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Abstract

We develop a general framework to study contests, containing the well-known models of Tullock (1980) and Lazear & Rosen (1981) as special cases. The contest outcome depends on players' effort and skill, the latter being subject to symmetric uncertainty. The model is tractable, because a symmetric equilibrium exists under general assumptions regarding production technologies and skill distributions. We construct a link between our contest model and expected utility theory and exploit this link to revisit important comparative statics results of contest theory and show how these can be overturned. Finally, we apply our results to study optimal workforce composition.

Keywords: contest theory, symmetric equilibrium, heterogeneity, risk, decision theory

JEL classification: C72, D74, D81, J23, M51

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1 Introduction

In a contest, two or more players invest effort or other costly resources to win a prize. Many economic interactions can be modeled as a contest. Promotions, for example, represent an important incentive in many firms and organizations. Employees exert effort to perform better than their colleagues and, thus, to be considered for promotion to a more highly paid position within the firm. Litigation can also be understood as a contest, in which the different parties spend time and resources to prevail in court. Procurement is a third example, where different firms invest resources into developing a proposal or lobbying politicians, thereby increasing the odds of being selected, receiving some rent in return.

Players participating in contests are typically heterogeneous in some respect. For instance, employees differ with respect to their skills, the litigant parties differ with respect to the quality of the available evidence, and firms differ with respect to their capabilities of designing a proposal. When accounting for such heterogeneity in contest models, equilibria are often asymmetric, meaning that players choose different levels of efforts. Due to this asymmetry, contests between heterogeneous players are typically difficult to analyze, and researchers have often imposed rather strict assumptions to keep the analyses tractable.

In this paper, we provide a novel framework to study contests between (possibly) heterogeneous players. Under general assumptions about the production technology and skill distributions, the class of contests we study has a symmetric equilibrium in which all players exert the same effort. This makes the contest much easier to investigate and allows us to study behavior in situations that proved to be intractable in other contest models.

In the class of contests that we focus on, the outcome of the contest depends on players' skills and their efforts. The skill distributions of the competing players (including the means) are common knowledge, whereas the exact skill realizations are generally (symmetrically) unknown (as, e.g., in Holmström 1982). In the example of the promotion contest, a player's expected ability may be commonly known (e.g., if the player's education, prior work experience, or CV is known and serves as a signal of ability), whereas the exact ability level for the particular job is unknown (e.g., since there is some uncertainty regarding how education translates into workplace performance and job match). Similar arguments apply to the other examples presented earlier. We assume that a player's skill and effort determine the player's "contribution" to the contest and that the player with the highest contribution wins the contest. Heterogeneity among players is accounted for by allowing the statistical distributions of possible skill realizations to be different for the competing players. Our model is general and contains the well-known models by Tullock (1980) and Lazear & Rosen (1981) as special cases.

We make three primary contributions.

First, we show that in a two-player contest, a symmetric (pure-strategy equal-effort) equilibrium exists under general assumptions about the production technology (i.e., the function mapping skill and effort into a player's contribution to the contest) and individual skill distributions. The main requirement is that the production function is such that for any given positive effort level, a player's contribution to the contest is increasing in his skill. This is a weak requirement from the perspective of the most commonly used neoclassical production technologies, and also appears to be quite realistic.

Second, we construct a link between our contest model and standard models of decision-making under risk (expected utility theory). Exploiting this link, we revisit important comparative statics results of contest theory and show how these can be overturned. In particular, we analyze how equilibrium effort is affected by making the skill distributions of the competing players more heterogeneous, investigating both the role of differences in expected skill (conceptualized by first-order stochastic dominance) and the role of differences in the uncertainty of the skill distributions of the competing players (conceptualized by second-order stochastic dominance). The general message is that making contest participants more heterogeneous can increase equilibrium effort. To the best of our knowledge, these results have not been found in the contest literature before, and indeed contradict "standard" results (e.g., those from the Tullock contest and the Lazear-Rosen tournament). Thus, the comparative statics results derived from those standard models are not representative of the conclusions derived in the more general model.

Third, in two important special cases, we provide novel results on the existence of symmetric equilibria in our general setting when the number of players n is greater than two, and analyze how equilibrium effort is affected by changes in n. We show that when $n > 2$, and players have identical skill distributions, a symmetric equilibrium exists even in the presence of general production technologies. We also show that, given certain assumptions on players' skill distributions, a symmetric equilibrium exists when $n-1$ identical players compete against a player who has a higher expected ability. In both cases, we investigate the effect of increasing the number of players on equilibrium effort. Exploiting the fact that a contest with $n > 2$ players can be interpreted as a two-player contest in which every player competes against the strongest (i.e., the largest order statistic) of the other competitors, we find that increasing the number of contestants can increase equilibrium effort. This result can be understood by the fact that as the number of contestants increases, the strongest opponents grow stronger in the sense of first-order stochastic dominance, allowing us to apply our results from the two-player case.

We discuss the trade-offs involved and provide intuition for all of our results. We also discuss the implications for optimal workforce composition and certain real-world applications in the context of labor and personnel economics. For instance, our finding that efforts can increase if the skill distribution of one of the competing players becomes more uncertain (in the the sense of second-order stochastic dominance) has several interesting managerial implications. It indicates that contest organizers might wish to increase the uncertainty regarding the skills of certain players in order to induce higher effort. In a worker-firm context, employers could achieve this by, for instance, hiring an inexperienced worker for whom less prior information is available, or a minority worker with a skill level drawn from a distribution that generally tends to be more uncertain (as argued, e.g., by Bjerk 2008). This means that a diverse workforce might be desirable from the employer's point of view.

The paper is organized as follows. In the section 2 below, we discuss related literature. Section 3 introduces the contest model and discusses how our model nests the Tullock contest and the Lazear-Rosen tournament as special cases. Section 4 solves the two player model. In section 5, we analyze the two player case in greater detail and provide a set of important comparative statics results. Section 6 studies the n-player case and presents novel comparative statics results for this case. Section 7 discusses implications for organizational design and optimal workforce composition. Finally, section 8 concludes. The Appendix contains the proofs of all our results.

2 Related literature

There are three main approaches to the study of contests, the Tullock or ratio-form contest, the Lazear-Rosen tournament, and the all-pay auction.[1] In the Tullock contest, a player's winning probability is given by the player's contribution to the contest (which is a function of the player's effort and sometimes also of ability) divided by the total contribution to the contest of all players. The Tullock contest has been introduced to the literature by Tullock (1980).[2] It has been axiomatized in various settings by Skaperdas (1996), Clark & Riis (1998), and Münster (2009). The Lazear-Rosen tournament assumes that the player with the highest contribution to the contest wins with certainty, and contributions depend on effort, some random factors (e.g., luck), and possibly on abilities. The seminal paper is by Lazear & Rosen (1981) who apply the model in a labor-market context.[3] The all-pay auction, finally, makes the same assumption as the Lazear-Rosen tournament except that contributions to the contest are deterministic and do not depend on random factors; a detailed equilibrium characterization was developed by Baye et al. (1996).[4]

Most studies analyzing the Tullock contest and the Lazear-Rosen tournament impose assumptions that ensure that equilibria in pure strategies exist. In contrast, only mixed-strategy equilibria exist in the all-pay auction (when players are symmetrically informed about the decision situation). As we indicated in the introduction, and as we explain in more detail in section 3, the Tullock contest and the Lazear-Rosen tournament are special cases of our model, while the all-pay auction is not. One important contribution of our paper is to generalize the Tullock contest and the Lazear-Rosen tournament and show that important results of these models do not always extend to more general production functions and ability distributions.

[1]The theoretical contest literature has been surveyed in a number of books and papers. See, e.g., Konrad (2009) and Vojnović (2015) for recent textbooks and Chowdhury & Gürtler (2015), Chowdhury et al. (2019), and Fu & Wu (2019) for recent surveys.

[2]It has been further investigated by, e.g., Hillman & Riley (1989), Cornes & Hartley (2005), Fu & Lu (2009a,b), Schweinzer & Segev (2012), Chowdhury & Kim (2017), Giebe & Schweinzer (2014, 2015).

[3]Further contributions include Malcomson (1984), O'Keeffe et al. (1984), Lazear (1989), Schotter & Weigelt (1992), Hvide (2002), Grund & Sliwka (2005), Gürtler & Kräkel (2010), Schöttner & Thiele (2010), Gürtler & Gürtler (2015), and Imhof & Kräkel (2016).

[4]The complete-information all-pay auction (with mixed-strategy equilibria) is the most commonly used in contest theory, but a private-values version can be found as well. The all-pay auction has been further studied by, e.g., Barut & Kovenock (1998), Moldovanu & Sela (2001, 2006), Moldovanu et al. (2007), Cohen et al. (2008), Siegel (2009, 2010), Sela (2012), Morath & Münster (2013), Barbieri et al. (2014), and Fang et al. (2019).

These important results refer to how player heterogeneity, the extent of risk or uncertainty, and the number of players affects the effort exerted by competing players. For example, Schotter & Weigelt (1992) have shown that efforts are higher when players have homogeneous skills relative to when players are heterogeneous. The reason in their setting is that disadvantaged players tend to give up and reduce their effort, whereas the advantaged players can afford to reduce their effort. Moreover, several studies have shown that greater uncertainty regarding the contest outcome tends to reduce effort (see, e.g., Hvide 2002). Intuitively, if the contest outcome depends to a greater extent on random factors, effort has a lower impact on who becomes the winner and players reduce effort accordingly. Finally, in the seminal work of Tullock (1980), effort decreases in the number of players who participate in the contest, which has been attributed to a discouragement effect. If a player competes against many rivals, his chance of winning is relatively low and the player reduces effort in turn. Although all of these results seem highly intuitive, we find that they are sensitive to the choice of production technologies and ability distributions. In our general framework, different comparative statics results may emerge.

