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Abstract. We consider strategic issues in one-to-one matching with externalities. We show that no core (stable) mechanism is strategy-proof, extending an impossibility result of Roth (1982) obtained in the absence of externalities. Moreover, we show that there are no limits on successful manipulation of preferences by coalitions of men and women, in contrast with the result of Demange et al. (1987) obtained in the absence of externalities.

Keywords: One-to-one matching; externalities; stability; core; strategic manipulation

JEL classification: C71; C78; D62

1 Introduction

A rapidly growing literature following Sasaki and Toda (1996) extends the classical matching theory pioneered by Gale and Shapley (1962) to problems with externalities. In this literature individuals in the matching environment care about the matches of other individuals, as well. More formally, individuals have preferences over the set of matchings, instead of preferences over the set of potential partners. While Sasaki and Toda (1996) and some researchers in the field (e.g., Chowdhury, 2004; Hafalir, 2008; Mumcu and Saglam, 2008, 2010; Fisher and Hafalir, 2016) confined their analysis to marriage markets (or one-to-one matchings), the related literature has extended over the past years to many settings, including college admissions (e.g., Echenique and Yenmez, 2007), job markets (e.g., Dutta and Massó, 1997; Alcalde and Re-

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villa, 2004; Revilla, 2004; Klaus and Klijn, 2005; Bando, 2012, 2014), and housing markets (e.g., Mumcu and Saglam, 2007; Salgado-Torres, 2011).

In this paper, we consider strategic issues in one-to-one matching with externalities, using a stability concept earlier introduced by Mumcu and Saglam (2008, 2010) for one-to-one matching problems and modified by Bando (2012) for many-to-one matching problems (to ensure the existence of stable matchings). Despite the wide range of applications of the theory of matching with externalities, the strategic issues have not been studied much in the literature, where the main focus has invariably been the existence of stable matchings and algorithms that produce stable matchings. On the other hand, the stability concepts in these applications varies a lot. As in the classical matching theory, a matching is assumed to be stable under externalities if it is not individually or pairwise blocked. But, in case a matching is blocked and changed, the welfare of every individual in the society may be affected due to externalities. Therefore, whether or not some individuals will block a matching when externalities are present depends, among other things, what these deviating individuals expect will happen to the actions of the rest of the society about keeping or changing their matches. These expectations of the (potentially) deviating pairs are called estimation functions in Sasaki and Toda (1996). Formally, an estimation function consists of the set of all matchings a pair of man and woman expect to form if they pairwise block a matching. A pair of individuals can block a matching if each of them becomes better off under all matchings predicted by his/her estimation function. Sasaki and Toda (1996) showed that any marriage market has a stable matching (but not necessarily a core matching) if and only if every individual has a universal estimation function, i.e. he/she believes all matchings are possible. Hafalir (2008) relaxed this extremely strong existence condition by endogenizing the estimation functions on the preferences of other individuals. Deviating slightly from this line, Mumcu and Saglam (2008, 2010) introduced a different notion of stability according to which the deviating individuals believe that the other individuals will not change their actions while blocking occurs. They showed that under this stronger stability concept, for any marriage market with externalities where there are at least three individuals (involving at least one pair of man and woman), one can always find a preference profile under which the stable set, and consequently the core, is empty. Moreover, the core and the stable set may be different if the society involves at least two pairs. Mumcu and Saglam (2010) further characterized conditions on preferences that are sufficient for the existence of a nonempty stable set in any society.

Using the stability concept used by Mumcu and Saglam (2010), we show in this paper that in the presence of externalities no core (stable) mechanism is strategy-proof, extending an impossibility result of Roth (1982) obtained in the absence of externalities. Moreover, we show that there are no limits on successful manipulation of preferences by coalitions of men and women, in contrast with the result of Demange et al. (1987) obtained in the absence of externalities.

