Effects of credit limit on efficiency and welfare in a simple general equilibrium model

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Abstract

We consider a simple general equilibrium model with two agents under the presence of financial market imperfections: agents can borrow to realize their productive project up to the level of debt whose repayment reaches a fraction of the project's value (so-called *credit limit*). After characterizing the whole set of equilibria, we investigate the connection between credit limit, (individual and social) welfare, and efficiency. We also compute the optimal credit limit which maximizes the social welfare function.

JEL Classifications: D53, E44, G10.
Keywords: General equilibrium, credit limit, welfare, efficiency.

1 Introduction

In an environment where there exists lack of contract enforcement, collateralized debts arise naturally for the lender to secure her loans. Enterprise Surveys (2018), conducted by the World Bank and its partners, provide a database of firms in 139 countries. According to Enterprise Surveys (2018), in the average level, 53.6% of firms need a loan and 79.1% of loans require collateral. Such collateral requirements are considered as endogenous borrowing constraints, which depend on the values of the assets and also the possible losses associated with the reallocation of those assets in case of default. This source of financial friction has been of great interest to both theoretical and empirical macroeconomists, mostly in examining the role of collateral constraints in the dynamics of the business cycles since the seminal papers Kiyotaki and Moore (1997), Geanakoplos and Zame (2002). Our paper aims to contribute to this literature

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by addressing two basic questions: (1) What are the effects of credit limit on individual and social welfares? and (2) What is the optimal level of the credit constraint?

For this purpose, we construct a tractable two-agent, two-period general equilibrium model with financial frictions. Agents differ in initial wealth, productivity, and credit limit. They have two ways of investing: buying capital (to realize their production project) or buying financial assets. Agents can borrow and then pay back in the next period. Following Kiyotaki (1998, 2011), the debtor is required to put her project as collateral in order to borrow: in case that she does not repay, the creditor can seize the collateral. Due to the lack of commitment, the creditor can only obtain a fraction of the value of the project. Anticipating the possibility of default, the creditor limits the amount of credit so that the debt repayment will not exceed a fraction f_i of the debtor's project value. This fraction can be interpreted as the *credit limit* of agent i.[1]

In general equilibrium models with financial frictions, borrowing constraints may occasionally bind. This makes the computation of equilibria challenging, especially in multi-period and/or stochastic models (see Brumm, Kubler and Scheidegger (2017) for an excellent survey). Thanks to the simplicity of our model, we can completely characterize the whole set of equilibria. In general, the more productive one (say, agent 1) will borrow from the unproductive one (say, agent 2). However, the productive agent's borrowing constraint is not binding if and only if its credit limit f_1 is sufficiently high. More interestingly, we show that the productive agent will borrow all initial wealth of the unproductive agent if and only if f_1 reaches a middle-level. If f_1 is low, borrowing constraint of agent 1 is binding but she can only borrow a part of initial wealth of the unproductive agent who still produces in this case. The main difference between our paper and Kiyotaki (1998) concerns the equilibrium interest rate: in Kiyotaki (1998), it is constant over time (equals to the rate of return on investment of productive agents) while it depends on the credit limit f_1 and other fundamentals in our model.

After characterizing all possible equilibria, we examine the effects of the credit limit on the welfare of agents and equilibrium efficiency.

First, we show that the welfare or consumption of the unproductive agent (lender) is increasing in credit limit but that of the productive agent (borrower) displays an inverted U-shape as a function of credit limit f_1. Let us explain the economic intuition. Credit limits influence agents' production level and borrowing as well as repayment. When credit limit f_1 is low, an increase in this credit limit leads to increases in both the agent 1's production level and her repayment with the production level increasing faster than the repayment. Hence, her consumption and welfare increase. Nevertheless, once credit limit reaches a middle level, this agent borrows all initial wealth of the unproductive agent, and hence cannot increase further her production level. By contrast, the repayment always increases in the credit limit once the credit constraint binds. Therefore, the productive agent's welfare is decreasing in her credit limit once the latter reaches the middle level.

From the individual welfare analysis, we can derive properties of the social welfare function defined as a weighted sum of individual welfares. If the lender's initial wealth

[1] The reader is referred to Matsuyama (2007), Quadrini (2011), Brunnermeier, Eisenbach and Sannikov (2013) for more complete reviews on the macroeconomic effects of financial frictions and to Buera et al. (2015) for the relationship between entrepreneurship and financial frictions.

is low and/or her weight is sufficiently high, then the social welfare is an increasing function of the productive agent's credit limit. Otherwise, the social welfare displays an inverted U-shape as a function of the credit limit f_1.

In our simple economy, we also manage to figure out the optimal level of the credit limit which maximizes the social welfare function. We find out that this optimal level is a non-decreasing function of the lender's productivity, initial wealth and the weight assigned to the lender's utility in the social welfare function. Notice that borrowing constraints may be binding even the credit limit is at the optimal level. Moreover, the optimal credit limit may not be at the maximum level. However, when the social welfare function is the aggregate output, which is increasing in credit limit, the optimal level of the credit limit must be the highest possible.

Our welfare analysis is related to Obiols-Homs (2011). Indeed, Obiols-Homs (2011) considers a general equilibrium model with an exogenous borrowing limit (consumers can borrow an amount which is bounded from below by an exogenous parameter). Obiols-Homs (2011) shows that there is a neighborhood of borrowing limit, in which the welfare of the borrower decreases when its borrowing limit increases.

There are important differences between our paper and Obiols-Homs (2011). First, the mechanism of Obiols-Homs (2011) relies on a different role of credit. While credit is in need of households to smooth their consumption in Obiols-Homs (2011), it is demanded by firms who want to finance their productive investment in our mechanism. Second, borrowing limit of each agent in our model is endogenous while it is exogenous in Obiols-Homs (2011). We show that models with endogenous constraints as in our model and those with exogenous borrowing limits as in Obiols-Homs (2011) may trigger important differences not only in terms of equilibrium outcomes but also on the relationship of credit limit and social welfare. Indeed, if in our model we replace credit constraints by exogenous borrowing constraints (as in Obiols-Homs (2011)), then we have that: (1) the equilibrium indeterminacy may arise and (2) both individual consumptions are increasing in the exogenous credit limits (which implies that an inverted-U relationship between the credit limit and the borrower's consumption does not appear). Third, we can compute all types of equilibrium (thanks to our model's tractability) and hence provide a more general picture by explicitly characterizing conditions under which the individual and social welfares increase or decrease in the credit limit. Furthermore, we complement the analysis of Obiols-Homs (2011) by computing the optimal level of the credit limit.

Our last avenue of contribution concerns the equilibrium efficiency. We provide a necessary and sufficient condition for an equilibrium to be Pareto efficient. This condition is characterized by agents' credit limit, initial wealth, and productivity. The economy is more likely to be efficient if the productive agent's credit limit, initial wealth and productivity are high. Two points should be mentioned: (1) an equilibrium with binding borrowing constraints may be efficient or inefficient, and (2) a level of credit limit may lead to an equilibrium efficiency but does not necessarily maximize the social welfare.

Our finding on the efficiency of equilibrium outcomes has a link with Theorem 2 in Gottardi and Kubler (2015) who provide a necessary and sufficient condition for the existence[2] of a Pareto-efficient equilibrium in a stochastic exchange economy with

[2] Gottardi and Kubler (2015) say that Pareto-efficient equilibria exist for an economy if there are

collateral constraints and without aggregate uncertainty. Although our model is deterministic, we introduce production (any agent can produce by using their technology). In Theorem 2 in Gottardi and Kubler (2015), the collateral requirements play no role while the credit limit in our setting role plays a crucial role on the equilibrium efficiency. It should be noticed that we can, furthermore, fully characterize all economies where efficient or inefficient equilibria arise.

The rest of this paper is organized as follows. Section 2 presents our framework. In Section 3, we characterize equilibria and then compare outcomes of models with and without credit constraints. Section 4 investigates the welfare effects of the credit limit and computes the optimal level of the credit limit while Section 5 studies the efficiency of equilibrium outcomes. Section 6 concludes. Technical proofs are gathered in Appendices.

2 A two-agent economy with credit constraints

We consider a two-period economy with two agents. There is no uncertainty and there is a single good (numéraire) which can be consumed or used to produce. Each agent i has exogenous wealth (S_i units of good) at the initial date and decides how much good for production and investment in the financial market in order to maximize her wealth in the next period. Since each agent lives for two periods, this wealth is also the agent's consumption.

On the one hand, if agent i wants to realize her productive project, she buys k_i units of physical capital at the initial date to produce $F(k_i)$ units of good at the second date, where F_i is her production function.

On the other hand, she can invest in a financial asset with real return r. Denote a_i the amount that the agent i invests in the financial asset. She can also borrow and then pay back ra_i in the next period. In the spirit of Kiyotaki (1998, 2011), we assume that the debtor is required to put her project as collateral in order to borrow: If she does not repay, the creditor can seize the collateral. Due to the lack of commitment (or just because the debtor is not willing to help the creditor take the whole value of the debtor's project), the creditor can only obtain a fraction f_i of the total value of the project. Anticipating the possibility of default, the creditor limits the amount of credit so that the debt repayment will not exceed a fraction f_i (called "credit limit") of the debtor's project value.

To sum up, the maximization problem of agent i can be described as follows:

$$(P_i) \qquad c_i = \max_{(k_i,a_i)} [F_i(k_i) - ra_i] \tag{1a}$$

$$\text{subject to :} \qquad 0 \le k_i \le S_i + a_i \tag{1b}$$

$$ra_i \le f_i F_i(k_i). \tag{1c}$$

The better the commitment, the higher value of f_i, the larger the set of feasible allocations of the agent i. The setup (1c) is also supported by Enterprise Surveys (2018). Indeed, the ratio $1/f_i$ of the agent i in our paper seems to represent the agent i's value of collateral needed for a loan (% of the loan amount) in Enterprise Surveys

initial distributions for which the competitive equilibrium is Pareto efficient.

(2018). Following Enterprise Surveys (2018), the average of credit limits f_i (data of 133 countries) is closed to 0.555. So, it is natural to make the following assumption.

Assumption 1. $f_i \in [0, 1)$ *for* $i = 1, 2$

Remark 1. *In terms of modeling, our model environment can be viewed as a simplified two-date version of Kiyotaki (1998) with alternative technology. In Kiyotaki (1998), the time is of infinite horizon, the utility function is logarithmic, and he provides analyses around the steady state. Our model is simpler but we can characterize the whole set of equilibrium and provide full comparative statics.*

It should be noticed that constraint (1c) is different from condition (3) in Kiyotaki and Moore (1997). Indeed, the borrower's repayment is assumed not to exceed the market value of her land quantity in Kiyotaki and Moore (1997) while being restricted to be at most the market value of the borrower's project under our assumption.