3 Model description

Consider a contest between two risk-neutral players $i \in \{1,2\}$ who compete for a single prize of value $V > 0$. Both players simultaneously choose effort $e_i \geq 0$, and the cost of effort $c(e_i)$ is increasing, strictly convex, and satisfies $c(0) = 0$. Player i's ability or skill is denoted by θ_i. There is uncertainty about skills, which means that θ_i is a random variable.[5] The realization of θ_i is not known to any of the players (not even player i). It is commonly known, however, that θ_i is independently and continuously distributed according to the pdf f_i (with cdf F_i) with finite mean μ_i. The supports of θ_i, $i \in \{1,2\}$, overlap on a subset of $\mathbb{R}$ with positive measure. The production of player i, and hence his or her *contribution* to the contest, is given by the production function $g(\theta_i, e_i)$. Importantly, we assume that $\frac{\partial g}{\partial \theta_i} > 0$ for all $e_i > 0$ which means (realistically) that each player's contribution to the contest is increasing with respect to his/her skill, for a given level of effort.

Player i wins the contest if and only if his or her contribution is strictly higher than the contribution of the opponent player, namely, $g(\theta_i, e_i) > g(\theta_k, e_k)$ (with $k \in \{1,2\} \setminus \{i\}$).[6] We denote by $P_i(e_i, e_k)$ player i's probability of winning the contest (as a function of the efforts of both players) and we define the expected payoff as $\pi_i(e_i, e_k) := P_i(e_i, e_k)V - c(e_i)$. We also define $\hat{e} := c^{-1}(V)$ and $E := [0, \hat{e}]$. A player's equilibrium effort will always belong to the set E as the probability of winning is bounded above by unity.

We impose the following assumption:

Assumption 1. *The primitives of the model are such that (i) $\pi_i(e_i, e_k)$ is continuously differentiable and (ii) the pure-strategy Nash-equilibrium efforts are positive and characterized by the*

[5] In our analysis, we always refer to θ_i as the player's skill. Of course, θ_i could also account for any other random variable affecting a player's production.

[6] Notice that $g(\theta_i, e_i) = g(\theta_k, e_k)$ happens with probability zero.

first-order conditions to the players' problems of maximizing $\pi_i(e_i, e_k)$, for all $i, k \in \{1,2\}, i \neq k$.

The validity of the first-order approach is typically ensured by imposing additional assumptions on the primitives of the model that guarantee that the objective functions π_i are quasi-concave and increasing at $e_i = 0$. Previous papers in the contest-theory literature, however, have shown that the first-order approach may be valid even when the objective functions are neither quasi-concave nor increasing at $e_i = 0$ (see, e.g., Figure 1 in Schweinzer & Segev 2012). As we do not want to rule out such cases, we simply assume that the Nash-equilibrium efforts are characterized by the players' first-order conditions to their maximization problems without restricting the shape of π_i too strongly. Each of the theoretical results we present will be accompanied by at least one example for which we verify Assumption 1. Finally, we assume that there exist $\bar{e}_i, \breve{e}_i \in int\ E$ such that $\frac{\partial \pi_i(e_i, e_k)}{\partial e_i}|_{e_1=e_2=\bar{e}_i} < 0$ and $\frac{\partial \pi_i(e_i, e_k)}{\partial e_i}|_{e_1=e_2=\breve{e}_i} > 0$. The assumption ensures that the first-order condition to player i's maximization problem can in principle be fulfilled in a symmetric equilibrium.

Below we illustrate by example which contests, distributions and technologies are included in the class of contests which we study in our model.

Tullock contest The well-studied Tullock contest represents a special case of our model. This is easily illustrated using the results in Jia (2008), who considers a contest with a multiplicative production technology, in which player i wins if and only if $\theta_i e_i$ is highest among all players. It is shown that if θ_i is distributed according to the pdf

$$f_i(x) = \gamma_i m x^{-(m+1)} \exp\left\{-\gamma_i x^{-m}\right\} I_{(x>0)},$$

then player i wins the contest with probability

$$P_i(e_1, e_2) = \frac{\gamma_i e_i^m}{\sum_{j=1}^2 \gamma_j e_j^m},$$

where $\gamma_j \geq 0$ for both players j and $m > 0$. Hence, in our model, if $g(\theta_i, e_i) = \theta_i e_i$, and θ_i is distributed according to the above pdf, then we obtain the Tullock contest-success function.

Lazear-Rosen tournament (additive noise) Assuming the production technology $g(\theta_i, e_i) = \theta_i + e_i$, our model includes the standard Lazear-Rosen tournament model (in the original Lazear & Rosen 1981, it is assumed that $\mu_i = 0$). We provide several new results for this well-known setting.

General production technologies Our framework is quite general with respect to the production technology $g(\theta_i, e_i)$. For example, feasible technologies include the CES production function $g(\theta_i, e_i) = \left(\alpha\theta_i^\rho + \beta e_i^\rho\right)^{\frac{1}{\rho}}$, with $\alpha, \beta > 0$ (except for the limiting case of perfect complements). Thus, the case of perfect substitutes, $\rho = 1$, is included as well as technologies where effort and ability are complements to different degrees, such as the standard Cobb-Douglas technology $g(\theta_i, e_i) = \theta_i^\alpha e_i^\beta$, with $\alpha, \beta > 0$ (when ρ approaches zero).

Distributions of ability In our framework, standard continuous distributions can be employed with both finite and infinite supports. Moreover, the distributions can be different for the two players. Examples are the (truncated) Normal distribution, the Exponential distribution, Student's t distribution, the Cauchy distribution, and the Uniform distribution.

4 Model solution

We are particularly interested in pure-strategy Nash equilibria in which both players choose the same level of effort. The following lemma provides a necessary and a sufficient condition for such a symmetric equilibrium to exist.

Lemma 1. *A necessary condition for a symmetric equilibrium to exist, in which both players choose the same effort, is that* $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_1=e_2=e}$ *is the same for* $i,k \in \{1,2\}, i \neq k$ *and at least one* $e \in int\ E$. *A sufficient condition for a symmetric equilibrium to exist, is that* $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_1=e_2=e}$ *is the same for* $i,k \in \{1,2\}, i \neq k$ *and all* $e \in int\ E$.

We investigate the circumstances under which a symmetric equilibrium exists. We make use of Lemma 1 and we prove existence of a symmetric equilibrium by checking the sufficient condition. Since the condition depends on the winning probability, we specify this probability first. For $e > 0$, we define the function $g_e : \mathbb{R} \to \mathbb{R}$ by $g_e(x) = g(x,e)$ with g_e^{-1} as the corresponding inverse. This notation can be motivated by the fact that the event of player i winning over player k can be written as

$$\begin{aligned} & g(\theta_k, e_k) < g(\theta_i, e_i) \\ \Leftrightarrow\quad & g_{e_k}(\theta_k) < g_{e_i}(\theta_i) \\ \Leftrightarrow\quad & \theta_k < g_{e_k}^{-1}\left(g_{e_i}(\theta_i)\right). \end{aligned}$$

Considering all potential realizations of θ^i and θ^k, the winning probability of player i is

$$P_i(e_i,e_k) = \int_{\mathbb{R}} F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right) f_i(x)\,dx.$$

Theorem 1. *A symmetric equilibrium in which both players choose the same level of effort always exists.*

Theorem 1 states that, even if the players are asymmetric (i.e., $f_1 \neq f_2$), there always exists a symmetric equilibrium of the contest game. This result is of great importance since it allows a tractable analysis of contests between asymmetric players in a variety of different settings.

The intuition for the theorem is as follows. A marginal increase in player i's effort affects the outcome of the contest only in cases where both players' contributions to the contest are identical ($g(\theta_i,e_i) = g(\theta_j,e_j)$). If, on the other hand, $g(\theta_i,e_i) > g(\theta_j,e_j)$, player i would win the contest for sure, implying that a marginal increase in e_i would not affect the contest outcome. The same is true if $g(\theta_i,e_i) < g(\theta_j,e_j)$. Here, a marginal increase in e_i would not suffice to overcome player i's disadvantage and player i would lose despite the higher effort. The two players do not know θ_1 and θ_2, so they do not know whose contribution to the contest is higher. But they infer that marginally increasing effort has an effect on the contest outcome only if $g(\theta_1, e_1) = g(\theta_2, e_2)$. In a symmetric equilibrium with $e_1 = e_2 > 0$, the latter equation is equivalent to $\theta_1 = \theta_2$.[7] This means that players' abilities are the same in all cases which matter in the sense that a marginal increase in effort has an effect on the contest outcome. Notice that this holds true even if f_1 and f_2 differ. As the players have the same technology, and the same cost function, $\theta_1 = \theta_2$ implies that they have the same incentive to exert effort and a symmetric equilibrium always exists.

To see this formally, notice that the derivative of player i's winning probability with respect to e_i is

$$\frac{\partial P_i(e_i, e_k)}{\partial e_i} = \int_{\mathbb{R}} f_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right) \frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(x)\right) f_i(x)\, dx.$$

Defining $a_e : \mathbb{R} \to \mathbb{R}$ by $a_e(x) = \frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(x)\right)\Big|_{e_1=e_2=e}$ and recognizing that $g_{e_k}^{-1}(g_{e_i}(x))|_{e_1=e_2=e} = x$, allows us to write the first-order condition for the optimal effort level for player i in a symmetric equilibrium as follows:

$$V \int_{\mathbb{R}} a_e(x) f_k(x) f_i(x)\, dx = c'(e) \Longleftrightarrow V \int_{\mathbb{R}} a_e(x) f_1(x) f_2(x)\, dx = c'(e).$$

The key thing to notice is that this condition does not depend on i, and a mathematically equivalent condition would have been obtained had we started from the perspective of player k.

