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Break and sustain bifurcations of S_N-invariant equidistant economy

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This paper aims at the elucidation of the bifurcation mechanism of an equidistant economy in Economic Geography. An attention is paid to the existence of invariant solutions that retain their spatial patterns when the bifurcation parameter changes. Theoretical results on symmetry-breaking bifurcation of the symmetric group S_N, which describes the symmetry of this economy, is combined with the mechanism of sustain bifurcation of invariant patterns that is inherent to the economy. The stability of bifurcating branches is investigated theoretically to demonstrate that most of them are asymptotically unstable. Among a plethora of theoretically possible spatial patterns, those which actually become stable for spatial economic models are investigated numerically. The solution curves of the economy are shown to display a complicated mesh-like structure, which looks like threads of warp and weft.

Keywords: Bifurcation; equidistant economy; group-theoretic bifurcation theory; invariant pattern; replicator dynamics; spatial economic model; stability.

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1. Introduction

An equidistant economy is an important spatial platform in Economic Geography. Bifurcation mechanism of this economy has come to be investigated and, in turn, to observe a complicated mesh-like network of equilibrium curves [Gaspar *et al.*, 2019b]. This paper aims to elucidate the mechanism of this network employing dual methodologies: (1) invariant patterns for the replicator dynamics [Ikeda *et al.*, 2018b, 2019a,b] and (2) a group-theoretic analysis for the symmetric group S_N [Golubitsky & Stewart, 2002; Elmhirst, 2004].

We would refer to a *break point* and a *sustain point*, which are well known to exist in a two-place economy [Fujita *et al.*, 1999]. A break point arises from the underlying symmetry of the system. Bifurcation analysis of a symmetric field is a well matured topic [Golubitsky *et al.*, 1988; Ikeda & Murota, 2019] and the bifurcation mechanism of an equidistant economy with an arbitrary number of places is readily available through the bifurcation analysis of the symmetric group S_N [Golubitsky & Stewart, 2002; Elmhirst, 2004]; all bifurcating patterns from the uniform state were obtained and were proved to be asymptotically unstable. We would like to extend this analysis to secondary and further bifurcations to arrive at a more complete view of the progress of agglomerations via bifurcations.

It has come to be acknowledged that there exist invariant patterns that retain their spatial distribution when the bifurcation parameter (transportation cost) changes. Invariant patterns of a racetrack economy and of a lattice economy with the replicator dynamics, which is the most popular in economics, were found and were employed to elucidate their bifurcation mechanisms [Ikeda *et al.*, 2018b, 2019a,b]. In this paper, we march on to find invariant patterns of an equidistant economy with the replicator dynamics, and, in turn, to investigate the mechanism of sustain bifurcation for these invariant patterns. Most of them are found to be asymptotically unstable.

In the numerical bifurcation analysis of a symmetric system, it is customary to obtain the uniform solution and direct, secondary, tertiary bifurcating solutions successively, and, in turn, to assemble a family of solution curves. In this paper, in view of the existence of invariant solutions, we employ the following innovative bifurcation analysis procedure to find stable equilibria [Ikeda *et al.*, 2019a]: (1) Obtain all invariant patterns and investigate their stability and, in turn, (2) find bifurcating curves connecting invariant solutions and investigate their stability.

We consider an equidistant economy with an arbitrary number of places in a theoretical analysis and up to eight places in a numerical analysis of spatial economic models, called the FO model and the Pf model [Forslid & Ottaviano, 2003; Pflüger, 2004]. The solution curves are shown to display complicated mesh-like structures, which look like threads comprising weft of invariant patterns and warp of non-invariant ones. Almost all bifurcating equilibria are proved and found to be asymptotically unstable.

The present study could contribute to Economic Geography, in which economic agglomeration is studied mostly for a simple spatial platform of a two-location economy [Fujita *et al.*, 1999; Baldwin *et al.*, 2003] and three equidistant places [Fujita *et al.*, 1999; Castro *et al.*, 2012; Commendatore *et al.*, 2015; Gaspar *et al.*, 2018, 2019a]. Krugman's Core-Periphery model [Krugman, 1991] was extended to show that if agglomeration (dispersion) is stable (unstable) with two regions it is also stable (unstable) with three regions [Castro *et al.*, 2012]. The stability analysis of well-known invariant patterns (core-periphery and uniform state) of the FO model was conducted [Gaspar *et al.*, 2018]. The direct bifurcation leading from a uniform state of the Pf model has been shown to lead to the particular two-level hierarchy state, comprising one large place and $N-1$ small places [Gaspar *et al.*, 2019a]. Other works in Economic Geography have considered an arbitrary number of equidistant regions under different settings, but they do not provide a complete account of the network of possible equilibria [Puga, 1999; Tabuchi *et al.*, 2005; Oyama, 2009; Zeng & Uchikawa, 2014].

This paper is organized as follows: A spatial economic model with the replicator dynamics is presented in Section 2. Invariant patterns are obtained in Section 3. Bifurcation mechanism of an equidistant economy is advanced in Section 4. Stability of bifurcating branches is studied in Section 5. Numerical bifurcation analyses of spatial economic models are carried out in Section 6.

2. Spatial economic model with the replicator dynamics

A spatial economic model with the replicator dynamics is presented and its steady-state solutions are classified. While the theoretical framework of this paper is efficacious for analyzing general spatial economic models, detailed aspects of payoff functions are defined in accordance with the FO model [Forslid & Ottaviano, 2003] and the Pf model [Pflüger, 2004] among many alternatives. These two models serve as concrete examples of spatial economic models to be used in the investigation of the stability of agglomeration patterns (Section 6).

2.1. *Spatial economic model*

Assume that there are N (≥ 3) regions and mobile agents (workers or firms, entrepreneurs) that can choose where to locate from N regions. Let $\boldsymbol{h} = \{h_i \mid i = 1, \ldots, N\}$ denote the spatial distribution of agents. It is assumed that $\sum_{i=1}^{N} h_i = 1$, thereby the state space is the probability simplex. The payoff (utility or profit) for locating in region i is given by a function v_i of the spatial distribution of mobile agents $\boldsymbol{h}$ and a parameter $\phi \in (0, 1)$ that represents the freeness of transport between the regions. A continuous C^1 function $\boldsymbol{v} : \mathbb{R}^N \times (0, 1) \to \mathbb{R}_+^N$ thus defines a general spatial economic model with N regions. An *equilibrium* is defined as a spatial distribution of agents $\boldsymbol{h}$ that satisfies the following conditions:

$$\begin{cases} v^* - v_i(\boldsymbol{h}, \phi) = 0 & \text{if} \quad h_i > 0, \\ v^* - v_i(\boldsymbol{h}, \phi) \geq 0 & \text{if} \quad h_i = 0, \end{cases} \tag{1}$$

such that $\sum_{i=1}^{N} h_i = 1$, where v^* denotes the equilibrium payoff level.

The economic backbones of the payoff function $\boldsymbol{v}$ for the FO and Pf models are briefly summarized in the following. There are two factors of production and two sectors. The two factors are skilled and unskilled labor and the workers supply one unit of each type of labor inelastically. The total endowment of skilled and unskilled workers is H and L, respectively, with H being normalized to unity ($H = 1$). The skilled worker is mobile across places and h_i denotes the number of them located in the place i. The unskilled worker is immobile and equally distributed across all places (i.e., the number of unskilled workers in each place is $\ell = L/N$). The two sectors consist of agriculture (abbreviated by A) and manufacturing (abbreviated by M). The A-sector output is homogeneous and each unit is produced using a unit of unskilled labor under perfect competition. The M-sector output is a horizontally differentiated product that is produced using both skilled and unskilled labor under increasing returns to scale and Dixit-Stiglitz monopolistic competition. There are three major parameters for the models: σ expresses the constant elasticity of substitution between any two manufactural goods, μ denotes the constant expenditure share on industrial varieties, and L that represents the endowment of immobile workers.

