Mixed Bundling in Oligopoly Markets

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Abstract

This paper provides a framework for studying competitive mixed bundling with an arbitrary number of firms. We examine both a firm's incentive to introduce mixed bundling and equilibrium tariffs when all firms adopt the mixed-bundling strategy. We develop a method to derive the equilibrium prices, and also offer a simple approximation of the equilibrium prices when the number of firms is large. In the duopoly case, relative to separate sales, mixed bundling has ambiguous impacts on prices, profit and consumer surplus. While with many firms mixed bundling lowers all prices, harms firms and benefits consumers under a mild condition.

Keywords: bundling, multiproduct pricing, oligopoly

JEL classification: D43, L13, L15

1 Introduction

There are many circumstances where consumers are offered a package of products at a discounted price relative to the sum of the component prices. This selling strategy is called "mixed bundling". Examples include software suites, TV-internet-phone bundles, insurance, banking services, theme park bundles, package tours, value meals, and so on. (In the extreme form of "pure bundling", all component

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products are sold in a package only and no individual products are available for purchase.) Except for some obvious reasons such as the cost savings in production and transactions, bundling can be a strategy to price discriminate and extract more surplus from consumers (e.g. Stigler (1968), and Adams and Yellen (1976)), or a strategy to deter the entry of potential competitors (e.g. Whinston (1990), and Nalebuff (2004)). The impact of bundling on market performance is a classic question which has received wide attention.

There is a substantial literature on bundling in the monopoly case,[1] but in many examples of bundling there are several competing multiproduct firms. This paper studies mixed bundling as a price-discrimination strategy in a competitive environment. The existing research on competitive mixed bundling such as Matutes and Regibeau (1992), Anderson and Leruth (1993), Reisinger (2004), Thanassoulis (2007), and Armstrong and Vickers (2010) focuses on the duopoly case. The usual setup is a two-dimensional Hotelling model where consumers with different preferences for the products are distributed on a square.[2] The literature often highlights the case (e.g. when consumers are uniformly distributed and have unit demand for each product) where mixed bundling intensifies competition and benefits consumers relative to separate sales. More generally, however, mixed bundling in duopoly has ambiguous impacts on price, profit and consumer surplus as we will emphasize in this paper.[3]

This is the first paper which studies competitive mixed bundling with an arbitrary number of firms.[4] It makes three contributions. First, we propose a random-utility framework for studying competitive mixed bundling. Compared to the spatial

[1] See, for example, Stigler (1968), Schmalensee (1984), and Fang and Norman (2006) for pure bundling, and Adams and Yellen (1976), Long (1984), McAfee, McMillan, and Whinston (1989), and Chen and Riordan (2013) for mixed bundling.

[2] Armstrong and Vickers (2010) is the most general study so far. They consider a general consumer distribution on the Hotelling square, and also allow for the existence of an exogenous shopping cost and consider elastic demand and general nonlinear pricing schedules.

[3] There are also works on competitive pure bundling. See, for example, Matutes and Regibeau (1988), Economides (1989), Kim and Choi (2015), Zhou (2017), and Hurkens, Jeon, and Menicucci (2019). See Section 7 in Stole (2007) and Section 4 in Armstrong (2016) for surveys of the literature on competitive bundling.

[4] In other aspects, however, we focus on the usual case where consumers have unit demand for each product and do not face an exogenous shopping cost (though it is not hard to introduce).

approach, this framework is more convenient to use when there are more than two multiproduct firms. In the duopoly case, our model can be converted into a two-dimensional Hotelling model such that we can compare our results with those in the existing literature.[5] Second, we explain why the problem of mixed bundling is much harder once we go beyond duopoly, and we develop an approach to characterize the demand and the equilibrium prices when all firms adopt the mixed-bundling strategy. When the number of firms is large, the equilibrium prices have a simple approximation under a mild condition. Third, we argue that the impact of mixed bundling on market performance can qualitatively defend on the number of firms in the market.

Section 2 introduces the model and studies the benchmark case of separate sales. Section 3 examines a firm's individual incentive to adopt mixed bundling starting from the separate-sales benchmark. The problem can be formulated as a monopoly problem with a random outside option, and the particular structure of the outside option in a symmetric competition environment enables us to obtain some clean results. For instance, for a given valuation distribution (regardless of its correlation structure), each firm has a unilateral incentive to introduce mixed bundling when there are only two firms,[6] or when there are many firms and a certain tail-behavior condition is satisfied.

Sections 4 and 5 characterize the demand and the equilibrium pricing schedule when all firms use the mixed-bundling strategy. The main challenge is how to calculate a firm's demand when there are more than two firms in the market. From a particular firm a consumer can buy both products, one product only, or nothing. In the third option, the consumer can buy both products from a single rival firm to take advantage of its bundling discount, or mix and match across all rival firms to assemble a better bundle. Which is better depends on whether the best matched

[5] Anderson and Leruth (1993) also use a random-utility framework in their duopoly model, but they focus on the logit setting where the utility shock follows the extreme value distribution. Another important difference is that in our model the utility shock is at the level of individual products, while in Anderson and Leruth (1993) the utility shock is at the bundle level (so that a consumer might like both products individually but dislike the package of the two products). Zhou (2017) uses a similar random-utility framework with an abtritary number of firms as in this paper to study competitive pure bundling.

[6] A similar result is also derived by Armstrong and Vickers (2010) in their Hotelling setup.

products among the rival firms are from the same firm or two different firms, and also depends on the magnitude of the bundling discount. We find a way to calculate the demand and derive the necessary conditions for equilibrium prices.

However, due to the complexity of the demand system, further analytical progress is made only in the duopoly case and the case with many firms. In the duopoly case, relative to separate sales how bundling affects prices is sensitive to the underlying consumer valuation distribution. The existing literature highlights the example with a uniform consumer distribution in the Hotelling square where bundling reduces all prices. However, it is also easy to find examples where bundling raises single-product prices and lowers the bundle price, or where bundling raises all prices. In the case with many firms, we offer a simple approximation of the equilibrium prices under a mild condition. The approximated single-product price approaches (from below) the price in the separate-sales benchmark, and the approximated bundling discount is equal to half of the single-product markup in the separate-sales benchmark. Therefore, all prices drop in the bundling regime relative to separate sales, and if the production cost is zero the approximate pricing scheme features "50% off for the second product". We explain this difference between the two cases by treating the bundling discount as an exogenous shopping cost. It is shown that a small shopping cost always induces firms to lower their prices, while a larger one has a less clear impact.

Section 6 examines the welfare impacts of mixed bundling relative to separate sales. With the assumption of full market coverage, bundling must harm total welfare as it causes too much one-stop shopping and so sub-optimal match between consumers and products. In the duopoly case, since bundling affects prices in an ambiguous way, its impacts on profit and consumer surplus are also ambiguous. The existing literature emphasizes the case where mixed bundling harms firms and benefits consumers, but there are also cases where the opposite is true, or where mixed bundling harms both firms and consumers. When the number of firms is large, however, our approximation result implies that firms suffer and consumers benefit from mixed bundling. Section 7 concludes. All omitted proofs are presented in the appendix.

2 The model and the benchmark

Consider a market where each consumer needs to buy two products 1 and 2. The measure of consumers is normalized to one. There are $n \geq 2$ firms, each supplying both products. The unit production cost of each product is normalized to zero, so we can regard prices as markups. Each product is horizontally differentiated across firms (e.g., each firm produces a different variety of the product), but there is no product compatibility issue and consumers can freely mix and match. We adopt a multiproduct version of the random utility framework in Perloff and Salop (1985) to model product differentiation. Let $\mathbf{X}_l^k \equiv (X_{1,l}^k, X_{2,l}^k)$, $k = 1, \cdots, n$, denote the random match utilities of firm k's two products for consumer l. We assume that $\mathbf{X}_l^k$ is i.i.d. across consumers (e.g. consumers have idiosyncratic tastes for the products from different firms), and is also i.i.d. across firms (so firms are *ex ante* symmetric). In the following we therefore suppress the subscript l and superscript k whenever there is no confusion. Suppose $\mathbf{X}$ is distributed according to a common joint cumulative distribution function (cdf) $F(x_1, x_2)$. F has a full-dimensional support $S \subset \mathbb{R}^2$ and a bounded, differentiable, and everywhere strictly positive probability density function (pdf) $f(x_1, x_2)$. Let $F_i(x)$ and $f_i(x)$, $i = 1, 2$, be the marginal cdf and pdf of X_i, and let $[\underline{x}_i, \overline{x}_i]$ be its support (where $\underline{x}_i = -\infty$ and $\overline{x}_i = \infty$ are allowed). We sometimes consider the "i.i.d." case where we further have $F(x_1, x_2) = F_1(x_1)F_2(x_2)$ and $F_1 = F_2$, i.e., the case when the two products in each firm are symmetric and have independent match utilities.

We consider a discrete-choice framework where the incremental utility from having more than one variety of a product is zero and so a consumer only wants to buy one variety of each product.[7] We also assume that a consumer has unit demand for her preferred variety of each product. If a consumer consumes two products with match utilities (x_1, x_2) (which can be purchased from different firms) and makes a total payment T, she obtains surplus $(x_1 + x_2) - T$.[8]

[7] This assumption is made in all the papers on competitive bundling, though it is not always without loss of generality. For example, reading another article on the same subject in a different newspaper, or reading another chapter on the same topic in a different textbook, sometimes improves utility. There are works on consumer demand which extend the usual discrete choice model by allowing consumers to consume multiple versions of a product (see, e.g., Gentzkow (2007)).

[8] Most of bundling papers assume such an additive utility function (which is compatible with

If a firm sells its two products separately, it chooses a price vector $(\hat{p}_1, \hat{p}_2)$. Let $\hat{P} \equiv \hat{p}_1 + \hat{p}_2$ be the associated bundle price. If a firm adopts the mixed-bundling strategy, it chooses a pair of single-product prices (p_1, p_2) together with a bundling discount δ. Let $P \equiv p_1 + p_2 - \delta$ be the associated bundle price. In either regime the timing is that firms choose their prices simultaneously, and then consumers make their choices after observing all the prices and match utilities. As often assumed in the literature on competitive bundling, the market is fully covered (i.e., all consumers buy both products). This will be the case if consumers do not have outside options, or if on top of the above match utilities, consumers have a sufficiently high basic valuation for each product.