Defining $r_{e,i} : \mathbb{R} \to \mathbb{R}$ given by $r_{e,i}(x) = a_e(x) f_i(x)$, the above condition can be written as:

$$V \int_{\mathbb{R}} r_{e,i}(x) f_k(x)\, dx = c'(e). \tag{1}$$

The reason for introducing the notation $r_{e,i}$ is that it describes, for a given realization of θ_k, the marginal effect of effort on the probability of outperforming the rival player. Accordingly, the integral on the LHS of the re-written condition above reflects the average marginal effect of effort on the probability of winning across all realizations of θ_k. The properties of the function $r_{e,i}$ will be of great importance for the comparative-statics results that we derive in the next sections. We therefore wish to emphasize that it depends both on the production technology (via a_e) and on the skill distribution (via f_i).

We end this section with an illustrative example in which we calculate the equilibrium for two different skill distributions. Consider the multiplicative production technology $g(\theta_i, e_i) = \theta_i e_i$, and the cost function $c(e_i) = e_i^2/2$. Assume further that the skill distribution of player 1 follows a Uniform distribution on $[1, 2]$, and the skill distribution of player 2 is given by the

[7] Notice that, while the event $\theta_1 = \theta_2$ happens with zero probability, the corresponding density, affecting players' incentive to exert effort, is strictly positive.

Student-t distribution on support $(-\infty,\infty)$, with one degree of freedom, such that:

$$f_1(s) = \begin{cases} 1 & 1 \le s \le 2 \\ 0 & \text{otherwise} \end{cases}, \qquad f_2(x) = \frac{1}{\pi(1+x^2)}, \; x \in \mathbb{R}.$$

The event of player 1 winning is described by $g(\theta_1, e_1) > g(\theta_2, e_2) \iff \theta_2 < g_{e_2}^{-1}(g_{e_1}(\theta_1)) = \theta_1 e_1/e_2$. The probability of that event, and its first derivative with respect to e_1 are

$$P_1(e_1,e_2) = \int_{-\infty}^{\infty} F_2\left(x\frac{e_1}{e_2}\right) f_1(x)dx,$$
$$\frac{\partial P_1(e_1,e_2)}{\partial e_1} = \int_{-\infty}^{\infty} f_2\left(x\frac{e_1}{e_2}\right)\left(\frac{x}{e_2}\right) f_1(x)dx.$$

The first-order condition of player 1's maximization problem is

$$\frac{\partial P_1(e_1,e_2)}{\partial e_1} V = e_1.$$

In a symmetric equilibrium with $e := e_1 = e_2$, this can now be written as

$$V \int_{-\infty}^{\infty} f_2(x) x f_1(x) dx = e^2.$$

For player 2 we obtain the same expression. Using our distributional assumptions, the left-hand side becomes

$$V \int_{-\infty}^{\infty} f_2(x) x f_1(x) dx = V \int_1^2 \frac{x}{\pi(1+x^2)} dx = V \frac{1}{2\pi} \log\left(\frac{5}{2}\right).$$

We thus have a symmetric equilibrium, and the corresponding effort is $e^* = \sqrt{\frac{V\log(\frac{5}{2})}{2\pi}} \approx 0.38\sqrt{V}$.

5 Comparative statics results in the two player model

In this section, we investigate the consequences of making players more heterogeneous (in terms of the statistical properties of their skill distributions) on the incentive to exert effort. To facilitate the analysis, we need one additional assumption.

Assumption 2. *The primitives of the model are such that $q : E \to \mathbb{R}$, defined by*

$$q(e) = V \int_{\mathbb{R}} r_{e,i}(x) f_k(x) dx - c'(e), \quad i,k \in \{1,2\}, i \neq k$$

is monotonically decreasing.

Assumption 2 is not very strict and is always satisfied if $\int_{\mathbb{R}} r_{e,i}(x) f_k(x) dx$ is non-increasing in e because of the assumed strict convexity of c. To give a specific example, consider the CES production function $g(\theta_i, e_i) = \left(\alpha\theta_i^\rho + \beta e_i^\rho\right)^{\frac{1}{\rho}}$, with $\alpha, \beta > 0$ and $\rho \le 1$. Here $a_e(x) = \frac{\beta}{\alpha}\left(\frac{x}{e}\right)^{1-\rho}$, implying that $\int_{\mathbb{R}} a_e(x) f_1(x) f_2(x) dx = e^{\rho-1} \int_{\mathbb{R}} \frac{\beta}{\alpha} x^{1-\rho} f_1(x) f_2(x) dx$. For this specification, Assumption 2 is satisfied in all cases where the worker has an incentive to exert positive effort (i.e., $\int_{\mathbb{R}} \frac{\beta}{\alpha} x^{1-\rho} f_1(x) f_2(x) dx > 0$). Furthermore, the assumption ensures that effort is always increasing in the prize and that the considered equilibrium is unique in the class of symmetric equilibria.

5.1 First-order stochastic dominance

A standard result in contest theory is that heterogeneity among players with respect to their skills reduces the incentive to exert effort (see, e.g., Schotter & Weigelt 1992, or Observation 1 in the survey by Chowdhury et al. 2019). In our framework, this standard result is potentially reversed, as we will now show.

Consider a contest with two players with skills drawn from two distributions with expected values μ_k and μ_i, respectively. If, from the outset, $\mu_k > \mu_i$ and the difference $\mu_k - \mu_i$ is increased, then the two players become more heterogeneous in terms of their expected skill. Based on this idea, we proceed by investigating the consequences of making players more heterogeneous in the sense of first-order stochastic dominance. The following definition makes clear what we mean when we say that one contest is more heterogeneous than another contest in a *first-order* sense.

Definition 1. *Let μ_k and μ_i refer to the expected values of the skill distributions in an initial contest (F_k, F_i), $k, i \in \{1,2\}, k \neq i$. Players in a contest with skill distributions $(\tilde{F}_k, F_i)$, are said to be more heterogeneous (with respect to their skills) relative to players in the initial contest with skill distributions (F_k, F_i), in a first-order sense, if either of the following conditions hold:*

(i) $\mu_k \geq \mu_i$ and $\tilde{F}_k$ dominates F_k in the sense of first-order stochastic dominance.

(ii) $\mu_k \leq \mu_i$ and $\tilde{F}_k$ is dominated by F_k in the sense of first-order stochastic dominance.

We are now in a position to derive our second main result. Due to Assumption 2, equilibrium effort increases if a change in the primitives of the model leads to an increase in $\int_{\mathbb{R}} r_{e,i}(x) f_k(x) dx$. Interestingly, this expression has the same structure as a decision maker's expected utility in decision theory (e.g., Levy 1992), where the function $r_{e,i}$ is replaced by the decision maker's utility function. Since the structure of the problems is the same, we can make extensive use of results from decision theory in our analysis. We obtain the following theorem.

Theorem 2. *Consider two contests with skill distributions $(\tilde{F}_k, F_i)$ and (F_k, F_i) and let $\tilde{e}^*$ and e^* denote the associated (symmetric) equilibrium efforts. Then, $\tilde{e}^* > e^*$ if either one of the following statements hold:*

(i) $r_{e,i}(x)$ is monotonically increasing in x for all values of e and $\tilde{F}_k$ dominates F_k in the sense of first-order stochastic dominance.

(ii) $r_{e,i}(x)$ is monotonically decreasing in x for all values of e and $\tilde{F}_k$ is dominated by F_k in the sense of first-order stochastic dominance.

The intuition behind Theorem 2 is as follows. For a given realization of θ_k, $r_{e,i}$ describes the marginal effect of effort on the probability of outperforming the rival player. If $r_{e,i}$ is monotonically increasing, then effort has a greater effect on the outcome of the contest for large values of θ_k relative to smaller ones. Hence, if $\tilde{F}_k$ first-order stochastically dominates F_k, larger values of θ_k are relatively more likely under distribution $\tilde{F}_k$ and effort has a greater overall impact on the contest outcome. As an immediate consequence, players have a higher incentive to exert effort. The interesting observation is that this is not only the case for player k (whose ability distribution is changed), but also for the opponent player i. The same intuition applies if $r_{e,i}$ is decreasing. Here, the incentive to exert effort increases if player k becomes weaker (in expectation).

Note that Theorem 2 holds independently of whether $\mu_k \leq \mu_i$ or $\mu_k \geq \mu_i$. Combining Definition 1 with Theorem 2, we have the following corollary.

Corollary 1. *Effort can be higher when contestants are more heterogeneous in a first-order sense.*

There is one small caveat to Corollary 1 that we should highlight. If e^* increases as contestants become more heterogeneous, then a symmetric equilibrium in which both players exert positive effort will fail to exist if the heterogeneity between players becomes too large. The reason is that the weaker player would eventually receive a negative payoff, meaning that this player would prefer to choose zero effort.

We illustrate the theorem and its corollary with two examples. In the first example, $r_{e,i}(x)$ is monotonically increasing and effort gets higher as player k becomes stronger. In the second example, $r_{e,i}(x)$ is monotonically decreasing and effort gets higher as player k becomes weaker. The examples also serve to illustrate that $r_{e,i}(x)$ depends on both the production technology and the skill distribution.

