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Tanaka, Yasuhito

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Tax reduction for full-employment and debt dynamics: A Keynesian analysis by mathematics and simulation

Yasuhito Tanaka

Faculty of Economics, Doshisha University,
Kamigyo-ku, Kyoto, 602-8580, Japan.
E-mail:yatanaka@mail.doshisha.ac.jp

Abstract

We examine the effects of a fiscal policy by tax reduction, which realizes full-employment from a state of under-employment or with deflationary GDP gap, on the debt-to-GDP ratio, using a continuous time and a discrete time debt dynamics. We show that the larger the growth rate of real GDP by tax reduction is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is, and an aggressive fiscal policy by tax reduction for full-employment can reduce the debt-to-GDP ratio. Therefore, full-employment can be realized with smaller debt-to-GDP ratio than before the tax reduction policy. However, for this result we need that the marginal propensity to consume is fairly large. We also consider a condition to realize full-employment within one year without increasing debt-to-GDP ratio.

Keywords: tax reduction, full-employment, debt-to-GDP ratio, continuous and discrete time debt dynamics

JEL Classification No.: E62.

1. Introduction

Watts and Sharpe (2016) presented a discrete time version of a dynamic analysis of debt-to-GDP ratio, and showed that an aggressive fiscal policy by an increase in the government expenditure can reduce the debt-to-GDP ratio. Generalizing their model, using continuous time and discrete time debt dynamics, we present analyses of a fiscal policy by tax reduction which realizes full-employment from a state of under-employment or with deflationary GDP gap[1]. Under-employment state arises due to aggregate demand shortage. As discussed in Mitchell et al. (2019) sustained unemployment imposes significant costs such as loss of current national output and income, skill loss, and so on. Therefore, it is valuable that full-employment is realized in the short term.

We consider time (or periods) required to realize full-employment for a continuous time model and a discrete time model of debt dynamics, and examine the debt-to-GDP ratio at the time when full-employment is realized. The government reduces the tax revenue to increase consumption and accelerate the economic growth until full-employment is realized. The magnitude of tax reduction depends on the target growth rate of real GDP over ordinal growth, the ratio of the tax revenue to real GDP, and the magnitude of multiplier effects.

In the next section we consider steady states of continuous time and discrete time debt dynamics. In Section 3 we analyze the effects of tax reduction to realize full-employment. In Section 4 we present some graphical simulations based on plausible assumptions of variables.

Let g be the growth rate of the full-employment real GDP, ρ be the extra growth rate of real GDP over g by tax reduction policy (the growth rate of real GDP ratio is $g + \rho$) in a state of under-employment, and γ be the extra (negative) growth rate of the tax revenue over g by tax reduction (the growth rate of the tax revenue is $g + \gamma$). The main results are as follows.

1. The larger the value of ρ is, the faster the full-employment state is realized. (Figure 1 and 10)

2. The larger the value of ρ is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is, that is, the more aggressive the fiscal policy is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is. (Figure 4 and 13)

   > The reason for this result is as follows. The smaller the value of ρ is, the longer the time we need to realize full-employment is. On the other hand, as shown in 4 below (or Proposition 1 and 3), the ratio of the tax revenue to real GDP at the time when full-employment is realized does not depend on ρ. Therefore, when ρ is small, the accumulated budget deficit including burden of interest is large.

3. When the value of ρ is larger than the critical value, the tax reduction to realize full-employment reduces the debt-to-GDP ratio. (Figure 6 and 15)

4. The ratio of the tax revenue to real GDP at the time when full-employment is realized does not depend on the values of ρ and γ. (Proposition 1 and 3)

[1] In other papers we have presented analyses of fiscal policy by an increase in the government expenditure to realize full-employment using a continuous or a discrete version of debt dynamics.

5. If full-employment is realized within one year, the debt-to-GDP ratio and the ratio of budget surplus to GDP satisfy the steady state condition, and the propensity to consume c satisfies the following condition

$$c > \frac{1}{1+2d(0)},$$

in the continuous time model, or

$$c > \frac{1}{1+d(0)},$$

in the discrete time model, the debt-to-GDP ratio at the time when full-employment is realized is smaller than that before the tax reduction policy $d(0)$. (Proposition 2, 4)

The main conclusion of this paper is that full-employment can be realized by an aggressive tax reduction policy with smaller debt-to-GDP ratio than before the fiscal policy. However, as we show in the concluding remark, for our results the marginal propensity to consume should be fairly large.

An increase in real GDP through an increase in consumption by tax reduction may induce a rise in the interest rate. Since the higher the interest rate is, the larger the debt-to-GDP is (Subsection 4.1.9 and 4.2.9), we need an appropriate monetary policy which maintains the low interest rate.

Since the multiplier of an increase in the government expenditure is larger than the multiplier of tax reduction, a fiscal policy by an increase in the government expenditure is more effective than tax reduction as a policy to realize full-employment, and in that case we do not require that the marginal propensity to consume is fairly large. However, a policy for full-employment should finish its role when the full-employment state is realized to avoid inflation. Therefore, for example, public investment with long-term expense is not appropriate for a policy to realize full-employment, and tax reduction may be more usable.

2. Debt dynamics and steady state

We consider a continuous time version and a discrete time version of debt dynamics. The variables are as follows. t denotes a time in a continuous time model, and it denotes a period in a discrete time model.

c: marginal propensity to consume, $0 < c < 1$[2],
τ: marginal tax rate at $t = 0$, $0 < \tau < 1$,
τ': marginal tax rate at $t > 0$, $0 < \tau' < 1$,
$Y(0)$: real GDP at $t = 0$,

[2] About consumption functions in a dynamic Keynesian model please see Otaki (2007) and Otaki (2009). In Appendix A we present a derivation of multiplier by an overlapping generations model of consumption.

$Y(t)$: real GDP at t, $t > 0$,
$Y_m(0)$: full-employment real GDP at $t = 0$,
$Y_m(t)$: full-employment real GDP at t, $t > 0$,
$\zeta = \frac{Y_m(0)}{Y(0)}$, $\zeta > 1$,
$\tilde{t}$: the time (or period) at which full-employment is realized, $\tilde{t} > 0$,
$G(0)$: government expenditure at $t = 0$,
$G(t)$: government expenditure at t, $t > 0$,
$T(0)$: tax revenue at $t = 0$,
$T(t)$: tax revenue at t, $t > 0$,
$\eta(0) = \frac{T(0)}{Y(0)}$: ratio of tax revenue to real GDP (or average tax rate) at $t = 0$,
$\eta(t) = \frac{T(t)}{Y(t)}$: ratio of tax revenue to real GDP (or average tax rate) at t, $t > 0$,
$B(0)$: government budget surplus at $t = 0$,
$B(t)$: government budget surplus at t, $t > 0$,
$b(0) = \frac{B(0)}{Y(0)}$,
$D(0)$: government debt at $t = 0$,
$D(t)$: government debt at t, $t > 0$,
$d(0) = \frac{D(0)}{Y(0)}$,
$d(t) = \frac{D(t)}{Y(t)}$,
d^*: the steady state value of $d(t)$,
g: the growth rate of the full-employment real GDP, $g > 0$
ρ: the extra growth rate of real GDP by tax reduction policy, $\rho > 0$,
γ: the extra (negative) growth rate of tax revenue by tax reduction policy, $\gamma < 0$,
r: interest rate.

$\gamma < 0$ means tax reduction. There are various ways of tax reduction. For example, lump-sum tax reduction (lump-sum subsidy) or reduction of the marginal tax rate, that is, $\tau' < \tau$. In this paper we consider a general formulation of tax reduction that the growth rate of the tax revenue is reduced by $|\gamma|$ from the ordinary growth rate g. Thus, the growth rate of the tax revenue is $g + \gamma$ with $\gamma < 0$. We assume that τ' is determined independently of γ.

Note that in our model the multiplier of tax reduction is $\frac{c}{1-c}$ not $\frac{c(1-\tau')}{1-c(1-\tau')}$. About the multipliers please see Appendix B and C.

The unit of time is a year. We assume $g + \rho > r$. $D(0)$ and $D(t)$ in a discrete time model denote the debts of the government at the ends of period 0 and period t.

2.1. Continuous time model

We examine a steady state of continuous time debt dynamics. At the steady state

$$Y(t) = e^{gt}Y(0),\ G(t) = e^{gt}G(0),\ T(t) = e^{gt}T(0).$$

Thus,

$$B(t) = T(t) - G(t) = e^{gt}B(0).$$

The derivative of $D(t)$ with respect to t is

$$D'(t) = rD(t) - B(t).$$

$D(t)$ is calculated as

$$\begin{aligned} D(t) =& e^{rt}D(0) - \int_0^t e^{r(t-s)}B(s)ds = e^{rt}D(0) - \int_0^t e^{r(t-s)}e^{gs}B(0)ds \\ =& e^{rt}D(0) - e^{rt}B(0)\int_0^t e^{(g-r)s}ds = e^{rt}D(0) - e^{rt}B(0)\left[\frac{e^{(g-r)s}}{g-r}\right]_0^t \\ =& e^{rt}D(0) - e^{rt}B(0)\frac{e^{(g-r)t}-1}{g-r}. \end{aligned}$$

$e^{r(t-s)}$ denotes a burden of interest between e and t. Since $Y(t) = e^{gt}Y(0)$,

$$\frac{D(t)}{Y(t)} = e^{(r-g)t}\frac{D(0)}{Y(0)} - e^{(r-g)t}\frac{B(0)}{Y(0)}\frac{e^{(g-r)t}-1}{g-r}.$$

Therefore, the debt-to-GDP ratio at time t is obtained as follows.

$$d(t) = e^{(r-g)t}d(0) - e^{(r-g)t}b(0)\frac{e^{(g-r)t}-1}{g-r} = e^{(r-g)t}d(0) - b(0)\frac{1-e^{(r-g)t}}{g-r}.$$

At the steady state

$$d(t) = d(0) = d^*.$$

Then,

$$d^* = \frac{1}{1-e^{(r-g)t}}\left[b(0)\frac{1-e^{(r-g)t}}{r-g}\right] = \frac{b(0)}{r-g}. \quad (1)$$

2.2. Discrete time model

Next we examine a steady state of discrete time debt dynamics.