For convenience, we introduce a few pieces of notation. Denote by

$$Y_i \equiv \max\{X_i^k\}_{k=1}^{n-1}$$

the match utility of the best product i among $n-1$ firms. The joint cdf of (Y_1, Y_2) is $G(y_1, y_2) \equiv F(y_1, y_2)^{n-1}$, and let $g(y_1, y_2)$ be the associated joint pdf. The marginal cdf of Y_i is $G_i(y) \equiv F_i(y)^{n-1}$.

Let

$$Z_i \equiv X_i - Y_i$$

be the valuation for a firm's product i relative to the best product i among its competitors. Its support is $[\underline{x}_i - \overline{x}_i, \overline{x}_i - \underline{x}_i]$. When firms sell their products separately and charge the same prices, a consumer will buy that firm's product i if and only if $Z_i > 0$. The joint cdf of (Z_1, Z_2) is

$$H(z_1, z_2) \equiv \int_S F(y_1 + z_1, y_2 + z_2) dG(y_1, y_2) \ ,$$

and let $h(z_1, z_2)$ be the associated joint pdf. (Whenever there is no confusion we suppress the integral region S thereafter.) The marginal cdf of Z_i is $H_i(z_i) \equiv \int F_i(y + z_i) dG_i(y)$, and let $h_i(z_i)$ be the associated marginal pdf. In particular, due to firm symmetry, we have

$$H_i(0) = 1 - \frac{1}{n} \quad \text{and} \quad h_i(0) = \int f_i(y) dF_i(y)^{n-1} \ . \tag{1}$$

perfect complements under the assumption of full market coverage). There is research which studies bundling of substitutes or complements (see, e.g., Long (1984), Armstrong (2013), and Haghpanah and Hartline (2019)).

Here $H_i(0)$ is the chance that a firm's product i is worse than the best product i among its $n-1$ competitors, and $h_i(0)$ is the density of consumers who are indifferent between this firm's product i and the best product i among its competitors.

We first report the equilibrium prices in the benchmark regime of separate sales. Since firms compete on each product separately, the market for each product is an independent Perloff-Salop model where only the marginal distribution of that product's match utility matters. Consider the market for product i, and let $\hat{p}_i$ be the (symmetric) equilibrium price.[9] Suppose a firm deviates to price $\hat{p}_i'$, while other firms stick to the equilibrium price $\hat{p}_i$. Then the demand for the deviation firm's product i is

$$q_i(\hat{p}_i') = \Pr[X_i - \hat{p}_i' > Y_i - \hat{p}_i] = 1 - H_i(\hat{p}_i' - \hat{p}_i) \ .$$

In equilibrium the demand is $q_i(\hat{p}_i) = \frac{1}{n}$ due to firm symmetry (which is also easy to verify by using (1)).

The deviation firm's profit from product i is $\hat{p}_i' q_i(\hat{p}_i')$, and for $\hat{p}_i$ to be the equilibrium price the profit should be maximized at $\hat{p}_i' = \hat{p}_i$. From the first-order condition we derive

$$\hat{p}_i = \frac{1}{nh_i(0)} \ , \tag{2}$$

where $h_i(0)$ is defined in (1). Henceforth, we assume that this first-order condition is also sufficient for defining the equilibrium price. This is the case, for example, when f_i is log-concave (see Caplin and Nalebuff (1991)).[10] In the example of uniform distribution with $F_i(x) = x$, we have $h_i(0) = 1$ and so $\hat{p}_i = \frac{1}{n}$. In the example of extreme value distribution with $F_i(x) = e^{-e^{-x}}$ (which generates the logit model), we have $h_i(0) = \frac{n-1}{n^2}$ and so $\hat{p}_i = \frac{n}{n-1}$. Generally, $\hat{p}_i$ decreases in n if f_i is log-concave (see, e.g., Anderson, de Palma, and Nesterov (1995), and Zhou (2017)), and $\lim_{n\to\infty} \hat{p}_i = 0$ if and only if $\lim_{x\to\overline{x}_i} \frac{f_i(x)}{1-F_i(x)} = \infty$ (see Zhou (2017)).

[9] In the duopoly case Perloff and Salop (1985) have shown that the pricing game has no asymmetric equilibrium. Beyond duopoly Caplin and Nalebuff (1991) show that there is no asymmetric equilibrium in the logit model. More recently Quint (2014) proves a general result (see Lemma 1 there) which implies that our pricing game has no asymmetric equilibrium if f_i is log-concave.

[10] Many often used distributions such as uniform, normal, logistic, and extreme value have a log-concave density. Caplin and Nalebuff (1991) provide a weaker sufficient condition which requires f_i to be $-\frac{1}{n+1}$-concave for a given n.

3 Incentive to use mixed bundling

We first examine, starting from separate sales, whether a firm has a unilateral incentive to introduce a small bundling discount (so that separate sales cannot be an equilibrium outcome). We need another two pieces of notation: the cdf of Z_i, conditional $Z_j = z_j$ where $j \neq i$, is

$$H_i(z_i|z_j) \equiv \int_{-\infty}^{z_i} h_i(t_i|z_j)dt_i \ ,$$

where $h_i(t_i|z_j) \equiv h(t_i, z_j)/h_j(z_j)$ is the conditional pdf of Z_i.

Suppose a firm unilaterally deviates from separate sales and introduces a *small* bundling discount $\delta > 0$ (but keeps its single-product prices the same as in the separate-sales equilibrium). Figure 1 below depicts how this small deviation affects consumer demand in the space of (z_1, z_2), where Ω_i, $i = 1, 2$, indicates consumers who buy only product i from the firm in question and Ω_b indicates consumers who buy both products from it.

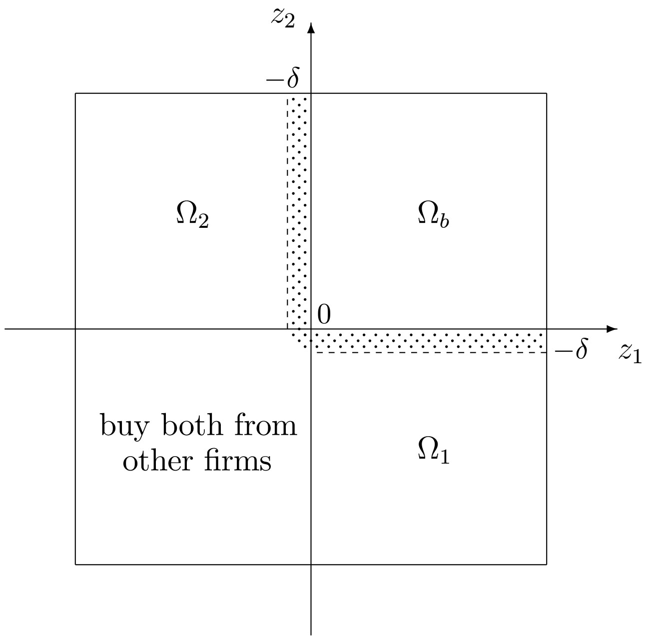

Figure 1: The impact of a small bundling discount on demand

The negative effect of the deviation is that the deviation firm earns δ less from the consumers who buy both products from it. In the regime of separate sales, the measure of those consumers is

$$\Omega_b = 1 - H_1(0) - H_2(0) + H(0,0) = \frac{2}{n} - 1 + H(0,0) \ ,$$

where we have used $H_i(0) = 1 - \frac{1}{n}$. So the (first-order) loss from the deviation is $\delta\Omega_b$.

The positive effect of the deviation is that more consumers buy both products from the deviation firm, i.e., the region Ω_b expands as indicated on the graph. Those consumers on the two shaded rectangle areas switch from buying only one product to buying both products from the deviation firm, and those on the small shaded triangle area switch from buying nothing to buying both products from the deviation firm.

Notice that the small triangle area is a second-order effect when δ is small (given our distribution conditions), so only the two rectangle areas matter. The measure of consumers on the vertical rectangle area is $\delta \int_0^\infty h(0, z_2)dz_2$, and the deviation firm now makes an extra profit $\hat{p}_1 - \delta$ from each of them. Similarly, the measure of consumers on the horizontal rectangle area is $\delta \int_0^\infty h(z_1, 0)dz_1$, and the deviation firm now makes an extra profit $\hat{p}_2 - \delta$ from each of them. Thus, the (first-order) gain from the small deviation is

$$\hat{p}_1\delta \int_0^\infty h(0, z_2)dz_2 + \hat{p}_2\delta \int_0^\infty h(z_1, 0)dz_1 = \frac{\delta}{n}[2 - H_1(0|0) - H_2(0|0)] \ , \qquad (3)$$

where the equality used

$$\hat{p}_1 \int_0^\infty h(0, z_2)dz_2 = \frac{1}{nh_1(0)} \times h_1(0) \int_0^\infty h_2(z_2|0)dz_2 = \frac{1}{n}[1 - H_2(0|0)] \ ,$$

and similarly

$$\hat{p}_2 \int_0^\infty h(z_1, 0)dz_1 = \frac{1}{n}[1 - H_1(0|0)] \ .$$

The small deviation is profitable if the gain in (3) is greater than the loss $\delta\Omega_b$. Therefore, we have the following result:

Proposition 1 *Starting from separate sales with prices defined in (2), each firm has a strict unilateral incentive to introduce mixed bundling if*

$$n[1 - H(0, 0)] > H_1(0|0) + H_2(0|0) \ . \qquad (4)$$

(i) For a given distribution F, (4) holds if $n = 2$, or if n is sufficiently large and $\lim_{z_i \to \overline{x}_i - \underline{x}_i} \frac{h(z_1,z_2)}{h_1(z_1)h_2(z_2)} > 0$.
(ii) For a given n, (4) holds if X_1 and X_2 are independent, negatively dependent (in the sense that $\Pr(X_i > a|X_{-i} > b)$ *is decreasing in b for any a), or limitedly positively dependent (in the sense that* $\Pr(X_i > a|X_{-i} > b) \geq \Pr(X_i > a)$ *for any a and b, and* $\frac{d}{dt}H_i(0|H_j^{-1}(t)) > -1$ *for* $t \in [1 - \frac{1}{n}, 1]$*).*

Proof. (i) When $n = 2$, we have $Y_i = X_i$. Then conditional on $Z_i = 0$ (i.e., $X_i^1 = X_i^2$), X_j^1 and X_j^2 should share the same conditional distribution. This implies $H_1(0|0) = H_2(0|0) = 1/2$. Meanwhile, $H(0,0)$ is always strictly less than $H_i(0) = 1/2$. Then

$$2[1 - H(0,0)] > 2[1 - H_i(0)] = 1 = H_1(0|0) + H_2(0|0) \ .$$

The proof for the large-n result is longer and is provided in the appendix.