The literature that deals with strategic issues in matching with externalities is quite sparse. The most related works, to the best of our knowledge, are Klaus and Klijn (2005), Kucuksenel (2011), and Kojima et al. (2013). Of these, Klaus and Klijn (2005) considered couples markets in many-to-one (hospital-doctor) matching problems and provided a maximal domain of preferences (the class of weakly responsive preferences) that ensures the existence of stable matchings. Given this domain, they also showed that there exists no stable matching mechanism under which reporting the true preferences is always a dominant strategy for every couple in the market, whereas the stable matching mechanism that yields the hospital-optimal matching makes it a dominant strategy for each hospital to report its true preferences. Kucuksenel (2011) considered college admissions problems allowing students to have preferences over other students attending to the same college, and proposed a sequential mechanism that implements the core. More recently, Kojima et al. (2013) reconsidered the couples problem in hospital-doctor markets, in the absence of the sufficient condition provided by Klaus and Klijn (2005), and showed that if the matching market (the number of hospitals) is large enough, then the probability that a stable matching does not exist can be made arbitrarily small, and also the truth-telling can arise as an approximate equilibrium behavior for all players in the market, namely doctors, couples, and hospitals. All of these works, being concerned with some real-life practices, consider externalities only among a group of individuals who are on the same side of the market. As a matter of fact, in Klaus and Klijn (2005) and Kojima et al. (2013) individuals face no externalities if they are not couples, and in Kucuksenel (2011) students face no externalities if they do not attend to the same college. Besides, externalities among hospitals (or among colleges) and cross externalities among hospitals and doctors (or colleges and students) are not considered, either. To make a contrast with these studies, we should say that our treatment of externalities is more general, as we allow any kind of externalities between any two individuals in the society regardless whether they are on the same side, or on opposite sides, of the matching market.

The rest of the paper is organized as follows: In Section 2 we formally present our basic structures and in Section 3 we give our results. Finally, we conclude in Section 4.

2 Basic Structures

We consider a society that involves a set of men, M and a set of women, W, which are non-empty, finite, and disjoint. Let $N = M \cup W$. A matching is a one-to-one function $\mu : N \to N$ such that for any $i \in N$ we have $\mu^2(i) = i$, for any $m \in M$ if $\mu(m) \neq m$ then $\mu(m) \in W$, and for any $w \in W$ if $\mu(w) \neq w$ then $\mu(w) \in M$. Let $\mathcal{M}^N$ denote the set of all matchings for the society N.

Given any matching μ and any pair of individuals m and w, we denote by $\mu_{m,w}$ a particular matching that is obtained from μ by marrying m and w. If these two individuals are not already married under μ, then their spouses (if any) also become unmarried under $\mu_{m,w}$, while the spouses of all other individuals (if any) remain the same. Also, for any $i \in N$, we denote by $\mu_{i,i}$ a matching obtained from μ by divorcing i from his/her spouse $\mu(i)$ (if any) and keeping them both unmarried under $\mu_{i,i}$, while holding the spouses of all other individuals (if any) unchanged.

Each individual has a complete, transitive, and strict preference relation over the matchings in $\mathcal{M}^N$. Let $\mathcal{P}_i$ denote the set of such preference relations for individual $i \in N$ and let $\mathcal{P} = \prod_{i \in N} \mathcal{P}_i$. For any $\mu, \mu' \in \mathcal{M}^N$, $i \in N$, and $P_i \in \mathcal{P}_i$, we write $\mu P_i \mu'$ to say that individual i strictly prefers μ to μ'. Also, for any $k = 1, 2, \ldots, |\mathcal{M}^N|$ and $i \in N$, we write $(P_i[k])$ to denote the k^{th} ranked matching from top with respect to preference relation P_i of individual i. We denote a marriage market by a triple (M, W, P). Also, we say that a market (M, W, P) has externalities if $P \in \mathcal{P}_i$.

A matching μ is *acceptable* to individual i if either $\mu P_i \mu_{i,i}$ or $\mu = \mu_{i,i}$. An individual *individually blocks* a matching μ if μ is not acceptable to him/her. A matching is *individually rational* if it is acceptable to each individual. We say that (m, w) is a *blocking pair* for a matching μ if m and w are not matched to each other at μ and both prefer the matching $\mu_{m,w}$, where they are matched to each other, to μ. A matching is *stable* if it is individually rational and if there are no blocking pairs for it. Let $\mathcal{S}(M, W, P)$ denote the set of all stable matchings for the marriage market (M, W, P).

A set $M' \cup W'$ of men and women, where $M' \subseteq M$ and $W' \subseteq W$, is a *blocking coalition* for a matching $\mu \in \mathcal{M}^N$ if there exists another matching $\mu' \in \mathcal{M}^N$ such that

i) at μ' each individual i in $M' \cup W'$ becomes either single or matched to an individual (of opposite gender) in $M' \cup W'$,

ii) the spouse $\mu(i)$ to which individual $i \in M' \cup W'$ is matched at μ becomes unmatched at μ' if $\mu(i)$ is outside $M' \cup W'$,

iii) if individual i is outside the coalition $M' \cup W'$, then his/her spouse (if any) is the same at μ and μ', and

iv) $\mu' P_i \, \mu$ for all $i \in M' \cup W'$.

