mathbb{E}\Big[X_2^2\Big|X_1 = x\Big] = -\phi^{(2)}_{X_2|x}(0)$, we evaluate (I.4) at $r = 0$ and get

$$\begin{aligned}
\varphi_{\boldsymbol{X}}(t,0) &= \exp\{-\sigma_1|t| - ia\sigma_1\beta_1 t\ln|t| + it\mu_1\},\\
A_1/2 &= \sigma_1^2\Big((\kappa_1^2 - a^2q_0^2)H_c(0) + 2a\kappa_1 q_0 H_s(0)\Big)\\
&\qquad\qquad + 2a\lambda_1\sigma_1^2\Big(-aq_0H_c(1) + \kappa_1 H_s(1)\Big) - a^2\lambda_1^2\sigma_1^2 H_c(2),\\
iA_2/2 &= \sigma_1\Big(-ak_1H_c(0) + \kappa_2 H_s(0)\Big) - a\lambda_2\sigma_1 H_c(1),\\
A_3/2 &= \sigma_1\Big((\sigma_1\kappa_2 + a\mu_1 k_1)H_c(0) + (\sigma_1 a k_1 - \mu_1\kappa_2)H_s(0)\Big)\\
&\qquad + a\sigma_1\Big((\lambda_2\mu_1 - a\sigma_1\beta_1 k_1)H_c(1) + \sigma_1(\lambda_2 + \beta_1\kappa_2)H_s(1)\Big) - a^2\sigma_1^2\beta_1\lambda_2 H_c(2),
\end{aligned}$$

where $k_1 = \sigma_1^{-1}\int_{S_2}(s_2/s_1)^2 s_1\ln|s_1|\Gamma(d\boldsymbol{s})$, and the $H_c$'s and $H_s$'s are defined at Lemma J.2. Using the result of the same Lemma under $\beta_1 \neq 0$ and $\beta_1 = 0$, and regrouping the terms allows to retrieve the two formulae of Theorem 2.2.

## J Proof of Proposition 2.1 in the case $\alpha = 1$

**Case $\boldsymbol{\beta_1 \neq 0}$** The conditional second order moment when $\alpha = 1$ has a particular form. We only consider the case $|\beta_1| \neq 1$ and $x \longrightarrow +\infty$. Since $|x| \to +\infty$, we have $x - \mu_1 \sim x$ and we may assume that $\mu_1 = 0$. From Hardin et al. (1991), we know that $U(x) \sim x^{-1}$. Notice that

$$\begin{aligned}
W(x) = &\int_0^{+\infty} e^{-\sigma_1 t}(1+\ln t)^2\cos(a\sigma_1\beta_1 t\ln t)\cos(tx)dt\\
&- \int_0^{+\infty} e^{-\sigma_1 t}(1+\ln t)^2\sin(a\sigma_1\beta_1 t\ln t)\sin(tx)dt.
\end{aligned}$$

Because the factors of $\cos(tx)$ and $\sin(tx)$ are integrable, we have by the Riemann-Lebesgue Lemma that $W(x) \underset{x\to+\infty}{\longrightarrow} 0$. Having also

$$f_{X_1}(x) \sim \frac{\sigma_1(1+\beta_1)}{\pi}x^{-2},$$

we deduce the following limits

$$\begin{aligned}
\Big(2a\sigma_1 q_0(\lambda_1 - \beta_1\kappa_1) + 2(\kappa_1\lambda_1 - \lambda_2)x\Big)\frac{\sigma_1 U(x)}{\beta_1\pi f_{X_1}(x)}x^{-2} &\underset{x\to+\infty}{\longrightarrow} \frac{2(\kappa_1\lambda_1 - \lambda_2)}{(1+\beta_1)\beta_1},\\
\Big(\lambda_2 + \beta_1\kappa_2 - 2\kappa_1\lambda_1 + a^2\sigma_1\beta_1(\lambda_1^2 - \beta_1\lambda_2)W(x)\Big)\frac{\sigma_1 x^{-2}}{\pi f_{X_1}(x)} &\underset{x\longrightarrow+\infty}{\longrightarrow} \frac{\lambda_2 + \beta_1\kappa_2 - 2\kappa_1\lambda_1}{(1+\beta_1)\beta_1}.
\end{aligned}$$

Hence,

$$x^{-2}\mathbb{E}\Big[X_2^2\Big|X_1 = x\Big] \underset{x\to+\infty}{\longrightarrow} \frac{\lambda_2}{\beta_1} + \frac{2(\kappa_1\lambda_1 - \lambda_2)}{(1+\beta_1)\beta_1} + \frac{\lambda_2 + \beta_1\kappa_2 - 2\kappa_1\lambda_1}{(1+\beta_1)\beta_1} = \frac{\kappa_2 + \lambda_2}{1+\beta_1}$$