Some authors (Buera and Shin, 2013; Moll, 2014) set $k_i \leq \theta w_i$, where $w_i \geq 0$ is the agent i's wealth and interpret that θ measures the degree of credit frictions (credit markets are perfect if $\theta = \infty$ while $\theta = 1$ corresponds to financial autarky, where all capital must be self-financed by entrepreneurs). In our framework, S_i plays a similar role of wealth w_i in Buera and Shin (2013), Moll (2014).

Other authors (Kocherlakota, 1992; Obiols-Homs, 2011) consider exogenous borrowing limits by imposing a short sales constraint: $a_i \leq B$ for any i.

As we will show in Appendix A.2, our model and that with exogenous borrowing limits lead to important differences in terms of equilibrium outcomes.

Definition 1. *Let us consider the economy* $\mathcal{E}$*, characterized by a list of fundamentals* $\mathcal{E} \equiv (F_i, f_i, S_i)_{i=1,2}$*. A list* (r, a_1, a_2, k_1, k_2) *is an equilibrium if the two following conditions are satisfied (1) Agents' optimality: for each* $i \in \{1, 2\}$*, given* r*,* (a_i, k_i) *is a solution of the problem* (P_i)*, and (2) Financial market clearing:* $a_1 + a_2 = 0$*.*

Notice that under standard specifications, the existence of equilibrium is guaranteed.[3] To simplify the exposition and to get closed-form solutions, the main text will focus on the linear technology case.

Assumption 2. *Assume that* $F_i(K) = A_i K$ *for* $i = 1, 2$*.*

In Appendix A.3, we present the analyses under **Cobb-Douglas production functions**. In this case, although the solutions do not have closed-form (see Proposition 6), we prove that the main economic insights do not change.

3 Computing equilibrium outcomes

Before computing equilibrium, we study the individual problem (P_i). At optimal, we have $k_i = S_i + a_i$. So, $A_i k_i - r a_i = A_i S_i + a_i(A - r)$ and two constraints (1c-1b) become $a_i \geq -S_i$ and $(r - f_i A_i) a_i \leq f_i A_i S_i$. By consequence, we obtain the following result.

Lemma 1 (individual problem)**.** *Assume that* $F_i(K) = A_i K$*. The solution for agent* i*'s maximization problem is described as follows.*

[3] We can prove the equilibrium existence by applying the method in Bosi, Le Van and Pham (2018).

1. *If $r > A_i$, then agent i does not produce goods and invest all her initial wealth in the financial market: $k_i = 0, a_i = -S_i$.*

2. *If $A_i = r$, then the solutions for the agent's problem include all sets (k_i, a_i) such that $-S_i \leq a_i \leq f_i k_i$ and $k_i = a_i + S_i$.*

3. *If $A_i > r > f_i A_i$, then agent i borrows from the financial market and the borrowing constraint is binding.*

$$k_i = \frac{r}{r - f_i A_i} S_i, \quad a_i = \frac{f_i A_i}{r - f_i A_i} S_i. \tag{2}$$

4. *If $r \leq f_i A_i$, there is no solution.*[4]

Given the interest rate r, Lemma 1 draws the relationship between productivity and identification of the borrower/lender: An agent borrows from the financial market if and only if her productivity is high enough, in the sense that $A_i > r$. Moreover, she borrows the maximum level imposed on her, i.e, the borrowing constraint is binding.

The most interesting case corresponds to point 3 of Lemma 1 according to which the capital and asset holding of agent i are increasing in her TFP (A_i), initial wealth S_i and credit limit f_i but decreasing in the interest rate r. However, the interest rate is endogenously determined. The following result fully describes the general equilibrium effects of all fundamentals including credit limits on the agents' decision and their consumption.

Proposition 1 (characterization of general equilibrium)**.** *Let Assumption 1, 2 be satisfied. Assume that $A_1 > A_2$. There are only three different cases (each having a unique equilibrium and with the productive agent being the borrower).*

1. *If $f_1 \leq \frac{A_2}{A_1}\frac{S_2}{S_1+S_2}$, then the borrowing constraint of agent 1 is binding and there exists a unique equilibrium characterized by:*

$$\begin{aligned} \textit{Interest rate:} \quad & r = A_2 \\ \textit{Physical capital:} \quad & k_1 = \frac{A_2}{A_2 - f_1 A_1} S_1, \quad k_2 = -\frac{f_1 A_1}{A_2 - f_1 A_1} S_1 + S_2 \\ \textit{Financial asset:} \quad & a_1 = \frac{f_1 A_1}{A_2 - f_1 A_1} S_1, \quad a_2 = -\frac{f_1 A_1}{A_2 - f_1 A_1} S_1; \end{aligned}$$

The aggregate output and consumption of each agent are:

$$Y = A_2 S_2 + A_1 S_1 \frac{A_2 - f_1 A_2}{A_2 - f_1 A_1}, \quad c_1 = A_1 S_1 \frac{A_2 - f_1 A_2}{A_2 - f_1 A_1}, \quad c_2 = A_2 S_2.$$

[4] Indeed, if $r \leq f_i A_i$, then agent i may choose $a_i = +\infty$ and $k_i = +\infty$ and have $c_i = +\infty$. However, in the case of Cobb-Douglas technology, we have $f'(\infty) = 0$, then the feasible set is compact, which implies that the individual problem always has a solution (see Appendix A.3).

2. *If* $\frac{A_2}{A_1}\frac{S_2}{(S_1+S_2)} < f_1 < \frac{S_2}{S_1+S_2}$*, then the borrowing constraint of agent 1 is binding, and there exists a unique equilibrium characterized by:*

$$\begin{aligned}
\textit{Interest rate:} \quad & r = f_1 A_1\Big(1 + \frac{S_1}{S_2}\Big) \\
\textit{Physical capital:} \quad & k_1 = S_1 + S_2, \quad k_2 = 0 \\
\textit{Financial asset:} \quad & a_1 = S_2, \quad a_2 = -S_2.
\end{aligned}$$

The aggregate output and consumption of each agent are:

$$Y = A_1(S_1 + S_2), \quad c_1 = A_1(1 - f_1)(S_1 + S_2), \quad c_2 = f_1 A_1(S_1 + S_2).$$

3. *If* $f_1 \geq \frac{S_2}{S_1+S_2}$*, then the borrowing constraint is not binding, and there exists a unique equilibrium characterized by:*

$$\begin{aligned}
\textit{Interest rate:} \quad & r = A_1 \\
\textit{Physical capital:} \quad & k_1 = S_2 + S_1, \quad k_2 = 0 \\
\textit{Financial asset:} \quad & a_1 = S_2; \quad a_2 = -S_2
\end{aligned}$$

The aggregate output and consumption of each agent are:

$$Y = A_1(S_1 + S_2), \quad c_1 = A_1 S_1, \quad c_2 = A_1 S_2.$$

Proof. See Appendix A.1.1.[5] □

Proposition 1 figures out all possible cases and compute equilibrium in each case. From this, we can explicitly express the equilibrium outcomes in terms of fundamentals. Before going further, we introduce some notations:

- $f_1^{**} \equiv \frac{S_2}{S_1+S_2}$. This is the threshold of credit limit above which borrowing constraint of agent 1 is not binding. Agent 1's borrowing constraint is binding if $f_1 < f_1^{**}$.
- $f_1^{*} \equiv \frac{A_2}{A_1}\frac{S_2}{S_1+S_2}$. We have $f_1^* < f_1^{**}$ because $A_1 > A_2$. This is the threshold of credit limit above which agent 1's borrowing constraint is binding and this agent borrow all initial wealth of agent 2 (i.e. $k_1 = S_1 + S_2$).

When the credit limit f_1 is in the interval (f_1^*, f_1^{**}), agent 1's borrowing constraint is binding and all initial wealth of agent 2 is lent to agent 1, so that this agent uses all physical capital of the economy for production. The higher (resp., lower) the productivity A_1 (resp., A_2), the lower the threshold f_1^*, and therefore the smaller the interval (f_1^*, f_1^{**}).

[5]In any case, agent 1 (the agent with higher productivity) borrows, and agent 2 (the one with lower productivity) lends in the financial market. Therefore borrowing constraint of agent 2 is not binding. Consequently, f_2 is irrelevant to the equilibrium outcomes.

We now present explicit formulas of equilibrium interest rate and aggregate output:

$$r = r(f_1) \equiv \begin{cases} A_2 & \text{if} \quad f_1 \leq f_1^* \\ f_1 A_1\Big(1+\dfrac{S_1}{S_2}\Big) & \text{if} \quad f_1^* < f_1 < f_1^{**} \\ A_1 & \text{if} \quad f_1 \geq f_1^{**} \end{cases}, \tag{3}$$

$$Y = Y(f_1) \equiv \begin{cases} A_2S_2 + A_1S_1\dfrac{A_2 - f_1A_2}{A_2 - f_1A_1} & \text{if} \quad f_1 \leq f_1^* \\ A_1(S_1+S_2) & \text{if} \quad f_1^* < f_1. \end{cases} \tag{4}$$

We see that the interest rate r is in $[A_2, A_1]$ and Y belongs to $[A_1S_1 + A_2S_2, A_1(S_1 + S_2)]$. Both r and Y are increasing in the credit limit f_1. It is possible to prove that $r \geq f_1A_1$, and hence agent 1's consumption cannot be infinity (see Lemma 1).

Our result is related to Kiyotaki (1998) who considers an infinite-horizon model with two agents (unproductive and productive agents having linear production functions, as in our model) and logarithmic utility functions. The main difference concerns the equilibrium interest rate. Although the model in Kiyotaki (1998) is of infinite horizon, the interest rate is constant over time (it equals the rate of return on investment of productive agents; this corresponds to the case $r = A_2$ in our model). However, it depends on the credit limit f_1 in our model (see (3)).