At the steady state

$$Y(t) = (1+g)^tY(0),\ G(t) = (1+g)^tG(0),\ T(t) = (1+g)^tT(0).$$

Thus,

$$B(t) = T(t) - G(t) = (1+g)^tB(0).$$

Then, $D(t)$ is calculated as

$$D(t) =(1+r)^tD(0) - \sum_{s=1}^{t}(1+r)^{t-s}B(s) = (1+r)^tD(0) - B(0)\sum_{s=1}^{t}(1+r)^{t-s}(1+g)^s$$
$$=(1+r)^tD(0) - (1+r)^tB(0)\sum_{s=1}^{t}\left(\frac{1+g}{1+r}\right)^s$$
$$=(1+r)^tD(0) - (1+r)^tB(0)\frac{\left(\frac{1+g}{1+r}\right)^{t+1} - \left(\frac{1+g}{1+r}\right)}{\left(\frac{1+g}{1+r}\right) - 1}$$
$$=(1+r)^tD(0) - B(0)\frac{(1+g)^{t+1} - (1+g)(1+r)^t}{g-r}.$$

$(1+r)^{t-s}$ denotes a burden of interest between e and t. Since $Y(t) = (1+g)^tY(0)$,

$$\frac{D(t)}{Y(t)} = \left(\frac{1+r}{1+g}\right)^t\frac{D(0)}{Y(0)} - \frac{1+g}{g-r}\frac{B(0)}{Y(0)}\left[1-\left(\frac{1+r}{1+g}\right)^t\right].$$

Therefore, the debt-to-GDP ratio at time t is obtained as follows.

$$d(t) = \left(\frac{1+r}{1+g}\right)^t d(0) - \frac{1+g}{g-r}b(0)\left[1-\left(\frac{1+r}{1+g}\right)^t\right].$$

At the steady state

$$d(t) = d(0) = d^*.$$

Then[3],

$$d^* = \frac{1}{1-\left(\frac{1+r}{1+g}\right)^t}\frac{1+g}{r-g}b(0)\left[1-\left(\frac{1+r}{1+g}\right)^t\right] = \frac{1+g}{r-g}b(0). \tag{2}$$

3. Tax reduction for full-employment

3.1. Continuous time model

We assume that there exists a deflationary GDP gap, that is, $Y(0)$ is smaller than the full-employment real GDP, $Y_m(0)$, at time 0. Thus, $\zeta > 1$. Since $Y_m(t)$ increases at the rate g,

$$Y_m(t) = e^{gt}Y_m(0).$$

[3]If we define $D(0)$ and $D(t)$ as the government debt at the beginnings of period 0 and period t, then we have

$$d^* = \frac{1}{r-g}b(0).$$

The government reduces the tax to increase the growth rate of real GDP from g to $g+\rho$ so as to realize full-employment. Then,

$$Y(t) = e^{(g+\rho)t}Y(0).$$

Suppose that at time $\tilde{t}$

$$e^{(g+\rho)\tilde{t}}Y(0) = e^{g\tilde{t}}Y_m(0),$$

that is, full-employment is realized at $\tilde{t}$. Then, we have

$$e^{\rho\tilde{t}} = \zeta.$$

$\tilde{t}$ is obtained as follows.

$$\tilde{t} = \frac{\ln\zeta}{\rho}. \tag{3}$$

The larger the value of ρ is, the faster the full-employment state is realized.

The tax revenue at t is written as

$$T(t) = e^{(g+\gamma)t}T(0).$$

The increase in real GDP over the ordinary growth is brought by the *multiplier effects* of tax reduction. Therefore, we have the following equation.

$$\left[e^{(g+\rho)\tilde{t}} - e^{g\tilde{t}}\right]Y(0) = \frac{c}{1-c}\left[-e^{(g+\gamma)\tilde{t}} + e^{g\tilde{t}}\right]T(0). \tag{4}$$

About derivation of (4) please see Appendix B. This means

$$e^{\rho\tilde{t}} - 1 = \frac{c}{1-c}(-e^{\gamma\tilde{t}} + 1)\eta(0).$$

Since $e^{\rho\tilde{t}} = \zeta$,

$$\zeta - 1 = \frac{c}{1-c}\left(1 - e^{\gamma\tilde{t}}\right)\eta(0).$$

Thus,

$$e^{\gamma\tilde{t}} = -\frac{1-c}{\eta(0)c}(\zeta - 1) + 1,$$

and so

$$\gamma\frac{\ln\zeta}{\rho} = \ln\left[-\frac{1-c}{\eta(0)c}(\zeta - 1) + 1.\right]$$

From this

$$\gamma = \frac{\rho\ln\left[-\frac{1-c}{\eta(0)c}(\zeta - 1) + 1\right]}{\ln\zeta}. \tag{5}$$

If we use approximation of a logarithmic function, $\ln x = x - 1$, we obtain

$$\gamma = \frac{-(1-c)\rho}{\eta(0)c}. \tag{6}$$

Note that $g+\rho$ is the average growth rate of real GDP in $0 < t \leq \tilde{t}$. On the other hand, $g+\gamma$ is the real growth rate of the tax revenue. The government determines γ so that real GDP at $\tilde{t}$ is equal to $e^{(g+\rho)\tilde{t}}Y(0)$.

$B(t)$ is the sum of the budget surplus (or deficit) growing at the rate g from $B(0)$ and the budget surplus (or deficit) brought by the tax reduction policy. It is written as

$$B(t) = e^{gt}B(0) + \left[e^{(g+\gamma)t} - e^{gt}\right]T(0) = e^{gt}B(0) + \left[e^{(g+\gamma)t} - e^{gt}\right]\eta(0)Y(0).$$

The derivative of $D(t)$ with respect to t is

$$D'(t) = rD(t) - B(t) = rD(t) - e^{gt}B(0) - \left[e^{(g+\gamma)t} - e^{gt}\right]\eta(0)Y(0).$$

Thus,

$$\begin{aligned} D(t) =& e^{rt}D(0) - B(0)\int_0^t e^{(t-s)r}e^{gs}ds - \eta(0)Y(0)\int_0^t e^{(t-s)r}[e^{(g+\gamma)s} - e^{gs}]ds \\ =& e^{rt}D(0) - e^{rt}B(0)\int_0^t e^{(g-r)s}ds - e^{rt}\eta(0)Y(0)\int_0^t [e^{(g+\gamma-r)s} - e^{(g-r)s}]ds. \end{aligned}$$

Let $t = \tilde{t}$. Since

$$Y(\tilde{t}) = e^{(g+\rho)\tilde{t}}Y(0),$$

we get

$$\begin{aligned} d(\tilde{t}) =& e^{(r-g-\rho)\tilde{t}}d(0) - e^{(r-g-\rho)\tilde{t}}b(0)\int_0^{\tilde{t}} e^{(g-r)s}ds - e^{(r-g-\rho)\tilde{t}}\eta(0)\int_0^{\tilde{t}} [e^{(g+\gamma-r)s} - e^{(g-r)s}]ds \\ =& e^{(r-g-\rho)\tilde{t}}d(0) - e^{(r-g-\rho)\tilde{t}}b(0)\left[\frac{e^{(g-r)s}}{g-r}\right]_0^{\tilde{t}} - e^{(r-g-\rho)\tilde{t}}\eta(0)\left[\frac{e^{(g+\gamma-r)s}}{g+\gamma-r} - \frac{e^{(g-r)s}}{g-r}\right]_0^{\tilde{t}} \\ =& e^{(r-g-\rho)\tilde{t}}d(0) - e^{(r-g-\rho)\tilde{t}}b(0)\left[\frac{e^{(g-r)\tilde{t}} - 1}{g-r}\right] - e^{(r-g-\rho)\tilde{t}}\eta(0)\left[\frac{e^{(g+\gamma-r)\tilde{t}} - 1}{g+\gamma-r} - \frac{e^{(g-r)\tilde{t}} - 1}{g-r}\right]. \end{aligned}$$

Thus,

$$\begin{aligned} d(\tilde{t}) =& e^{(r-g-\rho)\tilde{t}}d(0) - b(0)\left[\frac{e^{-\rho\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g-r}\right] \\ & - \eta(0)\left[\frac{e^{(\gamma-\rho)\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g+\gamma-r} - \frac{e^{-\rho\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g-r}\right]. \end{aligned} \tag{7}$$

From (7)

$$\begin{aligned} d(\tilde{t}) - d(0) = & \left[e^{(r-g-\rho)\tilde{t}} - 1\right]d(0) - b(0)\left[\frac{e^{-\rho\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g-r}\right] \\ & - \eta(0)\left[\frac{e^{(\gamma-\rho)\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g+\gamma-r} - \frac{e^{-\rho\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g-r}\right]. \end{aligned} \tag{8}$$

Because $e^{(r-g-\rho)\tilde{t}} - 1 < 0$ by $g+\rho > r$, (8) is decreasing with respect to $d(0)$.

$\eta(t) = \frac{T(t)}{Y(t)}$ is the ratio of the tax revenue to real GDP. GDP grows at the rate $g+\rho$, on the other hand the tax revenue grows at the rate $g+\gamma$. At $t = \tilde{t}$ we have

$$\eta(\tilde{t}) = \frac{T(\tilde{t})}{Y(\tilde{t})} = \frac{e^{(g+\gamma)\tilde{t}}}{e^{(g+\rho)\tilde{t}}}\eta(0) = e^{(\gamma-\rho)\tilde{t}}\eta(0).$$

From (3) and (5), we get

$$\eta(\tilde{t}) = e^{\left(\frac{\ln\left[\frac{-(1-c)}{\eta(0)c}(\zeta-1)+1\right]}{\ln\zeta}-1\right)\rho\frac{\ln\zeta}{\rho}}\eta(0) = \frac{\frac{-(1-c)}{\eta(0)c}(\zeta-1)+1}{\zeta}\eta(0).$$

This is constant, that is, it does not depend on ρ and γ. We have shown the following result.

