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Abstract

We consider a cost allocation problem arising from a hub network problem design. Finding an optimal hub network is NP-hard, so we start with a hub network that could be optimal or not. Our main objective is to divide the cost of such network among the nodes. We consider two cases. In the one-way flow case, we assume that the cost paid by a set of nodes depends only on the flow they send to other nodes (including nodes outside the set), but not on the flow they receive from nodes outside. In the two-way flow case, we assume that the cost paid by a set of nodes depends on the flow they send to other nodes (including nodes outside the set) and also on the flow they receive from nodes outside. In both cases, we study the core and the Shapley value of the corresponding cost game.

Keywords: game theory, hub network, cost allocation, core, Shapley value.

1 Introduction

Hub networks play a fundamental role in modelling telecommunication, transportation, and parcel delivery systems. Assume that there are users located at different geographical nodes who need to send a certain flow of data or goods to each other through costly connections. A planner needs to locate an optimal number of hub facilities at some nodes so that each non-hub node is connected to exactly one hub and all the hubs are connected

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to one another at a reduced cost (due to economies of scale). Hence, the optimal flow of data/goods between any pair of origin-destination nodes has a length of at most four: It must go from the point of origin to its assigned hub (when the origin is not itself a hub), then to the hub assigned to the destination (if it is a different node) and finally to the destination (again, if it is not itself a hub). This topology is applied to Internet connections (Bailey, 1997), telecommunications between local networks (Greenfield, 2000), satellite communication (Helme and Magnanti, 1989), airline networks (Bryan and O'Kelly, 1999; Yang, 2009), small package delivery (Sim et al., 2009), and biofuel supply chains (Roni et al., 2017).

Several classes of hub problems have been studied. Recent examples include Contreras et al. (2017), Jankovic et al. (2017), Azizi (2018), Azizi et al. (2018), Alumur et al. (2018), and Güden (2018).

The main issue addressed in these papers is the study of algorithms for computing optimal ways of sending goods between the nodes in such a way that the total cost is minimized. Of course, location of the hubs plays a relevant role in the minimization problem. See Alumur and Kara (2008) and Farahani et al. (2013) for surveys on this literature.

Once we have computed the optimal (or quasi optimal) hub network, another issue is how to divide the cost associated with such hub network among the nodes. This question has been successfully addressed in several kinds of problems. We mention some of them. Guardiola et al. (2009) study production-inventory problems where players share production processes and warehouse facilities. Bergantiños et al. (2014), Bergantiños and Kar (2010), Bogomolnaia and Moulin (2010), Dutta and Mishra (2012), Trudeau (2012) and Trudeau and Vidal-Puga (2017) consider the cost of connecting agents to a source. Moulin (2014) considers users that need to connect a pair of target nodes in a network. Perea et al. (2009) consider the problem of sharing the profits associated with a supply chain problem. Alcalde-Unzu et al. (2015) consider the cost of cleaning a river.

Our paper is specially related with two of these problems. Bergantiños et al. (2014) study source connection problems where a group of nodes require a service that can only be provided from a source. Computing an optimal tree for such problems is NP hard. Then, given a tree, which can be optimal or not, the authors study the problem of dividing the cost of such tree among the nodes. We follow a similar approach because we divide the cost of a hub network among the nodes. Perea et al. (2009) consider a profit sharing problem associated with a supply chain problem. There is a network and three kinds of agents over the network: suppliers, intermediary centres, and retailers. The delivery of good from a supply node to a demand node generates a profit. Besides, the transportation of goods through the network has an associated cost. The authors associate a cooperative game to such problems and study the core. Finally, they propose an allocation in the core, which is not characterized. In this paper, we also study a transportation problem in a network. We associate a cooperative game, we study the core, and we propose an allocation in the core based on the Shapley value (Shapley, 1953). Besides, we provide an axiomatic characterization of such allocation.

However, few papers have studied the cost sharing issue associated with hub problems. We mention three of them.

Skorin-Kapov (1998) studies p-hub allocation problems, where p hubs must be optimally allocated. Several cooperative games are considered depending on who the agents are (nodes or pairs of nodes) and what coalitions can do (whether they must use the optimal network for the whole problem or can construct the optimal network of the reduced problem induced by the coalition). He studies the core of such games. Some games have an empty core but others have not. Finally, he considers the nucleolus of such games.

Skorin-Kapov (2001) studies hub-like networks, which involve a p-hub median problem where direct connection between nodes is possible. Moreover, there are savings when the traffic is high. He defines several associated cooperative games where the set of agents are the links. He shows that some of them can have an empty core, in other cases the core is a singleton, and in other cases it has many points.

Matsubayashi et al. (2005) consider the case where the number of hubs to be located is arbitrary, there is a cost of opening a hub, and there is a congestion cost associated with nodes (the greater the flow through a node, the greater its cost). They also define an associated cooperative game and study its core. In the cooperative game, players are the nodes and the characteristic function is defined assuming that each coalition cooperatively constructs a network. Moreover, each coalition assumes that the rest of the nodes do not establish any hub nodes and the coalition can determine the routing of all the traffic generated by the other nodes. Given this, they prove that the core could be empty, but they find a sufficient condition for the non-emptiness of the core and propose an allocation in the core when this sufficient condition is satisfied.