A matching is called a *core matching* if there are no blocking coalitions for it. Let $\mathcal{C}(M, W, P)$ denote the set of all core matchings (or simply the core) for the marriage market (M, W, P). Apparently, $\mathcal{C}(M, W, P) \subseteq \mathcal{S}(M, W, P)$. As shown by Mumcu and Saglam (2010), for any society with at least three individuals there exists a preference profile under which the stable set, and consequently the core, is empty. In addition, if the society involves at least four individuals, then the core and the stable set may be different.

Now, we are ready to consider strategic issues. Consider a marriage market where the matching of individuals is determined by a central authority, based on a list of preference relations that individuals state for themselves. A *mechanism*, Γ, is a procedure which determines a matching for each marriage market (M, W, P). If the list of preference relations reported by the individuals is $Q = \prod_{i \in N} Q_i$ where Q_i is the report of individual i, the mechanism produces a matching $\Gamma[Q]$. If $\Gamma[Q]$ is always stable with respect to Q, the mechanism Γ is called a *stable matching mechanism*. Moreover, if $\Gamma[Q]$ is always in the core with respect to Q, the mechanism Γ is called a *core mechanism*. Also, a mechanism Γ is called *strategy-proof* if for all $P \in \mathcal{P}$ and for each $i \in N$ it is true that $\Gamma[P] \, P_i \, \Gamma[Q_i, P_{-i}]$ for all $Q_i \in \mathcal{P}_i$.

3 Results

Roth (1982) shows that for any marriage market where preference relations are strict and no externalities are present, there is no stable matching mechanism (equivalently, no core mechanism) which is strategy-proof. This result remains to hold under externalities.

Proposition 1. *For any marriage market where $|M||W| \geq 2$ and externalities are present, there is no core mechanism which is strategy-proof.*

Proof. First consider the case $|M||W| = 2$. Without loss of generality, assume $M = \{m_1, m_2\}$ and $W = \{w_1\}$. The possible matchings are

$$\mu_1 = w_1 m_2; \qquad \mu_2 = m_1 w_1; \qquad \mu_3 = m_1 m_2;$$

where at each matching the listed two individuals denote the matches of m_1 and m_2, respectively. Consider the preference profile $P \in \mathcal{P}$ given by

$$P_{m_1} = \mu_2 \mu_1 \mu_3; \qquad P_{m_2} = \mu_1 \mu_2 \mu_3; \qquad P_{w_1} = \mu_1 \mu_2 \mu_3.$$

It is easy to check that $S(M, W, P) = C(M, W, P) = \{\mu_1, \mu_2\}$. So, any stable mechanism must choose μ_1 or μ_2 when the preference report is P. Suppose the mechanism chooses μ_1. If m_1 misreports his/her preference relation as $Q_{m_1} = \mu_2 \mu_3 \mu_1$ while everyone else makes truthful revelations, then at the reported profile $(Q_{m_1}, P_{m_2}, P_{w_1})$, μ_2 becomes the unique matching in the core, which is preferred by m_1 to μ_1 at his/her true preference relation P_{m_1}. So, it is not a dominant strategy for all individuals to truthfully reveal their preference relations. Similarly, if the mechanism chooses μ_2 when P is reported, then m_2 can profitably misrepresent his/her preference relation as $Q_{m_2} = \mu_1 \mu_3 \mu_2$ to force the mechanism to select μ_1.

To show that the above result generalizes to the case of $|M||W| \geq 2$, we consider any society $N' = M' \cup W' \supset N$ and pick any $\mu \in \mathcal{M}^N$. We define the extended matching $\mu^{N'}$ over $\mathcal{M}^{N'}$ in such a way that all newcomers become single. That is,

$$\mu^{N'}(j) = \begin{cases} \mu(j) & \text{if } j \in N \\ j & \text{if } j \in N' \backslash N. \end{cases}$$

Following Mumcu and Saglam (2010), we will construct a preference profile P' of N' in such a way that the core (and the stable set) for the marriage markets (M, W, P) and (M', W', P') will be the same. So, let P' satisfy

(i) for all $k \in \{1, 2, \ldots, |\mathcal{M}^N|\}$,

$$(P'_i[k]) = \begin{cases} (P_i[k])^{N'} & \text{if } i \in N \\ (P_j[k])^{N'} & \text{if } i \in N' \backslash N, \end{cases}$$

for some $j \in N$ which is of the same gender as i;
(ii) for all $\mu' \in \mathcal{M}^{N'}$, for all $i \in N' \backslash N$ and for all $k \in \{1, 2, \ldots, |\mathcal{M}^N|\}$ such that $\mu' \neq (P'_i[k])$, it is true that $\mu'_{i,i} \, P'_i \, \mu'$ if $\mu' \neq \mu'_{i,i}$.