Proposition 1. *In the continuous time model the ratio of the tax revenue to real GDP at the time when full-employment is realized does not depend on the values of ρ and γ.*

If $d(0)$ and $b(0)$ have the steady state values, that is, $b(0) = (r-g)d(0)$, then (8) is rewritten as

$$\begin{aligned} d(\tilde{t}) - d(0)\big|_{b(0)=(r-g)d(0)} &= \left[e^{(r-g-\rho)\tilde{t}} - 1\right]d(0) - (r-g)d(0)\left[\frac{e^{-\rho\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g-r}\right] \\ &\quad - \eta(0)\left[\frac{e^{(\gamma-\rho)\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g+\gamma-r} - \frac{e^{-\rho\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g-r}\right] \\ &= \left[e^{-\rho\tilde{t}} - 1\right]d(0) - \eta(0)\left[\frac{e^{(\gamma-\rho)\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g+\gamma-r} - \frac{e^{-\rho\tilde{t}} - e^{(r-g-\rho)\tilde{t}}}{g-r}\right]. \end{aligned}$$

Suppose $\tilde{t} = 1$, that is, full-employment is realized within one year. Then, with $e^{\rho} = \zeta$,

$$d(\tilde{t}) - d(0)\big|_{b(0)=(r-g)d(0)} = \left(\frac{1}{\zeta} - 1\right)d(0) - \frac{1}{\zeta}\eta(0)\left[\frac{e^{\gamma} - e^{r-g}}{g+\gamma-r} - \frac{1-e^{(r-g)}}{g-r}\right].$$

We use the following approximation for exponential functions for small x;

$$e^x = 1 + x + \frac{1}{2}x^2.$$

Then, with an approximation $\ln x = x - 1$, we obtain

$$d(\tilde{t}) - d(0)\big|_{b(0)=(r-g)d(0)} = \frac{1}{\zeta}(1-\zeta)d(0) - \frac{1}{2\zeta}\eta(0)\left[(\gamma + r - g) - (r-g)\right] = \frac{1}{\zeta}(1-\zeta)d(0) - \frac{1}{2\zeta}\eta(0)\gamma.$$

From (6)

$$\eta(0)\gamma = -\frac{1-c}{c}\rho.$$

We apply the following approximation to $\zeta = e^{\rho}$,

$$e^x = 1 + x.$$

Then,

$$1 - \zeta = -\rho.$$

Therefore, we obtain

$$d(\tilde{t}) - d(0)\Big|_{b(0)=(r-g)d(0)} = \frac{\rho}{\zeta}\left(-d(0) + \frac{1-c}{2c}\right).$$

If

$$\frac{1-c}{2c} < d(0),$$

we have $d(\tilde{t}) - d(0) < 0$. This condition is rewritten as

$$c > \frac{1}{1 + 2d(0)}.$$

Proposition 2. *In the continuous time model, if the full-employment state is realized within one year, $d(0)$ and $b(0)$ have the steady state values, and the propensity to consume c satisfies the following condition*

$$c > \frac{1}{1 + 2d(0)}.$$

then the debt-to-GDP ratio at the time when the full-employment state is realized is smaller than that before the fiscal policy.

3.2. Discrete time model

We assume that there exists a deflationary GDP gap, and $\zeta > 1$. Since $Y_m(t)$ increases at the rate g,

$$Y_m(t) = (1+g)^t Y_m(0).$$

The government reduces the tax to increase the growth rate of real GDP from g to $g + \rho$ so as to realize full-employment. Then,

$$Y(t) = (1 + g + \rho)^t Y(0).$$

Suppose that at period $\tilde{t}$

$$(1 + g + \rho)^{\tilde{t}} Y(0) = (1+g)^{\tilde{t}} Y_m(0),$$

that is, full-employment is realized at $\tilde{t}$. We admit any positive real number for $\tilde{t}$. Then, we have

$$\left(\frac{1 + g + \rho}{1 + g}\right)^{\tilde{t}} = \zeta. \tag{9}$$

$\tilde{t}$ is obtained as follows.

$$\tilde{t} = \frac{\ln \zeta}{\ln \frac{1+g+\rho}{1+g}} = \frac{\ln \zeta}{\ln(1+g+\rho) - \ln(1+g)}. \tag{10}$$

The larger the value of ρ is, the smaller the value of $\tilde{t}$ is, that is, the faster the full-employment state is realized.

The tax revenue at t is written as

$$T(t) = (1+g+\gamma)^t T(0).$$

The increase in real GDP over the ordinary growth is brought by the multiplier effects of tax reduction. Therefore,

$$\left[(1+g+\rho)^{\tilde{t}} - (1+g)^{\tilde{t}}\right] Y(0) = \frac{c}{1-c}\left[-(1+g+\gamma)^{\tilde{t}}T(0) + (1+g)^{\tilde{t}}T(0)\right]. \tag{11}$$

About derivation of (11) please see Appendix C. It is rewritten as

$$\left(\frac{1+g+\rho}{1+g}\right)^{\tilde{t}} - 1 = \zeta - 1 = \frac{c}{1-c}\left[-\left(\frac{1+g+\gamma}{1+g}\right)^{\tilde{t}} + 1\right]\eta(0).$$

Thus,

$$\left(\frac{1+g+\gamma}{1+g}\right)^{\tilde{t}} = -\frac{1-c}{\eta(0)c}(\zeta - 1) + 1.$$

From this

$$\tilde{t}\ln\frac{1+g+\gamma}{1+g} = \ln\left[\frac{-(1-c)}{\eta(0)c}(\zeta - 1) + 1\right]. \tag{12}$$

γ is obtained from this and (10). If we use approximation of a logarithmic function, $\ln x = x - 1$, we obtain

$$\gamma = \frac{-(1-c)\rho}{\eta(0)c}. \tag{13}$$

If $\tilde{t} = 1$, (11) implies (13) without approximation. Note that similarly to the case of continuous time debt dynamics $g + \rho$ is the average growth rate of real GDP in $0 < t \leq \tilde{t}$.

$B(t)$ is the sum of the budget surplus (or deficit) growing at the rate g from $B(0)$ and the budget surplus (or deficit) brought by the tax reduction policy. It is written as

$$\begin{aligned} B(t) =& (1+g)^t B(0) + \left[(1+g+\gamma)^t - (1+g)^t\right] T(0) \\ =& (1+g)^t B(0) + \left[(1+g+\gamma)^t - (1+g)^t\right] \eta(0) Y(0). \end{aligned}$$

Therefore,

$$
\begin{aligned}
D(t) =& (1+r)^t D(0) - B(0) \sum_{s=1}^{t} (1+r)^{t-s}(1+g)^s \\
& - \eta(0)Y(0) \left[\sum_{s=1}^{t} (1+r)^{t-s}(1+g+\gamma)^s - \sum_{s=1}^{t} (1+r)^{t-s}(1+g)^s \right] \\
=& (1+r)^t D(0) - (1+r)^t B(0) \sum_{s=1}^{t} \left(\frac{1+g}{1+r} \right)^s \\
& - (1+r)^t \eta(0) Y(0) \left[\sum_{s=1}^{t} \left(\frac{1+g+\gamma}{1+r} \right)^s - \sum_{s=1}^{t} \left(\frac{1+g}{1+r} \right)^s \right].
\end{aligned}
$$

Let $t = \tilde{t}$. Since

$$
Y(\tilde{t}) = (1+g+\rho)^{\tilde{t}} Y(0),
$$

we get

$$
\begin{aligned}
d(\tilde{t}) =& \left(\frac{1+r}{1+g+\rho} \right)^{\tilde{t}} d(0) - \left(\frac{1+r}{1+g+\rho} \right)^{\tilde{t}} b(0) \sum_{s=1}^{\tilde{t}} \left(\frac{1+g}{1+r} \right)^s \\
& - \left(\frac{1+r}{1+g+\rho} \right)^{\tilde{t}} \eta(0) \left[\sum_{s=1}^{\tilde{t}} \left(\frac{1+g+\gamma}{1+r} \right)^s - \sum_{s=1}^{\tilde{t}} \left(\frac{1+g}{1+r} \right)^s \right] \\
=& \left(\frac{1+r}{1+g+\rho} \right)^{\tilde{t}} d(0) - \left(\frac{1+r}{1+g+\rho} \right)^{\tilde{t}} b(0) \frac{\left[\left(\frac{1+g}{1+r} \right)^{\tilde{t}+1} - \left(\frac{1+g}{1+r} \right) \right]}{\frac{1+g}{1+r} - 1} \\
& - \left(\frac{1+r}{1+g+\rho} \right)^{\tilde{t}} \eta(0) \left\{ \frac{\left[\left(\frac{1+g+\gamma}{1+r} \right)^{\tilde{t}+1} - \left(\frac{1+g}{1+r} \right) \right]}{\frac{1+g+\gamma}{1+r} - 1} - \frac{\left[\left(\frac{1+g}{1+r} \right)^{\tilde{t}+1} - \left(\frac{1+g}{1+r} \right) \right]}{\frac{1+g}{1+r} - 1} \right\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d(\tilde{t}) =& \left(\frac{1+r}{1+g+\rho} \right)^{\tilde{t}} d(0) - \left(\frac{1+r}{1+g+\rho} \right)^{\tilde{t}} b(0) \frac{(1+g) \left[\left(\frac{1+g}{1+r} \right)^{\tilde{t}} - 1 \right]}{g-r} \qquad (14) \\
& - \left(\frac{1+r}{1+g+\rho} \right)^{\tilde{t}} \eta(0) \left\{ \frac{(1+g+\gamma) \left[\left(\frac{1+g+\gamma}{1+r} \right)^{\tilde{t}} - 1 \right]}{g+\gamma-r} - \frac{(1+g) \left[\left(\frac{1+g}{1+r} \right)^{\tilde{t}} - 1 \right]}{g-r} \right\}.
\end{aligned}
$$

From (14)

$$d(\tilde{t})-d(0)=\left[\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}-1\right]d(0) \tag{15}$$

$$-\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}b(0)\frac{(1+g)\left[\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1\right]}{g-r}$$

$$-\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}\eta(0)\left\{\frac{(1+g+\gamma)\left[\left(\frac{1+g+\gamma}{1+r}\right)^{\tilde{t}}-1\right]}{g+\gamma-r}-\frac{(1+g)\left[\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1\right]}{g-r}\right\}.$$

Because $\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}-1<0$ by $g+\rho>r$, (15) is decreasing with respect to $d(0)$.