The three papers mentioned above follow a similar approach. The first step is to consider a class of hub problems, the next is to associate a cooperative game with each problem in the class, and the last one is to study the core of such problems. If the core is nonempty, an allocation in the core could be considered as a nice way of sharing the cost among agents.

Our paper also focuses on the cost sharing issue. We consider two cases. In the first case (called one-way flow) we assume that the cost paid by a set of nodes S depends only on the flow they send to other nodes (including nodes outside S), but not on the flow they receive from nodes outside S. For instance, when you use your mobile phone inside your country the amount you pay depends only on the calls you make. This is the most common case in practice and it has been considered in Skorin-Kapov (1998, 2001) and Matsubayashi et al. (2005). In the second case (called two-way flow) we assume that the cost paid by a set of nodes S depends on the flow they send to other nodes (including nodes outside S) and also on the flow they receive from nodes outside S. For instance, if you are travelling and you use your mobile phone outside your country, the amount you pay depends on the calls you make and also on the calls you receive.[1] On the other hand, you usually have to pay a minimum fare even in case you do not make calls, and only receive them. As far as we know this is the first paper considering this case.

Our main contributions are twofold. First, we study the existence of core allocations. Second, unlike Skorin-Kapov (1998, 2001) and Matsubayashi et al. (2005), we also characterize axiomatically rules that belong to the core and also satisfy other nice properties.

We now summarize our results for the one-way flow case. We consider two cooperative games associated with each hub problem and related to those presented by Skorin-Kapov (1998). In both games, the set of agents is the set of nodes. Nevertheless, the way in which we compute the cost game is different. In the first game, we consider that a network (which can be optimal or not) has been already constructed and nodes can use only the hubs associated with such network. Thus, we define the cost of a subset of nodes S as the cost of sending the flow of all agents in S using only the hubs available in the network. In the second game, we consider that each set of nodes can construct the hub network they want. Thus, we define the cost of a subset of nodes S as the cost of the optimal network we need for sending the flow of all agents in S.

We study the cores of both games. The core of the first game has many points. In any allocation in the core, each node pays the cost of sending its flow and the cost of any hub is divided in any way among the nodes that use such hub. As opposed, the core of the second game could be empty.

We study the Shapley value of the first game. In particular, we prove that the Shapley value corresponds to the allocation where each node pays the cost of sending its flow and the cost of any hub is divided equally among the nodes that use it. Thus, the Shapley value belongs to the core. We also provide two axiomatic characterizations of it. The first one uses core selection and equal treatment on hubs. Core selection says that the allocation must be in the core. Suppose that the cost of a hub increases. Consider a pair of agents such that both use the hub. Equal treatment on hubs says that the allocation to both agents change in the same amount. Alternatively, consider a pair of agents such that no one use the hub. Equal treatment on hubs also says that their cost allocations change in the same amount. The

[1] Assume that you are from Spain but you are in Argentina. If you receive a phone call from Spain, some phone companies consider it as an international call. The cost of this international call is divided by your company in two parts. The people who phone you pay the cost of a local call (inside Spain) and you pay the difference between the cost of an international call and the cost of a local call.

second characterization uses positivity (no node can obtain profits), equal treatment of hubs, independence of irrelevant hubs (nodes are not affected by a change in the cost of hubs that they do not use), and independence of irrelevant flows (if the flow between two nodes increases, then the other nodes should not be affected).

We now summarize our results for the two-way flow case. The study is similar to the one-way flow case. We associate two games with this setting. The first game is concave and hence its core is non empty. It consists of the convex hull of the vector of marginal contributions. As opposed, the second game may have an empty core.

We study the Shapley value of the first game. Since the game is concave, it belongs to the core. We prove that the Shapley value corresponds to the allocation where the cost of sending flow between any pair of nodes is divided equally between both nodes. Besides, the cost of each hub is divided equally between the nodes that use the hub. Finally, we provide two axiomatic characterizations. The first one uses core selection, equal treatment on hubs, and equal treatment on flows (if there is flow between a pair of nodes and it increases then both nodes must be affected in the same way). The second characterization uses positivity, independence of irrelevant hubs, independence of irrelevant flows, equal treatment on hubs, and equal treatment on flows.

The paper is organized as follows. In Section 2, we present the model. In Section 3, we study the one-way flow case, where the nodes are only interested in sending or receiving flow, but not both. In Section 4, we study the two-way flow case, where the nodes are interested in both sending and receiving flow. In Section 5 we give some examples. In Section 6, we present the conclusions.

2 The model

We consider situations where a group of agents, located at different locations, want to send and receive some specific good, which is sent through a costly network. Besides, we should locate some hubs at the agents' locations. All hub agents are connected to each other and each non-hub agent is connected only to a hub agent. We now introduce the model formally.