Remark 2 (Economy without credit constraints)**.** *Consider an economy without credit constraints (in the sense that constraint (1c) is removed). Under Assumptions 1, 2, there exists a unique equilibrium determined by:* $\bar{r} = A_1, \bar{k}_1 = S_1 + S_2, \bar{k}_2 = 0, \bar{a}_1 = S_2, \bar{a}_2 = -S_2$. *The aggregate output and consumption of each agent are:*

$$\bar{Y} = A_1(S_1 + S_2), \quad \bar{c}_1 = A_1S_1, \quad \bar{c}_2 = A_1S_2.$$

Corollary 1 (With versus without credit constraints)**.** *Consider the economy with credit constraints.*

1. *When the borrowing constraint of the borrower binds, the equilibrium interest rate is smaller than that in case without borrowing constraints:* $r \leq \bar{r}$.

2. *When the borrowing constraint of the borrower binds, the lender's consumption is smaller than that in the case without borrowing constraints:* $c_2 \leq \bar{c}_2$.

3. *When the borrowing constraint of the borrower binds, her consumption is greater than that in the case without borrowing constraints:* $c_1 \geq \bar{c}_1$.

4. *When the borrowing constraint of the borrower binds, the aggregate output in the case under borrowing constraint is smaller than or equal to that in the case without borrowing constraint:* $Y \leq \bar{Y}$. *The equality occurs with higher value of* f_1 *(i.e., when* $f_1 > f_1^*$*).*

In the following sections, we will explore the equilibrium efficiency and welfare analyses.

4 Welfare analysis

This section explores the (individual and social) welfare analyses. Following Proposition 1, we can provide the explicit formula of the consumption of agent 1 (borrower)

$$c_1 = c_1(f_1) \equiv \begin{cases} A_1 S_1 \dfrac{A_2 - f_1 A_2}{A_2 - f_1 A_1} & \text{if} \quad f_1 \leq f_1^* \\ A_1(1 - f_1)(S_1 + S_2), & \text{if} \quad f_1^* < f_1 < f_1^{**} \\ A_1 S_1 & \text{if} \quad f_1 \geq f_1^{**} \end{cases} \tag{5}$$

and that of agent 2 (lender)

$$c_2 = c_2(f_1) \equiv \begin{cases} A_2 S_2 & \text{if} \quad f_1 \leq f_1^* \\ f_1 A_1 (S_1 + S_2) & \text{if} \quad f_1^* < f_1 < f_1^{**} \\ A_1 S_2 & \text{if} \quad f_1 \geq f_1^{**}. \end{cases} \tag{6}$$

4.1 Individual welfare

According to (6), the consumption of agent 2 (lender) is increasing in f_1. However, the consumption of agent 1 (borrower) is not. Indeed, following (5), if the credit limit imposed on this agent is sufficiently strict in the sense that $f_1 \leq f_1^*$, her consumption is an increasing function of f_1. For medium values of f_1, i.e, $f_1^* < f_1 < f_1^{**}$, the borrower's consumption is decreasing in f_1. When f_1 is high ($f_1 > f_1^{**}$), the consumption is $A_1 S_1$ which does not depend on f_1.

To understand the mechanism behind this result, let us decompose c_1 into two terms

$$c_1(f_1) = A_i k_i - r a_i \equiv \begin{cases} A_1 \dfrac{A_2 S_1}{A_2 - f_1 A_1} - A_2 \dfrac{f_1 A_1 S_1}{A_2 - f_1 A_1} & \text{if} \quad f_1 \leq f_1^* \\ A_1(S_1 + S_2) - \dfrac{f_1 A_1 (S_1 + S_2)}{S_2} S_2, & \text{if} \quad f_1^* < f_1 < f_1^{**} \\ A_1(S_1 + S_2) - A_1 S_2, & \text{if} \quad f_1 \geq f_1^{**} \end{cases}$$

where the first term in each case is her production $A_1 k_1$ while the second term is her repayment $r a_1$. We write $c_1(f_1)$ because the consumption c_1 depends on f_1.

Notice that credit limits influence agents' production level and borrowing as well as repayment. When the credit limit f_1 is strict ($f_1 \leq f_1^*$), if f_1 increases, then the agent 1's production level $A_1 k_1$ increases faster than her repayment $R a_1$: $\frac{\partial(A_1 k_1)}{\partial f_1} \geq \frac{\partial(r a_1)}{\partial f_1}$. So, her consumption increases. However, once the credit limit exceeds f_1^*, the physical capital k_1 equals $S_1 + S_2$ (the aggregate capital of the economy), and the production level of agent 1 cannot increase anymore when f_1 increases.[6] On the contrary, the repayment always increases in f_1. As a result, agent 1's consumption c_1 decreases in

[6]Notice that under Cobb-Douglas production functions, the more productive agent does not borrow all initial wealths of the less productive agent because the less productive agent still produces (Inada's condition holds). However, we can prove that the more productive agent's production is increasing and strictly concave in f_1. By consequence, our main insight does not change: the agent 1's consumption first increases and then decreases when f_1 increases (see Appendix A.3 for more details).

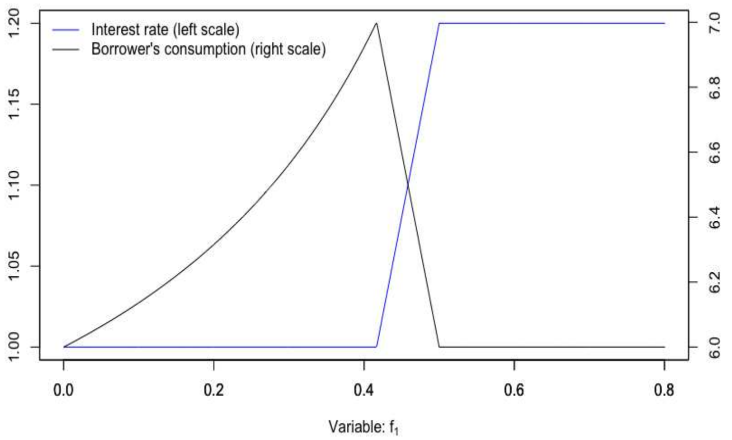

Figure 1: *Effects of credit limit* f_1

$f_1 \in (f_1^*, f^{**})$. Last, when f_1 is high enough ($f_1 \geq f_1^{**}$), equilibrium outcomes no longer depend on f_1.

Figure 1 illustrates our result on the non-monotonicity of agent 1's consumption. In this figure, parameters are $A_1 = 1.2, A_2 = 1, S_1 = S_2 = 5$. We observe that $c_1(0) = c_1(f_1^{**}) = A_1S_1 \leq c_1(f_1)$ for any $f_1 \in (0, f_1^{**})$, where the threshold $f_1^{**} = 0.5$.

Our finding is closely related to Obiols-Homs (2011). Indeed, Obiols-Homs (2011) investigates the effects of exogenous borrowing limit on welfare. He shows that when borrowing limit belongs to a *neighborhood* of the benchmark borrowing limit, then the welfare of borrower is decreasing in borrowing limit. Our added-value to Obiols-Homs (2011) is that we can compute the whole set of equilibria (thanks to our model's simplicity) and hence provide insights on how individual welfare is affected as the level of financial frictions varies.

The main difference in terms of setup between Obiols-Homs (2011) and our paper is that Obiols-Homs (2011) considers an exogenous borrowing limit (borrowing constraint in Obiols-Homs (2011) is under form $a_i \leq a^*$) while we take into account credit constraints. Notice that constraint (1c) can be rewritten as $a_i \leq \frac{f_iF_i(k_i)}{r}$ (this is a borrowing constraint but the borrowing limit $\frac{f_iF_i(k_i)}{r}$ is endogenous).

As we will prove in Appendix A.2, these two setups have different implications. For example, both individual consumptions are increasing in the model with exogenous borrowing limits $\bar{a}_i$. However, in the model with credit constraint (1c), the consumption of borrower has an inverted U–sharp form and that of lender is increasing in credit limit. Moreover, in a model with exogenous borrowing limits, multiple equilibria may arise while in the model with credit constraint (1c), there is a unique equilibrium.

4.2 Social welfare and optimal credit limit

In this section, we study how social welfare depends on the credit limit. Since the credit limits (f_i) are proxies of the financial development, it is interesting to compute the optimal level of credit limit which maximizes the social welfare.

Assume that the utility function of agent i is $u_i(c_i)$. We can define the social welfare

$$\mathcal{W} \equiv \sum_{i=1}^{2} \gamma_i u_i(c_i) \tag{7}$$

where $\gamma_i \geq 0$ is the weight assigned to agent i's utility. The function $\mathcal{W}$ would depend on credit limits. However, since borrowing constraint of borrower is binding, the welfare $\mathcal{W}$ does not depend on f_2. So, we write $\mathcal{W} = \mathcal{W}(f_1)$.

Let us recall that when $f_1 \geq f_1^{**}$, the equilibrium outcomes and the social welfare do not depend on f_1. Therefore, we only investigate the impact of f_1 when $f_1 \leq f_1^{**}$.

The following results (whose proofs are presented in Appendix A.1.2) show the impact of f_1 on the social welfare and the optimal level of f_1.

Proposition 2. *Assume that $u_i(c_i) = \frac{c_i^{1-\sigma}}{1-\sigma}$ if $\sigma \in (0,1)$, and $u_i(c_i) = ln(c_i)$ if $\sigma = 1$.*

1. *If $\frac{S_2}{S_1+S_2} \leq x_2$, then $\mathcal{W}(f_1)$ is increasing in f_1, where $x_2 \equiv \frac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}$.*
2. *If $\frac{A_2}{A_1}\frac{S_2}{S_1+S_2} < x_2 < \frac{S_2}{S_1+S_2}$, then $\mathcal{W}(f_1)$ is increasing on $[0, x_2]$ and decreasing on $(x_2, \frac{S_2}{S_1+S_2})$.*
3. *If $\frac{A_2}{A_1}\frac{S_2}{S_1+S_2} \geq x_2$, then $\mathcal{W}(f_1)$ is increasing on $(0, \frac{A_2}{A_1}\frac{S_2}{S_1+S_2})$ and decreasing on $(\frac{A_2}{A_1}\frac{S_2}{S_1+S_2}, \frac{S_2}{S_1+S_2})$.*

So, the optimal level of the credit limit is determined by

$$\hat{f}_1 = \begin{cases} \frac{A_2}{A_1}\frac{S_2}{S_1+S_2} & \text{if } \frac{A_2}{A_1}\frac{S_2}{S_1+S_2} \geq x_2 \\ x_2 & \text{if } \frac{A_2}{A_1}\frac{S_2}{S_1+S_2} < x_2 < \frac{S_2}{S_1+S_2} \\ \frac{S_2}{S_1+S_2} & \text{if } \frac{S_2}{S_1+S_2} \leq x_2. \end{cases} \tag{8}$$

Proposition 3. *Assume that $u_i(c_i) = c_i$ for any i. Then, we have*

1. *If $\gamma_2 \geq \gamma_1$, then $\mathcal{W}(f_1)$ is increasing in f_1.*
2. *If $\gamma_1 > \gamma_2$, then $\mathcal{W}(f_1)$ is increasing on $[0, f_1^*]$ and decreasing on (f_1^*, f_1^{**}), where recall that $f_1^{**} \equiv \frac{S_2}{S_1+S_2}$ and $f_1^* \equiv \frac{A_2}{A_1}\frac{S_2}{S_1+S_2} < f_1^{**}$.*

So, the optimal level of f_1 is determined by

$$\hat{f}_1 = \begin{cases} f_1^* & \text{if } \gamma_1 > \gamma_2 \\ f_1^{**} & \text{if } \gamma_2 \geq \gamma_1. \end{cases} \tag{9}$$

When $u_i(c_1) = c_i$ and $\gamma_1 = \gamma_2$, the social welfare function is exactly the aggregate output which is proved to be increasing in the credit limit f_1. Our interesting point is that the social welfare function can be increasing or have inverted U-shape form, depending not only on the form of the social welfare function but also on the distribution of productivity and endowments of agents $((A_i)_i, (S_i)_i)$. Even we chose the same weights $\gamma_1 = \gamma_2$, Proposition 2 suggests that the social welfare function can be increasing or have inverted U-shape form; this happens when the utility function is strictly concave.