$\eta(t)=\frac{T(t)}{Y(t)}$ is the ratio of the tax revenue to real GDP. GDP grows at the rate $g+\rho$, on the other hand the tax revenue grows at the rate $g+\gamma$. At $t=\tilde{t}$ we have

$$\eta(\tilde{t})=\frac{T(\tilde{t})}{Y(\tilde{t})}=\left(\frac{1+g+\gamma}{1+g+\rho}\right)^{\tilde{t}}\eta(0)=\left(\frac{1+g}{1+g+\rho}\right)^{\tilde{t}}\left(\frac{1+g+\gamma}{1+g}\right)^{\tilde{t}}\eta(0).$$

From (9) and (12), we get

$$\eta(\tilde{t})=\frac{1}{\zeta}\left[\frac{-(1-c)}{\eta(0)c}(\zeta-1)+1\right].$$

This is constant, that is, it does not depend on ρ and γ. We have shown the following result.

Proposition 3. *In the discrete time model the ratio of the tax revenue to real GDP at the time when full-employment is realized does not depend on the values of ρ and γ.*

If $d(0)$ and $b(0)$ have the steady state values, that is, $b(0)=\frac{r-g}{1+g}d(0)$, then (15) is

$$\begin{aligned}d(\tilde{t})-d(0)\big|_{b(0)=\frac{r-g}{1+g}d(0)}&=\left[\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}-1\right]d(0)+\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}d(0)\left[\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1\right]\\&-\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}\eta(0)\left\{\frac{(1+g+\gamma)\left[\left(\frac{1+g+\gamma}{1+r}\right)^{\tilde{t}}-1\right]}{g+\gamma-r}-\frac{(1+g)\left[\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1\right]}{g-r}\right\}\\&=\left[\left(\frac{1+g}{1+g+\rho}\right)^{\tilde{t}}-1\right]d(0)\\&-\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}\eta(0)\left\{\frac{(1+g+\gamma)\left[\left(\frac{1+g+\gamma}{1+r}\right)^{\tilde{t}}-1\right]}{g+\gamma-r}-\frac{(1+g)\left[\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1\right]}{g-r}\right\}.\end{aligned}$$

Suppose $\tilde{t} = 1$, that is, the full-employment state is realized within one year. Then, with (13) we obtain

$$d(\tilde{t}) - d(0)\Big|_{b(0)=\frac{r-g}{1+g}d(0)} = \left[\left(\frac{1+g}{1+g+\rho}\right) - 1\right] d(0) - \left(\frac{\gamma}{1+g+\rho}\right)\eta(0) = -\frac{\rho}{1+g+\rho}\left(d(0) - \frac{1-c}{c}\right)$$
$$= \frac{1}{1+g+\rho}.$$

If $\frac{1-c}{c} < d(0)$, or

$$c > \frac{1}{1+d(0)},$$

we have $d(t) - d(0) < 0$. Note that if $\tilde{t} = 1$ (13) holds without approximation of logarithmic functions. We have shown the following proposition.

Proposition 4. *In the discrete time model, if the full-employment state is realized within one year, $d(0)$ and $b(0)$ have the steady state values, and the propensity to consume c satisfies the following condition*

$$c > \frac{1}{1+d(0)},$$

then the debt-to-GDP ratio at the time when the full-employment state is realized is smaller than that before the fiscal policy.

4. Graphical simulations

4.1. Continuous time model

We present some simulation results. First we consider the continuous time model. Assume the following values for the variables.

> $c = 0.6$, $\tau = 0.25$, $\alpha = 0.28$, $\eta(0) = 0.26$, $g = 0.025$, $r = 0.015$, $b(0) = -0.015$ and $\zeta = 1.15$.

We assume that g and r are constant, and $g > r$[4]. However, in Subsection 4.1.9 we consider a case where $r > g$. We do not assume that $d(0)$ and $b(0)$ have the steady state values described in (1). But, in Subsection 4.1.8 we consider a case where $d(0)$ and $b(0)$ have the steady state values.

[4]In Mitchell et al. (2019) (pp. 357-358) it is stated that when $g > r$, there exists a stable steady state value of the debt-to-GDP ratio. Also see Wray (2016).

4.1.1. Relation between ρ and $\tilde{t}$

In addition to the above assumptions we assume $d(0) = 0.5$. Figure 1 represents the relation between ρ and $\tilde{t}$. From (3) the larger the value of ρ is, the smaller the value of $\tilde{t}$ is, that is, the faster the full-employment state is realized. Therefore, the more aggressive the fiscal policy is, the faster the full-employment state is realized. For example, when $\rho = 0.05$, $\tilde{t} \approx 2.8$, when $\rho = 0.1$, $\tilde{t} \approx 1.5$.

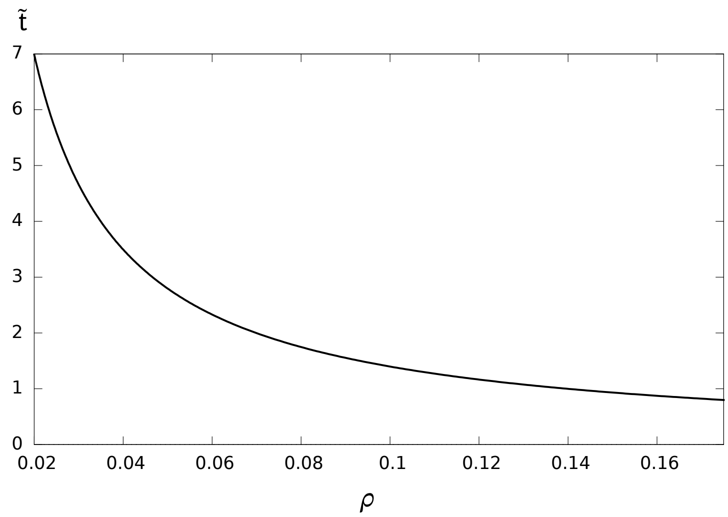

Figure 1: The relation between ρ and $\tilde{t}$

4.1.2. Relation between ρ and γ

We assume $d(0) = 0.5$. Figure 2 represents the relation between the value of ρ and the value of γ according to (5). The larger the value of ρ is, the smaller the value of γ is. For example, when $\rho = 0.05$, $\gamma \approx -0.2$, when $\rho = 0.1$, $\gamma \approx -0.35$.

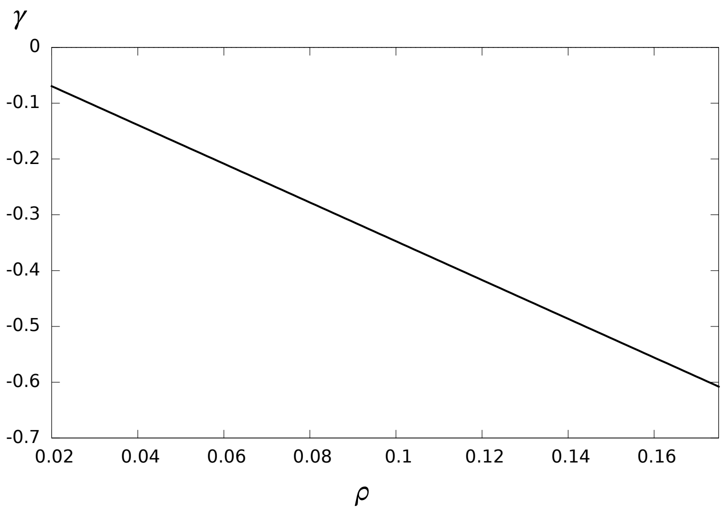

Figure 2: The relation between ρ and γ

4.1.3. Relation between ρ and $g + \gamma$

Again we assume $d(0) = 0.5$. Figure 3 represents the relation between the value of ρ and the value of $g+\gamma$. The larger the value of ρ is, the smaller the value of $g+\gamma$ is. For example, when $\rho = 0.05$, $g + \gamma \approx -0.18$, when $\rho = 0.1$, $g + \gamma \approx -0.33$.

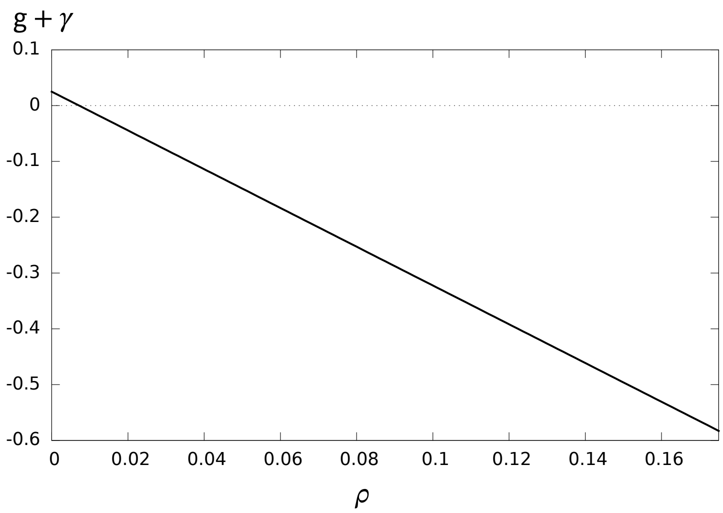

Figure 3: The relation between ρ and $g + \gamma$

4.1.4. Relation between ρ and $d(\tilde{t})$

We assume $d(0) = 0.5$. Figure 4 represents the relation between ρ and $d(\tilde{t})$ according to (7). The larger the value of ρ is, the smaller the value of $d(\tilde{t})$ is, that is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is.