(ii) When the two products have independent valuations, $H_i(0|0) = H_i(0)$ and $H(0,0) = H_1(0)H_2(0)$. Using $H_i(0) = 1 - \frac{1}{n}$, one can verify that the gain is twice the loss for any n. The proofs for the cases with dependent valuations are provided in the appendix. ■

Given other firms are selling their products separately at the equilibrium prices in the benchmark, a firm's problem of whether to use mixed bundling is essentially a monopoly problem where a consumer's net valuation for its product i is $X_i - (Y_i - \hat{p}_i)$. Here $Y_i - \hat{p}_i$ is regarded as a random outside option. Then our incentive results in part (ii) are closely related to the existing works on the profitability of mixed bundling in a monopoly setting. See, for example, McAfee, McMillan, and Whinston (1989), and Chen and Riordan (2013). (In particular, Chen and Riordan (2013) derived some similar conditions by using a copula approach. Our proof for the cases with dependent valuations closely follows their approach.) However, in our symmetric oligopoly setting $Y_i - \hat{p}_i$ is related to X_i in a particular way such that given the random outside option $Y_i - \hat{p}_i$, the optimal monopoly separate-sales price for product i is $\hat{p}_i$. This additional structure leads to the result for $n = 2$ (which has also been derived by Armstrong and Vickers (2010) in their Hotelling model), and the result for a sufficiently large n (which is new in the literature).

4 Demand with mixed bundling

Consider a symmetric mixed-bundling equilibrium (p_1, p_2, δ), where p_i is the single-product price for product i and δ is the bundling discount.[11] Suppose that a firm unilaterally deviates and offers a pricing schedule (p'_1, p'_2, δ'). Then for a consumer

[11] We restrict our attention to the case with $\delta < \min\{p_1, p_2\}$. Otherwise, the bundle would be cheaper than at least one individual product.

who values this firm's products at (x_1, x_2) and the best products from other firms at (y_1, y_2), she has the following purchase options:

(a) buy both products from the deviation firm, in which case her surplus is $x_1 + x_2 - (p_1' + p_2' - \delta')$;

(b) buy product 1 from the deviation firm but product 2 elsewhere, in which case her surplus is $x_1 + y_2 - p_1' - p_2$;

(c) buy product 2 from the deviation firm but product 1 elsewhere, in which case her surplus is $y_1 + x_2 - p_1 - p_2'$;

(d) buy both products from other firms, in which case her surplus is $\Theta - (p_1 + p_2 - \delta)$, where Θ is a random variable conditional on (y_1, y_2) as defined in (5) below.

When the consumer buys only one product, say, product i from some other firm, she will buy the best one with match utility y_i. When she buys both products from other firms, the situation is more complicated if $n \geq 3$. This is because now she may buy them together from one firm or separately from two different firms. For this reason, (y_1, y_2) is not a sufficient statistic for the match utilities from other firms. This is the main source of the complication in studying competitive bundling when we go beyond the duopoly model. To understand Θ, we need to discuss two cases:

First, if y_1 and y_2 are realized in the same firm, in option (d) the consumer will buy both products from that firm, and so $\Theta = y_1 + y_2$. Conditional on $Y_1 = y_1$ and $Y_2 = y_2$, this event occurs with probability

$$\lambda(y_1, y_2) \equiv \frac{(n-1)f(y_1, y_2)F(y_1, y_2)^{n-2}}{g(y_1, y_2)} \ ,$$

where the numerator is the probability in the density sense that $Y_1 = y_1$ and $Y_2 = y_2$ are realized in the same firm among $n-1$ ones, and the denominator is the joint pdf of (Y_1, Y_2), i.e. the probability that $Y_1 = y_1$ and $Y_2 = y_2$ in the density sense.[12] When $n = 2$, this probability is 1. When the two products at each firm have independent match utilities, this probability simplifies to $\frac{1}{n-1}$ as expected.

Second, with the rest of the probability $1 - \lambda(y_1, y_2)$, y_1 and y_2 are realized at two different firms. Then the consumer faces the trade-off between consuming better-matched products by two-stop shopping, in which case she gets surplus $y_1 + y_2 -$

[12] More explicitly, we have

$$g(y_1, y_2) = (n-1)F(y_1, y_2)^{n-3}[f(y_1, y_2)F(y_1, y_2) + (n-2)\frac{\partial F}{\partial y_1}\frac{\partial F}{\partial y_2}] \ .$$

(p_1+p_2), or enjoying the bundling discount by one-stop shopping, in which case she gets surplus $Y(y_1,y_2)-(p_1+p_2-\delta)$, where $Y(y_1,y_2)$ denotes the match utility of the best bundle among $n-1$ firms conditional on $Y_1=y_1$ and $Y_2=y_2$ being realized at different firms. (The cdf of $Y(y_1,y_2)$ is characterized in Lemma 1 below.) Hence, in this second case, $\Theta=\max\{Y(y_1,y_2),y_1+y_2-\delta\}$.

In sum, conditional on $Y_1=y_1$ and $Y_2=y_2$, we have

$$\Theta=\begin{cases} y_1+y_2 & \text{with probability } \lambda(y_1,y_2) \\ \max\{Y(y_1,y_2),y_1+y_2-\delta\} & \text{with probability } 1-\lambda(y_1,y_2) \end{cases}. \tag{5}$$

The simplest case is when $n=2$. Then y_1 and y_2 must be from the same firm and so $\Theta=y_1+y_2$ for sure. The problem can then be converted into a two-dimensional Hotelling model by using two "location" random variables $Z_1=X_1-Y_1$ and $Z_2=X_2-Y_2$. That is the model often used in the existing literature.

When $n\geq 3$, the situation is more complicated. We need to deal with one more random variable $Y(y_1,y_2)$ which is correlated with Y_1 and Y_2. Its cdf is reported in the following lemma (all omitted proofs can be found in the appendix):

Lemma 1 *When $n\geq 3$, the cdf of $Y(y_1,y_2)$ (the match utility of the best bundle among $n-1$ firms conditional on $Y_1=y_1$, $Y_2=y_2$, and they being realized at different firms), is*

$$\begin{aligned} L(y|y_1,y_2) &= \frac{F_1(y-y_2|y_2)}{F_1(y_1|y_2)}\frac{F_2(y-y_1|y_1)}{F_2(y_2|y_1)} \\ &\quad\times\frac{1}{F(y_1,y_2)^{n-3}}\left(F(y_1,y-y_1)+\int_{y-y_1}^{y_2}\int_{\underline{x}_1}^{y-x_2} f(x_1,x_2)dx_1dx_2\right)^{n-3} \end{aligned} \tag{6}$$

for $y\in[\max\{y_1+\underline{x}_2,\underline{x}_1+y_2\},y_1+y_2)$, where $F_i(y_i|y_j)$ is the conditional cdf of y_i.

With this result, we can calculate the expectation of any function $\phi(Y_1,Y_2,\Theta)$ (if exists) as follows:

$$\begin{aligned} \mathbb{E}[\phi(Y_1,Y_2,\Theta)] &= \int_{(y_1,y_2)}[\lambda(y_1,y_2)\times\phi(y_1,y_2,y_1+y_2) \\ &+(1-\lambda(y_1,y_2))\times\int_y \phi(y_1,y_2,\max\{y,y_1+y_2-\delta\})dL(y|y_1,y_2)]dG(y_1,y_2)\ . \end{aligned} \tag{7}$$

Given (y_1,y_2,θ), Figure 2 below describes how a consumer chooses among the four purchase options in the space of (x_1,x_2). As before, Ω_i, $i=1,2$, indicates the region where the consumer buys only product i from the deviation firm, and Ω_b indicates the region where the consumer buys both products from it. Then integrating the area of Ω_i over (y_1, y_2, θ) by using (7) yields the demand for the deviation firm's single product i, and integrating the area of Ω_b over (y_1, y_2, θ) yields the demand for its bundle.