According to (8) and (9), the optimal level of credit limit is a non-decreasing function of the lender's productivity, initial wealth and the weight assigned to the lender's utility in the social welfare function. Moreover, (8) and (9) leads to interesting points: (1) the optimal credit limit may be the level which makes borrowing constraint binding, and (2) the optimal credit limit may not be the maximum level (i.e., f_1^{**}).

5 Efficiency

This section aims to explore the efficiency of equilibrium outcomes. Following Malinvaud (1953), Alvarez and Jermann (2000), Becker, Dubey and Mitra (2014) we introduce some notions of efficiency.

Definition 2. *Consider an economy characterized by production functions and initial wealths $(F_i, S_i)_{i=1,2}$.*

1. *(Efficient production plan) A plan $(k_i)_i$ is said to be efficient if (1) it is feasible in the sense that $\sum_i k_i \leq \sum_i S_i$ and (2) there does not exist another feasible production plan $(k'_i)_i$ such that $\sum_i F_i(k'_i) > \sum_i F_i(k_i)$.*

2. *(Efficient allocation). An allocation $(c_i)_i$ is said to be efficient if (1) it is feasible in the sense that $\sum_i c_i \leq \sum_i F_i(k_i)$ with some feasible plan (k_i) and (2) there does not exist another feasible allocation $(c'_i)_i$ which Pareto dominates $(c_i)_i$.*

3. *(Constrained efficient allocation). An allocation $(c_i)_i$ is said to be constrained efficient if (1) it is efficient and (2) $c_i \geq A_i S_i \ \forall i = 1, 2$.*

Notice that $A_i S_i$ is the consumption of agent i if she cannot participate to the financial market. Thus, constrained efficiency requires that allocation is efficient and the well-being of every agent is not less than her autarkic welfare.

Let us consider an equilibrium of our economy $\mathcal{E} = (F_i, f_i, S_i)$ with credit constraints. One can prove that $(k_i)_i$ is an efficient production plan if and only if it is a solution of the following problem:

$$(PP): \quad F(S) \equiv \max_{(k_i)\geq 0} \sum_i F_i(k_i) \tag{10a}$$

$$\text{subject to}: \quad \sum_i k_i \leq S \equiv \sum_i S_i. \tag{10b}$$

The consumption allocation $(c_i)_i$ is efficient if and only if $\sum_i c_i = F(S)$. It is constrained efficient if and only if $\sum_i c_i = F(S)$ and $c_i \geq A_i S_i \ \forall i$.

The simplicity of our framework allows us to easily characterize the efficient production plans and Pareto efficient allocations of equilibrium. According to (3, 4, 5, 6), we obtain the following result.

Proposition 4. *Let assumptions in Proposition 1 be satisfied and consider the economy with credit constraints. The following statements are equivalent:*

1. *The production plan of equilibrium is efficient.*

2. *The consumption allocation of equilibrium is efficient.*

3. *The consumption allocation of equilibrium is constrained efficient.*

4. $f_1 \geq f_1^* \equiv \frac{A_2}{A_1}\frac{S_2}{S_1+S_2}$.

To sum up, consumption allocation or production plan of equilibrium is inefficient if and only if the credit limit f_1 is low (in the sense that $f_1 < f_1^* \equiv \frac{A_2}{A_1}\frac{S_2}{S_1+S_2}$). The higher productivity (A_1) and initial wealth (S_1) of the most productive agent (agent 1), the lower the threshold level $f_1^* \equiv \frac{A_2}{A_1}\frac{S_2}{S_1+S_2}$, and the easier we can obtain equilibrium efficiency.

Comparing Proposition 4 with Proposition 3 and Proposition 2, we see that although relaxing credit limit so that $f_1 \geq f_1^*$ helps to achieve an efficient equilibrium, this level may not be optimal in the sense that it maximizes the social welfare. It is noticed that when $f_1 \in [f_1^*, f_1^{**})$, the equilibrium is efficient but agent 1's borrowing constraint is binding.

Our result is related to Gottardi and Kubler (2015) who consider an exchange economy with complete markets and collateral constraints. Theorem 2 in Gottardi and Kubler (2015) gives a necessary and sufficient condition for *the existence of a Pareto-efficient equilibrium* with no aggregate uncertainty.[7] This condition is based on agents' endowments and Gottardi and Kubler (2015) require the Lucas tree's dividend in every state to be sufficiently large so that collateral constraints never bind.

There are some differences between our paper and Gottardi and Kubler (2015).

1. First, our model, on the one hand, is simpler than that of Gottardi and Kubler (2015) with deterministic and exogenous savings, but is more general on the other hand thanks to the introduction of productions where every agent may become entrepreneur.

2. Second, our necessary and sufficient condition ($f_1 \geq \frac{A_2}{A_1}\frac{S_2}{S_1+S_2}$) is based on the credit limit f_1, productivities, and wealths while in Theorem 2 in Gottardi and Kubler (2015) credit limits play no role.

3. Third, Theorem 2 in Gottardi and Kubler (2015) only considers the existence of a Pareto-efficient equilibrium while our Proposition 4 studies all kinds of equilibrium, including those with binding credit constraint.

6 Concluding remarks

We have constructed a very simple general equilibrium model with two heterogeneous agents and financial market imperfections which induces interesting results about (in-

[7]Theorem 2 in Gottardi and Kubler (2015): A necessary and sufficient condition for the existence of a Pareto-efficient equilibrium with no aggregate uncertainty is $(1-\beta)\sum_{h\in\mathcal{H}} \max_{s\in\mathcal{S}} \Big[e^h(s) + E\Big(\sum_{t=1}^{\infty}\beta^t e^h(s_t) \mid s_0 = s\Big)\Big] \leq \omega$, where $\mathcal{H}$ is the finite-set of agents, $\mathcal{S}$ is the set of states, ω is the aggregate consumption good, β is the rate of time preference, $e^h(s)$ is the endowment of agent h at state s.

dividual and social) welfare and efficiency. We have provided conditions under which relaxing credit limit has negative or positive effect on the individual and social welfares.

Our paper opens several research avenues in the future. One may introduce uncertainty and market incompleteness and then investigate the impact of this type of financial frictions on welfare and efficiency. Another line of research is to extend our analysis in a dynamic framework and investigate the effects of credit limits in the long run.

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A Appendix

Appendix A.1 presents formal proofs for the linear technology case. Appendix A.2 introduces a model with exogenous borrowing limit and compares it with that in the main text. Appendix A.3 provides analysis in a model with Cobb-Douglas technology.

A.1 Formal proofs - linear technology

A.1.1 Proof of Proposition 1

From Lemma 1, we see that, at equilibrium $Min\{A_1, A_2\} \leq r \leq Max\{A_1, A_2\}$. Under the assumption 1 ($A_2 < A_1$), we thus have $A_2 \leq r \leq A_1$. There are three cases, each having a unique equilibrium described as follows.

1. If $A_2 = r < A_1$, Lemma 1 implies that there exists an equilibrium determined by:

$$k_1 = \frac{r}{r - f_1 A_1} S_1, \quad a_1 = \frac{f_1 A_1}{r - f_1 A_1} S_1$$
$$k_2 = S_2 - \frac{f_1 A_1}{r - f_1 A_1} S_1, \quad a_2 = -\frac{f_1 A_1}{r - f_1 A_1} S_1$$

In this case, agent 2 is the lender and agent 1 is the borrower. We see that $ra_1 = f_1 A_1 k_1$, i.e the borrowing constraint is satisfied. We need to check the non-negative condition on k_1, k_2. It is easy to see that $k_1 \geq 0$. Then one more condition left to be checked, that is $k_2 \geq 0$. This condition is satisfied if and only if

$$S_2 - \frac{f_1 A_1}{r - f_1 A_1} S_1 \geq 0 \Leftrightarrow f_1 \leq \frac{A_2}{A_1} \frac{S_2}{S_1 + S_2}$$

2. If $A_2 < r < A_1$, based on Lemma 1, there exists an equilibrium determined by:

$$k_1 = \frac{r}{r - f_1 A_1} S_1, \quad k_2 = 0, \quad a_1 = \frac{f_1 A_1}{r - f_1 A_1} S_1, \quad a_2 = -S_2;$$

The equilibrium interest rate is given by $\frac{f_1A_1}{r-f_1A_1}S_1 = S_2$. So, we can compute $r = f_1A_1\Big(1 + \frac{S_1}{S_2}\Big)$. We see that $r \in (A_2, A_1)$ if and only if:

$$A_2 < f_1A_1\Big(1 + \frac{S_1}{S_2}\Big) < A_1 \Leftrightarrow \frac{A_2}{A_1}\frac{S_2}{S_1+S_2} < f_1 < \frac{S_2}{S_1+S_2}.$$

3. If $A_2 < r = A_1$, then there exists an equilibrium determined by $k_2 = 0, a_2 = -S_2, k_1 = S_1 + S_2, a_1 = S_2$. In this case, we need to verify the borrowing constraint of the borrower (agent 1): $ra_1 \leq f_1A_1k_1$, or $A_1a_1 \leq f_1A_1(S_1 + S_2)$. This condition is satisfied if and only if

$$S_2 \leq f_1(S_1+S_2) \Leftrightarrow f_1 \leq \frac{S_2}{S_1+S_2}$$

A.1.2 Proof of Proposition 3 and Proposition 2

We consider three kinds of utility functions.