Example 1. *Suppose that* $g(\theta,e) = \theta \cdot e$, $\theta_i \sim U[0,1]$, $\theta_k \sim U[1/4, 3/4]$, $\tilde{\theta}_k \sim U[5/16, 13/16]$, $c(e) = \frac{e^2}{2}$, $V = 1$. *Then* $e^* = 1/\sqrt{2} = 0.707$ *and* $\tilde{e}^* = 0.7506$.

Notice first that the positive slope of $r_{e,i}(x)$ in Example 1 is driven by the multiplicative production technology since it implies that $a_e(x) = x/e$ whereas the skill distribution is uniform. The intuition for the increase in effort is as follows. Suppose that, starting from a situation where two players have the same expected ability, one of the players becomes stronger and the contest becomes asymmetric as a result. The effect of a marginal increase in a player's effort on the contest outcome is positive only if players' contributions to the contest are exactly the same, as explained before. The increase in the player's strength increases both players' expected ability when conditioning on the relevant event where both players' contributions to the contest are equally high. Since ability and effort are complements in the multiplicative production technology, players thus have a higher incentive to exert effort. As mentioned before, it is interesting that this happens for both players and not only for the player whose ability distribution was changed.

Example 2. *Suppose that* $g(\theta,e)=\theta+e$, $\theta_i \sim Exp(4/3)$, $\theta_k \sim U[1/2,1]$, $\tilde{\theta}_k \sim U[7/16,15/16]$, $c(e)=\frac{e^2}{2}$, $V=1$. *Then* $e^*=0.499$ *and* $\tilde{e}^*=0.543$.

The negative slope of $r_{e,i}(x)$ in Example 2 is driven by the skill distribution (the decreasing pdf of the exponential distribution) since $a_e(x)=1$ with the given production technology. The intuition for the increase in equilibrium effort here is similar as for Example 1. If one of the players becomes weaker, then the expected value of both players' ability in the only relevant event when both players' contributions to the contest are exactly the same, is reduced. Since the pdf of the exponential distribution is decreasing, the players therefore "move" to higher values of the pdf, meaning that a marginal change in effort has a greater effect on the winning probability. Again, the incentive to exert effort is greater as a result.

Concluding this section, we note that the conditions in Theorem 2 are sufficient, but not necessary conditions for the result that effort can be higher when contestants are more heterogeneous. To illustrate this, we present an additional result based on normal distributions where we first determine the marginal winning probability in a situation with symmetric effort.

Proposition 1. *Suppose that* $\theta_i \sim \mathcal{N}(\mu_i,\sigma_i^2)$, $\theta_k \sim \mathcal{N}(\mu_k,\sigma_k^2)$, *and* $g(\theta,e)=\theta\cdot e$. *Then the marginal winning probability when* $e_1=e_2=e$ *is*

$$\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_1=e_2=e}=\frac{(\mu_i\sigma_k^2+\mu_k\sigma_i^2)\exp\{-\frac{(\mu_i-\mu_k)^2}{2(\sigma_i^2+\sigma_k^2)}\}}{(2\pi)^{\frac{1}{2}}(\sigma_i^2+\sigma_k^2)^{\frac{3}{2}}}.$$

In the above Proposition 1, it can be verified that $r_{e,i}(x)=a_e(x)f_i(x)$ is neither always increasing nor always decreasing, by virtue of the multiplicative production technology combined with the bell-shaped normal distribution. Nonetheless, as illustrated by the following example, equilibrium effort increases as players become more heterogeneous in the sense of increasing the distance $|\mu_i-\mu_k|$.

Example 3. *Consider Proposition 1 and assume that* $(\sigma_1,\sigma_2)=(1,1)$, $(\mu_1,\mu_2)=(\frac{1}{2},\frac{1}{2})$, $V=1$, *and* $c(e)=\frac{e^2}{2}$. *Then equilibrium effort is* $e^*=\frac{1}{2\sqrt[4]{\pi}}\approx 0.38$. *If we increase* μ_1 *from* $\frac{1}{2}$ *to* $\frac{3}{2}$, *keeping* μ_2 *constant, equilibrium effort increases to* $e^*=\frac{exp\{-0.125\}}{\sqrt{2}\sqrt[4]{\pi}}\approx 0.47$.

5.2 Second-order stochastic dominance

The studies by Hvide (2002), Kräkel & Sliwka (2004), Kräkel (2008), Gilpatric (2009), and DeVaro & Kauhanen (2016) investigate how "risk" or "uncertainty" affects players' incentive to exert effort in contests. One result that is common to all of these analyses is that in contests between homogeneous players, higher risk (as measured by a higher variance of the random variables capturing the uncertainty of the contest outcome) leads to lower efforts. We revisit this result in the context of our model and show that effort may increase as the ability distribution of one of the players becomes more uncertain (in the sense of second-order stochastic dominance). In the following definition we formalize what we mean when we say that one skill distribution is more uncertain than another one (see Rothschild & Stiglitz 1970 for details)

Definition 2. *The ability distribution $\tilde{F}_j$ is said to be more uncertain than the distribution F_j if $\tilde{F}_j$ is a mean-preserving spread of F_j. This is equivalent to $\tilde{F}_j$ being dominated by F_j in the sense of second-order stochastic dominance.*

Equipped with this definition, we can use well-known results from decision theory to obtain our next theorem:

Theorem 3. *Consider two contests with skill distributions $(\tilde{F}_k, F_i)$ and (F_k, F_i) and let $\tilde{e}^*$ and e^* denote the associated (symmetric) equilibrium efforts. Suppose that $\tilde{F}_k$ is more uncertain than F_k. Then, the following results hold:*

(i) If $r_{e,i}$ is convex for all values of e, then $\tilde{e}^* > e^*$

(ii) If $r_{e,i}$ is linear for all values of e, then $\tilde{e}^* = e^*$

(iii) If $r_{e,i}$ is concave for all values of e, then $\tilde{e}^* < e^*$

The intuition behind this result is as follows. Applying a mean-preserving spread to the distribution F_k shifts probability mass from the center to the tails of the distribution. If $r_{e,i}$ is convex, then effort has a greater impact on the outcome of the contest and the players have a higher incentive to exert effort. If $r_{e,i}$ is concave, then the opposite happens and equilibrium effort decreases. Finally, if $r_{e,i}$ is linear, effort remains unchanged.

Next, we define contestant heterogeneity in a *second-order* sense and we follow the structure of the corresponding definition of heterogeneity in a first-order sense. In that definition, we used the ranking of players' mean abilities to characterize the initial situation. In the new definition, we do so through the variances of the skill distributions of the competing players. Notice, however, that variance is not always a good measure of uncertainty or risk (see e.g, Rothschild & Stiglitz 1970). Therefore one should keep in mind, when applying the definition below, that higher variance entails higher uncertainty only for certain skill distributions (e.g., the normal distribution).

Definition 3. *Let Var_k and Var_i refer to the variances of the skill distributions in an initial contest (F_k, F_i), $k, i \in \{1,2\}, k \neq i$. Players in a contest with skill distributions $(\tilde{F}_k, F_i)$, are said to be more heterogeneous (with respect to their skills) relative to players in the initial contest with skill distributions (F_k, F_i), in a second-order sense, if either of the following conditions hold:*

(i) $Var_k \geq Var_i$ and F_k dominates $\tilde{F}_k$ in the sense of second-order stochastic dominance.

(ii) $Var_k \leq Var_i$ and F_k is dominated by $\tilde{F}_k$ in the sense of second-order stochastic dominance.

Combining Theorem 3 with Definition 3, we have the following corallary.

Corollary 2. *Effort can be higher when contestants are more heterogeneous in a second-order sense.*

Notice that Theorem 3 also holds if players have the same expected ability, namely $\mu_i = \mu_k$. This means that, in a contest with two players who are expected to be equally able, higher uncertainty regarding players' abilities may increase the incentive to exert effort, in contrast to what the existing contest literature has shown.

The following example, set in the context of the Lazear-Rosen framework with an additive production technology (for which $a_e(x) = 1$), illustrates this result. We assume a convex pdf f_1 (an exponential distribution) to make the function $r_{e,1}$ convex. For player 2, we enlarge the support of a uniform distribution (on a subset of the support of f_1) to create a mean-preserving spread.

Example 4. *Suppose the additive production function $g(\theta_i, e_i) = \theta_i + e_i$, the parameter $V = 1$, and the cost function $c(e) = \frac{e^2}{2}$. Suppose player 1's ability θ_1 follows the exponential distribution with parameter $\lambda = 1$, while player 2's ability is uniformly distributed on* $[0.5, 1.5]$ *(notice that these distributional assumptions imply $\mu_1 = \mu_2 = 1$). The equilibrium effort is then $e^* \approx 0.38$. Now, consider a mean-preserving spread of player 2's uniform distribution, enlarging the support to* $[0, 2]$*. Then effort increases to $e^* \approx 0.43$.*

We are thus able to demonstrate, by example, that increasing uncertainty can increase effort in a contest in which both players have the same expected ability. In order to further illustrate the results of Theorem 3, we consider a linear function $r_{e,i}$. In this case, we confirm that equilibrium effort (and payoff) is not affected as ability becomes more uncertain.