The goods of both sectors are transported. The transportation of A-sector goods is frictionless (cost free), while the transportation of M-sector goods demands iceberg costs. That is, for each unit of M-sector goods transported from place i to $j (\neq i)$, only a fraction $1/\tau_{ij} < 1$ arrives; intra-region transport is frictionless, i.e., $\tau_{ii} = 1$ for all i. The main assumption in the present paper is that the transport cost between all pairs of regions are the same, i.e., *equidistant economy.*

Assumption 1. $\quad \tau_{ij} = \tau > 1$ for all $i \neq j$.

By this assumption, we can define the freeness of transport parameter $\phi = \tau^{1-\sigma} \in (0, 1)$ so as to characterize the interregional transport cost structure of the FO and Pf models. The trade cost increases or decreases when ϕ approaches to 0 or 1, respectively. We employ ϕ as the bifurcation parameter.

The payoff functions for the FO and Pf models are given by the following:

$$[\text{FO}] \quad v_i(\boldsymbol{h}, \phi) = \Delta_i^{\frac{\mu}{\sigma-1}} w_i, \tag{2}$$

$$[\text{Pf}] \quad v_i(\boldsymbol{h}, \phi) = \ln \Delta_i^{\frac{\mu}{\sigma-1}} + w_i, \tag{3}$$

where $\Delta_i = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} h_j$ and

$$\text{[FO]} \quad \boldsymbol{w} = \frac{\ell\mu}{\sigma}\left(\boldsymbol{I} - \frac{\mu}{\sigma}\boldsymbol{M}\,\mathrm{diag}[\boldsymbol{h}]\right)^{-1}\boldsymbol{M}\boldsymbol{1}_N, \tag{4}$$

$$\text{[Pf]} \quad \boldsymbol{w} = \frac{\mu}{\sigma}\boldsymbol{M}(\boldsymbol{h} + \ell\,\boldsymbol{1}_N), \tag{5}$$

where $\boldsymbol{M} = [\tau_{ji}^{1-\sigma}/\Delta_j]$; $\boldsymbol{I}$ is the N-dimensional identity matrix; $\boldsymbol{1}_N = \underbrace{(1,\ldots,1)}_{N \text{ times}}$ is the N-dimensional all-one vector.

2.2. *Replicator dynamics*

It is customary in economics to replace the problem to obtain *stable* spatial equilibria in (1) by another problem to find a set of stable steady-state solutions of the replicator dynamics [Taylor & Jonker, 1978]:

$$\frac{d\boldsymbol{h}}{dt} = \boldsymbol{F}(\boldsymbol{h}, \phi), \tag{6}$$

where $\boldsymbol{F}(\boldsymbol{h}, \phi) = \{F_i(\boldsymbol{h}, \phi) \mid 1 \le i \le N\}$, and

$$F_i(\boldsymbol{h}, \phi) = \{v_i(\boldsymbol{h}, \phi) - \bar{v}(\boldsymbol{h}, \phi)\}h_i. \tag{7}$$

Here, $\bar{v} = \sum_{i=1}^{N} h_i v_i$ is the average utility. Steady-state solutions (rest points) $(\boldsymbol{h}^*(\phi), \phi)$ of the replicator dynamics (6) are defined as those points which satisfy the static governing equation

$$\boldsymbol{F}(\boldsymbol{h}^*, \phi) = \boldsymbol{0}. \tag{8}$$

A steady-state solution is stable if every eigenvalue of the Jacobian matrix $J(\boldsymbol{h}^*, \phi) = \partial\boldsymbol{F}/\partial\boldsymbol{h}(\boldsymbol{h}^*, \phi)$ has a negative real part and is unstable if at least one eigenvalue has a positive real part. A *stable equilibrium*, which is the main target of this paper, is defined as a stable steady-state solution of (8) with non-negative populations $h_i \ge 0$ $(1 \le i \le N)$; it is known that such a solution satisfies the equilibrium condition (1) of an underlying spatial economic model [Sandholm, 2010].

Steady-state solutions are classified into an *interior solution*, for which all regions have positive population, and a *corner solution*, for which some regions have zero population. A corner solution can be expressed, without loss of generality, by an appropriate permutation of components of $\boldsymbol{h}$, as

$$\boldsymbol{h} = (\boldsymbol{h}_m^+, \boldsymbol{0}_n) \qquad (1 \le m \le N = 1;\ m + n = N) \tag{9}$$

with $\boldsymbol{h}_m^+ = \{h_i > 0,\ \sum_{1=1}^{m} h_i = 1 \mid 1 \le i \le m\} \in \mathbb{R}_+^m$ and the n-dimensional zero vector $\boldsymbol{0}_n = \underbrace{(0,\ldots,0)}_{n \text{ times}}$.

3. Invariant patterns

Invariant patterns of an equidistant economy are presented. Steady-state solutions that satisfy the static governing equation $\boldsymbol{F}(\boldsymbol{h}, \phi) = \boldsymbol{0}$ in (8) form solution curves $(\boldsymbol{h}(\phi), \phi)$ parameterized by ϕ. In general, the spatial pattern $\boldsymbol{h}(\phi)$ varies with ϕ along a solution curve. By contrast, there can be a special solution curve $(\boldsymbol{h}(\phi), \phi) = (\bar{\boldsymbol{h}}, \phi)$ that has a constant spatial pattern $\boldsymbol{h}(\phi) = \bar{\boldsymbol{h}}$ along the curve by virtue of the product form (7) of the replicator dynamics. Such pattern $\bar{\boldsymbol{h}}$ is called herein an *invariant pattern*. The curve of an invariant pattern exists for any $\phi \in (0, 1)$. In contrast, a pattern $\boldsymbol{h}(\phi)$ that varies with ϕ is called a *non-invariant pattern* and might or might not be a solution for a given ϕ.

In connection with invariant patterns, we consider a *core–periphery* pattern

$$\boldsymbol{h}_m^{\mathrm{CP}} = \frac{1}{m}\,(\mathbf{1}_m, \mathbf{0}_n) \qquad (1 \le m \le N-1; \quad m + n = N) \tag{10}$$

with $\mathbf{1}_m = \underbrace{(1, \ldots, 1)}_{m \text{ times}}$. This pattern is a special form of the corner solution (9) with a two-level hierarchy: the identical population $1/m$ is agglomerated to m core places, while other n peripheral places have no populations.

An equidistant economy has a series of invariant patterns, including: the uniform state

$$\boldsymbol{h}^{\mathrm{uniform}} = \frac{1}{N}\mathbf{1}_N$$

and core–periphery patterns in (10) (see Proposition 1 below).