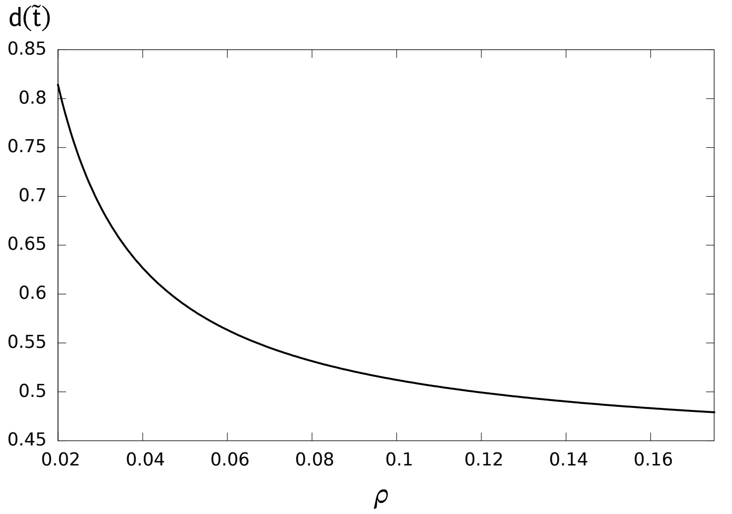

Figure 4: The relation between ρ and $d(\tilde{t})$

4.1.5. Relation between γ and $d(\tilde{t})$

We assume $d(0) = 0.5$. Figure 5 represents the relation between γ and $d(\tilde{t})$. The smaller the value of γ (or the larger the absolute value of γ) is, the smaller the value of $d(\tilde{t})$ is, that is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is.

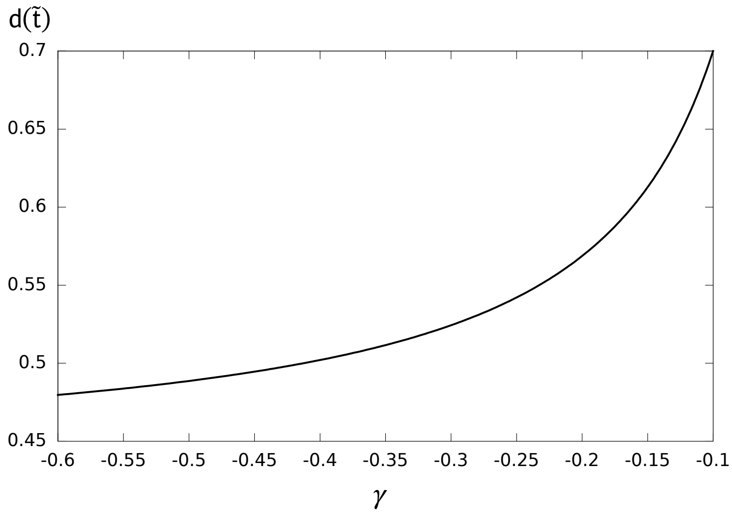

Figure 5: The relation between γ and $d(\tilde{t})$

4.1.6. Relation between ρ and $d(\tilde{t}) - d(0)$

We assume $d(0) = 0.5$. Figure 6 represents the relation between ρ and $d(\tilde{t}) - d(0)$, which is the difference between the debt-to-GDP ratio at $\tilde{t}$ and that at $t = 0$, according to (8). The larger the value of ρ is, the smaller the value of $d(\tilde{t}) - d(0)$ is. If ρ is larger than about 0.12, the debt-to-GDP ratio at $t = \tilde{t}$ is smaller than that at $t = 0$, that is, the aggressive tax reduction policy to realize full-employment reduces the debt-to-GDP ratio.

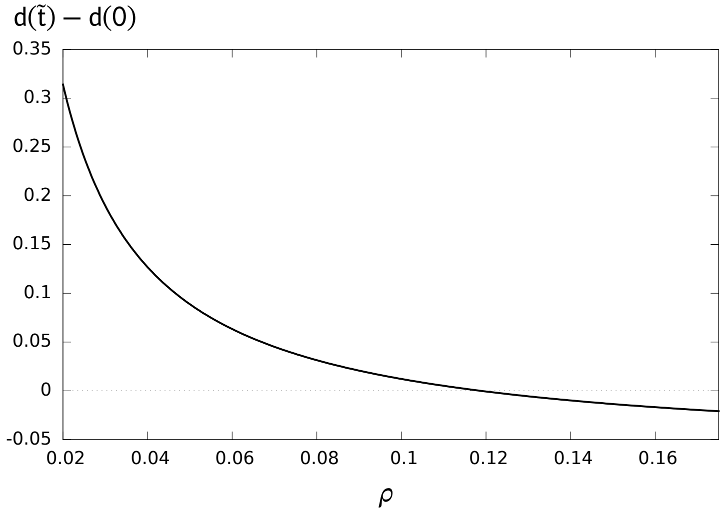

Figure 6: The relation between ρ and $d(\tilde{t}) - d(0)$

4.1.7. Relation between γ and $d(\tilde{t}) - d(0)$

We assume $d(0) = 0.5$. Figure 7 represents the relation between γ and $d(\tilde{t}) - d(0)$, which is the difference between the debt-to-GDP ratio at $\tilde{t}$ and that at $t = 0$. The smaller the value of γ is, the smaller the value of $d(\tilde{t}) - d(0)$ is. If γ is smaller than about -0.42, the debt-to-GDP ratio at $t = \tilde{t}$ is smaller than that at $t = 0$, that is, the aggressive tax reduction policy to realize full-employment reduces the debt-to-GDP ratio.

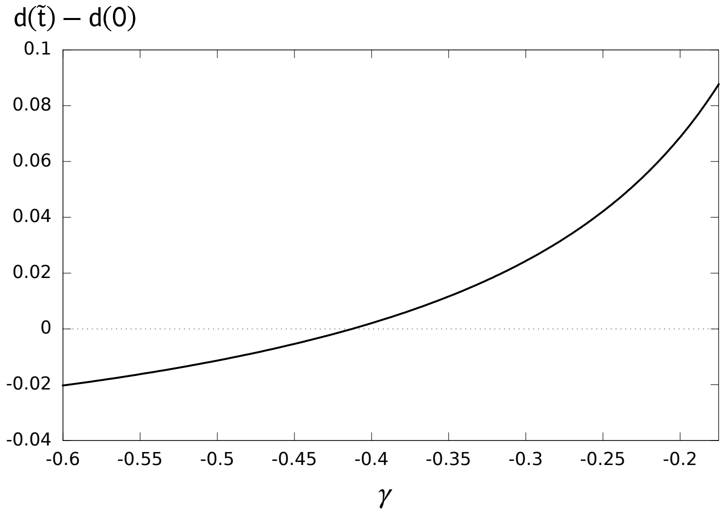

Figure 7: The relation between γ and $d(\tilde{t}) - d(0)$

4.1.8. Relation between ρ and $d(\tilde{t}) - d(0)$ when $d(0)$ and $b(0)$ have the steady state values

We assume $b(0) = (r - g)d(0)$. the values of other variables are the same as those in the previous cases. In Figure 8 we compare the relation between ρ and $d(\tilde{t}) - d(0)$ in this case and that when $b(0) = -0.015$.

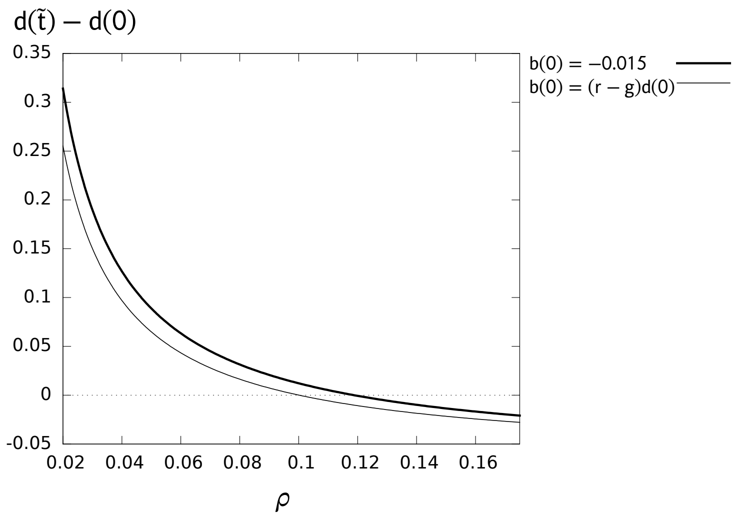

Figure 8: The relation between ρ and $d(\tilde{t}) - d(0)$ when $b(0) = -0.015$ and $(r - g)d(0)$

If $d(0)$ and $b(0)$ have the steady state values, the debt-to-GDP ratio at the time when full-employment is realized is more likely smaller than that at $t = 0$ than the case where $b(0) = -0.015$. It is because $-0.015 < (r - g)d(0)$.

4.1.9. Relation between ρ and $d(\tilde{t}) - d(0)$ with low and high interest rates

We assume $r = 0.035$. The values of other variables are the same as those in the previous cases. In Figure 9 we compare the relation between ρ and $d(\tilde{t}) - d(0)$ in the case of low interest rate and that in the case of high interest rate.

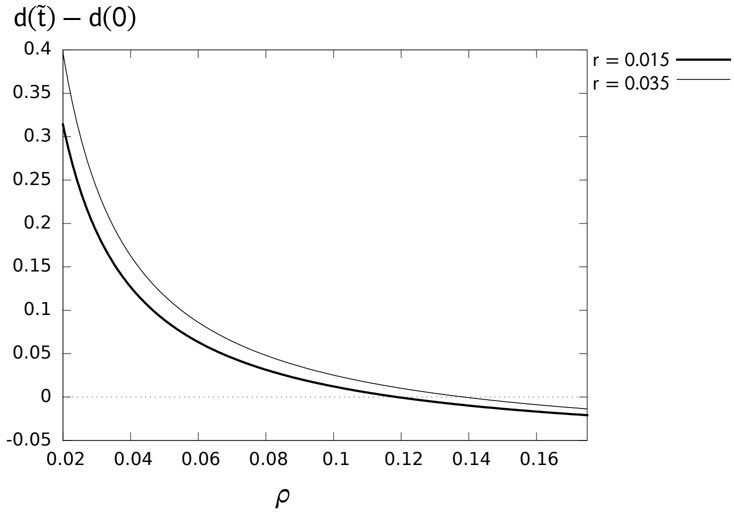

Figure 9: The relation between ρ and $d(\tilde{t}) - d(0)$ with low and high interest rates

With higher interest rate the debt-to-GDP ratio at the time when full-employment is realized is less likely smaller than that at time 0 than the case with low interest rate.