$N = \{1, \ldots, n\}$ is a finite set of nodes (also called agents).

$C = (c_{ij})_{i,j \in N}$ is a cost matrix. For each $i, j \in N$, c_{ij} is the cost of sending a unit of flow from node i to node j. We assume $c_{ii} = 0$, $c_{ij} = c_{ji} \geq 0$ and $c_{ik} \leq c_{ij} + c_{jk}$ for all $i, j, k \in N$.

$F = (f_{ij})_{i,j \in N}$ is the flow matrix. For each $i, j \in N$, f_{ij} represents the amount of flow from node i to node j. We assume $f_{ij} \geq 0$ and $f_{ii} = 0$ for all $i, j \in N$. Notice that we do not assume $f_{ij} = f_{ji}$, *i.e.* the flow is not necessarily symmetric.

Each coordinate in $d = (d_i)_{i \in N}$ indicates the cost of maintaining or constructing a hub at the respective node. We assume $d_i \geq 0$ for all $i \in N$.

Scalar $\alpha \in [0, 1]$ is the discounting factor of the cost when flow goes between a pair of hubs. Namely, if node i and node j are both hubs, then the cost of sending a unit of flow from node i to node j is αc_{ij} (instead of c_{ij}).[2]

The first issue is to locate an optimal number of hubs, selected from the set of nodes. Besides, each non-hub is linked to exactly one hub and all the hubs are connected to each other. The triangle inequality $c_{ik} \leq c_{ij} + c_{jk}$ assures that the optimal path origin-destination uses at most two hubs. When there is a hub in node $i \in N$, we say with some abuse of notation that node i is a hub. Otherwise, we say that node i is a non-hub.

A *hub network* on N is determined by a nonempty set $H \subseteq N$ and a function $h : N \setminus H \longrightarrow H$ such that $h(i)$ is the hub linked to non-hub i. Let $\mathcal{H}$ be the set of all hub networks on N. For notational convenience, we write $h(i) = i$ when $i \in H$, so that h is a function from N onto H. Besides, we also write $\overline{h}$ for the network associated with the function h. Namely

$$\overline{h} = \{\{i, h(i)\} : i \in N \setminus H\}.$$

Thus, given two nodes $i, j \in N$, flow from node i to node j goes first from node i to hub $h(i)$, then to hub $h(j)$ and finally to node j ($i = h(i)$ and/or $h(i) = h(j)$ and/or $h(j) = j$ are possible).

The *cost* of a hub network h is given by

$$\sum_{i \in N} \sum_{j \in N} \left(c_{ih(i)} + \alpha c_{h(i)h(j)} + c_{h(j)j} \right) f_{ij} + \sum_{i \in H} d_i.$$

For simplicity, we denote

$$\lambda_{ij}^h = \left(c_{ih(i)} + \alpha c_{h(i)h(j)} + c_{h(j)j} \right) f_{ij}$$

so that the cost is

$$\sum_{i \in N} \sum_{j \in N} \lambda_{ij}^h + \sum_{i \in H} d_i.$$

A hub network $h \in \mathcal{H}$ where

$$\min_{h \in \mathcal{H}} \left\{ \sum_{i \in N} \sum_{j \in N} \lambda_{ij}^h + \sum_{i \in H} d_i \right\}$$

is reached is called *optimal*. Since $\mathcal{H}$ is finite, there is always at least one optimal hub network.

[2] A generalization would be to assume that these costs are given by another cost matrix $C^h = \left(c_{ij}^h\right)_{i,j \in N}$ with $c_{ij}^h \leq c_{ij}$ for all $i, j \in N$. In our case, $C^h = \alpha C$.

We define a *hub network problem* as a tuple $P = (N, C, F, d, \alpha, h)$, where h is a hub network.

Notice that we do not assume h to be an optimal hub network. We know that computing an optimal hub network is *NP*-hard. Thus, in many practical situations we use heuristics to decide the hub network h to be constructed. Hence, we do not know exactly if such hub network is optimal or not. We make a very weak assumption on h, all hubs are needed in order to send the flow. Namely, for all $k \in H$ there exist $i, j \in N$ with $f_{ij} > 0$ and $k \in \{h(i), h(j)\}$. If h is an optimal hub network the assumption holds. In case h has been obtained using some heuristics, it seems reasonable to assume that the heuristics can detect when a hub is needed.

We now define $c(P)$ as the cost associated with the hub network h. Namely,

$$c(P) = \sum_{i \in N} \sum_{j \in N} \lambda_{ij}^{h} + \sum_{i \in H} d_i. \tag{1}$$

In many cases after finding an optimal (or quasi optimal) hub network, we need to divide the cost of such network among the nodes. A *rule* is a function R that assigns to each hub network problem P an allocation $R(P) \in \mathbb{R}^N$ satisfying

$$\sum_{i \in N} R_i(P) = c(P). \tag{2}$$

Our aim is to study the cost allocation problem generated by each hub network problem P. We are interested in studying allocations both fair and stable. The idea is to propose desirable properties and try to find a rule satisfying many of them.