Proposition 1. *The uniform state $\boldsymbol{h}^{\mathrm{uniform}} = \frac{1}{N}\mathbf{1}_N$ and the core–periphery pattern in* (10) *are invariant patterns for an equidistant economy.*

Proof. For the uniform sate, we have $(v_i - \bar{v})h_i = 0$ $(1 \le i \le N)$ since $v_1 = \cdots = v_N = \bar{v}$; accordingly, this state always satisfies the static governing equation $\boldsymbol{F}(\boldsymbol{h}^*, \phi) = \boldsymbol{0}$ in (8). For the core–periphery pattern $\boldsymbol{h}_m^{\mathrm{CP}} = (\frac{1}{m}\mathbf{1}, \mathbf{0}_n)$, we have $(v_i - \bar{v})h_i = 0$ $(m+1 \le i \le N)$ for zero components $\mathbf{0}_n$ of $\boldsymbol{h}_m^{\mathrm{CP}}$. For the components $\frac{1}{m}\mathbf{1}$, we have $v_1 = v_2 = \cdots = v_m$ and

$$\bar{v} = \sum_{i=1}^{m} h_i v_i + \sum_{i=m+1}^{N} h_i v_i = \left(\sum_{i=1}^{m} \frac{1}{m}\right) v_1 + \sum_{i=m+1}^{N} 0 \times v_i = v_1.$$

Then $(v_i - \bar{v})h_i = (v_1 - \bar{v})h_1 = 0$ $(1 \le i \le m)$. Thus the core–periphery pattern is a steady-state solution for any ϕ. ■

4. Bifurcation mechanism

The bifurcation mechanism of sustain points of an equidistant economy is investigated as a novel contribution of this paper, whereas that of break (symmetry-breaking) points of the uniform state [Golubitsky & Stewart, 2002; Elmhirst, 2004] is also included to make the discussion self-contained.

4.1. *Break bifurcation from an equidistant state*

The mechanism of the direct bifurcation from the uniform state $\boldsymbol{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$ of an N-equidistant economy was elucidated by the bifurcation analysis of a symmetric group S_N labeling the symmetry of this economy [Golubitsky & Stewart, 2002; Elmhirst, 2004]. This analysis is briefly presented consistently with our formulation. We consider a steady-state bifurcation, and does not refer to a Hopf bifurcation.

The uniform state has the Jacobian matrix of the form:

$$J = A_N(a,b) = \begin{pmatrix} a & b & \cdots & b \\ b & a & b & \vdots \\ \vdots & b & \ddots & b \\ b & \cdots & b & a \end{pmatrix} \tag{11}$$

with

$$a = \frac{\partial}{\partial h_i}(v_i - \bar{v}) \quad (1 \le i \le N); \qquad b = \frac{\partial}{\partial h_j}(v_i - \bar{v}) \quad (1 \le i, j \le N;\ i \ne j).$$

That is, all the diagonal entries are a and all the off-diagonals are b. When $a = b$, this state encounters the direct bifurcation point with $(N-1)$-times repeated zero eigenvalues of the Jacobian matrix, at which a number of two-level hierarchy states (Proposition 2):

$$\boldsymbol{h}_m = (\underbrace{u,\ldots,u}_{m\text{ times}},\underbrace{v,\ldots,v}_{n\text{ times}}) = (u\mathbf{1}_m, v\mathbf{1}_n) \qquad (1 \le m \le N-1;\quad m+n=N;\quad um+vn=1;\ u,v>0) \tag{12}$$

branch in the incremental directions:

$$\delta\boldsymbol{h}_m = w\left(\mathbf{1}_m, -\frac{m}{n}\mathbf{1}_n\right) \qquad (1 \le m \le N-1;\quad m+n=N;\ w \in \mathbb{R}). \tag{13}$$

Thus N places split into m places with an identical population size and n places with another size. A branch is called *symmetric* if $\delta\boldsymbol{h}$ and $-\delta\boldsymbol{h}$ denote the same state up to a permutation of place numbers, and is called *asymmetric* if they do not.

Proposition 2. *The two-level hierarchy states in* (12) *branch in the directions in* (13) *at a bifurcation point of the uniform (equidistant) state. The branch is symmetric if $n = m$ (N even) and is asymmetric otherwise.*

Proof. See the Appendix and a reference [Elmhirst, 2004]. ■

4.2. *Bifurcation from a two-level hierarchical state*

The two-level hierarchy state $\boldsymbol{h}_m = (u\mathbf{1}_m, v\mathbf{1}_n)$ in (12) can reach a break point or a sustain point. At a sustain point of this state, where either $u\mathbf{1}_m$ or $v\mathbf{1}_n$ vanishes, this state exits to a corner solution expressing the core–periphery pattern in (10):

$$\boldsymbol{h}_m^{\text{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n),\ \frac{1}{n}(\mathbf{0}_m, \mathbf{1}_n) \qquad (1 \le m \le N-1;\ m+n=N).$$

In the discussion of a break point, we refer to the Jacobian matrix of the two-level hierarchy state, which takes the form:

$$J = \begin{pmatrix} A_m(a,b) & eE_{mn} \\ fE_{nm} & A_n(c,d) \end{pmatrix},$$

where $A_m(a,b)$ and $A_n(c,d)$ are defined similarly to $A_N(a,b)$ in (11), $E_{mn} = \mathbf{1}_m{}^\top \mathbf{1}_n$ is an $m \times n$ matrix with all entries being equal to 1, and a, b, ..., f are constants. A secondary bifurcation takes place at a break point with $a = b$ or $c = d$.

We hereafter focus on the case of $c = d$, at which a series of three-level hierarchy states (Proposition 3):

$$\boldsymbol{h}_p = (u\mathbf{1}_m, v\mathbf{1}_p, w\mathbf{1}_{n-p}) \qquad (1 \leq p \leq n-1;\ m+n=N;\ um+vp+w(n-p)=1;\ u,v,w>0) \tag{14}$$

branch in the directions:

$$\delta\boldsymbol{h}_p = w\,(\mathbf{0}_m, (n-p)\mathbf{1}_p, -p\mathbf{1}_{n-p}) \qquad (1 \leq p \leq n-1;\ m+n=N;\ w \in \mathbb{R}). \tag{15}$$

Thus n identical places split into p places with an identical population size and $n-p$ places with another size $(1 \leq p \leq n-1)$. Note that another case $a = b$ can be treated similarly.

Proposition 3. *At a break point of the two-level hierarchy state in* (12)*, the three-level hierarchy states in* (14) *branch in the directions in* (15)*. The branch is symmetric if $p = n/2$ (n even) and asymmetric otherwise.*

Proof. See the Appendix. ∎

The three-level hierarchy state in (14) can exit to the corner solution at a sustain point or undergo further bifurcations to arrive at an aggregated inner state with an s-level hierarchy $(2 \leq s \leq N)$.

$$\boldsymbol{h}_{m_1,\ldots,m_s} = (u_1\mathbf{1}_{m_1}, \ldots, u_s\mathbf{1}_{m_s}) \tag{16}$$

with $\sum_{i=1}^{s} m_i = N$ and $\sum_{i=1}^{s} u_i m_i = 1$. Bifurcations can proceed until reaching a completely aggregated inner state: $\boldsymbol{h} = \{h_i \mid h_1 > h_2 > \cdots > h_N > 0\}$.

4.3. *Bifurcation from a core-periphery pattern*

In the discussion of the bifurcation from the core–periphery pattern $\boldsymbol{h}^{\mathrm{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n)$ in (10), we refer to its Jacobian matrix

$$J = \begin{pmatrix} A_m(a,b) & eE_{mn} \\ O & cI_n \end{pmatrix} \qquad (1 \leq m \leq N-1) \tag{17}$$

with I_n being an $n \times n$ identity matrix and

$$c = v_i - \bar{v} \qquad (m+1 \leq i \leq N).$$

The critical point of this pattern is either a break point for $a = b$ with singular $A_m(a,b)$ or a sustain point for $c = 0$ with singular cI_n in (17).