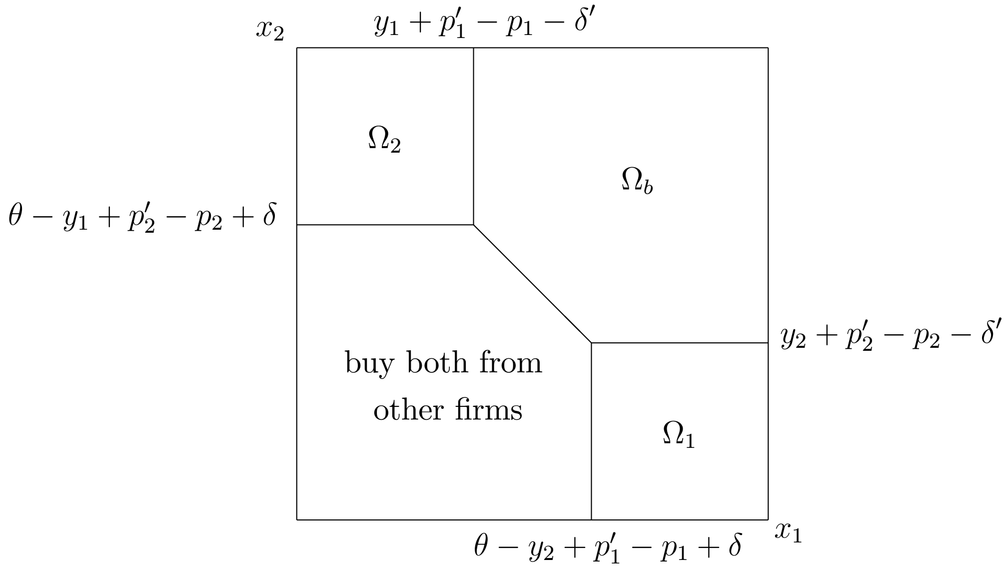

Figure 2: The pattern of consumer choice

From Figure 2, we can see that the equilibrium demand for a firm's single product 1 is

$$\Omega_1(\delta) \equiv \mathbb{E}[\int_{\underline{x}_2}^{y_2-\delta} \int_{\theta-y_2+\delta}^{\overline{x}_1} f(x_1, x_2) dx_1 dx_2] \ , \tag{8}$$

and the equilibrium demand for a firm's single product 2 is

$$\Omega_2(\delta) \equiv \mathbb{E}[\int_{\theta-y_1+\delta}^{\overline{x}_2} \int_{\underline{x}_1}^{y_1-\delta} f(x_1, x_2) dx_1 dx_2] \ . \tag{9}$$

(The expectations are taken over (y_1, y_2, θ).) Given full market coverage, the equilibrium demand depends only on the bundling discount δ but not on the single-product prices p_1 and p_2. Let $\Omega_b(\delta)$ be the equilibrium demand for a firm's bundle. Then we should have

$$\Omega_i(\delta) + \Omega_b(\delta) = \frac{1}{n} \ . \tag{10}$$

This is because with full market coverage all consumers buy product i, so $1/n$ of them should buy it from a particular firm (either via single product purchase or via bundle purchase). This also implies that $\Omega_1(\delta) = \Omega_2(\delta)$, even when the two products are asymmetric.

5 Equilibrium mixed-bundling prices

5.1 The general case

To characterize the equilibrium prices, it is useful to introduce a few more pieces of notation:

$$
\begin{aligned}
\alpha_1 &\equiv \mathbb{E}[\int_{\theta-y_1+\delta}^{\overline{x}_2} f(y_1-\delta,x_2)dx_2] \ , \ \beta_1 \equiv \mathbb{E}[\int_{\underline{x}_1}^{y_1-\delta} f(x_1,\theta-y_1+\delta)dx_1] \ ; \\
\alpha_2 &\equiv \mathbb{E}[\int_{\theta-y_2+\delta}^{\overline{x}_1} f(x_1,y_2-\delta)dx_1] \ , \ \beta_2 \equiv \mathbb{E}[\int_{\underline{x}_2}^{y_2-\delta} f(\theta-y_2+\delta,x_2)dx_2] \ ; \\
\gamma &\equiv \mathbb{E}[\int_{y_1-\delta}^{\theta-y_2+\delta} f(x_1,\theta-x_1)dx_1] \ . \qquad (11)
\end{aligned}
$$

(The expectations are all taken over (y_1, y_2, θ).) Notice that all α_i, β_i and γ are functions of δ only. The economic meaning of these notations will be clear below. In particular, $\alpha_i + \beta_j + \gamma$ is the density of marginal consumers who stop buying product i from a firm when it unilaterally increases its price p_i, and $\alpha_1 + \alpha_2 + \gamma$ is the density of marginal consumers who switch to buying both products from a firm when it unilaterally increases its discount δ_i.[13]

To derive the necessary conditions for (p_1, p_2, δ) to be the equilibrium pricing schedule, let us consider a few local deviations:

First, suppose a firm unilaterally raises its bundling discount to $\delta' = \delta + \varepsilon$ (where $\varepsilon > 0$ is small) while keeps its single-product prices unchanged. Then conditional on (y_1, y_2, θ), Figure 3a below describes how this small deviation affects consumer choices: Ω_b expands because now more consumers buy both products from the deviation firm. The marginal consumers who change their purchase behavior are distributed on the shaded areas.

[13] Note that θ can depend on δ, so it is not true that $\Omega_i'(\delta) = -(\alpha_j + \beta_j)$ except when $n = 2$. This implies that beyond the duopoly case, the first-order conditions for equilibrium prices which we derive below cannot be written in terms of equilibrium demand functions and their derivatives.

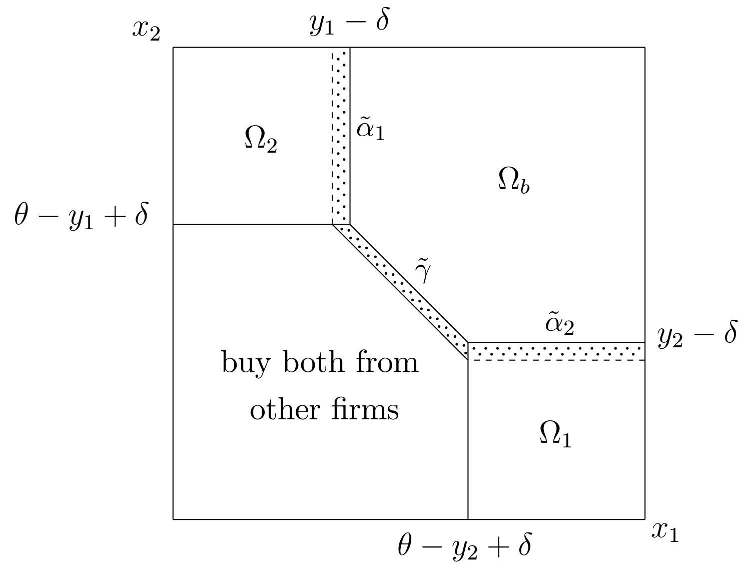

Figure 3a: Price deviation and consumer choice I

Here

$$\tilde{\alpha}_1 = \int_{\theta-y_1+\delta}^{\overline{x}_2} f(y_1-\delta, x_2)dx_2 \text{ and } \tilde{\alpha}_2 = \int_{\theta-y_2+\delta}^{\overline{x}_1} f(x_1, y_2-\delta)dx_1$$

are the densities of marginal consumers along the line segments $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ on the graph, respectively, and

$$\tilde{\gamma} = \int_{y_1-\delta}^{\theta-y_2+\delta} f(x_1, \theta-x_1)dx_1$$

is the density of marginal consumers along the diagonal line segment on the graph. Integrating them over (y_1, y_2, θ) yields the previously introduced notation: $\mathbb{E}[\tilde{\alpha}_i] = \alpha_i$ and $\mathbb{E}[\tilde{\gamma}] = \gamma$. For the marginal consumers on the vertical shaded area (which has a measure of $\varepsilon\tilde{\alpha}_1$), they switch from buying only product 2 to buying both products from the deviation firm, and so the firm makes $p_1-\delta-\varepsilon$ extra profit from each of them. Similarly, the deviation firm makes $p_2-\delta-\varepsilon$ extra profit from each of the marginal consumers on the horizontal shaded area (which has a measure of $\varepsilon\tilde{\alpha}_2$). For those marginal consumers on the diagonal shaded area (which has a measure of $\varepsilon\tilde{\gamma}$), they switch from buying both products from other firms to buying both from the deviation firm. So the deviation firm makes $p_1+p_2-\delta-\varepsilon$ extra profit from each of them. The only negative effect of the deviation is that those consumers on Ω_b who were already purchasing both products at the deviation firm now pay ε less. The sum of all these effects integrated over (y_1, y_2, θ) should be equal to zero in equilibrium. After all the second-order effects being discarded, this yields the following first-order condition:

$$\alpha_1(p_1-\delta)+\alpha_2(p_2-\delta)+\gamma(p_1+p_2-\delta)=\Omega_b(\delta)\ , \tag{12}$$

where $\Omega_b(\delta)$ is defined in (10) and α_i and γ are defined in (11).

Second, suppose a firm unilaterally raises its stand-along price p_1 to $p_1'=p_1+\varepsilon$ and its bundling discount to $\delta'=\delta+\varepsilon$ (such that its bundle price remains unchanged). Figure 3b below describes how this small deviation affects consumer choices: Ω_1 shrinks because now fewer consumers buy a single product 1 from the deviation firm.

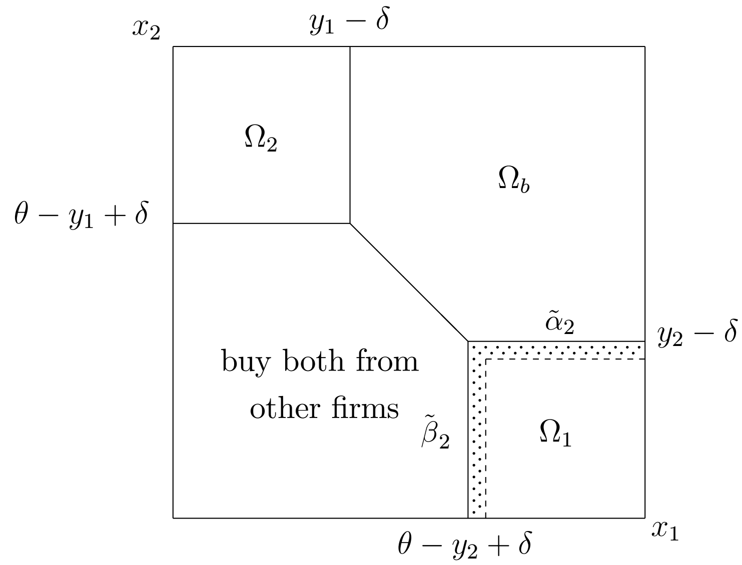

Figure 3b: Price deviation and consumer choice II

Here

$$\tilde{\beta}_2=\int_{\underline{x}_2}^{y_2-\delta}f(\theta-y_2+\delta,x_2)dx_2$$

is the density of marginal consumers along the line segment $\tilde{\beta}_2$ on the graph. Integrating it over (y_1,y_2,θ) yields the previously introduced notation: $\mathbb{E}[\tilde{\beta}_2]=\beta_2$. For those marginal consumers on the horizontal shaded area (which has a measure of $\varepsilon\tilde{\alpha}_2$), they switch from buying only product 1 to buying both products from the deviation firm. So the firm makes $p_2-\delta$ extra profit from each of them. For those marginal consumers on the vertical shaded area (which has a measure of $\varepsilon\tilde{\beta}_2$), they switch from buying product 1 to buying nothing from the deviation firm. So the firm loses p_1 from each of them. The direct revenue effect of this deviation is that the firm earns ε more from each consumer on Ω_1. The sum of these effects integrated over (y_1, y_2, θ) should be equal to zero in equilibrium. This yields another first-order condition:

$$\alpha_2(p_2 - \delta) + \Omega_1(\delta) = \beta_2 p_1 \ , \tag{13}$$

where $\Omega_1(\delta)$ is defined in (8) and α_2 and β_2 are defined in (11).