It should be clear that $S(M', W', P') = S(M, W, P)$ and $C(M', W', P') = C(M, W, P)$. Thus, the result for $|M||W| = 2$ remains to hold. $\square$

Proposition 1 shows that there exists no core (or stable) mechanism which will never give any individual an incentive to misrepresent his or her preference relation. The below proposition which strengthens the above impossibility result states that there always exists at least one individual who will behave strategically under a core mechanism when all other individuals act truthfully.

Proposition 2. *Consider any marriage market* (M, W, P)*, where* $\min\{|M|, |W|\} = 2$ *and externalities are present, such that (i) there is more than one stable matching in the core, (ii) every individual is married at any core matching, (iii) the matching at which every individual is single is bottom ranked by at least two men and two women, (iv) if a matching outside the core is such that some individuals are married, then this matching is unacceptable to at least one of these married individuals in the society. If any core mechanism is applied to this market, then at least one individual can profitably misreport his or her preference relation, whenever the others report truthfully.*

Proof. Consider a marriage market (M, W, P) satisfying the hypotheses of the theorem. Suppose that when all individuals reveal their true preference relations in P, the core mechanism Γ selects matching ν. By assumption (i), the core of this market, $C(M, W, P)$, contains at least two stable matchings. Let j be some individual who does not top rank ν in his/her preference relation. (There must exist such an individual, since $|C(M, W, P)| \geq 2$ by assumption (i).) Let ν' denote the core matching that individual j prefers most. Suppose j misreports his/her preference relation as Q_j that satisfies the following properties:

(1) $(Q_j[1]) = \nu'$;
(2) if $\mu \in \mathcal{M}^N \backslash \{\nu'\}$ is such that $\mu(j) \neq j$, then $\mu_{j,j} \, Q_j \, \mu$;
(3) if $\mu = (P_j[|\mathcal{M}^N|])$ is such that $\mu(j) \neq j$, then $\mu = (Q_j[|\mathcal{M}^N|])$;

(4) if $\mu \in \mathcal{M}^N \backslash \{\nu'\}$ is such that $\mu \notin C(M, W, P)$ and $\mu(j) \neq j$, then $\mu_{j,j} \, Q_j \, \mu$.

Suppose that all individuals, except for j, truthfully report their preference relations. Denote the reported preference profile by $\hat{P} = (Q_j, P_{-j})$. First consider any matching μ in $C(M, W, P)\backslash\{\nu'\}$. By assumption (ii), $\mu \neq \mu_{k,k}$ for any $k \in N$. Then, property (2) implies $\mu_{j,j} \, Q_j \, \mu$, therefore $\mu \notin C(M, W, \hat{P})$. Now, consider the matching μ^s at which every individual is single. Since only one individual (individual j) in the society misreports his/her preference relation, assumption (iii) ensures (irrespective of the report of individual j) that there exist at least one man and one woman who rank μ^s at the bottom with respect to their preference relations. Since such a pair of man and woman can pairwise block μ^s by marrying to each other, $\mu^s \notin C(M, W, \hat{P})$. Finally, consider any matching $\mu \notin C(M, W, P) \cup \{\mu^s\}$. Clearly, some individuals are married at μ since $\mu \neq \mu^s$. If $\mu(j) \neq j$, then property (4) ensures that individual j blocks μ via $\mu_{j,j}$. If $\mu(j) = j$, then assumption (iv) ensures that there exists some individual $k \in N\backslash\{j\}$ such that $\mu(k) \neq k$ and $\mu_{k,k} \, P_k \, \mu$, implying that individual k will block μ. Therefore, $\mu \notin C(M, W, \hat{P})$. So far, we have proved that if $\mu \in \mathcal{M}^N\backslash\{\nu'\}$, then $\mu \notin C(M, W, \hat{P})$. Finally, property (1) of Q_j and the fact that $\nu' \in C(M, W, P)$ together ensure $\nu' \in C(M, W, \hat{P})$. So, we must have $\Gamma(\hat{P}) = C(M, W, \hat{P}) = \{\nu'\}$. Since $\nu' \, P_j \, \nu$ and $\nu = \Gamma(\hat{P})$, the report Q_j makes individual j better-off. □

A result by Demange et al. (1987) shows that in any marriage market where preference relations are strict and no externalities are present, no coalition of men and women can manipulate their preference relations so successfully that every member of the coalition prefers one of the new outcomes to every stable (core) outcome (with respect to the true preference relations). Below, we prove that such limits on successful manipulation do not exist in the presence of externalities.