Prior to the main discussion, we refer to the *half branch* that is inherent in the replicator dynamics. Recall that the branches for break points presented above do exist in both directions of $\delta\boldsymbol{h}_p$ and $-\delta\boldsymbol{h}_p$. By contrast, a branch exists only in one direction for a sustain point since a negative population is not allowed (Propositions 4 and 6 below); such a branch is called a *half branch*.

We start with the simplest core–periphery pattern: the full agglomeration $\boldsymbol{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$, which is an invariant pattern (Proposition 1). This full agglomeration only has a sustain point, at which a series of three-level hierarchy states:

$$\boldsymbol{h}_p = (1-pu, u\mathbf{1}_p, \mathbf{0}_{N-p-1}) \qquad (1 \leq p \leq N-1;\ 0 < u < 1/p) \tag{18}$$

branch in the directions:

$$\delta\boldsymbol{h}_p = w\left(-1, \frac{1}{p}\mathbf{1}_p, \mathbf{0}_{N-p-1}\right) \qquad (1 \leq p \leq N-1;\ w > 0). \tag{19}$$

Proposition 4. *The full agglomeration $\boldsymbol{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$ dose not have a limit point or a break point but has a sustain point with the half branches in* (18).

Proof. Since $a = \frac{\partial(v_1 - \bar{v})}{\partial h_1} = -v_1(< 0)$ is always negative and does not become singular, a limit point or a break point does not exist. The proof for the half branches of the sustain point is similar to that for Proposition 3. ■

Other core–periphery patterns ($m \geq 2$) have both break and sustain points, which lead to an emergence of three-level hierarchy states, as expounded in the following propositions, the proofs of which are similar to that for Proposition 3.

Proposition 5. *At a break point of the core–periphery pattern in* (10)*, branches with a three-level hierarchy:*

$$\boldsymbol{h}_p = (u\mathbf{1}_p, v\mathbf{1}_{m-p}, \mathbf{0}_n) \qquad (1 \leq p \leq m-1;\ up + v(m-p) = 1;\ u, v > 0) \tag{20}$$

emerge. The branch is symmetric if $p = m/2$ *(*m *even) and asymmetric otherwise.*

Proposition 6. *At a sustain point of the core–periphery pattern in* (10)*, there emerge half branches with a three-level hierarchy:*

$$\boldsymbol{h}_p = (u\mathbf{1}_m, v\mathbf{1}_p, \mathbf{0}_{n-p}) \qquad (1 \leq p \leq n-1;\ um + vp = 1;\ u, v > 0). \tag{21}$$

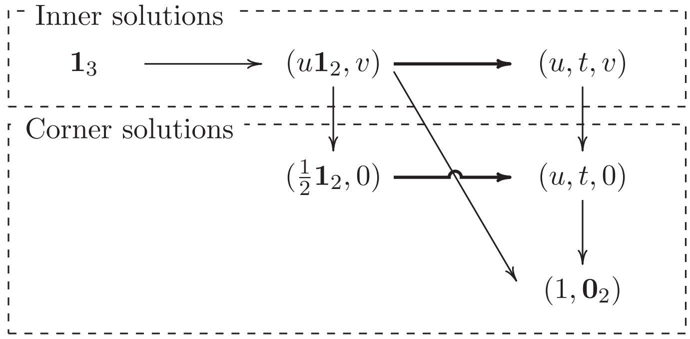
The three-level hierarchy corner states in (20) and (21) can encounter break and sustain points successively to arrive at an aggregated state with an s-level hierarchy ($2 \leq s \leq N$).

$$\boldsymbol{h}_{m_1,\ldots,m_s} = (u_1\mathbf{1}_{m_1}, \ldots, u_{s-1}\mathbf{1}_{m_{s-1}}, \mathbf{0}_{m_s}) \tag{22}$$

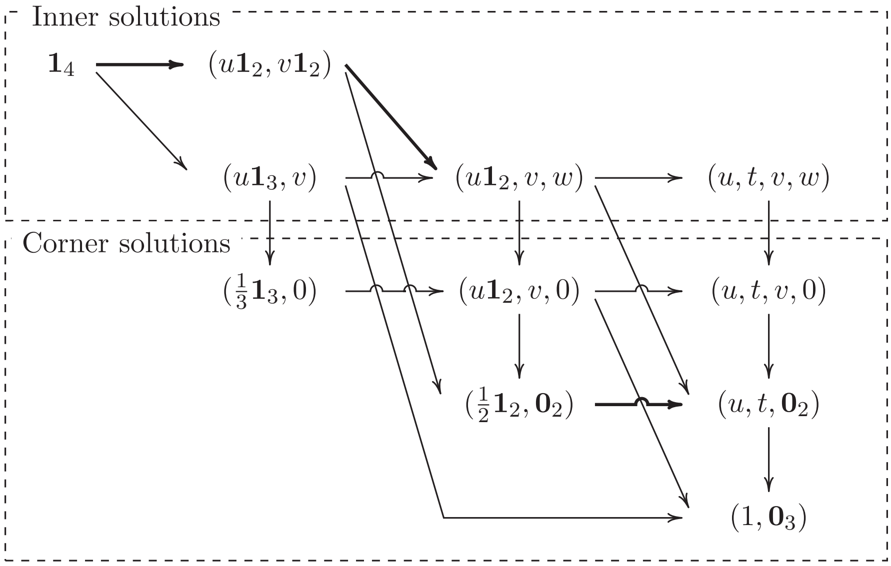
with $\sum_{i=1}^{s} m_i = N$ and $\sum_{i=1}^{s-1} u_i m_i = 1$.

4.4. *Simple examples*

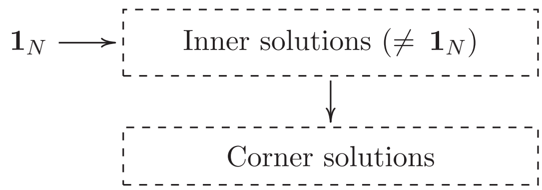
As simple examples of the bifurcation mechanism presented above, we advance the hierarchies of spatial patterns of an equidistant economy shown in Fig. 1a,b for $N = 3$ and $N = 4$, respectively. A symmetric branch is expressed by a thick arrow and an asymmetric one by a thin one. For each number of places, the subhierarchy for inner solutions at the top is connected to that for corner solutions at the bottom. There is a recurrent property: the hierarchy of $N = 3$ becomes the subhierarchy of corner solutions for $N = 4$; for an arbitrary number N of places, the subhierarchy of corner solutions is given by the hierarchy of $N - 1$ places (see Fig. 1c). By virtue of this recurrent property, the hierarchy grows rapidly as N increase, and, in turn, the bifurcation mechanism becomes progressively complicated.



(a) $N = 3$



(b) $N = 4$



(c) An arbitrary N

Fig. 1. Bifurcation mechanism of an equidistant economy expressed by hierarchies of spatial patterns. A symmetric branch is expressed by a thick arrow and an asymmetric one by a thin one.

5. Asymptotic stability of branches

Asymptotic stability of branches is investigated for spatial patterns of interest, such as the uniform state, the full agglomeration, and core–periphery patterns. The stability of branches from the uniform state is well known as explained below [Elmhirst, 2004].