Example 5. *Suppose the additive production function $g(\theta_i, e_i) = \theta_i + e_i$, the parameter $V = 1$, and the cost function $c(e) = \frac{e^2}{2}$. Suppose further that abilities are uniformly distributed. Player 1's ability has support* $[0, 2]$ *while player 2's has support* $[0.5, 1.5]$*. Then equilibrium effort is $e^* = 0.5$, and equilibrium payoffs are $\pi_i(e^*, e^*) = 0.375$. Now, consider a mean-preserving spread of player 2's uniform distribution, enlarging the support to* $[0, 2]$*. Then both efforts and payoffs in equilibrium remain unchanged.*

6 The case of more than two players ($n > 2$)

In the contest literature as well as in applications, how effort depends on the number of contestants is of interest. In order to address this question, we turn to the case of $n > 2$ contestants.[8]

6.1 The $n = 2$ main result does not extend to $n > 2$

In the case of $n > 2$ players with different skill distributions, the equilibrium in our model is generally no longer symmetric. A player i will only win the contest if he/she beats all of his/her

[8]Contests with more than two players have been studied by, e.g., Tullock (1980), Nalebuff & Stiglitz (1983), Hillman & Riley (1989), Chen (2003), and Münster (2007).

opponents. Essentially, each player is thus competing against the best of the other players, that is, the largest order statistic, and therefore faces a different "rival" in the contest. This introduces an asymmetry into the model that was absent in the 2-player case which generally leads to an asymmetric equilibrium. To see this formally, suppose, for simplicity, that $g(\theta_i, e_i) = \theta_i + e_i$, implying that $a_e(x) = 1$ (the following intuition also holds for general production technologies). Then, using a similar reasoning as in the two-player case (see section 4), the marginal probability of winning for player 1 and player 2 in a symmetric equilibrium can be written, respectively, as:

$$\int_{\mathbb{R}} f_1(x) \frac{d}{dx} \left(F_2(x) \prod_{i=3}^{n} F_i(x) \right) dx$$

and

$$\int_{\mathbb{R}} f_2(x) \frac{d}{dx} \left(F_1(x) \prod_{i=3}^{n} F_i(x) \right) dx.$$

Applying the product differentiation rule, the marginal winning probabilities can be restated as

$$\int_{\mathbb{R}} \left(f_1(x) f_2(x) \prod_{i=3}^{n} F_i(x) + f_1(x) F_2(x) \frac{d}{dx} \left(\prod_{i=3}^{n} F_i(x) \right) \right) dx$$

and

$$\int_{\mathbb{R}} \left(f_2(x) f_1(x) \prod_{i=3}^{n} F_i(x) + f_2(x) F_1(x) \frac{d}{dx} \left(\prod_{i=3}^{n} F_i(x) \right) \right) dx.$$

The first term in each of the expressions corresponds to the situation, in which players 3 to n perform worse than players 1 and 2 so that the n-player contest collapses to a contest between players 1 and 2. Here, the marginal winning probabilities are the same, as shown in the analysis of the 2-player contest. The second term in each of the expressions corresponds to the situation, in which player 2 (first expression) or player 1 (second expression) performs worse than player 1 (first expression) or player 2 (second expression) so that the contest boils down to a contest between the considered player and the strongest one of the players 3 to n.

Setting the first expression equal to the second, we obtain

$$\int_{\mathbb{R}} f_1(x) F_2(x) \frac{d}{dx} \left(\prod_{i=3}^{n} F_i(x) \right) dx = \int_{\mathbb{R}} f_2(x) F_1(x) \frac{d}{dx} \left(\prod_{i=3}^{n} F_i(x) \right) dx$$

$$\Leftrightarrow \quad \int_{\mathbb{R}} \left(\frac{f_1(x)}{F_1(x)} - \frac{f_2(x)}{F_2(x)} \right) F_1(x) F_2(x) \frac{d}{dx} \left(\prod_{i=3}^{n} F_i(x) \right) dx = 0.$$

In general, the latter equality is not fulfilled and a symmetric equilibrium does not exist. A symmetric equilibrium would occur if the skill distribution associated with each player had the same *constant reversed hazard rate* (i.e., $f_i(x)/F_i(x) = const$ for all $i \in \{1, ..., n\}$, see e.g., Chandra & Roy 2001), in which case we could solve the general case for more than two players in the same way as we did for the two-player case. To the best of our knowledge, however, there are no known examples of parametric distributions with constant reversed hazard rates on $(-\infty, \infty)$

(see e.g., Rinne 2014). This implies that our solution method generally cannot be extended to the case of $n > 2$ players. Therefore, we mainly look at the case of homogeneous players in this section (maintaining the generality of our assumed production technology). We also examine a special case of our model which demonstrates that a symmetric equilibrium can exist even when players are *asymmetric* (in the sense of having different skill distributions) and the number of players n is greater than two.

6.2 The case of homogeneous players

Consider a contest in which all players have the same skill distribution $f_1 = f_2 = ... = f_n =: f$.

Proposition 2. *In an n-player contest with homogeneous skill distributions, a symmetric Nash equilibrium with $e_1^* = ... = e_n^* =: e^*$ exists and is characterized by*

$$V \int_{\mathbb{R}} r_{e^*}(x)(n-1)(F(x))^{n-2} f(x)\,dx = V \int_{\mathbb{R}} r_{e^*}(x) \left(\frac{d}{dx} (F(x))^{n-1} \right) dx = c'\left(e^*\right).$$

Notice that $(F(x))^{n-1}$ describes the cdf of the largest order statistic out of a group of $n-1$ players. The condition from the proposition therefore illustrates what we claimed before: the n-player contest boils down to a two-player contest, in which every player competes against the strongest of the other players.

A particular focus in the literature has been on the relation between effort and the number of competitors. Early studies of the n-player Tullock contest with $\gamma_1 = ... = \gamma_n$, $m = 1$, and linear effort cost found that equilibrium effort is given by $e^* = \frac{n-1}{n^2} V$, so that effort is decreasing in n (e.g., Tullock 1980, Hillman & Riley 1989). With a convex cost function (as in our setting), the condition would change to $e^* c'(e^*) = \frac{n-1}{n^2} V$, but effort would still be decreasing in n. The result can be explained by a discouragement effect. If a player competes against many rivals, his/her chance of winning is relatively low and the player reduces effort in turn.

In what follows, we study the relationship between effort and the number of competitors in our framework. To do so, we need to extend Assumption 2 to the n-player case.

Assumption 3. *The primitives of the model are such that $q_n : E \to \mathbb{R}$, defined by*

$$q_n(e) = V \int_{\mathbb{R}} r_e(x) \left(\frac{d}{dx} (F(x))^{n-1} \right) dx - c'(e),$$

is monotonically decreasing.

We observe that, in addition to the discouragement effect mentioned before, there is also an encouragement effect, inducing players to increase their effort as they compete against more players. This is reflected by the term $(n-1)$ in $\int_{\mathbb{R}} r_{e^*}(x)(n-1)(F(x))^{n-2} f(x)\,dx$ in Proposition 2 above. As we will show, the encouragement effect might dominate, opening up for the possibility that effort increases as the number of competitors increase. In our proof, we use the fact that increasing n leads to a distribution of the largest order statistic that first-order stochastically dominates the original distribution. We can thus invoke Theorem 2 to study the effects of an increase in n on equilibrium effort.

Theorem 4. *i) If $r_e(x)$ is monotonically increasing in x for all values of e, then e^* is increasing in n.*
ii) If $r_e(x)$ is monotonically decreasing in x for all values of e, then e^* is decreasing in n.
iii) If $r_e(x)$ is constant for all values of e, then e^* is independent of n.

Notice that similar to what was mentioned in connection to Corollary 1, there is a small caveat to part (i) of Theorem 4. If e^* is increasing in n, a symmetric equilibrium in which all players exert positive effort will fail to exist if n becomes so large that $V/n < c(e^*)$, as (some) players would prefer to choose an effort of zero.

We conclude this subsection with an example to illustrate the potentially positive relationship between effort and the number of players in the context of the well-known Lazear-Rosen model that is a special case of our model.

Example 6. *Consider a contest with an additive production function $g(\theta_i, e_i) = \theta_i + e_i$, $\mu_i = 1$, $V = 1$, and a cost function given by $c(e) = \frac{e^2}{2}$. Suppose θ_i is distributed according to the modified ($\mu_i = 1$) reflected exponential distribution with parameter $\lambda = 0.5$. With two players, effort is $e^* = 0.25$. With three players, effort increases to $e^* = 0.33$.*

6.3 A contest with one player who is more highly skilled

We now turn to a special case of our contest model with $n > 2$, for which we obtain a symmetric equilibrium even when players have asymmetric skill distributions. For this purpose, suppose that θ_i can be written as $\theta_i = t_i + \varepsilon_i$, where $t_2 = ... = t_n =: t$ and $t_1 > t$, so that player 1 is more able than the other players in expectation. We define $\Delta t := t_1 - t$. The ε_i are i.i.d. and follow the *reflected exponential distribution*, defined by the following pdf and cdf.

$$f_i(x) = \begin{cases} \lambda \cdot \exp\{\lambda x\}, \text{ for } x < 0 \\ 0, \text{ for } x \geq 0, \end{cases}$$

with $\lambda > 0$, and

$$F_i(x) = \begin{cases} \exp\{\lambda x\}, \text{ for } x < 0 \\ 1, \text{ for } x \geq 0. \end{cases}$$

In this case, the reversed hazard rate $\frac{d}{dx} \log F_i(x) = \frac{f_i(x)}{F_i(x)}$ is constant and equal to λ on the support of f_i which is $(-\infty, 0]$. We can show the following result:

Proposition 3. *Consider an n-player contest where players have i.i.d. skill terms ε_i drawn from the reflected exponential distribution. Suppose that one player in expectation is more highly skilled than the other $n-1$ players who share the same expected skill. Then, a symmetric equilibrium exists in which all players choose the same equilbrium effort e^*.*

Notice that no particular assumptions are being imposed on the production technology in Proposition 3. To understand why a symmetric equilibrium exists, consider the condition

$$\int_{\mathbb{R}} \left(\frac{f_1(x)}{F_1(x)} - \frac{f_2(x)}{F_2(x)} \right) F_1(x) F_2(x) \frac{d}{dx} \left(\prod_{i=3}^{n} (F_i(x)) \right) dx = 0,$$

that we derived in Subsection 6.1. Notice that, for player 1, the upper bound of the support is t_1, whereas for the other players it is t (since the upper bound of the support of the reflected exponential distribution is zero). For $x < t$, we have $\frac{f_1(x)}{F_1(x)} = \frac{f_2(x)}{F_2(x)}$ since the reversed hazard rate is constant. For $x \geq t$, we have $\prod_{i=3}^{n} (F_i(x)) = 1 \Rightarrow \frac{d}{dx} \left(\prod_{i=3}^{n} (F_i(x)) \right) = 0$. Hence, the marginal winning probabilities are the same.