1. $u_i(c_i) = c_i$ for any i. We have

$$\mathcal{W}(f_1) = \gamma_1c_1 + \gamma_2c_2 = \begin{cases} \gamma_1A_1S_1\dfrac{A_2 - f_1A_2}{A_2 - f_1A_1} + \gamma_2A_2S_2 & \text{if} \quad f_1 \leq f_1^* \\ \gamma_1A_1(S_1+S_2) + (\gamma_2-\gamma_1)f_1A_1(S_1+S_2) & \text{if} \quad f_1^* < f_1 \leq f_1^{**} \end{cases}$$

   If $f_1 \leq f_1^*$, then $\mathcal{W}(f_1)$ is an increasing function in f_1.

   If $f_1^* < f_1 \leq f_1^{**}$, then $\mathcal{W}(f_1)$ is increasing in f_1 if and only if $\gamma_1 \leq \gamma_2$, and decreasing in f_1 otherwise.

2. $u_i(c_i) = \frac{c_i^{1-\sigma}}{1-\sigma}$ for any i, where $\sigma \in (0,1)$. In this case, we have

$$\mathcal{W}(f_1) = \begin{cases} \gamma_1\dfrac{\Big(A_1S_1\dfrac{A_2-f_1A_2}{A_2-f_1A_1}\Big)^{1-\sigma}}{1-\sigma} + \gamma_2\dfrac{(A_2S_2)^{1-\sigma}}{1-\sigma} & \text{if } f_1 \leq f_1^* \\ \gamma_1\dfrac{\Big(A_1(1-f_1)(S_1+S_2)\Big)^{1-\sigma}}{1-\sigma} + \gamma_2\dfrac{\Big(f_1A_1(S_1+S_2)\Big)^{1-\sigma}}{1-\sigma} & \text{if } f_1^* < f_1 \leq f_1^{**} \end{cases}$$

   If $f_1 \leq f_1^*$, then $\mathcal{W}(f_1)$ is increasing in f_1.

   If $f_1^* < f_1 \leq f_1^{**}$, then:

$$\mathcal{W}(f_1) = \gamma_1\frac{\Big(A_1(1-f_1)(S_1+S_2)\Big)^{1-\sigma}}{1-\sigma} + \gamma_2\frac{\Big(f_1A_1(S_1+S_2)\Big)^{1-\sigma}}{1-\sigma}$$
$$\mathcal{W}'(f_1) = [\gamma_2f_1^{-\sigma} - \gamma_1(1-f_1)^{-\sigma}]\big(A_1(S_1+S_2)\big)^{1-\sigma}$$

   We consider following sub-cases:

   - $f_1^{**} < x_2$, then $\mathcal{W}(f_1)$ is an increasing function in f_1 for all $f_1 \in (f_1^*, f_1^{**})$.
   - $f_1^* > x_2$, then $\mathcal{W}(f_1)$ is a decreasing function in f_1 for all $f_1 \in (f_1^*, f_1^{**})$.
   - $f_1^* < x_2 < f_1^{**}$, then $\mathcal{W}(f_1)$ is increasing in f_1 for $f_1 \in (f_1^*, x_2)$ and decreasing in f_1 for $f_1 \in (x_2, f_1^{**})$.

To sum up, we have that:

- If $f_i^{**} \leq x_2$, then $\mathcal{W}(f_1)$ is increasing in f_1.
- If $f_1^* < x_2 \leq f_i^{**}$, then $\mathcal{W}(f_1)$ is increasing in f_1 for $f_1 \in [0, x_2]$ and decreasing in f_1 for $f_1 \in (x_2, f_i^{**})$.
- If $f_1^* \geq x_2$, then $\mathcal{W}(f_1)$ is increasing in f_1 for $f_1 \in (0, f_1^*)$ and is decreasing in f_1 for $f_1 \in (f_1^*, f_1^{**})$

3. If $u_i(c_i) = ln(c_i)$ for any i. In this case, we have

$$\mathcal{W}(f_1) = \begin{cases} \gamma_1 ln\Big(A_1 S_1 \dfrac{A_2 - f_1 A_2}{A_2 - f_1 A_1}\Big) + \gamma_2 \ln(A_2 S_2) & \text{if} \quad f_1 \leq f_1^* \\ \gamma_1 ln\big(A_1(1-f_1)(S_1+S_2)\big) + \gamma_2 ln\big(f_1 A_1 (S_1+S_2)\big) & \text{if} \quad f_1^* < f_1 \leq f_1^{**} \end{cases}$$

If $f_1 \leq f_1^*$, then $\mathcal{W}(f_1)$ is increasing in f_1.

If $f_1^* < f_1 \leq f_1^{**}$, then we have

$$\mathcal{W}'(f_1) = \frac{\gamma_2}{f_1} - \frac{\gamma_1}{(1-f_1)} = \frac{\gamma_2 - f_1(\gamma_1+\gamma_2)}{f_1(1-f_1)}$$

Thus, $\mathcal{W}'(f_1) > 0$ if $f_1 < x_2 \equiv x_2$ and $\mathcal{W}'(f_1) < 0$ if $f_1 > x_2$. We consider following cases:

- $f_1^{**} \leq x_2$, then $\mathcal{W}(f_1)$ is an increasing function for all $f_1 \in (f_1^*, f_1^{**})$
- $f_1^* \geq x_2$, then $\mathcal{W}(f_1)$ is a decreasing function for all $f_1 \in (f_1^*, f_1^{**})$
- $f_1^* < x_2 < f_1^{**}$, then $\mathcal{W}(f_1)$ is an increasing function in (f_1^*, x_2) and decreasing function in (x_2, f_1^{**})

To sum up, we have that:

- If $f_1^{**} \leq x_2$, then $\mathcal{W}(f_1)$ is increasing in f_1.
- If $f_1^* < x_2 < f_1^{**}$, then $\mathcal{W}(f_1)$ is increasing in f_1 for $f_1 \in [0, \gamma_2]$ and decreasing in f_1 for $f_1 \in (\gamma_2, f_1^{**})$.
- If $f_1^* < x_2$, then $\mathcal{W}(f_1)$ is increasing in f_1 for $f_1 \in [0, f_1^*]$ and decreasing in f_1 for $f_1 \in \big(f_1^*, f_1^{**}\big)$.

A.2 A model with exogenous borrowing limit

To provide a sharper comparison in terms of equilibrium outcomes between our model and the setup with exogenous borrowing limit, we replace constraint (1c) by $a_i \leq \bar{a}_i$. The problem of agent i now becomes

$$(Q_i): \quad \pi_i = \max_{(k_i, a_i)} [F_i(k_i) - r a_i] \text{ subject to: } 0 \leq k_i \leq S_i + a_i \tag{A.1a}$$

$$\text{and } a_i \leq \bar{a}_i. \tag{A.1b}$$

Recall that in the problem (P_i) with credit constraint (1c), the bound of a_i depends on the future value of the investment project and on the interest rate. Consequently, agent i cannot borrow if her project is not productive. By contrast, under exogenous borrowing limit setup (the problem (Q_i)), agent i can always borrow an amount $\bar{a}_i$ whether she has a project or not.

Notice that $a_i \geq -S_i \; \forall i$. At optimal, we must have $k_i = S_i + a_i$. Then, $\pi_i = A_i S_i + (A_i - r)a_i$. Consequently, we obtain the following result which is similar to Lemma 1.

Lemma 2 (individual problem)**.** *The solution of the problem (Q_i) is given by the following.*

1. *If $A_i < r$, then agent i does not produce goods and invest all her wealth in the financial market: $k_i = 0, a_i = -S_i$.*
2. *If $A_i > r$, then agent i borrows from the financial market and the borrowing constraint is binding: $a_i = \bar{a}_i, k_i = S_i + \bar{a}_i$.*
3. *If $A_i = r$, then the solutions for the agent's problem include all sets (k_i, a_i) such that $-S_i \leq a_i \leq \bar{a}_i$ and $k_i = a_i + S_i$.*

Notice that, in each case, the allocation does not depend on the interest rate. This is the main difference between this model and one with credit constraint.

Proposition 5 (general equilibrium with two agents)**.** *Assume that there are two agents with production function $F_i(k) = A_i k_i \ \forall i$ and $A_1 > A_2$.*

1. *If $S_2 > \bar{a}_1$, then $r = A_2$.*
2. *If $S_2 = \bar{a}_1$, then any $r \in [A_2, A_1]$ is an equilibrium interest rate. Equilibrium indeterminacy arises.*
3. *If $S_2 < \bar{a}_1$, then $r = A_1$. (High borrowing limit $\bar{a}_1$.)*

Proof. See Appendix A.2.1 below. □

According to Proposition 5, we can compute individual consumptions and the aggregate

$$Y = \begin{cases} A_2 S_2 + A_1 S_1 + (A_1 - A_2)\bar{a}_1 & \text{if} \quad \bar{a}_1 < S_2 \\ A_1 S & \text{if} \quad \bar{a}_1 \geq S_2 \end{cases}$$

$$c_2 = \begin{cases} A_2 S_2 & \text{if} \quad \bar{a}_1 < S_2 \\ r\bar{a}_1 & \text{if} \quad \bar{a}_1 = S_2 \\ A_1 S_2 & \text{if} \quad \bar{a}_1 \geq S_2 \end{cases} \quad \text{and } c_1 = \begin{cases} A_1 S_1 + (A_1 - A_2)\bar{a}_1 & \text{if} \quad \bar{a}_1 < S_2 \\ A_1 S_1 + (A_1 - r)\bar{a}_1 & \text{if} \quad \bar{a}_1 = S_2 \\ A_1 S_1 & \text{if} \quad \bar{a}_1 \geq S_2 \end{cases}$$

where $r \in [A_2, A_1]$.

There are two main differences between the model with exogenous borrowing limits and that with credit constraints (1c).

- According to Proposition 5, multiple equilibria arises (but it is not totally generic because it only happens when $\bar{a}_1 = S_2$). However, our model with credit constraint (1c) has a unique equilibrium.
- Both individual consumptions are increasing in exogenous borrowing limits $\bar{a}_1$. However, our model with credit constraint (1c), the consumption of borrower has an inverted U–sharp form and that of lender is increasing in credit limit. The intuition is that the borrowing amount that an agent can borrow are exogenous in the problem (Q_i) while it is endogenous and depends on the interest rate r in the problem (P_i) with credit constraint (1c).