Third, suppose a firm slightly raises its stand-along price p_2 to $p_2' = p_2 + \varepsilon$ and its bundling discount to $\delta' = \delta + \varepsilon$ (such that its bundle price remains unchanged). (If the two products are symmetric, there is no need to consider this third deviation.) Then Ω_2 shrinks as described in Figure 3c below.

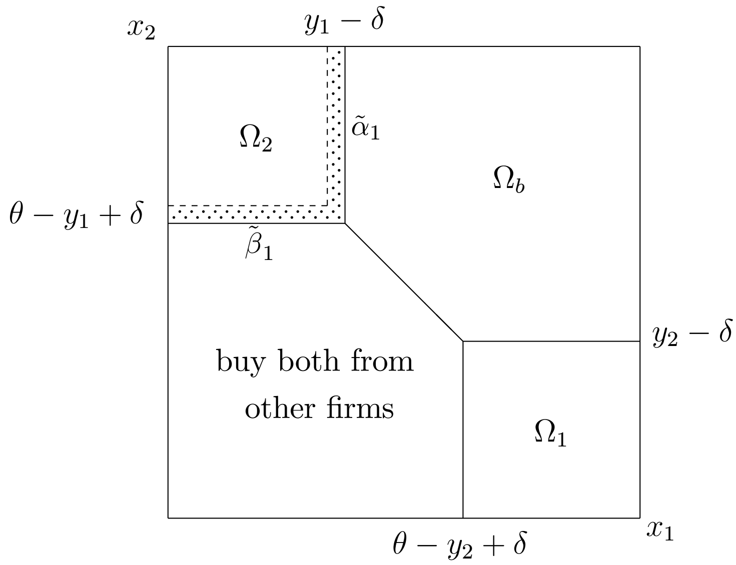

Figure 3c: Price deviation and consumer choice III

Here

$$\tilde{\beta}_1 = \int_{\underline{x}_1}^{y_1 - \delta} f(x_1, \theta - y_1 + \delta) dx_1$$

is the density of marginal consumers along the line segment $\tilde{\beta}_1$ on the graph. Integrating it over (y_1, y_2, θ) yields the previously introduced notation: $\mathbb{E}[\tilde{\beta}_1] = \beta_1$. A similar argument as before yields the third first-order condition:

$$\alpha_1(p_1 - \delta) + \Omega_2(\delta) = \beta_1 p_2 \ , \tag{14}$$

where $\Omega_2(\delta)$ is defined in (9) and α_1 and β_1 are defined in (11).

The following result rewrites the above three first-order conditions:

Proposition 2 *If a symmetric (pure-strategy) mixed-bundling equilibrium exists, the single-product prices p_1 and p_2 and the bundling discount δ must satisfy*

$$(\alpha_1 + \beta_2 + \gamma)p_1 + \gamma p_2 = \frac{1}{n} + (\alpha_1 + \gamma)\delta \ , \tag{15}$$

$$(\alpha_2 + \beta_1 + \gamma)p_2 + \gamma p_1 = \frac{1}{n} + (\alpha_2 + \gamma)\delta \ , \tag{16}$$

and

$$(\beta_2 - \alpha_1)p_1 + (\beta_1 - \alpha_2)p_2 = 2\Omega_1(\delta) - (\alpha_1 + \alpha_2)\delta \ . \tag{17}$$

Notice that (15) is derived from (12) and (13) by using (10), and (16) is derived from (12) and (14) by using (10). Adding (13) to (14) and using $\Omega_1(\delta) = \Omega_2(\delta)$ yield (17). The first two conditions are linear in p_1 and p_2. From them one can solve solve p_1 and p_2 as functions of δ. Substituting them into the third condition yields an equation of δ. These equations are more complicated than they appear because all α_i, β_i and γ are functions of δ.

Discussion: equilibrium existence. To prove the existence of a symmetric (pure-strategy) equilibrium,[14] we need to show that (i) the system of necessary conditions (15)-(17) has a solution, and (ii) the necessary conditions are also sufficient for defining the equilibrium prices. Unfortunately, both issues are hard to investigate. For the first one, we can prove it in the i.i.d. case when $n = 2$ or when n is sufficiently large under the log-concavity condition. For the second one, no analytical progress has been made in general even in the duopoly case. This is an unsolved problem in the literature on mixed bundling.

Discussion: δ as an exogenous shopping cost. It is useful to consider an alternative situation where firms do not offer a bundling discount but instead consumers face an exogenous shopping cost δ, i.e. if they buy the two products from two different firms, they need to pay an extra cost δ (which reflects, for instance, the extra travelling cost or the transaction cost of paying an additional bill). Such a shopping cost affects consumer behavior exactly the same as the bundling discount. Using a similar local-deviation argument as above, one can derive the first-order conditions for (p_1, p_2) to be the equilibrium prices:

$$(\alpha_1 + \beta_2 + \gamma)p_1 + \gamma p_2 = \frac{1}{n} \ , \tag{18}$$

[14] The existence of a symmetric mixed-strategy equilibrium is guaranteed by Theorem 1 in Becker and Damianov (2006) if we can impose a finite upper bound on prices.

and

$$(\alpha_2 + \beta_1 + \gamma)p_2 + \gamma p_1 = \frac{1}{n} \ . \tag{19}$$

This system is simpler and as we will see later, when the shopping cost is small, the solution to this system has a simple approximation. This will be helpful later on when we explain the intuition of the pricing result when the number of firms is large.

5.2 The i.i.d. case

The system of (15)-(17) is hard to deal with analytically. To make progress, henceforth we focus on the i.i.d. case where the two products at each firm are symmetric and have independent match utilities. Slightly abusing the notation, let $F(x)$ and $f(x)$ be the common cdf and pdf of X_i, and let

$$H(z) = \int F(y+z)dF(y)^{n-1} \quad \text{and} \quad h(z) = \int f(y+z)dF(y)^{n-1}$$

be the common cdf and pdf of $Z_i = X_i - Y_i$. (When $n = 2$, $h(z)$ is symmetric around zero.) Let p be the common single-product price, and let $\alpha = \alpha_i$ and $\beta = \beta_i$. Then the equilibrium conditions in (15)-(17) simplify to

$$p = \frac{1/n + (\alpha+\gamma)\delta}{\alpha + \beta + 2\gamma} \ , \tag{20}$$

and

$$(\beta - \alpha)\, p = \Omega_1(\delta) - \alpha\delta \ . \tag{21}$$

The expressions for α, β, γ, and $\Omega_1(\delta)$ are still complicated in general. However, they are simple in the duopoly case, and they also have simple approximations when δ is small (which can be shown to be true under a mild condition when the number of firms is large). Hence, in the following we study two polar cases.

The duopoly case. In the duopoly case, $\Theta = y_1 + y_2$ for sure, and so only the two random variables $Z_i = X_i - Y_i$, $i = 1, 2$, matter for the demand analysis given the assumption of full market coverage. Our random utility model can then be converted into a two-dimensional Hotelling model.

Using the symmetry of h one can check that $\alpha = \beta = h(\delta)[1 - H(\delta)]$ and $\Omega_1(\delta) = [1 - H(\delta)]^2$. Thus, (21) simplifies to

$$\delta = \frac{1 - H(\delta)}{h(\delta)} \ . \tag{22}$$

If $1 - H$ is log-concave (which is implied by the log-concavity of f), this equation has a unique positive solution $\delta > 0$. Meanwhile, (20) simplifies to

$$p = \frac{\delta}{2} + \frac{1}{4(\alpha + \gamma)} \tag{23}$$

with $\gamma = 2\int_0^\delta h(t)^2 dt$.

The following table reports the equilibrium prices for a few distributions:

	$\hat{p}$	p	δ	P
Uniform	0.5	0.57	1/3	0.81
Normal	1.77	1.85	1.06	2.63
Exponential	1	1.5	1	2

Table 1: Examples of price impact in duopoly

In both the uniform and the normal example, compared to the regime of separate sales, each single product becomes more expensive but the bundle becomes cheaper. (Note that in our uniform example $Z_i = X_i - Y_i$ has a triangle distribution, so it is different from the usual uniform example in the Hotelling model where consumers are uniformly distributed on the square.) In the exponential example, the bundle price remains unchanged, so that all prices weakly go up. Such an example has not appeared in the literature.

More generally, in the duopoly case we have the following result:

Proposition 3 *In the i.i.d. duopoly case, if* $1-H(z)$ *is log-concave (which is true if* f *is log-concave), the system of (22) and (23) has a unique solution with* $\delta < p$*, and the bundle price is no greater than in the regime of separate sales (i.e.,* $2p - \delta \leq 2\hat{p}$*).*

Proof. Notice that $\delta < p$ if $\alpha + \gamma < \frac{1}{2\delta}$. This is equivalent to

$$h(\delta)[1 - H(\delta)] + 2\int_0^\delta h(t)^2 dt < \frac{1}{2}\frac{h(\delta)}{1 - H(\delta)}$$

by using (22) and the definitions of α and γ. This inequality holds at $\delta = 0$. At the same time, it easy to check that both sides are increasing in δ given $1 - H(\delta)$ is log-concave, and the derivative of the right-hand side is at least twice the derivative of the left-hand side given $1 - H(\delta) < \frac{1}{2}$ for $\delta > 0$.