If we make a particular assumption on the production technology, we can obtain an expression for the marginal winning probability in the symmetric equilibrium, and calculate how effort depends on the number of players, as the following proposition demonstrates.

Proposition 4. *Consider the contest described in Proposition 3. Suppose the production technology takes the form $g(\theta_i, e_i) = \theta_i e_i$ and assume that t is chosen sufficiently large such that $n\lambda t - 1 > 0$. Then, the marginal winning probability in the symmetric equilibrium is equal to:*

$$\Psi(n) = \frac{(n-1)}{e^*} \exp\{-\lambda \Delta t\} \frac{(n\lambda t - 1)}{n^2}, \quad \textit{with} \quad \Psi'(n) > 0,$$

implying that equilibrium effort is increasing in effort n.

The above Proposition 4 further substantiates our finding that larger contests can lead to higher effort.

7 Implications for optimal workforce composition

Our analysis has implications for optimal workforce composition and organizational design. In particular, our results suggest that employers could find it desirable to employ a more heterogeneous workforce as an instrument to induce higher effort. In the paper, we have analyzed two ways of making players in a contest more heterogeneous. First, we have analyzed the effects of increasing the heterogeneity in workers' expected skills, and shown how this can increase equilibrium effort. This means that a firm could benefit (from the perspective of inducing higher effort) by hiring some workers with a high expected ability and some with a low expected ability, based on, for example, signals such as the quality of the institution where a college graduate received his/her degree. Second, we have shown how increased uncertainty regarding abilities of some players can increase equilibrium effort. Thus, a firm could benefit from hiring a mix of experienced workers (for whom the uncertainty regarding abilities is relatively small) and inexperienced workers (for whom the uncertainty regarding abilities is relatively large).

To see this more formally, suppose a firm already employs a worker with ability distribution F_1 and considers to hire another worker with ability distribution F_2. Moreover, assume that

$r_{e,1}(x)$ is monotonically decreasing and convex (for example, by assuming that the production function is given by $g(\theta, e) = \theta + e$ and skills are Exponentially distributed with parameter λ).[9] Then, the firm may gain from hiring another worker with a lower expected ability ($\mu_2 < \mu_1$), but where F_2 is more uncertain (meaning that worker 2 is drawn from a more uncertain skill distribution). This finding can be understood from the perspective of Theorem 2, that tells us that effort will be higher due to the lower expected ability of worker 2, combined with Theorem 3, which tells us that effort will be higher due to the larger uncertainty regarding the skill of player 2. In other words, hiring a worker with a lower expected ability, drawn from a more uncertain distribution, can induce higher effort. Theorem 2 and Theorem 3 also have other managerial implications as they indicate that employers may want to hire workers who have worked on different tasks in the past, to create uncertainty about workers' abilities. In a similar vein, it might be desirable to implement some kind of job rotation.

8 Concluding remarks

We have presented a novel framework to study contests between heterogeneous players. Under general assumptions about the production technology and the distribution of skills, we have shown that the contest has a symmetric equilibrium, in which all players exert the same effort. We have constructed a link between our contest model and standard models of decision-making under risk (expected utility theory) and exploited this link to revisit important comparative statics results of contest theory. We have shown that standard results in the literature are not robust to generalizations of the production technology or skill distributions. In particular, we have found that making skill distributions more heterogeneous (in terms of first and second moments), or increasing the number of contestants, can increase equilibrium effort. Finally, we have discussed optimal workforce composition and concluded that employers could find it desirable to increase the heterogeneity of the workforce in terms of the statistical properties of the skill distributions of the competing players.

A possible next step would be to use our framework to study additional aspects of tournament design. For instance, prior work has investigated strategic information revelation by the tournament designer (e.g., Aoyagi 2010, Ederer 2010, Gürtler et al. 2013). If the tournament designer possesses some private information about the players' abilities, he or she may decide to reveal some or all of this information to trigger higher effort by the players. Another example would be to allow for different prize structures in the n-player case and investigate how effort depends on the prize structure. For example, one alternative prize structure would be to award the prize V to the $n-1$ best-performing players, and a prize of zero to the worst performing player. This would change Theorem 4 since every player now would compete against the lowest order statistic associated with the opponent players. The lowest order statistic becomes

[9] An alternative skill distribution that would also be decreasing and convex would be a normal distribution that is truncated to the left at a point to the right of the second inflection point. Such a distribution could be motivated by the observation that abilities are often normally distributed and that, when employing worker 1, the firm tried to hire the most able applicant, meaning that abilities in the higher end of the distribution are most relevant (see e.g., Aguinis & O'Boyle Jr. 2014).

weaker as the number of players increase, implying that the relationship between effort and the number of players would change. Summing up, we believe that our new contest framework opens up many avenues for interesting future research.

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9 Appendix

All proofs and computations will be presented in the order in which they appear in the paper.

Proof of Lemma 1. In a pure-strategy Nash equilibrium, for given effort e_k of the other player, player i chooses e_i to maximize π_i. Denote player i's best response by $e_i^*(e_k)$. We know that $e_i^*(e_k) \in E$. Since E is compact and π_i is a continuous function of e_i, the maximum of π_i on E exists (by the Weierstrass theorem).

Denote the pair of Nash equilibrium efforts by (e_1^*, e_2^*). From Assumption 1, we know that $e_i^*(e_k^*) \in int\ E$ and that it is characterized by the first-order condition to the problem of maximizing π_i. This in turn implies that (e_1^*, e_2^*) is determined by $\frac{\partial P_i}{\partial e_i}(e_i^*, e_k^*)V = c'(e_i^*)$ for both $i \in \{1,2\}$.

In a symmetric equilibrium with effort $e^* \in int\ E$, the pair of first-order conditions becomes

$$\frac{\partial P_1(e_1, e_2)}{\partial e_1}|_{e_1=e_2=e^*} V - c'(e^*) = 0$$

and

$$\frac{\partial P_2(e_2, e_1)}{\partial e_2}|_{e_1=e_2=e^*} V - c'(e^*) = 0.$$

A necessary condition for both these conditions to be fulfilled (and thus, a symmetric equilibrium to exist), is that $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_1=e_2=e}$ is the same for both $i \in \{1,2\}$ and at least one $e \in int\ E$.

It remains to be shown that a sufficient condition for the existence of a symmetric equilibrium is that $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_1=e_2=e}$ is the same for both $i \in \{1,2\}$ and all $e \in int\ E$. Given this condition, we observe $\frac{\partial P_1(e_1,e_2)}{\partial e_1}|_{e_1=e_2=e} V - c'(e) = \frac{\partial P_2(e_2,e_1)}{\partial e_2}|_{e_1=e_2=e} V - c'(e)$ for all $e \in int\ E$. Since $\pi_i(e_i, e_k)$ is continuously differentiable, $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_1=e_2=e} V - c'(e)$ is a continuous function of e. Furthermore, recall that there exists $\bar{e}_i, \breve{e}_i \in int\ E$ such that $\frac{\partial \pi_i(e_i,e_k)}{\partial e_i}|_{e_1=e_2=\bar{e}_i} < 0$ and $\frac{\partial \pi_i(e_i,e_k)}{\partial e_i}|_{e_1=e_2=\breve{e}_i} > 0$. Hence, by the intermediate value theorem some $e^* \in int\ E$ with $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_1=e_2=e^*} V - c'(e^*) = 0$ exists.

□

Proof of Theorem 1. By Assumption 1, each player i chooses $e_i > 0$ with probability 1 in any equilibrium. Then, the function $g_e : \mathbb{R} \to \mathbb{R}$ defined by $g_e(x) = g(x, e)$ is monotonically increasing and, thus, invertible. The inverse, g_e^{-1}, is increasing as well. For the two (different) players $i, k \in \{1,2\}$, we observe

$$\begin{aligned} & g(\theta_i, e_i) < g(\theta_k, e_k) \\ \Leftrightarrow\quad & g_{e_i}(\theta_i) < g_{e_k}(\theta_k) \\ \Leftrightarrow\quad & \theta_i < g_{e_i}^{-1}\left(g_{e_k}(\theta_k)\right). \end{aligned}$$

Player k thus wins with probability

$$\int F_i\left(g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right) f_k(x)\, dx.$$

Differentiating with respect to e_k, we obtain

$$\int f_i\left(g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right)\left(\frac{d}{de_k} g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right) f_k(x)\, dx.$$

According to Lemma 1, a sufficient condition for a symmetric equilibrium to exist is that

$$\begin{aligned} & \int \left(\frac{d}{de_1} g_{e_2}^{-1}\left(g_{e_1}(x)\right)\Big|_{e_1=e_2=e} \right) f_1(x) f_2(x)\, dx \\ = \quad & \int \left(\frac{d}{de_2} g_{e_1}^{-1}\left(g_{e_2}(x)\right)\Big|_{e_1=e_2=e} \right) f_1(x) f_2(x)\, dx, \end{aligned}$$

for all $e \in int\, E$. Since $\frac{d}{de_1} g_{e_2}^{-1}\left(g_{e_1}(x)\right)\Big|_{e_1=e_2=e} = \frac{d}{de_2} g_{e_1}^{-1}\left(g_{e_2}(x)\right)\Big|_{e_1=e_2=e}$, this condition is always fulfilled. □

Proof of Theorem 2. Suppose that Assumption 2 holds, $r_{e,i}(x)$ is monotonically increasing in x and $\tilde{F}_k$ first-order stochastically dominates F_k. Denote the optimal effort levels for the two distributions by $\tilde{e}^*$ and e^*. Our goal is to show that $\tilde{e}^* > e^*$.