These points suggest that the forms of borrowing constraints (credit constraint or exogenous borrowing limit) matter for the equilibrium analysis.

A.2.1 Proof of Proposition 5

Since $\sum_i a_i = 0$, Lemma 2 implies that $r \in [A_2, A_1]$.

If $r = A_2$, then $r < A_1$ which implies that $a_1 = \bar{a}_1$ and $k_1 = S_1 + \bar{a}_1$. By using market clearing condition, we have $a_2 = -\bar{a}_1$, and hence $k_2 = S_2 - \bar{a}_1$. Therefore, we need condition $S_2 - \bar{a}_1 \geq 0$.

If $r = A_1$, then $r > A_2$ which implies that $k_2 = 0$, $a_2 = -S_2$. By using market clearing condition, we have $a_1 = S_2$, and hence $k_1 = S_1 + S_2$. We also need $a_1 \leq \bar{a}_1$, i.e., $S_2 \leq \bar{a}_1$.

If $r \in (A_2, A_1)$, then $a_1 = \bar{a}_1$ and $k_1 = S_1 + \bar{a}_1$. By using market clearing condition, we have $a_2 = -\bar{a}_1$, and hence $k_2 = S_2 - \bar{a}_1$. However, $A_2 < r$ implies that $k_2 = 0$. Therefore, we need condition $S_2 - \bar{a}_1 = 0$.

It is easy to verify that if $S_2 - \bar{a}_1 = 0$, then any $r \in [A_2, A_1]$ is an equilibrium interest rate. In this case, we have

$$c_2 = A_2k_2 - ra_2 = A_2S_2 + (A_2 - r)a_2 = rS_2 \tag{A.2}$$

$$c_1 = A_1k_1 - ra_1 = A_1S_1 + (A_1 - r)a_1 = A_1S_1 + (A_1 - r)S_2. \tag{A.3}$$

Notice that c_1 is decreasing in r.

Multiple equilibria arises but it is not totally generic because we need $S_2 = \bar{a}_1$.

A.3 A model with Cobb-Douglas technology

In this section, we complement our analysis in the main text by presenting the equilibrium analysis when technologies have Cobb-Douglas form (all formal proofs in this section are presented in the online appendix.) Our main insights do not change.

Assume that $F_i(k) = A_i k^\alpha \ \forall i$, the problem (P_i) becomes:

$$(P_i'^C): \quad c_i = \max_{k_i, a_i} [A_i k_i^\alpha - ra_i] \text{ subject to}: \quad 0 \leq k_i \leq S_i + a_i \tag{A.4}$$

$$\text{and } ra_i \leq f_i A_i k_i^\alpha \tag{A.5}$$

A.3.1 Partial equilibrium

Lemma 3. *The solutions for the maximization problem of agent i with Cobb-Glass technology and collateral constrains are described in the following cases:*

1. *If* $\left(\frac{r}{\alpha A_i S_i^{\alpha-1}}\right)^{\frac{1}{1-\alpha}} + \frac{f_i}{\alpha} \leq 1$, *then the borrowing constraint is binding. In this case, we have* $k_i^{1-\alpha} - \frac{S_i}{k_i^\alpha} = \frac{f_i A_i}{r}$ *and* $a_i = k_i - S_i$.

2. *If* $\left(\frac{r}{\alpha A_i S_i^{\alpha-1}}\right)^{\frac{1}{1-\alpha}} + \frac{f_i}{\alpha} > 1$ *then the credit constraint is not binding. In this case, we have* $k_i = \left(\frac{\alpha A_i}{r}\right)^{\frac{1}{1-\alpha}}$ *and* $a_i = k_i - S_i$. *Agent i lends if* $r \geq \alpha A_i S_i^{\alpha-1}$, *and borrows if* $r < \alpha A_i S_i^{\alpha-1}$.

Proof. See Appendix B.1. □

From Lemma 3, we see that when the credit limit is sufficiently hight, i.e, $f_i > \alpha$, then constraint is not binding, the capital use k_i of each agent depends on A and α. The more interesting case is when the credit limit is low, i.e, $f_i < \alpha$, then the credit constraint is binding when $\alpha A_i S_i^{\alpha-1}$ is high enough. The intuition is that, the higher productive the agent is, the more he would like to borrow from the financial market. When the productivity is sufficiently high, the credit constraint binds.

Assume that $\alpha A_i S_i^{\alpha-1} > r$ (this condition is satisfied if the TFP A_i is high enough). According to the Lemma 3, the credit constraint is binding if and only if the credit limit is lower than a critical threshold, $\bar{f}$, determined by

$$\bar{f} = \alpha\Big(1 - \big(\frac{r}{\alpha A_i S_i^{\alpha-1}}\big)^{\frac{1}{1-\alpha}}\Big)$$

$\bar{f}$ is decreasing in the interest rate r and the initial wealth S_i but increasing in the TFP A_i.

A.3.2 General equilibrium

It is easy to obtain the equilibrium when there is no credit constraint.

Lemma 4 (equilibrium without credit constraints)**.** *With Cobb-Douglass production function and no credit constraints, there exists a unique equilibrium determined by:*

$$\textit{Interest rate:} \quad r = \bar{r} \equiv \frac{\alpha\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^{1-\alpha}}{(S_1+S_2)^{1-\alpha}}$$

$$\textit{Allocation:} \quad k_i = \Big(\frac{\alpha A_i}{r}\Big)^{\frac{1}{1-\alpha}} = \frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}(S_1+S_2), \quad a_i = k_i - S_i$$

At equilibrium, agent i borrows if and only if his lowest marginal productivity in autarky ($\alpha A_i S_i^{\alpha-1}$) is greater than the other's.

The aggregate output and consumption of each agent are:

$$Y = \bar{Y} \equiv (S_1+S_2)^{\alpha}\Big((A_1)^{\frac{1}{1-\alpha}} + (A_2)^{\frac{1}{1-\alpha}}\Big)^{1-\alpha}$$

$$c_1 = \bar{c}_1 \equiv \left(\frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}\right)^{\alpha} A_1^{\frac{1}{1-\alpha}} - \frac{\alpha(A_1^{\frac{1}{1-\alpha}}S_2 - A_2^{\frac{1}{1-\alpha}}S_1)}{(S_1+S_2)^{1-\alpha}\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^{\alpha}}$$

$$c_2 = \bar{c}_2 \equiv \left(\frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}\right)^{\alpha} A_2^{\frac{1}{1-\alpha}} - \frac{\alpha(A_2^{\frac{1}{1-\alpha}}S_1 - A_1^{\frac{1}{1-\alpha}}S_2)}{(S_1+S_2)^{1-\alpha}\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^{\alpha}}$$

Notice that, if $f_i > \alpha$ for any i, then credit constraint of each agent does not bind and hence the equilibrium in the credit-constrained economy coincides with that in the economy without credit constraints.

Let us denote

$$\bar{r}_i \equiv \alpha A_i S_i^{\alpha-1}\big(1 - \frac{f_i}{\alpha}\big)^{1-\alpha}, \quad \hat{r}_2 \equiv \alpha A_2 S_2^{\alpha-1}$$

The following result figures out all possible equilibria.

Proposition 6 (general equilibrium with credit constraints)**.** *Without loss of generality, we assume that $\bar{r}_1 > \bar{r}_2$. At equilibrium, agent i borrows if and only if his lowest marginal productivity in autarky ($\alpha A_i S_i^{\alpha-1}$) is greater than the other's. Moreover, we have:*

1. *If $\bar{r} > r_1$, or equivalently*

$$\frac{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}}}\frac{S_1}{S_1+S_2} + \frac{f_1}{\alpha} > 1 \tag{A.6}$$

then no credit constraint is binding and the equilibrium coincides with that in the economy without credit constraints.

2. *If $\bar{r} \leq r_1$, or equivalently*

$$\frac{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}}} \frac{S_1}{S_1+S_2} + \frac{f_1}{\alpha} \leq 1 \tag{A.7}$$

then there exists an equilibrium whose interest rate is determined by:

$$f(r) \equiv \Big(S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big)^{1-\alpha} - \frac{S_1}{\Big(S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big)^{\alpha}} - \frac{f_1 A_1}{r} = 0 \tag{A.8}$$

The equilibrium allocation is given by

$$\textit{Physical capital:} \quad k_1^{1-\alpha} - \frac{S_1}{k_1^{\alpha}} = \frac{f_1 A_1}{r}, \quad k_2 = \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}} \tag{A.9}$$

$$\textit{Financial asset:} \quad a_1 = k_1 - S_1, \quad a_2 = k_2 - S_2. \tag{A.10}$$

In this equilibrium, $r \in (\hat{r}_2, \bar{r})$ and the credit constraint of agent 1 is binding. The aggregate output and consumption of each agent are:

$$\begin{aligned}
Y &= A_1\big[S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\big]^{\alpha} + A_2\big(\frac{\alpha A_2}{r}\big)^{\frac{\alpha}{1-\alpha}}, \\
c_1 &= A_1\big[S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\big]^{\alpha} - r\big[S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\big], \\
c_2 &= A_2\big(\frac{\alpha A_2}{r}\big)^{\frac{\alpha}{1-\alpha}} + r\Big(S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big)
\end{aligned}$$

Proof. See Appendix B.2. □

With sufficiently high value of f_1 (in the sense that (A.6) holds), the credit constraint is not binding, and we achieve the same equilibrium outcomes as in the case without credit constraints, which are independent of the value of f_1. The credit constraint of agent 1 is binding if and only if $\bar{r} \leq r_1$, i.e. the interest rate of the economy without friction is lower than the subjective interest rate of agent 1. This happens if (1) f_1 is sufficiently low, and/or (2) agent 1's relative wealth S_1/S_2 is low, and/or (3) agent 1's relative A_1/A_2 is high.

The impact of f_1 on the equilibrium outcomes when the credit constraint binds are summarized in the following result. The economic insights are in the same direction as in the case of linear technology.

Corollary 2 (interest rate and output)**.** *At equilibrium where the credit constraint is binding, the equilibrium interest rate r and the aggregate output are increasing functions of f_1.*

Proof. See Appendix B.3. □

Since when the credit constraint of agent 1 binds, $r \in (\hat{r}_2, \bar{r})$, we thus have $f_1 \in (\underline{f}_1, \bar{f}_1)$ in this case, where $\underline{f}_1 = f_1(\hat{r}_2)$, $\bar{f}_1 = f_1(\bar{r})$.