2.1 Properties

We consider two cases depending on the needs of the agents. In the *one-way flow* case, we assume that each agent is only interested in the outgoing flow (the case where each agent is only interested in the ingoing flow is analogous). In the *two-way flow* case, each agent is interested in both outgoing and ingoing flow.

We now define several properties. Since the definitions in the one-way and two-way are quite similar (some of them are exactly the same), we only define it once.

The first property says that no agent should obtain profit.

Positivity (*Pos*) For any hub network problem P and each $i \in N$, we have $R_i(P) \geq 0$.

The second property says that equal agents must pay the same. Since we are dealing with situations where h is given, we should consider such hub network when defining equal nodes. Thus, given a hub network problem P

we say that nodes i and j are *equal* when several conditions hold: First, $f_{ik} = f_{jk}$ for all $k \in N \setminus \{i, j\}$ (in the two-way case we should add the condition $f_{ki} = f_{kj}$ for all $k \in N \setminus \{i, j\}$). Second, $f_{ij} = f_{ji}$. Third, $i \in H$ if and only if $j \in H$ (namely, node i is a hub if and only if node j is a hub). Forth, $\{i, k\} \in \overline{h}$ if and only if $\{j, k\} \in \overline{h}$ (namely, if nodes i and j are nonhubs then both are connected to the same hub). Fifth, for each $\{i, k\}, \{j, k\} \in \overline{h}$, $c_{ik} = c_{jk}$.

Equal Treatment of Equals (ETE) For any hub network problem P and each pair of equal nodes $i, j \in N$, we have that $R_i(P) = R_j(P)$.

The next property says that if a node does not send any flow in the one-way flow case (respectively, a node does not send nor receive any flow in the two-way flow case), then it pays nothing.

Null Flow (NF) For any hub network problem P and each $i \in N$ such that $f_{ij} = 0$ for all $j \in N \setminus \{i\}$ (in the two-way we must add the condition $f_{ji} = 0$ for all $j \in N \setminus \{i\}$), we have that $R_i(P) = 0$.

The next property says that if the flow from node i to node j increases, then in the one-way flow case node i cannot pay less, whereas in the two-way flow case node i cannot pay less and also node j cannot pay less,

Flow Monotonicity (FM) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F', d, \alpha, h)$ such that there exist $i, j \in N$ satisfying $f_{ij} \geq f'_{ij}$ and $f_{kl} = f'_{kl}$ otherwise, then $R_i(P) \geq R_i(P')$ (in the two-way we must add the condition $R_j(P) \geq R_j(P')$).

The next property says that if the cost of a hub increases, then no node requiring such hub could pay less. Before giving the formal definition we introduce some notation.

For each $S \subseteq N$, let $H_S^{of} \subseteq H$ denote the set of hubs used for sending the flow of agents in S. Namely,

$$H_S^{of} = \{k \in H : \exists i \in S, j \in N \text{ with } f_{ij} > 0 \text{ and } k \in \{h(i), h(j)\}\}.$$

Notice that $H_S^{of} \subseteq H$ is the set of hubs used by nodes in S in the one-way flow case.

Given $i \in N$, we write H_i^{of} instead of $H_{\{i\}}^{of}$. Notice that $H_S^{of} = \bigcup_{i \in S} H_i^{of}$ for all $S \subseteq N$.

For each $S \subseteq N$, let $H_S^{tf} \subseteq H$ denote the set of hubs used for sending or receiving the flow of agents in S. Namely,

$$H_S^{tf} = H_S^{of} \cup \{k \in H : \exists i \in S, j \in N \text{ with } f_{ji} > 0 \text{ and } k \in \{h(i), h(j)\}\}.$$

Notice that $H_S^{tf} \subseteq H$ is the set of hubs used by nodes in S in the two-way case.

Given $i \in N$, we write H_i^{tf} instead of $H_{\{i\}}^{tf}$. Again, $H_S^{tf} = \bigcup_{i \in S} H_i^{tf}$ for all $S \subseteq N$.

Hub Monotonicity (HM) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F, d', \alpha, h)$ such that there exists $k \in N$ satisfying $d_k \geq d'_k$ and $d_j = d'_j$ otherwise, then for each agent i such that $k \in H_i^{of}$ (in the two-way flow case, we replace H_i^{of} by H_i^{tf}), we have that $R_i(P) \geq R_i(P')$.

The next property says that if the cost of a link increases, then the two agents located at its vertices could not pay less. This property coincides for the one-way and two-way flow cases.

Cost Monotonicity (CM) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C', F, d, \alpha, h)$ such that there exists $i, j \in N$ satisfying $c_{ij} \geq c'_{ij}$ and $c_{kl} = c'_{kl}$ otherwise, then we have that $R_i(P) \geq R_i(P')$ and $R_j(P) \geq R_j(P')$.

Assume that the cost of some hub d_k decreases. It is then clear that if h was an optimal hub network in the original problem it will be also optimal in the new problem. How should agents be affected? The next two properties provide an answer to this question.