**Case $\boldsymbol{\beta_1 = 0}$** From Hardin et al. (1991),

$$V(x) \longrightarrow -\frac{\pi}{2x},$$

hence,

$$2a\sigma_1\lambda_1\Big(a\sigma_1 q_0 - \kappa_1(x-\mu_1)\Big)\frac{V(x)}{\pi f_{X_1}(x)}x^{-2} \longrightarrow a\pi\lambda_1\kappa_1.$$

Moreover,

$$a\sigma_1\frac{F_{X_1}(x)-1/2}{f_{X_1}(x)}x^{-2} \longrightarrow \frac{1}{2}a\pi(\lambda_2 - 2\kappa_1\lambda_1).$$

It can be shown that $W(x) \longrightarrow 0$. Therefore,

$$x^{-2}\mathbb{E}\Big[X_2^2\Big|X_1 = x\Big] \underset{x\to+\infty}{\longrightarrow} \kappa_2 + \frac{1}{2}a\pi(\lambda_2 - 2\kappa_1\lambda_1) + a\pi\kappa_1\lambda_1 = \kappa_2 + \lambda_2$$

$\square$

**Lemma J.1** *For any $x > 0$ and $v > 0$*

$$|\ln x| \le \frac{1}{v}\Big(2 + x^v + x^{-v}\Big).$$

We provide here two Lemmas which are used in the proof of Theorem 2.2.

**Lemma J.2** *Let for any $n \ge 0$,*

$$H_c(n) = \int_0^{+\infty} e^{-\sigma_1 t}(1+\ln t)^n \cos\Big(t(x-\mu_1) + a\sigma_1\beta_1 t\ln t\Big)dt,$$
$$H_s(n) = \int_0^{+\infty} e^{-\sigma_1 t}(1+\ln t)^n \sin\Big(t(x-\mu_1) + a\sigma_1\beta_1 t\ln t\Big)dt.$$

*Then, if $\beta_1 \neq 0$,*

$$H_c(1) = \frac{1}{a\sigma_1\beta_1}\Big(\sigma_1 H_s(0) - (x-\mu_1)H_c(0)\Big), \qquad H_s(1) = \frac{1}{a\sigma_1\beta_1}\Big(1 - \sigma_1 H_c(0) - (x-\mu_1)H_s(0)\Big).$$

*If $\beta_1 = 0$,*

$$H_c(0) = \pi f_{X_1}(x),$$
$$H_s(0) = \frac{x-\mu_1}{\sigma_1}\pi f_{X_1}(x),$$
$$H_s(1) - \frac{x-\mu_1}{\sigma_1}H_c(1) = \frac{\pi F_{X_1}(x)}{\sigma_1}.$$

*Proof.* The equalities of Lemmas D.1-J.2 can be obtained by integrating by parts. We provide details for the last equality of Lemma J.2 when $\beta_1 = 0$. Integrating the exponential by parts, we obtain

$$H_s(1) = \frac{1}{\sigma_1} \int_0^{+\infty} e^{-\sigma_1 t} t^{-1} \sin\Big(t(x-\mu_1)\Big) dt + \frac{x-\mu_1}{\sigma_1} H_c(1)$$

Denote $A(x) = \int_0^{+\infty} e^{-\sigma_1 t} t^{-1} \sin\Big(t(x-\mu_1)\Big) dt$ for $x \in \mathbb{R}$ ($A$ is well defined since $e^{-\sigma_1 t} t^{-1} \sin\Big(t(x-\mu_1)\Big) \to x - \mu_1$ as $t \to 0$). It can be shown that we can derivate $A$ under the integral sign and get

$$A'(x) = \int_0^{+\infty} e^{-\sigma_1 t} \cos\Big(t(x-\mu_1)\Big) dt = \pi f_{X_1}(x),$$

Since $X_1$ is Cauchy distributed when $\alpha = 1$ and $\beta_1 = 0$,

$$A(x) = \pi F_{X_1}(x) + \text{const} = \text{Arctg}\Big(\frac{x-\mu_1}{\sigma_1}\Big) + \frac{\pi}{2} + \text{const},$$

and evaluating the integral form of $A$ at $\mu_1$, we deduce that const $= -\pi/2$. Thus, $A(x) = \pi\Big(F_{X_1}(x) - 1/2\Big)$. $\square$

## Additional References


Daley, D. J. and D., Vere-Jones. 2007. *An introduction to the theory of point processes*, Springer, New York, vol. 1.

Davis, R. and S., Resnick. 1985. Limit Theory for Moving Averages of Random Variables with Regularly Varying Tail Probabilities. *Annals of Probability*, 13, 179-195.

Leadbetter, M. R., and S., Nandagopalan. 1989. On exceedance point processes for stationary sequences under mild oscillation restrictions. *Lecture Notes in Statistics*, 51, 69-80.

Royden, H. L., and P. M., Fitzpatrick. 2010. *Real analysis*. Prentice Hall, fourth edition.

Titchmarsh, E. C. 1948. *Introduction to the theory of Fourier integrals*. Second edition, Oxford University Press.