4.2. Discrete time model

We present some simulation results about the discrete time model. Assume the following values for the variables.

$c = 0.65$, $\tau = 0.25$, $\alpha = 0.28$, $\eta(0) = 0.26$, $g = 0.025$, $r = 0.015$, $b(0) = -0.015$ and $\zeta = 1.15$.

We assume that g and r are constant, and $g > r$. However, in Subsection 4.2.9 we consider a case where $r > g$. We do not assume that $d(0)$ and $b(0)$ have the steady state values described in (2). But, in Subsection 4.2.8 we consider a case where $d(0)$ and $b(0)$ have the steady state values.

4.2.1. Relation between ρ and $\tilde{t}$

In addition to the above assumptions we assume $d(0) = 0.5$. Figure 10 represents the relation between ρ and $\tilde{t}$. From (10) the larger the value of ρ is, the smaller the value of $\tilde{t}$ is, that is, the faster the full-employment state is realized. Therefore, the more aggressive the fiscal policy is, the faster the full-employment state is realized. For example, when $\rho = 0.05$, $\tilde{t} \approx 3$, when $\rho = 0.1$, $\tilde{t} \approx 1.5$.

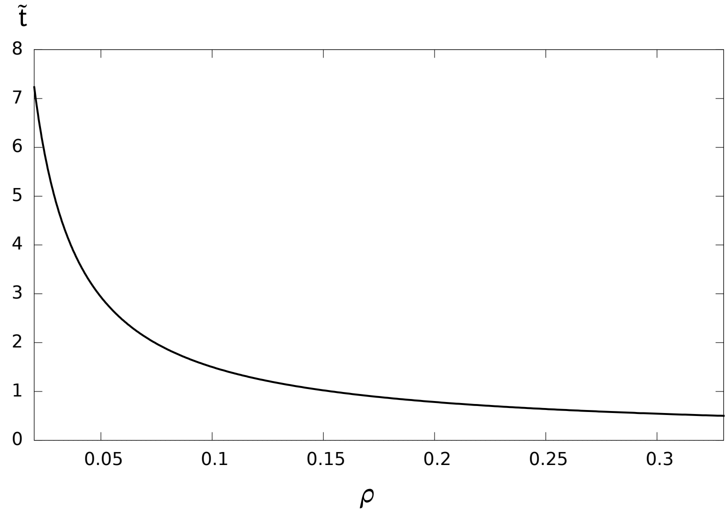

Figure 10: The relation between ρ and $\tilde{t}$

4.2.2. Relation between ρ and γ

We assume $d(0) = 0.5$. Figure 11 represents the relation between the value of ρ and the value of γ according to (12). The larger the value of ρ is, the smaller the value of γ is. For example, when $\rho = 0.05$, $\gamma \approx -0.15$, when $\rho = 0.1$, $\gamma \approx -0.24$.

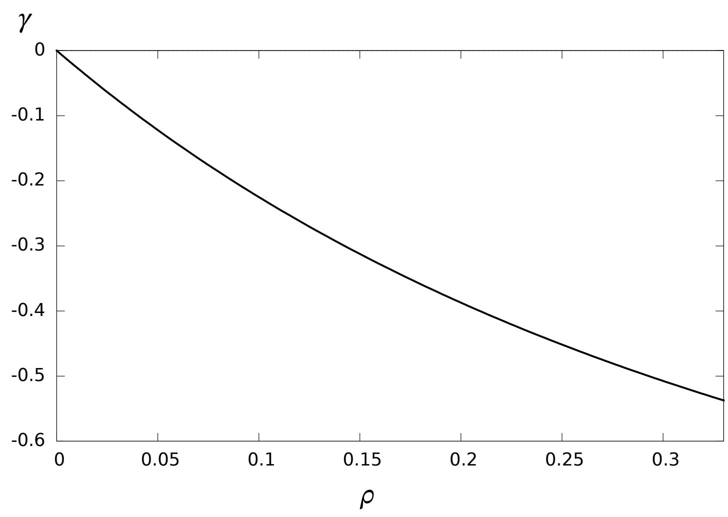

Figure 11: The relation between ρ and γ

4.2.3. Relation between ρ and $g + \gamma$

We assume $d(0) = 0.5$. Figure 12 represents the relation between the value of ρ and the value of $g + \gamma$. The larger the value of ρ is, the smaller the value of $g + \gamma$ is. For example, when $\rho = 0.05$, $g + \gamma \approx -0.13$, when $\rho = 0.1$, $g + \gamma \approx -0.22$.

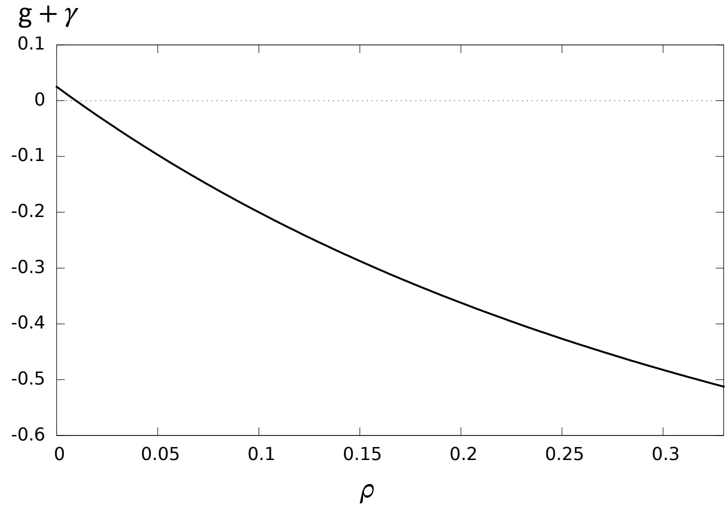

Figure 12: The relation between ρ and $g + \gamma$

4.2.4. Relation between ρ and $d(\tilde{t})$

We assume $d(0) = 0.5$. Figure 13 represents the relation between ρ and $d(\tilde{t})$ according to (14). The larger the value of ρ is, the smaller the value of $d(\tilde{t})$ is, that is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is.

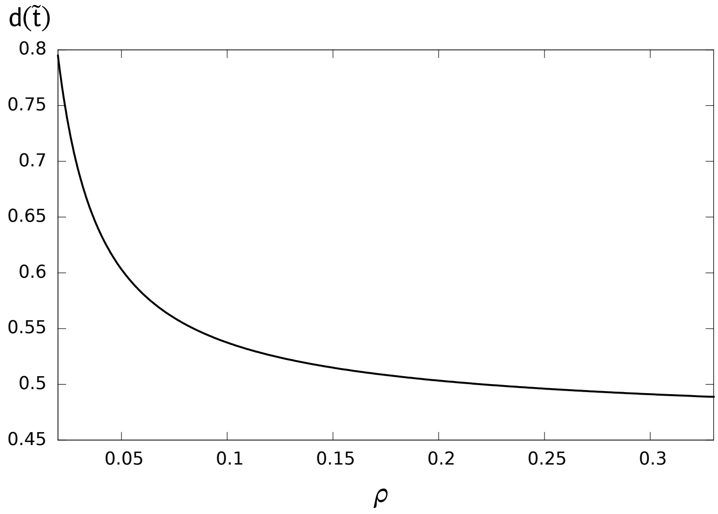

Figure 13: The relation between ρ and $d(\tilde{t})$

4.2.5. Relation between γ and $d(\tilde{t})$

We assume $d(0) = 0.5$. Figure 14 represents the relation between γ and $d(\tilde{t})$. The smaller the value of γ (or the larger the absolute value of γ) is, the smaller the value of $d(\tilde{t})$ is, that is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is.

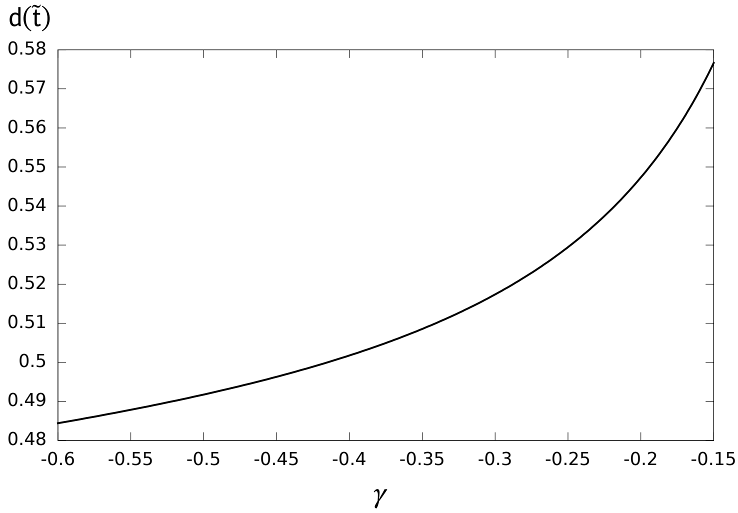

Figure 14: The relation between γ and $d(\tilde{t})$

4.2.6. Relation between ρ and $d(\tilde{t}) - d(0)$

We assume $d(0) = 0.5$. Figure 15 represents the relation between ρ and $d(\tilde{t}) - d(0)$, which is the difference between the debt-to-GDP ratio at $\tilde{t}$ and that at $t = 0$, according to (15).

The larger the value of ρ is, the smaller the value of $d(\tilde{t}) - d(0)$ is. If ρ is larger than about 0.22, the debt-to-GDP ratio at $t = \tilde{t}$ is smaller than that at $t = 0$, that is, the aggressive tax reduction policy to realize full-employment reduces the debt-to-GDP ratio.