The first one says that agents that use hub k or do not use hub k are affected in the same way.

Equal Treatment on Hubs (ETH) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F, d', \alpha, h)$ such that there exists $k \in N$ satisfying $d_k \geq d'_k$ and $d_j = d'_j$ otherwise, then for each pair of agents i, j such that $k \in H_i^{of} \cap H_j^{of}$ or $k \notin H_i^{of} \cup H_j^{of}$ (in the two-way flow case, we replace $k \in H_i^{of} \cap H_j^{of}$ by $k \in H_i^{tf} \cap H_j^{tf}$ and $k \notin H_i^{of} \cup H_j^{of}$ by $k \notin H_i^{tf} \cup H_j^{tf}$), we have that

$$R_i(P) - R_i\left(P'\right) = R_j(P) - R_j\left(P'\right).$$

The second property says that agents that do not use hub k are not affected.

Independence of Irrelevant Hubs (IIH) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F, d', \alpha, h)$ such that there exists $k \in N$ satisfying $d_k \geq d'_k$ and $d_j = d'_j$ otherwise, then $R_i(P) = R_i(P')$ for each agent i such that $k \notin H_i^{of}$ (in the two-way flow case, we replace $k \notin H_i^{of}$ by $k \notin H_i^{tf}$).

We now introduce a similar property to IIH but with flows instead of hubs. Assume that node i increases its flow to some other node j. In the one-way flow case, all nodes but i should not be affected. In the two-way flow case, all nodes but i and j should not be affected.

Independence of Irrelevant Flows (IIF) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F', d, \alpha, h)$ such that there exist $j, k \in N$ satisfying $0 < f'_{jk} \leq f_{jk}$ and $f'_{j'k'} = f_{j'k'}$ otherwise, then $R_i(P) = R_i(P')$ for each agent $i \in N \setminus \{j\}$ (in the two-way flow case, we replace $i \in N \setminus \{j\}$ by $i \in N \setminus \{j, k\}$).

The next property is not reasonable in the one-way flow case but it is in the two-way case. It says that a variation of flow affects the sender and the receiver in the same way.

Equal Treatment on Flows (ETF) For any pair of hub network problems $P = (N, C, F, d, \alpha, h)$ and $P' = (N, C, F', d, \alpha, h)$ such that there exist $k, l \in N$ satisfying $0 < f'_{kl} \leq f_{kl}$ and $f'_{ij} = f_{ij}$ otherwise, we have that

$$R_i(P) - R_i\left(P'\right) = R_j(P) - R_j\left(P'\right)$$

for all pair of agents i, j such that $\{i, j\} = \{k, l\}$ or $\{i, j\} \cap \{k, l\} = \emptyset$.

2.2 Cooperative game concepts

We finally introduce some well-known concepts of cooperative game theory.

A *cost game* is a pair $(N, \hat{c})$ where N is the set of agents and $\hat{c} : 2^N \to \mathbb{R}$ is a cost function satisfying $\hat{c}(\emptyset) = 0$. Each nonempty subset $S \subseteq N$ is called a *coalition*, and $\hat{c}(S)$ denotes the cost of providing the needs of all agents in S. Since $\hat{c}$ depends on N, we write $\hat{c}$ instead of $(N, \hat{c})$.

We say that $\hat{c}$ is *concave* if for all $l \in T \subset S \subseteq N$, we have $\hat{c}(S) - \hat{c}(S \setminus \{l\}) \leq \hat{c}(T) - \hat{c}(T \setminus \{l\})$.

We now introduce two well-known solution concepts in cooperative game theory: the core (Gillies, 1959) and the Shapley value (Shapley, 1953).

The *core* of a cost game $\hat{c}$ is defined as

$$Core(\hat{c}) = \left\{ y \in \mathbb{R}^N : \sum_{i \in N} y_i = \hat{c}(N) \text{ and } \sum_{i \in S} y_i \leq \hat{c}(S) \, \forall S \subset N \right\}.$$

The main motivation behind the core is "stability". Namely, to identify the set of allocations such that no coalition of agents has incentives to leave the grand coalition (N). It is well-known that, in general cost games, the core may be empty. Nevertheless, when the cost game is concave, the core is non-empty.

The Shapley value is defined as the allocation $Sh(\hat{c})$ such that

$$Sh_i(\hat{c}) = \sum_{S \subset N \backslash \{i\}} \frac{|S|!\,(n - |S| - 1)!}{n!} \left[\hat{c}(S \cup \{i\}) - \hat{c}(S)\right]$$

for each $i \in N$.

The main motivation behind the Shapley value is "fairnes". Namely, to provide a fair way of dividing the total cost among the agents. Shapley (1953) proved that the Shapley value is the unique allocation satisfying the following four properties: efficiency (the total cost is divided among the agents), symmetry (symmetric agents must pay the same), null agent (if an agent does not increase the cost of any coalition, such agent should pay nothing) and additivity (the allocation should be additive on the cost function). Later on, several authors provided other characterizations of the Shapley value which made it very popular as a fair outcome. It is well-known that the Shapley value can be outside the core even when the core is non-empty. Nevertheless, when the cost game is concave, the Shapley value belongs to the core.