Notice that the bundle price is $2p - \delta = \frac{1}{2(\alpha+\gamma)}$, and the bundle price in the regime of separate sales is $1/h(0)$. The former is weakly lower if $\alpha + \gamma \geq \frac{h(0)}{2}$, or more explicitly if

$$h(\delta)[1 - H(\delta)] + 2\int_0^\delta h(t)^2 dt \geq \frac{1}{2}h(0) \ .$$

This is true as the equality holds at $\delta = 0$ and the left-hand side is increasing in δ if and only if $1 - H(\delta)$ is log-concave. ■

From the proof, we can see that if $1 - H(\delta)$ is log-convex, the bundle will be actually more expensive than in the regime of separate sales. (Its price remains unchanged in the edge case of the exponential distribution as we have seen.) For instance, in the case of Pareto distribution with $F(x) = 1 - \frac{1}{x^2}$ where $x \in [1, \infty)$, one can check $\hat{p} = \frac{5}{8}$, $p \approx 1.48$ and $\delta \approx 1.37$, so the bundle price increases from 1.25 in the regime of separate sales to about 1.58 in the regime of mixed bundling.

One example which is often highlighted in the literature based on the Hotelling model is when $H(z)$ is a uniform distribution (which is possible in our setup only if we go beyond the i.i.d. case). In that case, all three prices go down compared to separate sales. Therefore, these examples suggest that in the duopoly case the impact of bundling on market prices is sensitive to the underlying consumer valuation distribution. We will see a similar observation concerning the impacts of bundling on profit and consumer welfare.

The case with many firms. We first report a useful approximation result when δ is small:

Lemma 2 *For a given n, if $\delta \approx 0$, we have*

$$\begin{aligned}
\alpha &\approx \frac{h(0)}{n} - \left(\frac{h'(0)}{n} + \frac{h(0)^2}{n-1}\right)\delta \ , \\
\beta &\approx \left(1 - \frac{1}{n}\right)h(0) + \left(\frac{h'(0)}{n} - h(0)^2\right)\delta \ , \\
\gamma &\approx \frac{n}{n-1}h(0)^2\delta \ , \\
\Omega_1(\delta) &\approx \frac{1}{n}\left(1 - \frac{1}{n}\right) - \frac{2}{n}h(0)\delta \ ,
\end{aligned} \tag{24}$$

where $h(0) = \int f(x)dF(x)^{n-1}$ and $h'(0) = \int f'(x)dF(x)^{n-1}$.

Note that $\alpha+\beta+\gamma = h(0)$, where $h(0)$ is the equilibrium (single-product) demand slope in the regime of separate sales. When n is large, under a mild condition we show in the appendix that the system of (20) and (21) has a solution with δ close to zero. Then the approximations in (24) can be used to solve the equilibrium prices.

Proposition 4 *Suppose $\frac{|f'(x)|}{f(x)}$ is uniformly bounded and $\lim_{n\to\infty}\hat{p}=0$, where $\hat{p}$ is the separate-sales price defined in (2). When n is large, the system of (20) and (21) has a solution with $\delta\in(0,\hat{p})$ and it can be approximated as*

$$p\approx\frac{1}{nh(0)}\frac{1+h(0)\delta}{1+\frac{n}{n-1}h(0)\delta};\quad \delta\approx\frac{1}{\frac{2h'(0)}{h(0)}+\frac{2n^2-3n+2}{n^2-n}nh(0)}.$$

Both the single-product price and the bundle price are lower than in the regime of separate sales.

For a large n, we can further simplify the approximations to $p\approx\hat{p}$ and $\delta\approx\frac{1}{2}\hat{p}$. That is, the single-product price is approximately equal to the price in the regime of separate sales and the bundling discount is approximately half of the single-product price. The mixed-bundling scheme in this limit case can thus be interpreted as "50% off for the second product".[15] This proposition also implies that when there are many firms in the market, mixed bundling tends to be pro-competitive relative to separate sales.

To understand the result that the single-product price drops, suppose now δ is an exogenous shopping cost. In the i.i.d. case, from (18) and (19) we know that the equilibrium price satisfies $(\alpha+\beta+2\gamma)p=\frac{1}{n}$. When δ is small, we have

$$p\approx\frac{1/n}{h(0)+\gamma}\approx\frac{1}{nh(0)}\frac{1}{1+\frac{n}{n-1}h(0)\delta}<\hat{p}$$

by using the approximations in (24) and $\alpha+\beta+\gamma\approx h(0)$. That is, introducing a small shopping cost in the regime of separate sales will induce firms to lower their prices. This is because the shopping cost generates a new margin γ (i.e., the diagonal boundary in the previous graphs) on which the consumers are doubly profitable. (The shopping cost also affects the other margins α and β, but it has no first-order effect on $\alpha+\beta+\gamma$ since it equals $h(0)$ when δ is small.) A similar economic force works when δ is a small endogenous bundling discount.

[15] When there is a positive production cost c for each product, we have $\delta\approx(\hat{p}-c)/2$, i.e., the bundling discount is approximately equal to half of the single product's markup.

This argument, however, can fail when δ becomes larger.[16] For example, when δ is sufficiently large, the situation will be as if all firms use a pure-bundling strategy. According to Zhou (2017), we know that pure bundling often induces higher market prices than in separate sales when n is above a threshold. Hence, in general whether δ increases or decreases the single-product prices also depends on the magnitude of δ. This discussion also points out an important difference between mixed bundling and pure bundling: with mixed bundling, δ is endogenous and it becomes small when there are many firms; while pure bundling is the same as having a fixed and sufficiently large δ regardless of the number of firms.

6 Impact of mixed bundling

Given the assumption of full market coverage, total welfare is determined only by the match quality between consumers and products. Since the bundling discount induces consumers to one-stop shop too often, mixed bundling must lower total welfare relative to separate sales. In the following, we discuss its impacts on industry profit and consumer surplus.

Let $\pi(p_1, p_2, \delta)$ denote the equilibrium industry profit. Then

$$\pi(p_1, p_2, \delta) = p_1 + p_2 - n\delta\Omega_b(\delta) \ .$$

Every consumer buys both products, but those who buy both from the same firm pay δ less. Thus, relative to separate sales the impact of mixed bundling on industry profit is

$$\begin{aligned} \Delta\pi \equiv \pi(p_1, p_2, \delta) - \pi(\hat{p}_1, \hat{p}_2, 0) &= (p_1 - \hat{p}_1) + (p_2 - \hat{p}_2) - n\delta\Omega_b(\delta) \\ &= P - \hat{P} + n\delta\Omega_1(\delta) \ , \end{aligned} \tag{25}$$

where we used (10) in the second equality.

Let $v(\tilde{p}_1, \tilde{p}_2, \tilde{\delta})$ denote the consumer surplus when all firms are charging single-product prices $(\tilde{p}_1, \tilde{p}_2)$ and offering a bundling discount $\tilde{\delta}$. Given full market coverage, an envelope argument implies that $v_i(\tilde{p}_1, \tilde{p}_2, \tilde{\delta}) = -1$, $i = 1, 2$, and $v_3(\tilde{p}_1, \tilde{p}_2, \tilde{\delta}) = n\Omega_b(\tilde{\delta})$, where the subscripts indicate partial derivatives. (This is because raising $\tilde{p}_i$

[16] When $n = 2$, it can be shown that p always decreases in the shopping cost δ when $1 - H$ is log-concave.

by ε will make every consumer pay ε more, and raising the discount $\tilde{\delta}$ by ε will save ε for every consumer who buy both products from the same firm.) Then relative to separate sales, the impact of mixed bundling on consumer surplus is

$$\begin{aligned}
\Delta v &\equiv v(p_1, p_2, \delta) - v(\hat{p}_1, \hat{p}_2, 0) \\
&= \int_{\hat{p}_1}^{p_1} v_1(\tilde{p}_1, p_2, \delta) d\tilde{p}_1 + \int_{\hat{p}_2}^{p_2} v_2(\hat{p}_1, \tilde{p}_2, \delta) d\tilde{p}_2 + \int_0^{\delta} v_3(\hat{p}_1, \hat{p}_2, \tilde{\delta}) d\tilde{\delta} \\
&= -(p_1 - \hat{p}_1) - (p_2 - \hat{p}_2) + n \int_0^{\delta} \Omega_b(\tilde{\delta}) d\tilde{\delta} \\
&= \hat{P} - P - n \int_0^{\delta} \Omega_1(\tilde{\delta}) d\tilde{\delta} \ , \qquad (26)
\end{aligned}$$

where we used (10) in the last equality. From (25) and (26), it is also clear that mixed bundling always harms total welfare since $\Omega_1(\tilde{\delta})$ decreases in $\tilde{\delta}$.

To make more progress we again focus on the i.i.d. case. In the duopoly case, we have seen that the impact of bundling on prices is ambiguous. This makes the impacts on industry profit and consumer surplus ambiguous as well. The following table reports a few examples by using the above formulas:

	$\Delta\pi$	Δv	$\Delta(\pi + v)$
Uniform	-0.16	0.10	-0.06
Normal	-1.10	0.93	-0.17
Exponential	0.07	-0.22	-0.15

Table 2: Examples of welfare impact in duopoly

The exponential example differs qualitatively from the uniform and the normal example. Armstrong and Vickers (2010) derived a sufficient condition for mixed bundling to harm firms and benefit consumers (see their Proposition 4). With our notation, the condition is $\frac{d}{dz}\frac{H(z)}{h(z)} \geq \frac{1}{4}$ for $z \leq 0$. The uniform example satisfies this condition, but the normal and the exponential example do not. To show another possibility where both firms and consumers suffer from bundling, let us consider a generalized Pareto distribution with $f(x) = (1 - ax)^{\frac{1}{a}-1}$ and $F(x) = 1 - (1 - ax)^{\frac{1}{a}}$, where $a \in [0, 1]$ and the support is $[0, \frac{1}{a}]$. The distribution becomes the exponential distribution when $a = 0$ and the uniform distribution when $a = 1$. Figure 4 below depicts how the impacts of mixed bundling in this example vary with a. When a is sufficiently large, mixed bundling harms firms and benefits consumers; when a

is sufficiently small, the opposite is true; in between mixed bundling harms both firms and consumers. Therefore, in the duopoly case the welfare impact of mixed bundling is ambiguous in general.