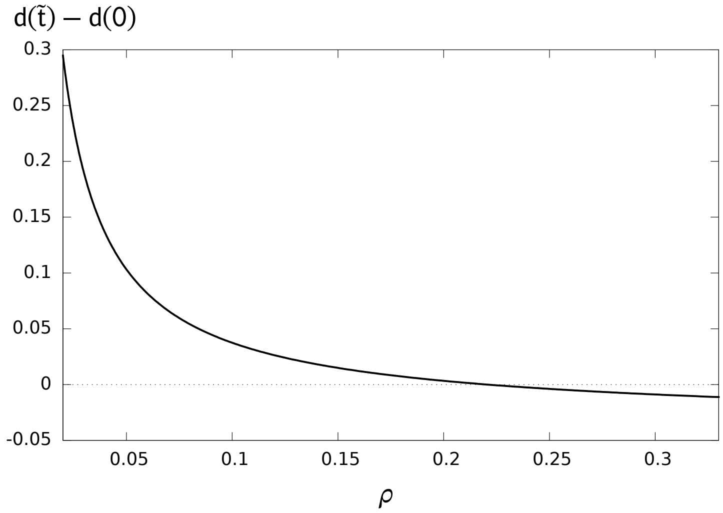

Figure 15: The relation between ρ and $d(\tilde{t}) - d(0)$

4.2.7. Relation between γ and $d(\tilde{t}) - d(0)$

We assume $d(0) = 0.5$. Figure 16 represents the relation between γ and $d(\tilde{t}) - d(0)$, which is the difference between the debt-to-GDP ratio at $\tilde{t}$ and that at $t = 0$. The smaller the value of γ is, the smaller the value of $d(\tilde{t}) - d(0)$ is. If γ is smaller than about -0.4, the debt-to-GDP ratio at $t = \tilde{t}$ is smaller than that at $t = 0$, that is, the aggressive tax reduction policy to realize full-employment reduces the debt-to-GDP ratio.

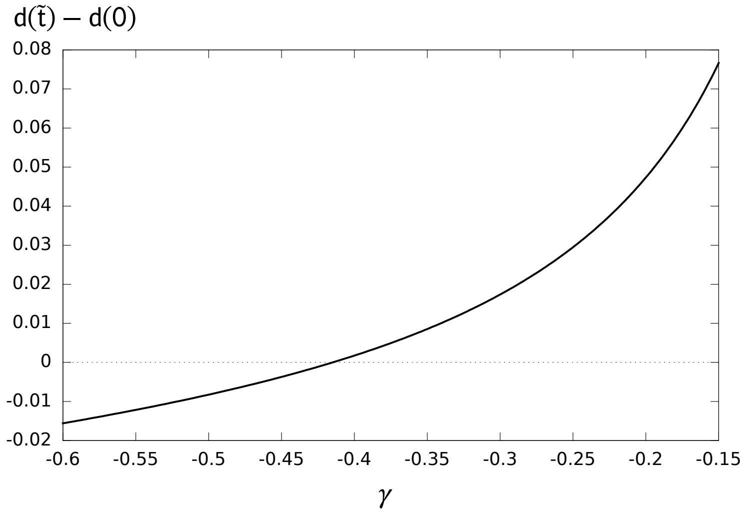

Figure 16: The relation between γ and $d(\tilde{t}) - d(0)$

4.2.8. Relation between ρ and $d(\tilde{t}) - d(0)$ when $d(0)$ and $b(0)$ have the steady state values

We assume $b(0) = \frac{r-g}{1+g}d(0)$. the values of other variables are the same as those in the previous cases. In Figure 17 we compare the relation between ρ and $d(\tilde{t}) - d(0)$ in this case and that when $b(0) = -0.015$.

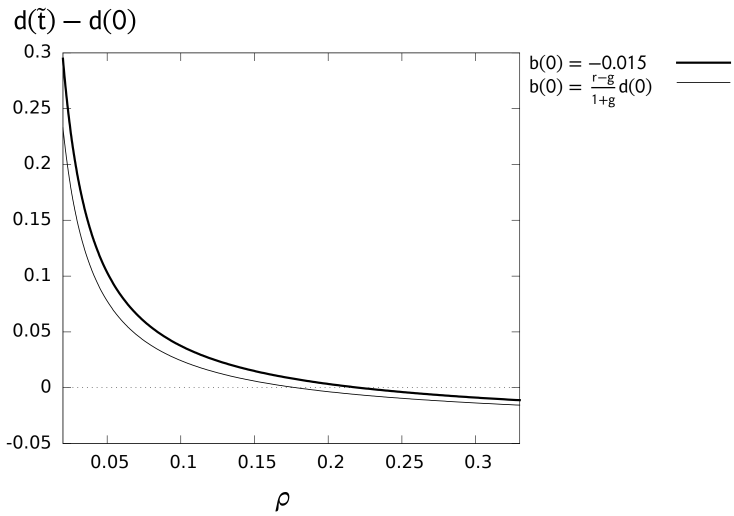

Figure 17: The relation between ρ and $d(\tilde{t}) - d(0)$ when $b(0) = -0.015$ and $\frac{r-g}{1+g}d(0)$

If $d(0)$ and $b(0)$ have the steady state values, the debt-to-GDP ratio at the time when full-employment is realized is more likely smaller than that at period 0 than the case where $b(0) = -0.015$. It is because $-0.015 < \frac{r-g}{1+g}d(0)$.

4.2.9. Relation between ρ and $d(\tilde{t}) - d(0)$ with low and high interest rates

We assume $r = 0.035$. The values of other variables are the same as those in the previous cases. In Figure 18 we compare the relation between ρ and $d(\tilde{t}) - d(0)$ in the case of low interest rate and that in the case of high interest rate.

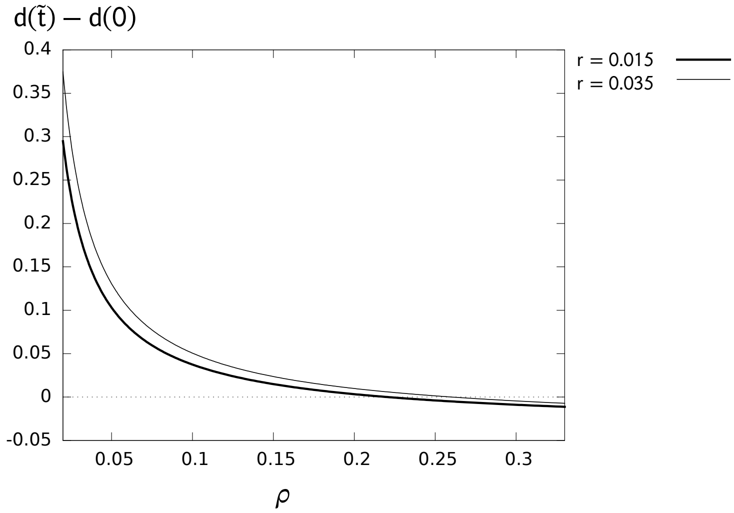

Figure 18: The relation between ρ and $d(\tilde{t}) - d(0)$ with low and high interest rates

With higher interest rate the debt-to-GDP ratio at the time when full-employment is realized is less likely smaller than that at time 0 than the case with low interest rate.

5. Concluding Remark

We have presented mathematical analyses and simulations of a fiscal policy by tax reduction which realizes full-employment from an under-employment state without increasing the debt-to-GDP ratio than before the tax reduction, using a continuous time model and a discrete time model of debt dynamics. Full-employment can be realized by a tax reduction policy with smaller debt-to-GDP ratio than before the tax reduction.

However, for our results the marginal marginal propensity to consume must be fairly large. Figure 19 depicts the relation of ρ and $d(\tilde{t}) - d(0)$ if $c = 0.45$ in the continuous time model and Figure 20 depicts that if $c = 0.5$ in the discrete time model.

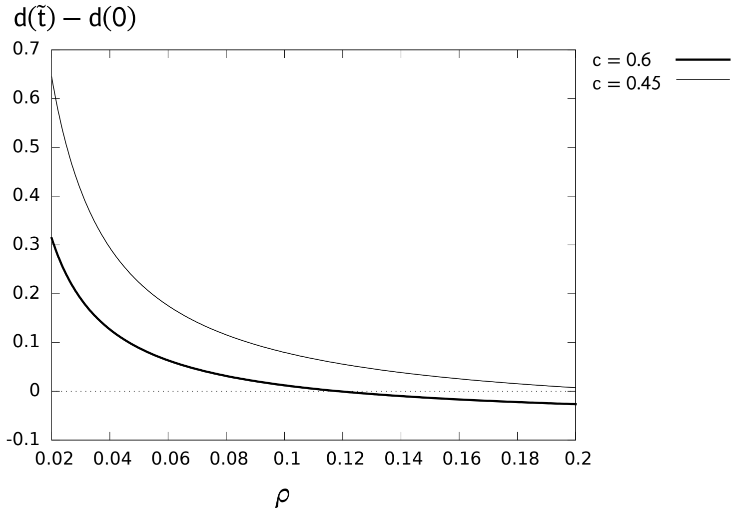

Figure 19: The relation between ρ and $d(\tilde{t}) - d(0)$ in the continuous time model

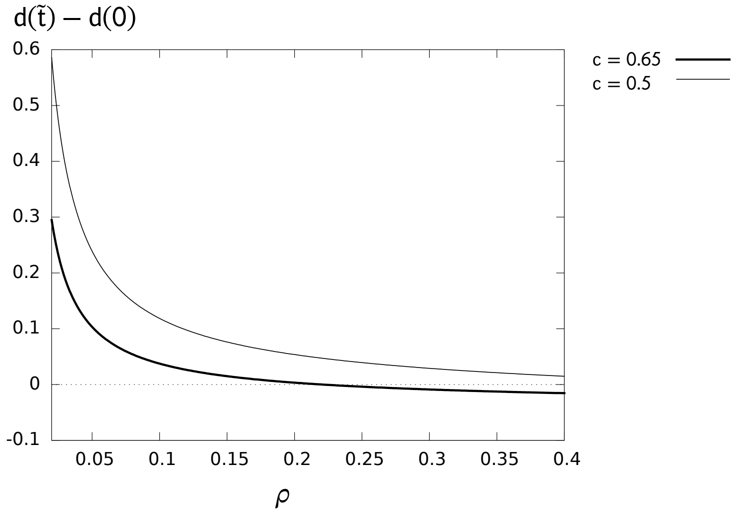

Figure 20: The relation between ρ and $d(\tilde{t}) - d(0)$ in the discrete time model

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The author declares that: (i) no support, financial or otherwise, has been received from any organization that may have an interest in the submitted work; and (ii) there are no other relationships or activities that could appear to have influenced the submitted work.