3 One-way flow

In this section we assume that nodes are interested only in the outgoing flow. Namely, the cost of a group of nodes depends only on the outgoing flow of such nodes. We first associate to each hub network problem a cost game. Later, we study its core and its Shapley value.

For each hub network problem P, we associate the cost game c_P^{of} where for each $S \subseteq N$, $c_P^{of}(S)$ is the cost of sending the flow of all nodes in S to all nodes through the hub network h. The cost game c_P^{of} models situations where the hub network h (with associated set of hubs H) has already been constructed. Thus, d could be considered as a vector of maintenance costs. This cost game is formally defined as

$$c_P^{of}(S) = \sum_{i \in S} \sum_{j \in N} \lambda_{ij}^h + \sum_{i \in H_S^{of}} d_i \tag{3}$$

for all $S \subseteq N$. When no confusion arises we write $c^{of}(S)$ instead of $c_P^{of}(S)$.

Remark 3.1 *Skorin-Kapov (1998) associates several games with each hub network problem. One of them, denoted as c_1, is closely related to c^{of}. In our model, when h has a fixed number of hubs and $d_i = 0$ for all i, c^{of} coincides with c_1. Thus, c^{of} can be considered as a generalization of c_1 to our model. Besides, Skorin-Kapov (1998) proves that the core of c_1 contains the single allocation where each agent pays the cost of sending its flow.*

3.1 The core

In the next theorem we prove that the core of c^{of} is the set of cost allocations in which each agent pays the cost of sending its flow. Besides, the cost of any hub is divided in any way among the agents that use the hub for sending their flow.

Theorem 3.1 *For each hub network problem P, the core of c^{of} is nonempty, and it is given by*

$$Core\left(c^{of}\right)=\left\{\begin{array}{c} x\in\mathbb{R}^N:\sum_{i\in N}x_i=c(P), x_i=\sum_{j\in N}\lambda_{ij}^h+y_i\forall i\in N \\ \text{where } y\in\mathbb{R}_+^N \text{ and } \sum_{i\in S}y_i\leq\sum_{i\in H_S^{of}}d_i\forall S\subset N \end{array}\right\}.$$

Proof. "$\supseteq$" is obvious. We now prove "$\subseteq$". Let $x\in Core\left(c^{of}\right)$. For each $i\in N$, we define $y_i=x_i-\sum_{j\in N}\lambda_{ij}^h$. Then, for each $i\in N$, x_i can be rewritten as $x_i=\sum_{j\in N}\lambda_{ij}^h+y_i$. Since, $x\in Core\left(c^{of}\right)$, for each $S\subset N$,

$$c^{of}(S)=\sum_{i\in S}\sum_{j\in N}\lambda_{ij}^h+\sum_{i\in H_S^{of}}d_i\geq\sum_{i\in S}x_i=\sum_{i\in S}\sum_{j\in N}\lambda_{ij}^h+\sum_{i\in S}y_i$$

and thus $\sum_{i\in S}y_i\leq\sum_{i\in H_S^{of}}d_i$. It only remains to prove that $y\in\mathbb{R}_+^N$. Suppose not. Let $j\in N$ be such that $y_j<0$. Thus,

$$\begin{aligned}\sum_{i\in N\setminus\{j\}}x_i&=\sum_{i\in N}x_i-x_j=c^{of}(N)-x_j\\&=\sum_{i\in N}\sum_{k\in N}\lambda_{ik}^h+\sum_{i\in H}d_i-\sum_{k\in N}\lambda_{jk}^h-y_j\\&=\sum_{i\in N\setminus\{j\}}\sum_{k\in N}\lambda_{ik}^h+\sum_{i\in H}d_i-y_j\\&>\sum_{i\in N\setminus\{j\}}\sum_{k\in N}\lambda_{ik}^h+\sum_{i\in H}d_i\end{aligned}$$

since $H_{N\setminus\{j\}}^{of}\subseteq H$,

$$\geq\sum_{i\in N\setminus\{j\}}\sum_{k\in N}\lambda_{ik}^h+\sum_{i\in H_{N\setminus\{j\}}^{of}}d_i=c^{of}(N\setminus\{j\})$$

which is a contradiction. ■

Skorin-Kapov (1998) also considers another game, denoted as c_1^*, which is obtained as c_1 but assuming that each coalition can build their optimal hub network. Namely, instead of using the hubs given by h, each coalition can locate hubs wherever they prefer. Skorin-Kapov (1998) proves that the core of c_1^* can be empty. Since, as argued in Remark 3.1, any cost game c^* is a particular case of some c^{of}, this negative result also holds in our model.