Corollary 3 (consumptions)**.** *Consider the case where the credit constraint imposed on the borrower binds.*

1. *The lender's consumption is an increasing function in f_1*

2. *There exist $\tilde{f}_1$ the borrower's consumption is an increasing function in f_1 for $f_1 \in (\underline{f}_1, \tilde{f}_1)$, and is a decreasing function in f_1 for $f_1 \in (\tilde{f}_1, \bar{f}_1)$*

Proof. See Appendix B.4. □

Corollary 4 (with versus without credit constraints)**.**

1. *Interest rate under collateral constraints is smaller than or equal to that in an unconstrained economy.*

2. *The lender's consumption under collateral constraints is smaller than or equal to that in an unconstrained economy.*

3. *There exists $\tilde{f}_1 < \bar{f}_1$ such that the borrower's consumption under collateral constraints is greater than or equal to that in an unconstrained economy at least with $f_1 \in (\tilde{f}_1 < \bar{f}_1)$*

4. *The aggregate output under collateral constraints is smaller than or equal to that in an unconstrained economy.*

Proof. See Appendix B.5. □

The following result is a direct consequence of point 4 of Corollary 4.

Corollary 5 (efficiency)**.** *The economy with credit constraints is efficient if and only if*

$$\frac{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}{A_i^{\frac{1}{1-\alpha}}} \frac{S_i}{S_1 + S_2} + \frac{f_i}{\alpha} \geq 1 \ \forall i. \tag{A.11}$$

B Online appendix: formal proofs - Cobb-Douglas technology case

B.1 Proof of Lemma 3

Assume that $F'(0) = 0$, then at optimum, $k_i > 0$. The Lagrangian function is $L = A_i k_i^\alpha - r a_i + \lambda_i(s_i + a_i - k_i) + \mu_i(f_i A_i k_i^\alpha - r a_i)$. An allocation (k_i, a_i) is a solution if and only if there exist λ_i, μ_i such that

$$[k_i] : (1 + \mu_i f_i)\alpha A_i k_i^{\alpha-1} = \lambda_i$$
$$[a_i] : (1 + \mu_i) r = \lambda_i, \quad \mu_i \geq 0, \text{ and } \mu_i(f_i A_i k_i^\alpha - a_i) = 0.$$

These equations imply that:

$$\frac{\alpha A_i k_i^{\alpha-1}}{r} = \frac{1 + \mu_i}{1 + f_i \mu_i} \geq 1 \tag{B.1}$$

Case 1: The credit constraint is binding: $f_i A_i k_i^\alpha = r a_i$. In this case, (k_i, a_i) is the solutions of the following equations:

$$k_i^{1-\alpha} - \frac{S_i}{k_i^\alpha} = \frac{f_i A_i}{r} \tag{B.2}$$
$$a_i = k_i - S_i. \tag{B.3}$$

We see that the left hand-side of the equation (B.2) is an increasing function in k_i. And, the left-hand side of (B.2) goes to $-\infty$ as k_i goes to 0. Hence, the equation (B.2) has a solution such that $0 < k_i \leq (\frac{\alpha A_i}{r})^{\frac{1}{1-\alpha}}$ if and only if:

$$\frac{\alpha A_i}{r} - S_i\Big(\frac{r}{\alpha A_i}\Big)^{\frac{\alpha}{1-\alpha}} \geq \frac{f_i A_i}{r} \Leftrightarrow \quad \Big(\frac{r}{\alpha A_i S_i^{\alpha-1}}\Big)^{\frac{1}{1-\alpha}} \leq 1 - \frac{f_i}{\alpha}.$$

From (B.2), we see that: $k_i - S_i = \frac{f_i A_i}{r} k_i^\alpha > 0$. Therefore, in this case, agent i is always a borrower. We also see that $k_i \leq (\frac{\alpha A_i}{r})^{\frac{1}{1-\alpha}}$ implies that $\frac{\alpha A_i k_i^{\alpha-1}}{r} \geq 1$, and therefore $\mu_i \geq 0$.

Case 2: $f_i A_i k_i^\alpha > r a_i$. We see that $\mu_i = 0$, and hence $\frac{\alpha A_i k_i^{\alpha-1}}{r} = 1$, i.e, $k_i = (\frac{\alpha A_i}{r})^{\frac{1}{1-\alpha}}$. It remains to check that this value of k_i satisfies the condition: $f_i A_i k_i^\alpha > a_i$, i.e.,

$$r\Big((\frac{\alpha A_i}{r})^{\frac{1}{1-\alpha}} - S_i\Big) < f_i A_i(\frac{\alpha A_i}{r})^{\frac{\alpha}{1-\alpha}} \Leftrightarrow \quad \Big(\frac{r}{\alpha A_i S_i^{\alpha-1}}\Big)^{\frac{1}{1-\alpha}} > 1 - \frac{f_i}{\alpha}.$$

So, the solution (k_i, a_i) is given by $k_i = (\frac{\alpha A_i}{r})^{\frac{1}{1-\alpha}}, a_i = k_i - S_i$. In this case, agent i borrows (i.e $a_i > 0$) if and only if $\alpha A_i S_i^{\alpha-1} > r$ and lends if and only if $\alpha A_i S_i^{\alpha-1} \leq r$. $\square$

B.2 Proof of Proposition 6

Under our assumption ($\bar{r}_1 > \bar{r}_2$), we have that: if $f_1 > \alpha$, then $f_2 < \alpha$. We consider three cases.

Case 1: $f_1 < \alpha$ and $f_2 < \alpha$. Since $\bar{r}_1 > \bar{r}_2$, we have

$$\alpha A_1 S_1^{\alpha-1}(1 - \frac{f_1}{\alpha})^{1-\alpha} > \alpha A_2 S_2^{\alpha-1}(1 - \frac{f_2}{\alpha})^{1-\alpha} \Leftrightarrow \Big(\frac{A_2}{A_1}\Big)^{\frac{1}{1-\alpha}} < \frac{S_2}{S_1}\frac{1 - \frac{f_1}{\alpha}}{1 - \frac{f_2}{\alpha}}$$

Let (r, a_1, a_2, k_1, k_2) be an equilibrium. There exists an agent whose credit constraint is not binding. According to point 1 in Lemma 3, we have $r > min(\bar{r}_1, \bar{r}_2) = \bar{r}_2$. So, we will consider two cases: $\bar{r}_2 < r \leq \bar{r}_1$ and $r > \bar{r}_1$.

Case 1.1: $r > \bar{r}_1$. According to point 1 in Lemma 3, no credit constraint is binding. Hence, we find the same equilibrium as in the case without credit constraint. However, we have to check that both credit constraints are satisfied, i.e., $ra_i \leq f_i A_i k_i^{\alpha}$ for $i = 1, 2$. This condition is satisfied if and only if:

$$\frac{\alpha\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^{1-\alpha}}{(S_1+S_2)^{1-\alpha}}\Big(\frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}(S_1+S_2) - S_i\Big) \leq f_i A_i\Big(\frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}(S_1+S_2)\Big)^{\alpha}$$

$$\Leftrightarrow (1-\frac{f_i}{\alpha})\frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}} \leq \frac{S_i}{S_1+S_2}.$$

Condition $r > \bar{r}_i$ is equivalent to:

$$r = \frac{\alpha\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^{1-\alpha}}{(S_1+S_2)^{1-\alpha}} > \alpha A_i S_i^{\alpha-1}\big(1-\frac{f_i}{\alpha}\big)^{1-\alpha} = \bar{r}_1 \Leftrightarrow \frac{S_i}{S_1+S_2} \quad > \frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}\big(1-\frac{f_i}{\alpha}\big).$$

Notice that since $\bar{r}_1 > \bar{r}_2$, we have $\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\left(1-\frac{f_1}{\alpha}\right)} < \frac{S_2}{A_2^{\frac{1}{1-\alpha}}\left(1-\frac{f_2}{\alpha}\right)}$. So, condition $\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\left(1-\frac{f_1}{\alpha}\right)} > \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$ implies that $\frac{S_2}{A_2^{\frac{1}{1-\alpha}}\left(1-\frac{f_2}{\alpha}\right)} > \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$.

Case 1.2: $\bar{r}_2 < r \leq \bar{r}_1$. According to Lemma 3, agent 1's credit constraint is binding and hence agent 2 is lender. We have $a_1 = k_1 - S_1, a_2 = k_2 - S_2$ and $k_1^{1-\alpha} - \frac{S_1}{k_1^{\alpha}} = \frac{f_1 A_1}{r}$, $k_2 = \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}$. The market clearing condition $k_1 + k_2 = S_1 + S_2$ implies that

$$k_1 = S_1 + S_2 - \Big(\frac{\alpha A_2}{r}\Big)^{\frac{1}{1-\alpha}}, \text{ i.e., } k_1 = k_1(r) \equiv S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}} \tag{B.4}$$

Since $k_1^{1-\alpha} - \frac{S_1}{k_1^{\alpha}} = \frac{f_1 A_1}{r}$, we have the following equation determining the equilibrium interest rate

$$f(r) \equiv \Big(S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big)^{1-\alpha} - \frac{S_1}{\Big(S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big)^{\alpha}} - \frac{f_1 A_1}{r} = 0. \tag{B.5}$$

Since $k_1(r)$ is increasing in r, the function $f(r)$ is increasing in r. We have $f(0) = -\infty$. Let r^* be defined by $S_1 + S_2 - \big(\frac{\alpha A_2}{r^*}\big)^{\frac{1}{1-\alpha}} = 0$. We have $f(r^*) = +\infty$. So, the equation $f(r) = 0$ has a unique solution and this solution is in $(0, r^*)$.

We have to now check that (i) $r \in (\bar{r}_2, \bar{r}_1]$, and (ii) $\mu_i \geq 0$, i.e., $\alpha A_1 k_1^{\alpha-1} \geq r$. Notice that condition $\bar{r}_1 > \bar{r}$ implies that $k_1(\bar{r}_1) > 0$. So, we have $\bar{r}_2 < \bar{r}_1 < r^*$.