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Appendices

A. Derivation of multiplier by an overlapping generations model

We consider a two-period (young and old) overlapping generations model under monopolistic competition according to Otaki (2007), Otaki (2009), Otaki (2011) and Otaki (2015). There is one factor of production, labor, and there is a continuum of goods indexed by $z \in [0, 1]$. Each good is monopolistically produced by Firm z. Consumers are born at continuous density $[0, 1] \times [0, 1]$ in each period. They can supply only one unit of labor when they are young.

A.1. Consumers

We use the following notations.

$c^i(z)$: consumption of good z at period i, $i = 1,2$.
$p^i(z)$: the price of good z at period i, $i = 1,2$.
$X^i = \left\{\int_0^1 c^i(z)^{1-\frac{1}{\eta}}dz\right\}^{\frac{1}{1-\frac{1}{\eta}}}$, $i = 1,2$, $\eta > 1$.
ξ: disutility of labor, $\xi > 0$.
$0 < \alpha < 1$.
W: nominal wage rate.
Π: profits of firms which are equally distributed to each consumer.
L: employment of each firm and the total employment.
L_f: population of labor or employment at the full-employment state.
y: labor productivity, $y \geq 1$.

δ is the definition function. If the consumer is employed, $\delta = 1$; if he is not employed, $\delta = 0$. The labor productivity is y, that is, y unit of the goods is produced by one unit of labor. The utility of consumers of one generation over two periods is

$$U(X^1, X^2, \delta, \xi) = \left(X^1\right)^{\alpha}\left(X^2\right)^{1-\alpha} - \delta\xi.$$

With the budget constraint

$$\int_0^1 p^1(z)c^1(z)dz + \int_0^1 p^2(z)c^2(z)dz = \delta W + \Pi.$$

$p^2(z)$ is the expectation of the price of good z at period 2. The Lagrange function is

$$\mathcal{L} = \left(X^1\right)^{\alpha}\left(X^2\right)^{1-\alpha} - \delta\xi - \lambda\left(\int_0^1 p^1(z)c^1(z)dz + \int_0^1 p^2(z)c^2(z)dz - \delta W - \Pi\right).$$

λ is the Lagrange multiplier. The first order conditions are

$$\alpha\left(X^1\right)^{\alpha-1}\left(X^2\right)^{1-\alpha}\left\{\int_0^1 c^1(z)^{1-\frac{1}{\eta}}dz\right\}^{\frac{\frac{1}{\eta}}{1-\frac{1}{\eta}}} c^1(z)^{-\frac{1}{\eta}} = \lambda p^1(z),$$

and

$$(1-\alpha)\left(X^1\right)^{\alpha}\left(X^2\right)^{-\alpha}\left\{\int_0^1 c^2(z)^{1-\frac{1}{\eta}}dz\right\}^{\frac{\frac{1}{\eta}}{1-\frac{1}{\eta}}} c^2(z)^{-\frac{1}{\eta}} = \lambda p^2(z).$$

They are rewritten as

$$\alpha\left(X^1\right)^{\alpha}\left(X^2\right)^{1-\alpha}\left\{\int_0^1 c^1(z)^{1-\frac{1}{\eta}}dz\right\}^{-1} c^1(z)^{1-\frac{1}{\eta}} = \lambda p^1(z)c^1(z), \tag{16}$$

and

$$(1-\alpha)\left(X^1\right)^{\alpha}\left(X^2\right)^{1-\alpha}\left\{\int_0^1 c^2(z)^{1-\frac{1}{\eta}}dz\right\}^{-1} c^2(z)^{1-\frac{1}{\eta}} = \lambda p^2(z)c^2(z). \tag{17}$$

From (16) and (17) we obtain

$$\begin{aligned}
&\alpha \left(X^1\right)^{\alpha} \left(X^2\right)^{1-\alpha} \left\{\int_0^1 c^1(z)^{1-\frac{1}{\eta}} dz\right\}^{-1} \int_0^1 c^1(z)^{1-\frac{1}{\eta}} dz \\
=&\alpha \left(X^1\right)^{\alpha} \left(X^2\right)^{1-\alpha} = \lambda \int_0^1 p^1(z)c^1(z)dz,
\end{aligned}$$

and

$$\begin{aligned}
&(1-\alpha) \left(X^1\right)^{\alpha} \left(X^2\right)^{1-\alpha} \left\{\int_0^1 c^2(z)^{1-\frac{1}{\eta}} dz\right\}^{-1} \int_0^1 c^2(z)^{1-\frac{1}{\eta}} dz \\
=&(1-\alpha) \left(X^1\right)^{\alpha} \left(X^2\right)^{1-\alpha} = \lambda \int_0^1 p^2(z)c^2(z)dz.
\end{aligned}$$

Thus, we get

$$\frac{\int_0^1 p^1(z)c^1(z)dz}{\int_0^1 p^2(z)c^2(z)dz} = \frac{\alpha}{1-\alpha},$$

$$\int_0^1 p^1(z)c^1(z)dz = \alpha(\delta W + \Pi),$$

and

$$\int_0^1 p^2(z)c^2(z)dz = (1-\alpha)(\delta W + \Pi).$$

Therefore, the aggregate demand of the younger generation is

$$\alpha(\delta W + \Pi).$$

The total aggregate demand is

$$\alpha(\delta W + \Pi) + G + M.$$

G is the government expenditure and M is consumption by the old generation. Since in the model of this appendix the goods are produced by only labor, the investments by firms are zero. The aggregate supply is

$$WL.$$

The profit of a firm is written as

$$\Pi = P^1 Ly - \delta WL.$$

Since the aggregate demand and supply are equal,

$$P^1 Ly = \alpha P^1 Ly + G + M.$$

IN the real terms

$$Ly = \frac{1}{1-\alpha}\left(\frac{G}{P^1} + \frac{M}{P^1}\right).$$

Therefore, we get the multiplier $\frac{1}{1-\alpha}$.

B. Derivation of (4)

Let $C(0)$ and $C(t)$ be the consumptions at $t = 0$ and t. Then,

$$Y(0) = C(0) + G(0),$$

$$Y(t) = C(t) + G(t).$$

For simplicity we omit export and import. We assume that the consumptions are

$$C(0) = c(Y(0) - T(0)) + A(0),$$

$$C(t) = c(Y(t) - T(t)) + A(t).$$

$A(0)$ and $A(t)$ are the constant parts of the consumptions. From them we obtain

$$Y(0) = c(Y(0) - T(0)) + A(0) + G(0).$$

and

$$\begin{aligned} Y(t) = e^{(g+\rho)t}Y(0) &= c(Y(t) - T(t)) + A(t) + G(t) \\ &= ce^{(g+\rho)t}Y(0) - ce^{(g+\gamma)t}T(0) + e^{gt}A(0) + e^{gt}G(0). \end{aligned} \tag{18}$$

$T(t)$ may depend on $Y(t)$. However, we assume that the government controls the tax system so that $T(t)$ satisfies

$$T(t) = e^{(g+\gamma)t}T(0)$$

given $T(0)$. The steady state values of the variables satisfy

$$e^{gt}Y(0) = c[e^{gt}Y(0) - e^{gt}T(0)] + e^{gt}A(0) + e^{gt}G(0). \tag{19}$$

From (18) and (19), we get

$$e^{(g+\rho)t}Y(0) - e^{gt}Y(0) = ce^{(g+\rho)t}Y(0) - ce^{(g+\gamma)t}T(0) - \left[ce^{gt}Y(0) - ce^{gt}T(0)\right].$$

This means

$$e^{(g+\rho)t}Y(0) - e^{gt}Y(0) = \frac{c}{1-c}\left[-e^{(g+\gamma)t}T(0) + e^{gt}T(0)\right].$$

C. Derivation of (11)

Similarly to the continuous time case, we have

$$Y(0) = c(Y(0) - T(0)) + A(0) + G(0),$$

and

$$\begin{aligned} Y(t) &= (1+g+\rho)^t Y(0) = c(Y(t) - T(t)) + A(t) + G(t) \\ &= c(1+g+\rho)^t Y(0) - c(1+g+\gamma)^t T(0) + (1+g)^t A(0) + (1+g)^t G(0). \end{aligned} \tag{20}$$

Similarly to the continuous time case $T(t)$ may depend on $Y(t)$. However, we assume that the government controls the tax system so that $T(t)$ satisfies

$$T(t) = (1+g+\gamma)^t T(0)$$

given $T(0)$. The steady state values of the variables satisfy

$$\begin{aligned}(1+g)^t Y(0) =& c\left[((1+g)^t Y(0) - (1+g)^t T(0)\right] + (1+g)^t A(0) \\ &+ (1+g)^t G(0).\end{aligned} \tag{21}$$

From (20) and (21),

$$\begin{aligned}(1+g+\rho)^t Y(0) - (1+g)^t Y(0) =& c(1+g+\rho)^t Y(0) - c(1+g+\gamma)^t T(0) \\ &- \left[c(1+g)^t Y(0) - c(1+g)^t T(0)\right].\end{aligned}$$

This means

$$(1+g+\rho)^t Y(0) - (1+g)^t Y(0) = \frac{c}{1-c}\left[-(1+g+\gamma)^t T(0) + (1+g)^t T(0)\right].$$