Proposition 7. *Under the assumption that the uniform state $\boldsymbol{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$ is stable until reaching the bifurcation point, all branches of this state are asymptotically unstable.*

As a novel contribution of this paper, we hereafter investigate the stability of half branches from a sustain point of the core–periphery pattern $\boldsymbol{h}_m^{\text{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n)$ $(m+n=N)$ in (10). We recall its Jacobian matrix in (17):

$$J = \begin{pmatrix} A_m(a,b) & eE_{mn} \\ O & cI_n \end{pmatrix} \qquad (1 \le m \le N-1) \tag{23}$$

and consider its sustain point at $\phi = \phi_{\text{c}}$ with a singular cI_n $(c=0)$ and a non-singular $A_m(a,b)$ $(a \ne b)$. Define incremental variables $(\boldsymbol{y}, \boldsymbol{x}, \psi)$ from this point by

$$\boldsymbol{h} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n) + (\boldsymbol{y}, \boldsymbol{x}), \qquad \phi = \phi_{\text{c}} + \psi$$

with $\boldsymbol{y} = (y_1, \ldots, y_m)$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$. We obtain the bifurcation equation

$$\boldsymbol{G} = \{G_i(\boldsymbol{x}, \psi) \mid 1 \le i \le n\} = \mathbf{0} \tag{24}$$

by expressing the static governing equation $\boldsymbol{F}(\boldsymbol{h}, \phi) = \mathbf{0}$ in (8) in terms of these incremental variables $(\boldsymbol{y}, \boldsymbol{x}, \psi)$ and eliminating $\boldsymbol{y}$ from the last n components of $\boldsymbol{F} = \mathbf{0}$ with the use of the first m components as $A_m(a,b)$ is non-singular.

At a sustain point of the core–periphery pattern $\boldsymbol{h}_m^{\text{CP}}$, there emerge a number of half branches with a three-level hierarchy ((21) in Proposition 6):

$$\boldsymbol{h}_p = (u\mathbf{1}_m, v\mathbf{1}_p, \mathbf{0}_{n-p}) \qquad (1 \le p \le n-1;\ um + vp = 1;\ u, v > 0), \tag{25}$$

which are associated with

$$\boldsymbol{x} = w\,(\mathbf{1}_p, \mathbf{0}_{n-p}) \qquad (w > 0). \tag{26}$$

By the analysis of the bifurcation equation (the Appendix), we see that there are asymptotic bifurcating ψ versus w curves:

$$\psi \approx -\frac{\beta + (p-1)\gamma}{\alpha}\, w \qquad (1 \le p \le N-1) \tag{27}$$

with expansion coefficients α, β, and γ of the bifurcation equation. The following lemma on the $n \times n$ Jacobian matrix $\hat{J}(\boldsymbol{x}, \psi) = \partial \boldsymbol{G}/\partial \boldsymbol{x}$ of the bifurcation equation plays a pivotal role in the description of the stability of half branches.

Lemma 1. *The eigenvalues of the Jacobian matrix $\hat{J}(\boldsymbol{x}, \psi) = \partial \boldsymbol{G}/\partial \boldsymbol{x}$ are real and are given asymptotically as*

$$\begin{cases} \lambda_1 \approx \{\beta + (p-1)\gamma\}w & (\text{repeated once}), \\ \lambda_2 \approx -(\gamma - \beta)w & (\text{repeated } p-1 \text{ times}), \\ \lambda_3 \approx (\gamma - \beta)w & (\text{repeated } n-p \text{ times}). \end{cases} \tag{28}$$

Proof. See the Appendix. ■

The associated half branches are stable if all eigenvalues in (28) are negative. Note that $p=1$ and $p=n$ are exceptional cases where λ_2 and λ_3 are absent, respectively. It is these exceptional cases where stable half branches exist as expounded below, unlike for a break point for the uniform state, for which all half branches are unstable (Proposition 7). In the description of stability, we employ the following assumption, which is in line with the numerical results of sustain bifurcation of the full agglomeration $\boldsymbol{h}^{\text{FA}} = (1, \mathbf{0}_{N-1})$ of spatial economic models (Section 6).

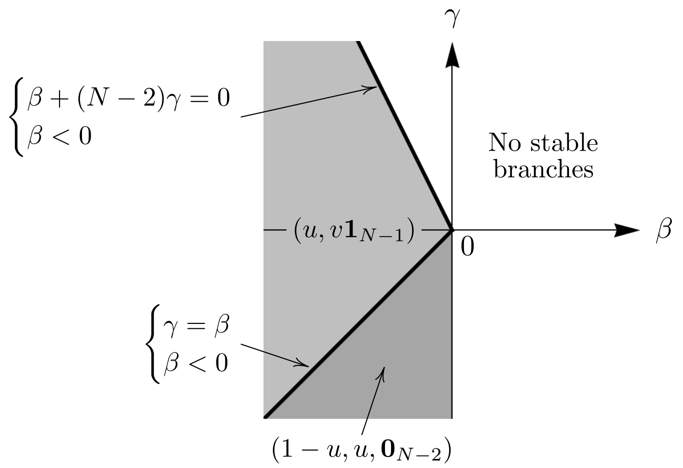


Fig. 2. Classification of stability of half branches of the full agglomeration $\boldsymbol{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$ in the parameter space (β, γ).

Assumption 2. The pre-bifurcation core–periphery pattern is stable for $\psi > 0$ ($\phi > \phi_{\mathrm{c}}$).

We first deal with the stability of half branches of the full agglomeration state $\boldsymbol{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$. For this state with $n = N-1$ in (24)–(28), there are a series of half branches in the directions $\delta\boldsymbol{h}_p = w\,(-1, \mathbf{1}_p, \mathbf{0}_{N-p-1})$ ($1 \le p \le N-1;\ w > 0$). Among this plethora of half branches, either zero or one of them is stable. Under Assumption 2, the stability of half branches of $\boldsymbol{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$ is classified into three distinct cases in the parameter space (β, γ) as shown in Fig. 2 (Proposition 8).

Proposition 8. *The stability of half branches of $\boldsymbol{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$ is classified into three distinct cases:*

i) A two-place $(1-u, u, \mathbf{0}_{N-2})$ is the only stable half branch and resides in $\psi < 0$ for $\gamma < \beta < 0$.

ii) A star-like pattern $\boldsymbol{h}^{\mathrm{star}} = (u, v\mathbf{1}_{N-1})$ is the only stable half branch and resides in $\psi < 0$ for $\beta < \min(\gamma, -(N-2)\gamma)$.

iii) All half branches are unstable for $\beta > 0$ or $-(N-2)\gamma < \beta < 0$.

Proof. See the Appendix. ■

We next deal with the stability of half branches of the core–periphery pattern $\boldsymbol{h}^{\mathrm{CP}}_m = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n)$ ($1 \le m \le N-1$).

Proposition 9. *The stability of half branches of $\boldsymbol{h}^{\mathrm{CP}}_m = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n)$ ($1 \le m \le N-1$) is classified into three distinct cases:*

i) A three-level hierarchy state $(u\mathbf{1}_m, v, \mathbf{0}_{n-1})$ is the only stable half branch and resides in $\psi < 0$ for $\gamma < \beta < 0$.

ii) A two-level hierarchy state $(u\mathbf{1}_m, v\mathbf{1}_n)$ is the only stable half branch and resides in $\psi < 0$ for $\beta < \min(\gamma, -(n-1)\gamma)$.

iii) All half branches are unstable for $\beta > 0$ or $-(n-1)\gamma < \beta < 0$.