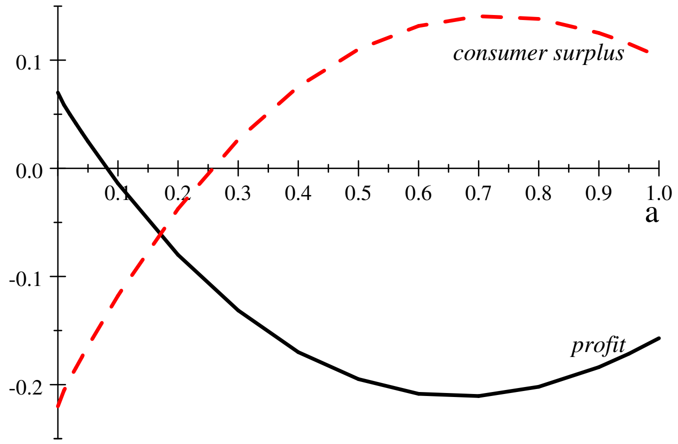

Figure 4: Welfare impacts with a generalized Pareto distribution

The welfare impacts become less dependent on the underlying distributions when n is large and the approximations in Proposition 4 apply. Since mixed bundling reduces all prices in that case relative to separate sales, it must harm firms and benefit consumers (as consumers can at least buy the same products as in the case of separate sales but at lower prices).

7 Conclusion

This paper has offered a random-utility framework for studying competitive mixed bundling in an oligopoly market. This approach is more convenient to use than the usual spatial approach in the existing literature when we need to deal with more than two firms. We have explained the source of difficulty in studying competitive mixed bundling beyond the duopoly case, and have developed a method to calculate demand and characterize the equilibrium prices. Analytical progress on the impacts of bundling on prices, profit and consumer welfare is only made in the duopoly case (where we derive some new results) and the case with many firms (where an approximation of the equilibrium prices is possible). We emphasize that the impact of bundling in the duopoly case is sensitive to the underlying consumer valuation distribution and so it is hard to draw general conclusions, while in the case with many firms bundling lowers market prices, harms firms and benefits consumers under a mild condition.

Appendix

Proof of Proposition 1: We prove the other sufficient conditions for (4). We use the copula approach introduced in Chen and Riordan (2013). (A classic reference on copula is Nelson, 2006.) Let $C(t_1, t_2)$ be the copula associated with the joint cdf H such that $H(z_1, z_2) = C(H_1(z_1), H_2(z_2))$. According to the Sklar's Theorem, such a copula exists uniquely for a given joint cdf if its marginal distributions are continuous. Therefore, a joint cdf can be represented by its marginal cdf's and a copula. A copula itself is a joint cdf on $[0, 1]^2$ with uniform marginal distributions, and it captures the dependence structure of the original distribution. Let $C_i(t_1, t_2)$ be the partial derivative with respect to t_i. Let $d(t) \equiv C(t, t)$ be the diagonal section of C, and it is increasing and uniformly continuous on $[0, 1]$. The following properties on copula are useful:

(a) $C(t_1, 0) = C(0, t_2) = 0$;

(b) $C(t_1, 1) = t_1$ and $C(1, t_2) = t_2$;

(c) $C_i(t_1, t_2)$ is the conditional distribution of t_{-i} given t_i;

(d) $\max\{0, 2t - 1\} \leq d(t) \leq t$.

We first claim that (4) is equivalent to

$$1 - d(t) > (1 - t)d'(t) \text{ at } t = 1 - \frac{1}{n} \ . \tag{27}$$

The definition of copula and $H_i(0) = 1 - \frac{1}{n}$ imply that $H(0, 0) = C(H_1(0), H_2(0)) = d(t)$ at $t = 1 - \frac{1}{n}$. Using the fact

$$h(z_1, z_2) = C_{12}(H_1(z_1), H_2(z_2))h_1(z_1)h_2(z_2) \tag{28}$$

and property (a), one can check that $H_1(0|0) = C_2(t, t)$ and $H_2(0|0) = C_1(t, t)$ at $t = 1 - \frac{1}{n}$. Then (4) can be written as $n(1 - d(t)) > C_1(t, t) + C_2(t, t)$ at $t = 1 - \frac{1}{n}$ which is equivalent to (27).

The large-n result. Let $t = 1 - \varepsilon$ with $\varepsilon \approx 0$. Then Taylor expansion, together with $d(1) = 1$ and $d'(1) = 2$ (both of which are from property (b)) imply that $1 - d(t) \approx 2\varepsilon - \frac{1}{2}d''(1)\varepsilon^2$ and $(1 - t)d'(t) \approx 2\varepsilon - d''(1)\varepsilon^2$. The former is greater whenever

$d''(1) > 0$. Notice that $d''(1) = C_{11}(1,1) + 2C_{12}(1,1) + C_{22}(1,1) = 2C_{12}(1,1)$ since $C_{ii}(1,1) = 0$ (which is again from property (b)). So $d''(1) > 0$ if and only if $C_{12}(1,1) > 0$, which is equivalent to the condition stated in the proposition according to (28).

The negative dependence result. Since $\Pr(X_i > a|X_{-i} > b)$ decreases in b for any a, for any given realization of (Y_i, Y_{-i}) we have $\Pr(X_i > a + Y_i|X_{-i} > b + Y_{-i})$ decreases in b. Then $\Pr(Z_i > a|Z_{-i} > b)$ decreases in b for any a. (This is called "right tail decreasing" in Nelson, 2006.) Corollary 5.2.6. in Nelson (2006) then implies that for any $t \in (0,1)$ we have

$$C_i(t,t) < \frac{t - C(t,t)}{1-t},\ i = 1, 2.$$

So $(1-t)d'(t) < 2(t - d(t))$. Then a sufficient condition for (27) is

$$1 - d(t) \geq 2(t - d(t)) \Leftrightarrow d(t) \geq 2t - 1\ .$$

This is always true given property (d).

The positive dependence result. As in the proof of Proposition 3 in Chen and Riordan (2013), (27) can be rewritten as

$$1 - 2t + d(t) + \int_t^1 (1-\xi)C_{11}(\xi,t)d\xi + \int_t^1 (1-\xi)C_{22}(t,\xi)d\xi > 0 \text{ at } t = 1 - \frac{1}{n}\ . \quad (29)$$

(This can be verified by using integration by parts and property (b).) Given $\Pr(X_i > a|X_{-i} > b) \geq \Pr(X_i > a)$ (which is called "positive quadrant dependence" in Nelson, 2006), we have $F(x_1,x_2) \geq F_1(x_1)F_2(x_2)$. This implies that $H(z_1,z_2) \geq H_1(z_1)H_2(z_2)$ and so $d(t) \geq t^2$ for any t. Also notice that $C_1(\xi,t) = H_2(H_2^{-1}(t)|H_1^{-1}(\xi)) = H_2(0|H_1^{-1}(\xi))$ at $t = 1 - \frac{1}{n}$. Then our condition on the conditional distribution implies that $C_{11}(\xi,t) > -1$ for $\xi \geq t = 1 - \frac{1}{n}$. Similarly, $C_{22}(t,\xi) > -1$ for $\xi \geq t = 1 - \frac{1}{n}$. Then the left-hand side of (29) is strictly greater than $(1-t)^2 - 2\int_t^1 (1-\xi)d\xi = 0$.

Proof of Lemma 1: For a given consumer, let $I(y_i)$, $i = 1, 2$, be the identity of the firm where y_i is realized. The lower bound of $Y(y_1,y_2)$ is from the fact that the lowest possible match utility of the bundle from firm $I(y_i)$ is $y_i + \underline{x}_j$. We now calculate the conditional probability of $Y(y_1,y_2) < y$. This event occurs if and only if all of the following three conditions are satisfied: (i) $y_1 + X_2^{I(y_1)} < y$, (ii)

$X_1^{I(y_2)} + y_2 < y$, and (iii) $X_1^k + X_2^k < y$ for all $k \neq I(y_1), I(y_2)$ among the $n-1$ competitors. Given y_1 and y_2, condition (i) holds with probability

$$\frac{F_2(y - y_1|y_1)}{F_2(y_2|y_1)} ,$$

since the cdf of $X_2^{I(y_1)}$ conditional on y_1 and $X_2^{I(y_1)} < y_2$ is $F_2(x_2|y_1)/F_2(y_2|y_1)$. Similarly, condition (ii) holds with probability

$$\frac{F_1(y - y_2|y_2)}{F_1(y_1|y_2)} .$$

One can also check (with the help of a graph) that the probability that $X_1^k + X_2^k < y$ holds for a firm other than $I(y_1)$ and $I(y_2)$, is

$$\frac{1}{F(y_1, y_2)} \left(F(y_1, y - y_1) + \int_{y-y_1}^{y_2} \int_{\underline{x}_1}^{y-x_2} f(x_1, x_2) dx_1 dx_2 \right) .$$

(The term in the bracket is the unconditional probability that (X_1^k, X_2^k) lies in the region where $X_i^k < y_i$ and $X_1^k + X_2^k < y$.) Conditional on y_1 and y_2, these three events are independent of each other. Therefore, the conditional probability of $Y(y_1, y_2) < y$ is as stated in (6).