Proposition 3. *For any marriage market where* $min\{|M|, |W|\} \geq 2$ *and externalities are present, there exist two preference profiles* P *and* $\hat{P}$*, where* $\hat{P}$ *differs from* P *for some coalition* G *of men and women, such that there exists a matching* μ *in* $C(M, W, \hat{P})$ *which is preferred to every matching in* $C(M, W, P)$ *under* P *by all members of* G*.*

Proof. First consider the case $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Clearly, $min\{|M|, |W|\} = 2$. The seven possible matchings are

$$\begin{array}{llll} \mu_1 = w_1, w_2; & \mu_2 = m_1, w_2; & \mu_3 = w_2, m_2; & \mu_4 = m_1, w_1; \\ \mu_5 = w_2, w_1; & \mu_6 = w_1, m_2; & \mu_7 = m_1, m_2; & \end{array}$$

and the preference profile is given by

$$\begin{array}{ll} P_{m_1} = \mu_3\mu_1\mu_4\mu_5\mu_2\mu_6\mu_7 & P_{m_2} = \mu_1\mu_4\mu_5\mu_2\mu_3\mu_6\mu_7 \\ P_{w_1} = \mu_1\mu_4\mu_5\mu_2\mu_3\mu_6\mu_7 & P_{w_2} = \mu_3\mu_4\mu_1\mu_5\mu_2\mu_6\mu_7. \end{array}$$

It is easy to check that $S(M, W, P) = C(M, W, P) = \{\mu_4\}$. Now suppose that the singleton coalition $\{m_1\}$ misrepresents his/her preference relation as $\hat{P}^{m_1} = \mu_1\mu_3\mu_4\mu_5\mu_2\mu_6\mu_7$. Define $\hat{P} = (\hat{P}^{m_1}, P^{-m_1})$. We then have $S(M, W, \hat{P}) = C(M, W, \hat{P}) = \{\mu_1, \mu_4\}$. Clearly, m_1 prefers $\mu_1 \in C(M, W, \hat{P})$ to μ_4, the unique matching in $C(M, W, P)$. The result immediately follows for any larger society $N' = M' \cup W'$ with $min\{|M'|, |W'|\} > 2$, if the all matchings in $\mathcal{M}^N$ are extended to $\mathcal{M}^{N'}$ such that all individuals in $N' \backslash N$ become single and the preference profile P' of N' is obtained from P by respecting the society N as in the proof of Proposition 1. □

When one of the genders has a unique member, we have a weaker result.

Proposition 4. *For any marriage market where $min\{|M|, |W|\} = 1$ and externalities are present, there exist two preference profiles P and $\hat{P}$, where $\hat{P}$ differs from P for some coalition G of men and women, such that there exists a matching μ in $C(M, W, \hat{P})$ which is weakly preferred to every matching in $C(M, W, P)$ under P by all members of G.*

Proof. Follows from the proof of Proposition 1, where one of the individuals in the society can manipulate his/her preference relation to make the core mechanism uniquely select his/her most preferred core matching. □

4 Conclusions

In this paper, we have considered strategic issues in marriage markets with externalities using the stability concept in Mumcu and Saglam (2008, 2010). We have showed that no core (stable) mechanism makes truth-telling always a dominant strategy for all individuals, extending an earlier result of Roth (1982) obtained in the absence of externalities. Also, we have found that there are no limits on successful manipulation of preferences by coalitions of men and women, in contrast with the result of Demange et al. (1987) obtained in the absence of externalities.

Future research can fruitfully extend our work to problems with many-to-one matching, and also study how our results would change in both one-to-one and many-to-one matching settings if the stability concept used in our paper were changed with one of the alternative concepts in the literature; namely, the stability concepts of Sasaki and Toda (1996) or Hafalir (2008) for one-to-one matching problems and the stability concept of Bando (2012) for many-to-one matching problems.

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