Proof. The proof for this case is similar to that of Proposition 8. ■

6. Numerical bifurcation analysis

This section provides numerical bifurcation analyses of $N = 3$, 4, and 8 equidistant places for spatial economic models: the FO and Pf models (Section 2). The values of the parameters in (2)–(5) are set as $(\sigma, \mu, \ell) = (6.0, 0.4, 1.0)$ for the FO model and $(\sigma, \mu, \ell) = (4.0, 0.6, 2.0)$ for the Pf model.

We employ the following innovative bifurcation analysis procedure to find stable equilibria [Ikeda *et al.*, 2019a]: (1) Obtain all invariant patterns and investigate their stability and, in turn, (2) find bifurcating equilibrium curves connecting invariant solutions and investigate their stability with reference to theoretical results in Sections 3–5.

Figure 3 reports the bifurcation diagrams for the FO and Pf models with $N = 3$, 4, 8. In each figure, the horizontal axis is the freeness of transport ϕ; the vertical axis is taken as $h_{\max}(\boldsymbol{h}) = \max_i\{h_i\}$. These are a series of horizontal lines ($h_{\max}$ = constant) expressing solution curves for invariant patterns without dependence on ϕ (Proposition 1):

$$h_{\max} = \begin{cases} \dfrac{1}{N}: & \text{uniform state } \boldsymbol{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N, \\ \dfrac{1}{m}: & \text{core-periphery pattern } \boldsymbol{h}_m^{\text{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n) \quad (2 \le m \le N-1), \\ 1: & \text{full agglomeration } \boldsymbol{h}^{\text{FA}} = (1, \mathbf{0}_{N-1}). \end{cases}$$

The solid (broken) curves corresponds to stable (unstable) steady-state solutions of the governing equation (8). The white circles ($\circ$) in the figures indicate break points, whereas the black disks ($\bullet$) sustain points. The double circle ($\circledcirc$) in Figure 3b is a limit point of ϕ.

As for the direct bifurcation from the uniform state $\boldsymbol{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$, at the break point A ($\circ$) that resides at the right end of the solid horizontal line with $h_{\max} = \frac{1}{N}$ of the stable uniform state, there emerged a number of two-level hierarchy states (Proposition 2):

$$\boldsymbol{h}_m = (u\mathbf{1}_m, v\mathbf{1}_n) \qquad (1 \le m \le N-1;\ m+n = N;\ um + vn = 1).$$

These states connect the break point A ($\circ$) of the uniform state $\boldsymbol{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$ with $N-1$ sustain points ($\bullet$) of core–periphery patterns in (10):

$$\boldsymbol{h}_m^{\text{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n) \qquad (1 \le m \le N-1;\ m+n = N).$$

As for the secondary bifurcation from the two-level hierarchy state, at a break point ($\circ$) with $h_{\max} = \frac{1}{m}$ $(1 \le m \le N-1)$, there emerged a number of branches with three-level hierarchy states ((20) in Proposition 5):

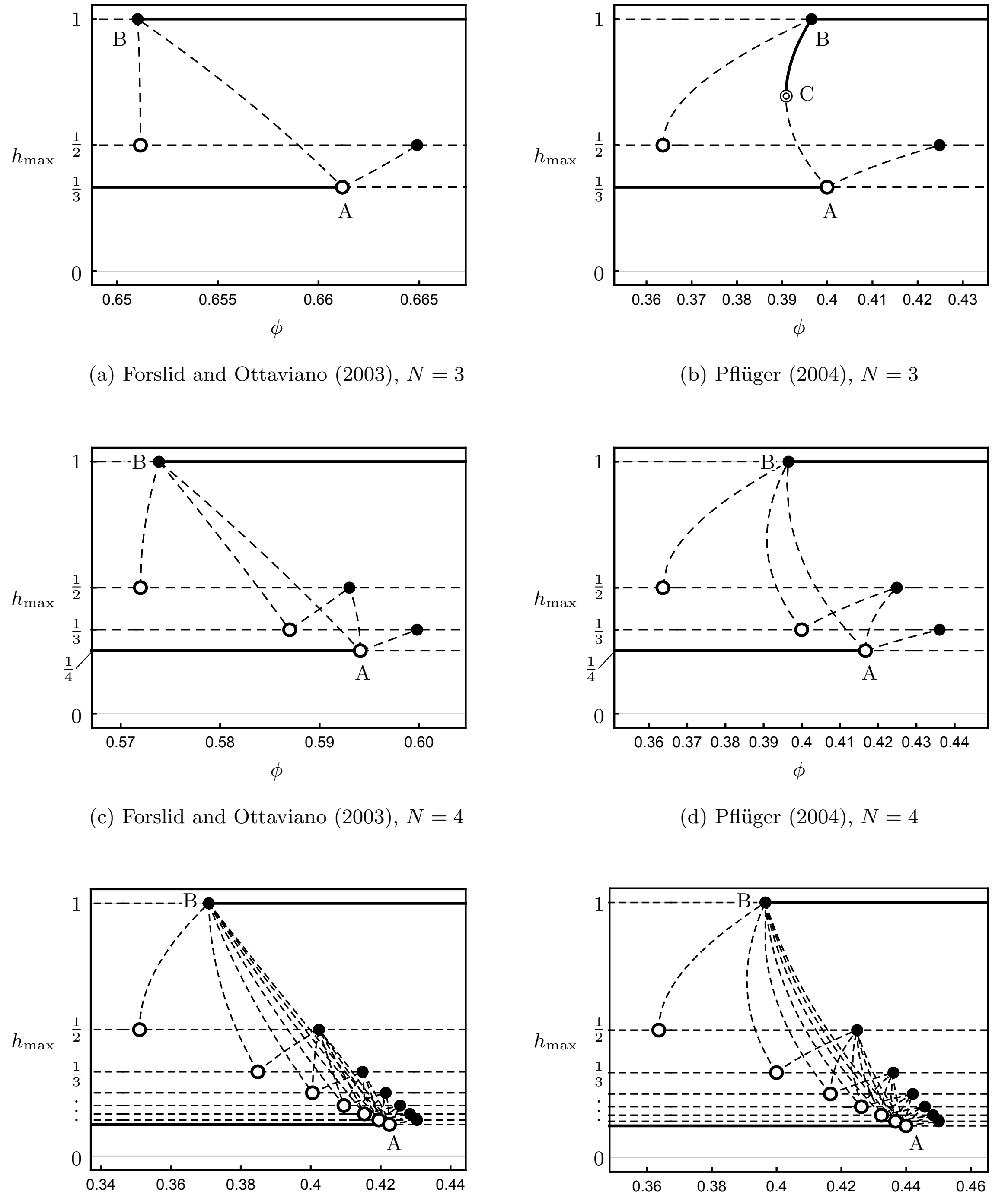
$$\boldsymbol{h}_p = (u\mathbf{1}_p, v\mathbf{1}_{m-p}, \mathbf{0}_n) \qquad (1 \le p \le m-1;\ up + v(m-p) = 1).$$

Each of these states connects a break point ($\circ$) with a sustain point ($\bullet$). Such pairs of break point and sustain point were encountered recurrently until reaching the full agglomeration $\boldsymbol{h}^{\text{FA}} = (1, \mathbf{0}_{N-1})$ at the sustain point B that resides at the left end of the solid horizontal line for the stable full agglomeration.

As we have seen, there are horizontal lines of invariant patterns and non-horizontal curves of non-invariant patterns that look like threads of warp and weft. This warp and weft structure observed herein is much clearer and systematic than that for the hexagonal lattice [Ikeda *et al.*, 2019a] possibly by virtue of a large symmetry of the symmetric group S_N.