Proof of Lemma 2: We first explain how to calculate $\mathbb{E}[\phi(Y_1, Y_2, \Theta)]$ defined in (7), where the expectation is taken over (Y_1, Y_2, Θ). Using (5) in the i.i.d. case, we have

$$\mathbb{E}[\phi(Y_1, Y_2, \Theta)] = \frac{1}{n-1} \int \phi(y_1, y_2, y_1 + y_2) dG$$

$$+ \frac{n-2}{n-1} \int \left(L(y_1 + y_2 - \delta | y_1, y_2) \phi(y_1, y_2, y_1 + y_2 - \delta) + \int_{y_1+y_2-\delta}^{y_1+y_2} \phi(y_1, y_2, y) dL(y|y_1, y_2) \right) dG ,$$

where $G(y_1, y_2) = F(y_1, y_2)^{n-1}$ and $L(y|y_1, y_2)$ is defined in (6). By integration by parts and using $L(y_1 + y_2 | y_1, y_2) = 1$, we can simplify this to

$$\mathbb{E}[\phi(Y_1, Y_2, \Theta)] = \int \phi(y_1, y_2, y_1 + y_2) dG - \frac{n-2}{n-1} \int \left(\int_{y_1+y_2-\delta}^{y_1+y_2} \frac{\partial}{\partial y} \phi(y_1, y_2, y) L(y|y_1, y_2) dy \right) dG .$$

Now let us derive the first-order approximation of α. (For our purpose, we do not need the higher-order approximations.) According to the formula above, we have

$$\alpha = \int f(y_1 - \delta)(1 - F(y_2 + \delta)) dG + \frac{n-2}{n-1} \int \varphi(y_1, y_2, \delta) dG , \quad (30)$$

where

$$\varphi(y_1, y_2, \delta) = \int_{y_1+y_2-\delta}^{y_1+y_2} f(y_1 - \delta) f(y - y_1 + \delta) L(y|y_1, y_2) dy \ .$$

When $\delta \approx 0$, we have $f(y_1 - \delta) \approx f(y_1) - \delta f'(y_1)$, so

$$\int f(y_1 - \delta) dG \approx \int f(y_1) dF(y_1)^{n-1} - \delta \int f'(y_1) dF(y_1)^{n-1} = h(0) - h'(0)\delta \ .$$

We also have $1 - F(y_2 + \delta) \approx 1 - F(y_2) - \delta f(y_2)$, so

$$\int (1 - F(y_2 + \delta)) dG \approx \int (1 - F(y_2)) dF(y_2)^{n-1} - \delta \int f(y_2) dF(y_2)^{n-1} = \frac{1}{n} - h(0)\delta \ .$$

To approximate $\int \varphi(y_1, y_2, \delta) dG$, notice that $\varphi(y_1, y_2, 0) = 0$ and $\varphi_3(y_1, y_2, 0) = f(y_1) f(y_2)$ since $L(y|y_1, y_2)$ is independent of δ and $L(y_1 + y_2|y_1, y_2) = 1$. Hence,

$$\int \varphi(y_1, y_2, \delta) dG \approx \delta \int f(y_1) f(y_2) dG = \delta h(0)^2 \ .$$

Substituting these approximations into (30) and discarding all higher order terms yields the approximation for α in (24). The other approximations can be derived similarly.

Proof of Proposition 4: We first show that when n is large, the system of (20) and (21) has a solution with a small δ under mild conditions.

Lemma 3 *Suppose* $\frac{|f'(x)|}{f(x)}$ *is uniformly bounded and* $\lim_{n\to\infty} \hat{p} = 0$*, where* $\hat{p} = \frac{1}{nh(0)}$ *is the separate sales price in (2). Then when* n *is sufficiently large, the system of (20) and (21) has a solution with* $\delta \in (0, \frac{1}{nh(0)})$*.*

Proof. Recall that (21) is

$$\frac{1/n + \delta(\alpha + \gamma)}{\alpha + \beta + 2\gamma} (\beta - \alpha) = \Omega_1(\delta) - \delta\alpha \ .$$

Denote the left-hand side by $\chi_L(\delta)$ and the right-hand side by $\chi_R(\delta)$. Notice that the assumption that $\frac{|f'(x)|}{f(x)}$ is uniformly bounded implies that $\frac{|h'(0)|}{h(0)}$ is uniformly bounded for any n.[17]

We first show that $\chi_L(0) < \chi_R(0)$. At $\delta = 0$, it is easy to verify that $\alpha = \frac{1}{n} h(0)$, $\beta = \left(1 - \frac{1}{n}\right) h(0)$, $\gamma = 0$ and $\Omega_1(0) = \frac{1}{n}\left(1 - \frac{1}{n}\right)$. Then

$$\chi_L(0) = \frac{1}{n}(1 - \frac{2}{n}) < \chi_R(0) = \frac{1}{n}(1 - \frac{1}{n}) \ .$$

[17] Suppose $\frac{|f'(x)|}{f(x)} < M$ for a constant $M < \infty$. Then $-Mf(x) < f'(x) < Mf(x)$, and so $-M \int f(x) dF(x)^{n-1} < \int f'(x) dF(x)^{n-1} < M \int f(x) dF(x)^{n-1}$ for any n. That is, $-Mh(0) < h'(0) < Mh(0)$ for any n, and so $\frac{|h'(0)|}{h(0)}$ is uniformly bounded.

Next, we show that $\chi_L(\delta) > \chi_R(\delta)$ at $\delta = \frac{1}{nh(0)}$ when n is sufficiently large. The condition $\lim_{n\to\infty} \hat{p} = 0$ implies that $\delta = \frac{1}{nh(0)} \approx 0$ when n is large. Replacing δ in (24) by $\frac{1}{nh(0)}$, we have

$$\alpha \approx \frac{h(0)}{n} - \left(\frac{h'(0)}{n} + \frac{h(0)^2}{n-1}\right)\frac{1}{nh(0)} = \frac{h(0)}{n} - \frac{h'(0)}{n^2 h(0)} - \frac{h(0)}{n(n-1)} \ .$$

Similarly,

$$\beta \approx \left(1 - \frac{1}{n}\right) h(0) + \left(\frac{h'(0)}{n} - h(0)^2\right)\frac{1}{nh(0)} = \left(1 - \frac{2}{n}\right) h(0) + \frac{h'(0)}{n^2 h(0)} \ ,$$

$$\gamma \approx \frac{nh(0)^2}{n-1}\frac{1}{nh(0)} = \frac{h(0)}{n-1} \ ,$$

$$\Omega_1(\delta) \approx \frac{1}{n}\left(1 - \frac{1}{n}\right) - \frac{2h(0)}{n}\frac{1}{nh(0)} = \frac{1}{n} - \frac{3}{n^2} \ .$$

Notice that in each expression we just replaced δ by $\frac{1}{nh(0)}$ and no further approximations have been made.

Notice that $\chi_L(\delta) > \chi_R(\delta)$ if and only if

$$\left[\frac{1}{n} + \delta(\alpha + \gamma)\right](\beta - \alpha) > \left[\Omega_1(\delta) - \delta\alpha\right](\alpha + \beta + 2\gamma) \ . \tag{31}$$

Using the above approximations, we have

$$\alpha + \gamma \approx \frac{2h(0)}{n} - \frac{h'(0)}{n^2 h(0)} \quad \text{and} \quad \beta - \alpha \approx \left(1 - \frac{3}{n}\right) h(0) + \frac{2h'(0)}{n^2 h(0)} + \frac{h(0)}{n(n-1)} \ .$$

Then the left-hand side of (31) equals

$$\begin{aligned} & \left[\frac{1}{n} + \frac{1}{nh(0)}\left(\frac{2h(0)}{n} - \frac{h'(0)}{n^2 h(0)}\right)\right] \times \left[\left(1 - \frac{3}{n}\right) h(0) + \frac{2h'(0)}{n^2 h(0)} + \frac{h(0)}{n(n-1)}\right] \\ = & \left[\frac{1}{n} + \frac{2}{n^2} - \frac{1}{n^3}\frac{h'(0)}{h(0)^2}\right] \times \left[h(0) - \frac{3}{n}h(0) + \frac{2h'(0)}{n^2 h(0)} + \frac{h(0)}{n(n-1)}\right] \\ \approx & \left(\frac{1}{n} - \frac{1}{n^2}\right) h(0) \ . \end{aligned}$$

(The final step is from discarding all higher order terms. This is valid given $\lim_{n\to\infty} \frac{1}{nh(0)} = 0$ and $\frac{|h'(0)|}{h(0)}$ is uniformly bounded for any n.)

Using the approximations, we also have

$$\Omega_1(\delta) - \delta\alpha \approx \frac{1}{n} - \frac{4}{n^2} + \frac{1}{n^2(n-1)} + \frac{h'(0)}{n^3 h(0)^2} \ ,$$

and

$$\alpha + \beta + 2\gamma \approx h(0) + \gamma \approx h(0) + \frac{h(0)}{n-1} = \frac{n}{n-1} h(0) \ ,$$

where we have used the fact that $\alpha + \beta + \gamma \approx h(0)$ when δ is small. Then the right-hand side of (31) equals

$$\left[\frac{1}{n} - \frac{4}{n^2} + \frac{1}{n^2(n-1)} + \frac{h'(0)}{n^3 h(0)^2}\right] \times \frac{n}{n-1} h(0) \approx \left(\frac{1}{n} - \frac{3}{n(n-1)}\right) h(0) \ .$$

(The final step is again from discarding all higher order terms.) Then it is ready to see that $\chi_L(\delta) > \chi_R(\delta)$ at $\delta = \frac{1}{nh(0)}$ when n is sufficiently large. This completes the proof of the lemma. ■

Given the system has a solution with a small δ when n is large, we can approximate each side of (21) around $\delta \approx 0$ by using (24) and discarding all higher order terms. Then one can solve

$$p \approx \frac{1}{nh(0)} \frac{1 + \delta h(0)}{1 + \frac{n}{n-1}\delta h(0)}; \quad \delta \approx \frac{1}{\frac{2h'(0)}{h(0)} + \frac{2n^2-3n+2}{n^2-n} nh(0)} \ .$$

It is clear that $p < \hat{p} = \frac{1}{nh(0)}$.

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