The proof proceeds by contradiction, so suppose $\tilde{e}^* \leq e^*$. Now observe that

$$\begin{aligned} & V \int r_{\tilde{e}^*,i}(x) \tilde{f}_k(x)\, dx - c'\left(\tilde{e}^*\right) \geq \\ & V \int r_{e^*,i}(x) \tilde{f}_k(x)\, dx - c'\left(e^*\right) > \\ & V \int r_{e^*,i}(x) f_k(x)\, dx - c'\left(e^*\right) = 0. \end{aligned}$$

The first inequality follows from $\tilde{e}^* \leq e^*$ together with Assumption 2. The second inequality follows from $r_{e,i}(x)$ being monotonically increasing and $\tilde{F}_k$ first-order stochastically dominating F_k (see, e.g., Levy 1992, p. 557). The equality follows since e^* is characterized by the first-order condition $V \int r_{e^*,i}(x) f_k(x)\, dx - c'(e^*) = 0$. We conclude that

$$V \int r_{\tilde{e}^*,i}(x) \tilde{f}_k(x)\, dx - c'\left(\tilde{e}^*\right) > 0,$$

showing that the first-order condition in the case of distribution $\tilde{F}_k$ cannot be fulfilled. This gives us the desired contradiction.

By an analogous argument we can show that $\tilde{e}^* > e^*$ also in the case where $r_{e,i}$ is monotonically decreasing in x for all $e > 0$ and F_k first-order stochastically dominates $\tilde{F}_k$, since in this case, $\int r_{e,i}(x) \tilde{f}_k(x)\, dx > \int r_{e,i}(x) f_k(x)\, dx$ for all $e > 0$ (see, e.g., Levy 1992, p. 557).

□

Proof of Proposition 1. Suppose that $g(\theta,e)=\theta e$. This means that

$$\left(\frac{d}{de_i}g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)\Big|_{e_1=e_2=e} = \frac{d}{de_i}\left(\frac{xe_i}{e_k}\right)\Big|_{e_1=e_2=e} = \frac{x}{e}.$$

In the considered situation, the marginal winning probability is

$$\frac{1}{2\pi\sigma_1\sigma_2}\int\frac{x}{e}\exp\left\{-0.5\tfrac{(x-\mu_1)^2}{\sigma_1^2}-0.5\tfrac{(x-\mu_2)^2}{\sigma_2^2}\right\}dx.$$

To prove the lemma, it needs to be shown that

$$\begin{aligned}
&\frac{1}{2\pi\sigma_1\sigma_2}\int x\exp\left\{-0.5\tfrac{(x-\mu_1)^2}{\sigma_1^2}-0.5\tfrac{(x-\mu_2)^2}{\sigma_2^2}\right\}dx \\
=\ &\frac{\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\exp\left\{-\tfrac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right\}}{(2\pi)^{0.5}\left(\sigma_1^2+\sigma_2^2\right)^{1.5}}.
\end{aligned}$$

Define

$$Z:=\frac{1}{2\pi\sigma_1\sigma_2}\int x\exp\left\{-0.5\tfrac{(x-\mu_1)^2}{\sigma_2^2}-0.5\tfrac{(x-\mu_1)^2}{\sigma_2^2}\right\}dx$$

and notice that we can state

$$\begin{aligned}
&0.5\tfrac{(x-\mu_1)^2}{\sigma_1^2}+0.5\tfrac{(x-\mu_2)^2}{\sigma_2^2} \\
=\ &\frac{\sigma_2^2\left(x-\mu_1\right)^2\left(\sigma_1^2+\sigma_2^2\right)+\sigma_1^2\left(x-\mu_2\right)^2\left(\sigma_1^2+\sigma_2^2\right)}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)} \\
=\ &\frac{\sigma_2^2\left(x^2-2x\mu_1+\mu_1^2\right)\left(\sigma_1^2+\sigma_2^2\right)+\sigma_1^2\left(x^2-2x\mu_2+\mu_2^2\right)\left(\sigma_1^2+\sigma_2^2\right)}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)} \\
=\ &\frac{x^2\left(\sigma_1^2+\sigma_2^2\right)-2x\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\left(\sigma_1^2+\sigma_2^2\right)+\left(\mu_1^2\sigma_2^2+\mu_2^2\sigma_1^2\right)\left(\sigma_1^2+\sigma_2^2\right)}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)} \\
=\ &\frac{x^2\left(\sigma_1^2+\sigma_2^2\right)^2-2x\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\left(\sigma_1^2+\sigma_2^2\right)+\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)^2}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)} \\
&-\frac{\left(\mu_1^2\sigma_2^4+2\mu_1\sigma_2^2\mu_2\sigma_1^2+\mu_2^2\sigma_1^4-\mu_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)-\mu_2^2\sigma_1^2\left(\sigma_1^2+\sigma_2^2\right)\right)}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)} \\
=\ &\frac{\left(x\left(\sigma_1^2+\sigma_2^2\right)-\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\right)^2+\left(\mu_1^2\sigma_1^2\sigma_2^2-2\mu_1\sigma_2^2\mu_2\sigma_1^2+\mu_2^2\sigma_1^2\sigma_2^2\right)}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)} \\
=\ &\frac{\left(x\left(\sigma_1^2+\sigma_2^2\right)-\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\right)^2}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)}+\frac{\left(\mu_1-\mu_2\right)^2}{2\left(\sigma_1^2+\sigma_2^2\right)}.
\end{aligned}$$

Using this, we obtain

$$
\begin{aligned}
Z &= \frac{\exp\left\{-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right\}}{2\pi\sigma_1\sigma_2}\int x\exp\left\{-\frac{\left(x(\sigma_1^2+\sigma_2^2)-(\mu_1\sigma_2^2+\mu_2\sigma_1^2)\right)^2}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}\right\}dx \\
&= \frac{\exp\left\{-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right\}}{(2\pi)^{0.5}\left(\sigma_1^2+\sigma_2^2\right)^{1.5}}\left(\sigma_1^2+\sigma_2^2\right)\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}\int x\exp\left\{-\frac{\left(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right\}dx.
\end{aligned}
$$

Now notice that

$$
\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}\int x\exp\left\{-\frac{\left(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right\}dx
$$

describes the mean of a normally distributed random variable with variance $\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2$ and mean $\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}$, hence

$$
\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}\int x\exp\left\{-\frac{\left(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right\}dx = \frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}.
$$

We obtain

$$
\begin{aligned}
Z &= \frac{\exp\left\{-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right\}}{(2\pi)^{0.5}\left(\sigma_1^2+\sigma_2^2\right)^{1.5}}\left(\sigma_1^2+\sigma_2^2\right)\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2} \\
&= \frac{\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\exp\left\{-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right\}}{(2\pi)^{0.5}\left(\sigma_1^2+\sigma_2^2\right)^{1.5}}.
\end{aligned}
$$

□

Proof of Theorem 3. Because of Assumption 2 and the condition characterizing equilibrium effort, we need to show that $\int r_{e,i}(x)\tilde{f}_k(x)dx > (=,<) \int r_{e,i}(x)f_k(x)dx$ if $r_{e,i}$ is convex (linear, concave). The corresponding proof is completely analogous to part a) of the proof of Theorem 2 in Rothschild & Stiglitz (1970, p.237). In the case of convex $r_{e,i}$, the inequality in their proof is reversed, while it is replaced by an equality if $r_{e,i}$ is linear.

□

Proof of Proposition 2. Player i wins the contest with probability

$$
\int \prod_{k\neq i} F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)f_i(x)dx.
$$

Differentiating with respect to e_i, we obtain

$$\int \left(\prod_{k\neq i} F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)\right)\left(\sum_{k\neq i} \frac{f_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)\left(\frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)}{F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)}\right) f_i(x)\,dx.$$

In a symmetric equilibrium with $e_1^* = \ldots = e_n^* =: e^*$, the marginal effect of effort on the probability of winning,

$$\int \left(\prod_{k\neq i} F(x)\right)\left(\sum_{k\neq i} \left(\frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(t+x)\right)\right)\bigg|_{e_1^*=\ldots=e_n^*=:e^*} \frac{f(x)}{F(x)}\right) f(x)\,dx,$$

must be the same for all i. This expression can be restated as

$$\int r_{e^*}(x)(n-1)(F(x))^{n-2} f(x)\,dx = \int r_{e^*}(x)\left(\frac{d}{dx}(F(x))^{n-1}\right) dx.$$

The latter expression is the same for all i.