As for the stability, it was confirmed that only the uniform state and the full agglomeration have some stable equilibria, whereas other invariant patterns are unstable for any values of the parameter ϕ. All the branches from the uniform state $\boldsymbol{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$ are unstable just after bifurcation (Proposition 7). The curve BC in Figure 3b for the Pf model is a stable non-invariant branch that is predicted in Proposition 8 ii); other cases do not include stable non-invariant curves at all.

Figure 4 depicts the hierarchy of spatial patterns for the present numerical analyses. It was observed that the hierarchy diagram of the $N = 3$ case is a "subset" of the $N = 4$ case. These hierarchies correspond to the subsets of theoretical hierarchies summarized by Fig. 1. As demonstrated by Fig. 3e,f for $N = 8$, the hierarchy grows rapidly in a systematic manner as N increases.



(a) Forslid and Ottaviano (2003), $N = 3$

(b) Pflüger (2004), $N = 3$

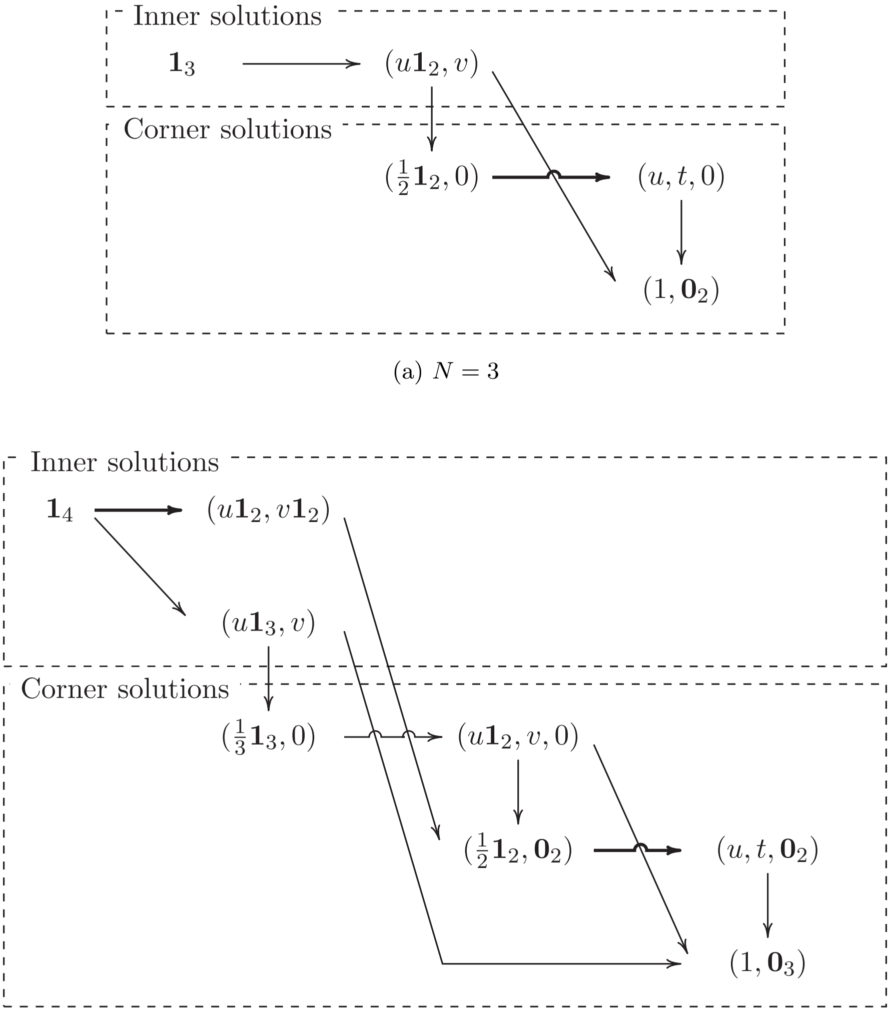
(c) Forslid and Ottaviano (2003), $N = 4$

(d) Pflüger (2004), $N = 4$

(e) Forslid and Ottaviano (2003), $N = 8$

(f) Pflüger (2004), $N = 8$

Fig. 3. Bifurcation diagrams for $N = 3$, 4, and 8 for the models by Forslid and Ottaviano (2003) and Pflüger (2004). Solid line: stable steady state; broken line: unstable steady state; $\circ$: break point; $\bullet$: sustain point; $\circledcirc$: limit point of ϕ.



(a) $N = 3$

(b) $N = 4$

Fig. 4. Hierarchies of spatial patterns in numerical analyses for $N = 3$ and 4. A symmetric branch is expressed by a thick arrow and an asymmetric one by a thin one.

7. Conclusions

A thorough study of bifurcation mechanism and stability of an equidistant economy has been conducted. As a novel contribution of this paper, we have investigated the bifurcation mechanism of sustain points of core–periphery patterns, whereas the bifurcation mechanism of the uniform state [Golubitsky & Stewart, 2002; Elmhirst, 2004] is included to make the discussion self-contained. By a theoretical study, exceptional stable branches have been found, whereas all other branches are unstable. The solution curves of this economy have complicated mesh-like structures, comprising invariant and non-invariant patterns, just like threads of warp and weft. This paper would contribute to the study of a spatial agglomeration in Economic Geography, in which the stability of spatial patterns is investigated in a model and parameter dependent manner, through the introduction of a methodology in group-theoretic bifurcation theory [Golubitsky *et al.*, 1988; Elmhirst, 2004; Ikeda *et al.*, 2018a].

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A. Proof of propositions and a lemma

Proof of Proposition 2: Consider the uniform state $\boldsymbol{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$, which is invariant to the symmetric group S_N, and a state with the symmetry of an axial subgroup $\mathrm{S}_m \times \mathrm{S}_n$ $(m+n=N)$. Denote by

$$\delta\boldsymbol{h} = (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n)$$

an incremental variable vector for this state with $\mathrm{S}_m \times \mathrm{S}_n$ symmetry. By S_m and S_n symmetries, we have

$$\alpha_1 = \cdots = \alpha_m = \alpha, \qquad \beta_1 = \cdots = \beta_n = \beta$$

for some variables α and β. By virtue of the orthogonality between subspaces for S_N and $\mathrm{S}_m \times \mathrm{S}_n$ $(m+n=N)$, we have

$$\boldsymbol{h}^{\text{uniform}}\ \delta\boldsymbol{h}^\top = \frac{1}{N}\mathbf{1}_N\ (\alpha\mathbf{1}_m, \beta\mathbf{1}_n)^\top = \frac{1}{N}(\alpha m + \beta n) = 0.$$

Hence $\beta = -\frac{m}{n}\alpha$ and

$$\delta\boldsymbol{h} = \alpha\left(\mathbf{1}_m, -\frac{m}{n}\mathbf{1}_n\right) \tag{A1}$$

spans a one-dimensional space. Then by the equivariant branching lemma [Golubitsky *et al.*, 1988; Ikeda & Murota, 2019], there exists a bifurcating solution in the direction (A1), i.e., (13). A bifurcating solution takes the form:

$$\boldsymbol{h} = \gamma\mathbf{1}_N + \delta\boldsymbol{h} = \left((\gamma+\alpha)\mathbf{1}_m, \left(\gamma - \alpha\frac{m}{n}\right)\mathbf{1}_n\right) = (u\mathbf{1}_m, v\mathbf{1}_n)$$

with $u = \gamma + \alpha$ and $v = \gamma - \alpha\frac{m}{n}$, thereby showing (12).