Nevertheless, we can study an intermediate situation. Assume that the optimal hub network is not unique. Thus, we should decide which one to construct. It could be the case that some agents prefer one over another. Thus, we can define the cost of a coalition as the minimum over all optimal hub networks. Namely, for each $S \subseteq N$,

$$c^*(S) = \min_{h \in \mathcal{H}, h \text{ is optimal}} \left\{ c^{of}_{P(h)}(S) \right\}$$

where $P(h)$ is the hub network problem induced by the optimal hub network h.

Next example shows that the core of c^* can be empty.

Example 3.1 *Let $N = \{1,2,3\}$, $c_{ij} = 1$ for all $i,j \in N$, $f_{12} = f_{23} = f_{31} = 1$, $f_{21} = f_{32} = f_{13} = 10$, $\alpha = 1$, and $d_i = 6$ for all $i \in N$. There exist three optimal hub networks $\{h^i\}_{i \in N}$, corresponding to putting a single hub in each node $i \in N$, respectively. We will see that each two-node coalition would prefer a different hub location. We start with coalition $\{1,2\}$. We compute $c^{of}_{P(h^1)}(\{1,2\})$, $c^{of}_{P(h^2)}(\{1,2\})$, and $c^{of}_{P(h^3)}(\{1,2\})$:*

$$\begin{aligned}
c^{of}_{P(h^1)}(\{1,2\}) &= \sum_{i \in \{1,2\}} \sum_{j \in N} \lambda^{h^1}_{ij} + \sum_{i \in H^{of}_{\{1,2\}}} d_i \\
&= \lambda^{h^1}_{12} + \lambda^{h^1}_{13} + \lambda^{h^1}_{21} + \lambda^{h^1}_{23} + d_1 \\
&= (c_{12})f_{12} + (c_{13})f_{13} + (c_{21})f_{21} + (c_{21} + c_{13})f_{23} + d_1 \\
&= (1)1 + (1)10 + (1)10 + (1+1)1 + 6 \\
&= 29.
\end{aligned}$$

$$\begin{aligned}
c^{of}_{P(h^2)}(\{1,2\}) &= \sum_{i \in \{1,2\}} \sum_{j \in N} \lambda^{h^2}_{ij} + \sum_{i \in H^{of}_{\{1,2\}}} d_i \\
&= \lambda^{h^2}_{12} + \lambda^{h^2}_{13} + \lambda^{h^2}_{21} + \lambda^{h^2}_{23} + d_2 \\
&= (c_{12})f_{12} + (c_{12} + c_{23})f_{13} + (c_{21})f_{21} + (c_{23})f_{23} + d_2 \\
&= (1)1 + (1+1)10 + (1)10 + (1)1 + 6 \\
&= 38.
\end{aligned}$$

$$\begin{aligned}
c^{of}_{P(h^3)}(\{1,2\}) &= \sum_{i \in \{1,2\}} \sum_{j \in N} \lambda^{h^3}_{ij} + \sum_{i \in H^{of}_{\{1,2\}}} d_i \\
&= \lambda^{h^3}_{12} + \lambda^{h^3}_{13} + \lambda^{h^3}_{21} + \lambda^{h^3}_{23} + d_3 \\
&= (c_{13} + c_{32})f_{12} + (c_{13})f_{13} + (c_{23} + c_{31})f_{21} + (c_{22})f_{23} + d_3 \\
&= (1+1)1 + (1)10 + (1+1)10 + (1)1 + 6 \\
&= 39.
\end{aligned}$$

Thus, coalition $\{1,2\}$ *would prefer the hub to be at* 1*, because*

$$c^{of}_{P(h^1)}(\{1,2\}) = 29 < \min\left\{c^{of}_{P(h^2)}(\{1,2\}), c^{of}_{P(h^3)}(\{1,2\})\right\}.$$

Analogously, coalition $\{1,3\}$ *would prefer to locate the hub at* 3*, because*

$$c^{of}_{P(h^3)}(\{1,3\}) = 29 < \min\left\{c^{of}_{P(h^1)}(\{1,3\}), c^{of}_{P(h^2)}(\{1,3\})\right\};$$

and coalition $\{2,3\}$ *would prefer to locate the hub at* 2*, because*

$$c^{of}_{P(h^2)}(\{2,3\}) = 29 < \min\left\{c^{of}_{P(h^1)}(\{2,3\}), c^{of}_{P(h^3)}(\{2,3\})\right\}.$$

Let x *be a core allocation. Then*

$$\begin{aligned} 100 &= 2c^*(N) = 2(x_1 + x_2 + x_3) \\ &= (x_1 + x_2) + (x_1 + x_3) + (x_2 + x_3) \\ &\leq c^{of}_{P(h^1)}(\{1,2\}) + c^{of}_{P(h^3)}(\{1,3\}) + c^{of}_{P(h^2)}(\{2,3\}) \\ &= 29 + 29 + 29 = 87, \end{aligned}$$

which is a contradiction.

Thus, the core of c^* *is empty.*

3.2 The Shapley value

We now study the Shapley value of c^{of}, which we call the *Shapley rule*. We first give an explicit formula. Later, we provide two axiomatic characterizations.