STEP 1. We firstly prove that $r > \bar{r}_2$. We have

$$k_1(\bar{r}_2) = S_1 + S_2 - \frac{S_2}{1-\frac{f_2}{\alpha}}.$$

If $k_1(\bar{r}_2) \leq 0$ then $k_1(\bar{r}_2) \leq 0 < k_1(r)$. This implies that $\bar{r}_2 < r$.
If $k_1(\bar{r}_2) > 0$, it is easy to see that $f(k_1(\bar{r}_2)) < 0$, and hence $\bar{r}_2 < r$.

STEP 2. We will prove that $\alpha A_1 k_1^{\alpha-1} \geq r$ and $r \leq \bar{r}_1$.

We see that $\alpha A_1 k_1^{\alpha-1} \geq r$ is equivalent to

$$k_1 = S_1 + S_2 - \Big(\frac{\alpha A_2}{r}\Big)^{\frac{1}{1-\alpha}} < \Big(\frac{\alpha A_1}{r}\Big)^{\frac{1}{1-\alpha}} \Leftrightarrow r \leq \bar{r}.$$

Since $\bar{r}_1 \geq \bar{r}$, it is sufficient to prove that $r \leq \bar{r}$. We will do so by proving that $f(\bar{r}) \geq 0$. One can check that this is equivalent to:

$$\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\big(1 - \frac{f_1}{\alpha}\big)} \leq \frac{S_1 + S_2}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}. \tag{B.6}$$

We now need to verify that under this condition, there will be an equilibrium mentioned in the part 2 of Proposition 2. We can do so by verifying budget constraints, first-order and market clearing conditions.

It should be noticed that $r \in (\hat{r}_2, \bar{r}]$ in this case.

Case 2: $f_1 < \alpha < f_2$. In this case, the credit constraint of agent 2 is non binding, and:

$$k_2 = \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}, \quad a_2 = k_2 - S_2.$$

Case 2.1. $r \leq \bar{r}_1$. This condition is satisfied if and only if $\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\big(1-\frac{f_1}{\alpha}\big)} > \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$. In this case, the credit constraint of agent 1 is also not binding, we get the same equilibrium outcomes as in the unconstrained model.

Case 2.2. $r > \bar{r}_1$. Using similar arguments as Case 1.2, we can find that the condition of parameters such that there exists an equilibrium in this case is

$$\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\big(1 - \frac{f_1}{\alpha}\big)} \leq \frac{S_1 + S_2}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}.$$

The equilibrium interest rate $r \in (\hat{r}_2, \bar{r}]$.

By combining Case 1 and Case 2, we see that:

If $\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\big(1-\frac{f_1}{\alpha}\big)} > \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$, no credit constraints bind, and the equilibrium outcomes coincide with the unconstrained economy.

If $\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\big(1-\frac{f_1}{\alpha}\big)} \leq \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$, credit constraint imposed on the borrower binds, and $r \in (\hat{r}_2, \bar{r}_1]$.

Case 3: $f_1 > \alpha, \quad f_2 > \alpha$, then no credit constraint is binding. We obtain the same general equilibrium outcomes as in the unconstrained economy.

Combine all three cases, we can derive results of Proposition 6.

□

B.3 Proof of Corollary 2

1. **Interes rate**. With Cobb-Douglas technology, when the credit constraint binds, and hence the equilibrium interest rate is determined by:

$$S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}} = \frac{f_1 A_1}{r}\big[S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\big]^{\alpha}$$
$$\Leftrightarrow f_1 A_1 = r\Big[S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big]^{1-\alpha} - S_1 r\Big[S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big]^{-\alpha}.$$

Denote $g(f_1,x) \equiv x\big[S_1+S_2-\big(\frac{\alpha A_2}{x}\big)^{\frac{1}{1-\alpha}}\big]^{1-\alpha} - S_1 x\big[S_1+S_2-\big(\frac{\alpha A_2}{x}\big)^{\frac{1}{1-\alpha}}\big]^{-\alpha} - f_1 A_1$. At equilibrium, we have $g(f_1,r)=0$ and we can compute

$$\frac{\partial g(f_1,r)}{\partial r} = \frac{\frac{\alpha S_1}{1-\alpha}\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}} + S_2}{S_1+S_2-\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}}\Big[S_1+S_2-\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big]^{-\alpha} > 0$$

$$\frac{\partial g(f_1,r)}{\partial f_2} = -A_1 < 0.$$

By the implicit theorem, $\frac{dr}{df_1} = -\frac{g'(f_1)}{g'(r)} > 0$, i.e, r is an increasing function in f_1.

2. **Output**. We have: $Y = A_1\big[S_1+S_2-\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\big]^{\alpha} + A_2\big(\frac{\alpha A_2}{r}\big)^{\frac{\alpha}{1-\alpha}}$. Then:

$$Y'(r) = \frac{\alpha}{1-\alpha}\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big[A_1\Big(S_1+S_2-\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big)^{\alpha-1} - A_2\Big].$$

Therefore, we see that

$$Y'(r) > 0 \Leftrightarrow \Big(S_1+S_2-\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big)^{\alpha-1}\frac{\alpha A_2}{r} - A_2 > 0$$
$$\Leftrightarrow r < \frac{\alpha\big((A_1)^{\frac{1}{1-\alpha}} + (A_2)^{\frac{1}{1-\alpha}}\big)^{1-\alpha}}{(S_1+S_2)^{1-\alpha}} \Leftrightarrow r < \bar{r}.$$

According to Proposition 6, the interest rate in the case where credit constraints binds is always smaller than or equal to that in case without credit constraint. We thus conclude that the output Y is in increasing in r, and as a result, is an increasing function of f_1.

□

B.4 Proof of Corollary 3.

1. We have: $c_2 = A_2\big(\frac{\alpha A_2}{r}\big)^{\frac{\alpha}{1-\alpha}} + r\big[S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\big]$. Then, $c_2'(r) = S_2\big[1-\big(\frac{\alpha A_2 S_2^{\alpha-1}}{r}\big)^{\frac{1}{1-\alpha}}\big]$. Since $\hat{r}_2 = \alpha A_2 S_2^{\alpha-1} \leq r$ at the equilibrium when credit constraint of the borrower binds, we thus have $c_2'(r) \geq 0$. As r is an increasing function in f_1, we thus see that the the lender's consumption is an increasing function in f_1.

2. Since $c_1 = A_1\big[S_1+S_2-\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\big]^{\alpha} - r\big[S_2-\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\big]$, we have

$$c_1'(r) = \frac{\alpha}{(1-\alpha)}\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big[\frac{A_1}{r}\Big(S_1+S_2-\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big)^{\alpha-1} - 1\Big] - S_2. \tag{B.7}$$

Let $P(r) \equiv \frac{A_1}{r}\Big(S_1+S_2-\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big)^{\alpha-1} - 1$. It is easy to see that $P(r)$ is a decreasing function in r. From Proposition 2, we know that when the credit constraint is binding, $r \in [\hat{r}_2, \bar{r})$. Hence, $P(r) \geq P(\bar{r})$ for any r. We also have $P(\bar{r}) = \frac{1}{\alpha} - 1 > 0$. Hence, $P(r) > 0$ for any $r \in [\hat{r}_2, \bar{r})$. This implies that $c_1'(r) = \frac{\alpha}{(1-\alpha)}\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big[\frac{A_1}{r}\Big(S_1+S_2-\big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big)^{\alpha-1} - 1\Big] - S_2$ is a decreasing function in r when $r \in (\hat{r}_2, \bar{r})$.

Let us recall that at equilibrium when credit constraint binds we have:

$$\frac{S_1}{A_1^{\frac{1}{1-\alpha}}(1-\frac{f_1}{\alpha})} \leq \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}.$$

One can derive from this condition that: $\frac{A_1}{A_2}\Big(\frac{S_2}{S_1}\Big)^{1-\alpha} > 1$. By consequence, at the point $r = \alpha A_2 S_2^{\alpha-1}$, we have

$$c_1'(\alpha A_2 S_2^{\alpha-1}) = \frac{S_2}{1-\alpha}\Big(\frac{A_1}{A_2}\big(\frac{S_2}{S_1}\big)^{1-\alpha} - 1\Big) > 0.$$

As we already proved that $Y'(\bar{r}) = 0$, and $c_2'(\bar{r}) > 0$, we have $c_1'(\bar{r}) = Y'(\bar{r}) - c_2'(\bar{r}) < 0$. Having proved that $c_1'(r)$ is a decreasing function in r for $r \in (\bar{r}_2, \bar{r})$, $c_1'(\hat{r}_2) > 0$, and $c_1'(\bar{r}) < 0$, we conclude that there exist a value $\tilde{r}$ such that $c_1'(r)$ is positive if $r \in (\bar{r}_2, \tilde{r})$ and negative if $r \in (\tilde{r}, \bar{r})$. As a result, $c_1(r)$ is increasing in r (or f_1) if $r \in (\bar{r}_2, \tilde{r})$ and is decreasing in r (or f_1) if $r \in (\tilde{r}, \bar{r})$.

□

B.5 Proof of Corollary 4.

1. The proof is straightforward. As already proved in Proposition 2, when the credit constraint imposed on the borrower binds, $r \in (\hat{r}_2, \bar{r}]$, where $\bar{r}$ is the interest rate in an economy without financial frictions.

2. As we already proved in Corollary 3, $c_2(r)$ is an increasing function in r. We also proved that $r \leq \bar{r}$ when the credit constraint on the borrower binds. Hence $c_2(r) \leq c_2(\bar{r})$, for any $r \in (\hat{r_2}, \bar{r}]$. Since $c_2(\bar{r})$ equals the lender's consumption in the case without credit constraint, it follows that the lender's consumption in case with credit constraints is always smaller than or equal to that in case without credit constraints.

3. From Corollary 3, we know that when credit constraint imposed on the borrower binds, there exist $\tilde{r} \in (\hat{r}_2, \bar{r}]$ such that, $c_1(r)$ is decreasing in r for $r \in [\tilde{r}, \bar{r}]$, and r is sufficiently high. Since r is an increasing function of f_1, we thus conclude that the borrower's consumption in a constrained economy could be higher than that in a non-constrained economy if the collateral is binding but not very strict.

4. As we proved in Corollary 3, $Y(r)$ is an increasing function in r for $r \in (\hat{r}_2, \bar{r}]$. Therefore $Y(r) \leq Y(\bar{r})$ for any $r \in (\hat{r}_2, \bar{r}]$.

One can easily check that $Y(\bar{r})$ equals the equilibrium aggregate output when there is no credit constraint. Thus, the aggregate output in the case with credit constraints is always smaller than or equal to that in case without credit constraints. □