The branch for (A1) is symmetric if $m = n$ since $\delta\boldsymbol{h} = \alpha\,(\mathbf{1}_m, -\mathbf{1}_m)$ and $-\delta\boldsymbol{h} = \alpha\,(-\mathbf{1}_m, \mathbf{1}_m)$ $(N = 2m)$ are identical up to the permutation. It is asymmetric otherwise since the number of positive components is different from that of negative components.

Proof of Proposition 3: Consider the uniform state $\boldsymbol{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$ with the symmetry of S_N, a two-level hierarchy state $\boldsymbol{h}^* = (u\mathbf{1}_m, v\mathbf{1}_n)$ with the symmetry of $\mathrm{S}_m \times \mathrm{S}_n$, and a three-level hierarchy state with the symmetry of $\mathrm{S}_m \times \mathrm{S}_{n_1} \times \mathrm{S}_{n_2}$ $(m+n=N;\ n_1+n_2=n)$. Denote by

$$\delta\boldsymbol{h} = (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_{n_1}, \gamma_1, \dots, \gamma_{n_2}) \tag{A2}$$

an incremental variable vector for this state with $\mathrm{S}_m \times \mathrm{S}_{n_1} \times \mathrm{S}_{n_2}$ symmetry. By S_m, S_{n_1}, and S_{n_2} symmetries, we have

$$\alpha_1 = \cdots = \alpha_m = \alpha, \qquad \beta_1 = \cdots = \beta_{n_1} = \beta, \qquad \gamma_1 = \cdots = \gamma_{n_1} = \gamma,$$

for some variables α, β, and γ. By virtue of the orthogonality between subspaces for S_N, $\mathrm{S}_m \times \mathrm{S}_n$, and $\mathrm{S}_m \times \mathrm{S}_{n_1} \times \mathrm{S}_{n_2}$, we have

$$\boldsymbol{h}^{\text{uniform}}\ \delta\boldsymbol{h}^\top = \frac{1}{N}\mathbf{1}_N\ (\alpha\mathbf{1}_m, \beta\mathbf{1}_{n_1}, \gamma\mathbf{1}_{n_2})^\top = \frac{1}{N}(\alpha m + \beta n_1 + \gamma n_2) = 0,$$
$$\boldsymbol{h}^*\ \delta\boldsymbol{h}^\top = (u\mathbf{1}_m, v\mathbf{1}_n)\ \ (\alpha\mathbf{1}_m, \beta\mathbf{1}_{n_1}, \gamma\mathbf{1}_{n_2})^\top = u\alpha m + v\,(\beta n_1 + \gamma n_2) = 0.$$

Thus we have $\alpha = 0$ and $\gamma = -\frac{n_1}{n_2}\beta$ and (A2) becomes

$$\delta\boldsymbol{h} = \beta\left(\mathbf{0}_m, \mathbf{1}_{n_1}, -\frac{n_1}{n_2}\mathbf{1}_{n_2}\right).$$

Since this is spanned by a one-dimensional space, by the equivariant branching lemma, there exists a bifurcating solution in this direction $\delta\boldsymbol{h}$, which leads to (15) by setting $\beta = w(n-p)$, $n_1 = p$, and $n_2 = n - p$. Equation (14) and symmetry/asymmetry of the branch can be proved similarly to the proof of Proposition 2.

Proof of Lemma 1: The asymptotic forms of G_i in (24) and its Jacobian matrix $\partial G_i/\partial x_j$ are given as follows: By virtue of a factored form (7) of the replicator dynamics, $G_i(\boldsymbol{x}, \psi)$ takes a special form:

$$G_i = x_i \cdot \hat{G}_i(\boldsymbol{x}, \psi) \qquad (1 \le i \le n).$$

We can expand $\hat{G}_i$ into a power series to arrive at

$$G_i = x_i \left(\alpha\psi + \sum_{j=1}^{n} \beta_j x_j \right)$$

for some constants α and β_i. By the symmetry (equivariance) of the system of equations G_i $(1 \le i \le n)$, a permutation $x_i \leftrightarrow x_j$ leads to a permutation $G_i \leftrightarrow G_j$. This entails $\beta_j = \beta$ $(j = i)$ and $\beta_j = \gamma$ $(j \ne i)$ for some constants β and γ. Then we have

$$G_i \approx x_i \left(\alpha\psi + \beta x_i + \gamma \sum_{j\ne i}^{n} x_j \right) \qquad (1 \le i \le n), \tag{A3}$$

and, in turn,

$$\frac{\partial G_i}{\partial x_j} \approx \begin{cases} \alpha\psi + 2\beta x_i + \gamma \displaystyle\sum_{j\ne i}^{n} x_j, & (i = j), \\ \gamma x_i, & (i \ne j). \end{cases} \tag{A4}$$

The use of the form $\boldsymbol{x} = w\,(\mathbf{1}_p, \mathbf{0}_{n-p})$ of a bifurcating branch in (26) in (A3) leads to

$$G_1 = \cdots = G_p \approx w\,\{\alpha\psi + (\beta + (p-1)\gamma)w\}, \qquad G_{p+1} = \cdots = G_n = 0.$$

Thus a set of equations $G_i = 0$ $(1 \le i \le n)$ is satisfied by the solution curve $\psi \approx -\frac{\beta+(p-1)\gamma}{\alpha}\,w$ in (27). Substituting $\boldsymbol{x} = w\,(\mathbf{1}_p, \mathbf{0}_{n-p})$ in (26) into (A4) and using (27), we obtain

$$\hat{J} = \left\{ \frac{\partial G_i}{\partial x_j} \right\} = w \begin{pmatrix} A_p(\beta,\gamma) & \gamma E_{pq} \\ O & (\gamma - \beta) I_{n-p} \end{pmatrix}.$$

The eigenvalues of the first diagonal block $wA_p(\beta,\gamma)$ give λ_1 and λ_2 and the eigenvalues of the second diagonal block $w(\gamma - \beta)I_{n-p}$ give λ_3 in (28), respectively.

Proof of Proposition 8: For $p = 2, \ldots, N-2$, there are all three eigenvalues and λ_2 and λ_3 have opposite signs; accordingly, the associated branches are unstable.

From (A4), the Jacobian matrix for the pre-bifurcation state reads $\hat{J}_{w=0} = \alpha\psi I_n$ and has an n-times repeated eigenvalue $\alpha\psi$. Since the pre-bifurcation state is stable for $\psi > 0$, we have $\alpha < 0$.

For i), by setting $p = 1$ in (28), we have the stability conditions: $\lambda_1 = \beta w < 0$ and $\lambda_3 = (\gamma - \beta)w < 0$, i.e., $\gamma < \beta < 0$ since $w > 0$. Then from $\alpha < 0$ and (27), which reduces to $\psi \approx -\frac{\beta}{\alpha}\,w$ for this case, we see that $\psi < 0$.

For ii), by setting $n = p = N-1$ in (28), we have the stability conditions: $\lambda_1 = \{\beta + (N-2)\gamma\}w < 0$ and $\lambda_2 = -(\gamma - \beta)w < 0$, i.e., $\beta < \gamma$ and $\beta < -(N-2)\gamma$. Then from (27), which reads $\psi \approx -\frac{\beta+(N-2)\gamma}{\alpha}\,w$ for this case, we see that $\psi < 0$.

To sum up, there is a unique stable branch for each of the cases i) and ii), whereas there are no stable branches in other cases, called iii). For iii), we have the remaining parameter space of $\beta > 0$ or $-(N-2)\gamma < \beta < 0$ in Fig. 2.