□

Proof of Theorem 4. i) We show that $\int r_e(x)\left(\frac{d}{dx}(F(x))^{n-1}\right) dx$ is increasing in n. If $n_1, n_2 \in \mathbb{N}$, with $n_1 > n_2$, then $(F(x))^{n_1-1}$ first-order stochastically dominates $(F(x))^{n_2-1}$, and the result follows from Theorem 2.

ii) Suppose that $r_e(x)$ is monotonically decreasing in x for all values of e, and let $n_1, n_2 \in \mathbb{N}$, with $n_1 > n_2$. It follows that $(F(x))^{n_1-1}$ first-order stochastically dominates $(F(x))^{n_2-1}$, as just mentioned, implying that

$$\int r_e(x)\left(\frac{d}{dx}(F(x))^{n_1-1}\right) dx < \int r_e(x)\left(\frac{d}{dx}(F(x))^{n_2-1}\right) dx.$$

iii) If $r_e(x)$ is constant for all values of e, we have

$$\int r_e(x)\left(\frac{d}{dx}(F(x))^{n-1}\right) dx = r_e(x)\int \left(\frac{d}{dx}(F(x))^{n-1}\right) dx = r_e(x),$$

which is independent of n. □

Proof of Proposition 3. Player i outperforms player k iff

$$\begin{aligned} & g_{e_i}(t_i+\varepsilon_i) > g_{e_k}(t_k+\varepsilon_k) \\ \Leftrightarrow\quad & \varepsilon_k < g_{e_k}^{-1}\left(g_{e_i}(t_i+\varepsilon_i)\right) - t_k. \end{aligned}$$

Hence, player i wins the contest with probability

$$\int \prod_{k\neq i} F_k\left(g_{e_k}^{-1}\left(g_{e_i}(t_i+x)\right) - t_k\right) f_i(x)\,dx.$$

Recall that the ε_j are i.i.d., so that we can drop the subscripts accompanying the cdfs and pdfs. In a symmetric equilibrium with $e_1^* = \ldots = e_n^* =: e^*$, the marginal effect of effort on the probability of winning,

$$\int \left(\prod_{k\neq i} F(t_i + x - t_k) \right) \left(\sum_{k\neq i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_i + x) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(t_i + x - t_k)}{F(t_i + x - t_k)} \right) f(x)dx,$$

must be the same for all i. For player 1, we have

$$\int \left(\prod_{k\neq 1} F(\Delta t + x) \right) \left(\sum_{k\neq 1} \left(\frac{d}{de_1} g_{e_k}^{-1} \left(g_{e_1}(t_1 + x) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(\Delta t + x)}{F(\Delta t + x)} \right) f(x)dx.$$

For some other player $i \in \{2, \ldots, n\}$, we have

$$\begin{aligned} \int \left(F(-\Delta t + y) \prod_{k\neq 1,i} F(y) \right) & \left(\left(\frac{d}{de_i} g_{e_1}^{-1} \left(g_{e_i}(t + y) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(-\Delta t + y)}{F(-\Delta t + y)} \right. \\ & \left. + \sum_{k\neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t + y) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(y)}{F(y)} \right) f(y)dy. \end{aligned}$$

The map $\phi_1 : \mathbb{R}_x \to \mathbb{R}_y$ given by $x \to y = \Delta t + x$ is a smooth diffeomorphism with $\det|\phi_1'(x)| = 1$. Applying the associated change of variables to the preceding expression, we obtain

$$\begin{aligned} \int \left(F(x) \prod_{k\neq 1,i} F(\Delta t + x) \right) & \left(\left(\frac{d}{de_i} g_{e_1}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(x)}{F(x)} \right. \\ & \left. + \sum_{k\neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(\Delta t + x)}{F(\Delta t + x)} \right) f(\Delta t + x)dx. \end{aligned}$$

The expressions for the two types of players can be restated as

$$\begin{aligned} & \int \left(F(x) \prod_{k\neq 1} F(\Delta t + x) \right) \left(\sum_{k\neq 1} \left(\frac{d}{de_1} g_{e_k}^{-1} \left(g_{e_1}(t_1 + x) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(\Delta t + x)}{F(\Delta t + x)} \right) \frac{f(x)}{F(x)} dx, \\ & \int \left(F(x) \prod_{k\neq 1} F(\Delta t + x) \right) \left(\left(\frac{d}{de_i} g_{e_1}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(x)}{F(x)} \right. \\ & \quad \left. + \sum_{k\neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(\Delta t + x)}{F(\Delta t + x)} \right) \frac{f(\Delta t + x)}{F(\Delta t + x)} dx. \end{aligned}$$

Notice that both expressions are equal to zero for $x \geq -\Delta t$. Hence, they can be restated as

$$\begin{aligned} & \int^{-\Delta t} \left(F(x) \prod_{k\neq 1} F(\Delta t + x) \right) \left(\sum_{k\neq 1} \left(\frac{d}{de_1} g_{e_k}^{-1} \left(g_{e_1}(t_1 + x) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(\Delta t + x)}{F(\Delta t + x)} \right) \frac{f(x)}{F(x)} dx, \\ & \int^{-\Delta t} \left(F(x) \prod_{k\neq 1} F(\Delta t + x) \right) \left(\left(\frac{d}{de_i} g_{e_1}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(x)}{F(x)} \right. \\ & \quad \left. + \sum_{k\neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Bigg|_{e_1^*=\ldots=e_n^*=e^*} \frac{f(\Delta t + x)}{F(\Delta t + x)} \right) \frac{f(\Delta t + x)}{F(\Delta t + x)} dx. \end{aligned}$$

For $x < -\Delta t$, we observe $\frac{f(x)}{F(x)} = \frac{f(\Delta t+x)}{F(\Delta t+x)} = \lambda$, and the expressions become

$$\lambda^2 \int^{-\Delta t} \left(F(x) \prod_{k\neq 1} F(\Delta t + x)\right) \left(\sum_{k\neq 1} \left(\frac{d}{de_1} g_{e_k}^{-1}\left(g_{e_1}(t_1+x)\right)\right)\Big|_{e_1^*=\ldots=e_n^*=e^*}\right) dx,$$
$$\lambda^2 \int^{-\Delta t} \left(F(x) \prod_{k\neq 1} F(\Delta t + x)\right) \left(\sum_{k\neq i} \left(\frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(t_1+x)\right)\right)\Big|_{e_1^*=\ldots=e_n^*=e^*}\right) dx,$$

which are the same. □

Proof of Proposition 4. As shown before, if $g(\theta_i, e_i) = \theta_i e_i$, we have

$$\left(\frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(t_i+x)\right)\right)\Big|_{e_1^*=\ldots=e_n^*=e^*} = \frac{(t_i+x)}{e^*}.$$

Thus,

$$\begin{aligned}
&\lambda^2 \int^{-\Delta t} \left(F(x) \prod_{k\neq 1} F(\Delta t + x)\right) \left(\sum_{k\neq i} \left(\frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(t_1+x)\right)\right)\Big|_{e_1^*=\ldots=e_n^*=e^*}\right) dx \\
=\ & \lambda^2 \int^{-\Delta t} (\exp\{\lambda x\} \cdot \exp\{(n-1)\lambda(\Delta t + x)\})(n-1) \frac{(t_1+x)}{e^*} dx \\
=\ & \frac{\lambda^2 (n-1)}{e^*} \int^{-\Delta t} \exp\{n\lambda y + (n-1)\lambda \Delta t\}(t_1 + y)\, dy.
\end{aligned}$$

The map $\phi_2 : \mathbb{R}_x \to \mathbb{R}_y$ given by $x \to y = -\Delta t + x$ is a smooth diffeomorphism with $\det|\phi_2'(x)| = 1$. Applying the associated change of variables to the integral, we obtain

$$\begin{aligned}
&\frac{\lambda^2 (n-1)}{e^*} \int^0 \exp\{n\lambda(x - \Delta t) + (n-1)\lambda\Delta t\}(t+x)\, dx \\
=\ & \frac{(n-1)}{e^*} \exp\{-\lambda \Delta t\}\, \lambda^2 \left(\int^0 x \exp\{n\lambda x\}\, dx + t \int^0 \exp\{n\lambda x\}\, dx\right).
\end{aligned}$$

Notice that

$$n\lambda \int^0 x \exp\{n\lambda x\}\, dx$$

is the mean of a random variable that is distributed according to the reflected exponential distribution with parameter $n\lambda$, hence

$$\begin{aligned}
& n\lambda \int^0 x \exp\{n\lambda x\}\, dx = -\frac{1}{n\lambda} \\
\Leftrightarrow\ & \int^0 x \exp\{n\lambda x\}\, dx = -\frac{1}{n^2 \lambda^2}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
n\lambda \int^0 \exp\{n\lambda x\}\, dx &= 1 \\
\int^0 \exp\{n\lambda x\}\, dx &= \frac{1}{n\lambda}.
\end{aligned}$$

It follows that

$$
\begin{aligned}
& \frac{(n-1)}{e^*} \exp\{-\lambda\Delta t\}\,\lambda^2 \left(\int^0 x \exp\{n\lambda x\}\,dx + t \int^0 \exp\{n\lambda x\}\,dx \right) \\
= \;& \frac{(n-1)}{e^*} \exp\{-\lambda\Delta t\}\,\lambda^2 \frac{(-1+n\lambda t)}{n^2\lambda^2} \\
= \;& \frac{(n-1)}{e^*} \exp\{-\lambda\Delta t\} \frac{(-1+n\lambda t)}{n^2}.
\end{aligned}
$$

Taking the derivative of the above expression w.r.t. n results in an expression that is positive for all parameters such as $n\lambda t - 1 > 0$. □