In next theorem we prove that in the Shapley rule each node pays the cost of sending its flow. Besides, the cost of any hub is divided equally among the nodes that use the hub for sending their flow.

Theorem 3.2 *For each hub network problem* P *and each* $i \in N$*,*

$$Sh_i\left(c^{of}_P\right) = \sum_{j\in N} \lambda^h_{ij} + \sum_{j\in H^{of}_i} \frac{d_j}{\left|\left\{k \in N : j \in H^{of}_k\right\}\right|}.$$

Proof. We consider several cost games. Let c^0 be defined as $c^0(S) = \sum_{i\in S}\sum_{j\in N} \lambda^h_{ij}$ for each $S \subseteq N$. For each $j \in N$, let c^j be defined as $c^j(S) = d_j$ if $j \in H^{of}_S$ and $c^j(S) = 0$ otherwise. Thus, for each $S \subseteq N$, $c^{of}(S) = c^0(S) + \sum_{j\in N} c^j(S)$. Since the Shapley value is additive on c, we have that for each $i \in N$, $Sh_i\left(c^{of}\right) = Sh_i\left(c^0\right) + \sum_{j\in N} Sh_i\left(c^j\right)$. Since c^0 is an additive game (there exists $a \in \mathbb{R}^N$ such that for each $S \subseteq N$, $c^0(S) = \sum_{j\in S} a_i$) we deduce that $Sh_i\left(c^0\right) = \sum_{j\in N} \lambda^h_{ij}$. For each $j \in N$, in the cost game c^j, all agents that use hub j (i.e. all $k \in N$ such that $j \in H_k^{of}$) are symmetric and the agents that do not use hub j are dummy. Thus, for each $j \in N$,

$$Sh_i\left(c^j\right) = \begin{cases} \frac{d_j}{\left|\left\{k \in N : j \in H_k^{of}\right\}\right|} & \text{if } j \in H_i^{of} \\ 0 & \text{otherwise,} \end{cases}$$

from where it is straightforward to check the result. ■

We now introduce a new property of rules which says that we must select an allocation in the core of the problem.

Core Selection (CS) For any hub network problem P, we have that

$$R(P) \in Core\left(c_P^{of}\right).$$

There are some relations between CS and some ot the properties introduced in Subsection 2.1.

Proposition 3.1 *(a) CS implies Pos.*

(b) Pos, IIH and IIF imply CS.

Proof. (a) Assume $x \in Core\left(c^{of}\right)$. Then, for all $i \in N$,

$$x_i = c^{of}(N) - \sum_{j \in N \setminus \{i\}} x_j \geq c^{of}(N) - c^{of}(N \setminus \{i\}) \geq \sum_{j \in N \setminus \{i\}} \lambda_{ij}^h \geq 0.$$

(b) Let R be a rule satisfying Pos, IIH, and IIF. Fix $S \subset N$. Let $\varepsilon > 0$ and define $P^{S,\varepsilon} = \left(N, C, F^{S,\varepsilon}, d^S, \alpha, h\right)$ as the problem obtained from P by turning all positive flows not used by S into ε and all hub costs not used by S into zero. Formally,

$$f_{ij}^{S,\varepsilon} = \begin{cases} \varepsilon & \text{if } i \notin S \text{ and } f_{ij} > 0 \\ f_{ij} & \text{otherwise} \end{cases}$$

and

$$d_k^S = \begin{cases} 0 & \text{if } k \notin H_S^{of} \\ d_k & \text{otherwise.} \end{cases}$$

Then, $c_{P^{S,\varepsilon}}^{of}(N) \leq \sum_{i \in S} \sum_{j \in N} \lambda_{ij}^h + \sum_{k \in H_S^{of}} d_k + a(P)\varepsilon$ where

$$a(P) = \left|\{f_{ij} : f_{ij} > 0\}\right| \max\left\{\frac{\lambda_{ij}^h}{f_{ij}} : f_{ij} > 0\right\}.$$

Now,

$$\sum_{i\in S} R_i(P) \overset{IIH+IIF}{=} \sum_{i\in S} R_i\left(P^{S,\varepsilon}\right) = c^{of}_{P^{S,\varepsilon}}(N) - \sum_{i\in N\setminus S} R_i\left(P^{S,\varepsilon}\right)$$
$$\overset{Pos}{\leq} c^{of}_{P^{S,\varepsilon}}(N) \leq \sum_{i\in S}\sum_{j\in N}\lambda^h_{ij} + \sum_{k\in H^{of}_S} d_k + a(P)\varepsilon$$
$$= c^{of}_P(S) + a(P)\varepsilon$$

which implies $\sum_{i\in S} R_i(P) \leq c^{of}_P(S)$ because $a(P)$ does not depend on ε. ■

CS does not imply neither IIH nor IIF. The rule in which each node pays the cost of sending its flow and the cost of each hub is paid equally by the nodes that use the most expensive hubs among those that use that hub satisfies CS but not IIH. The rule in which each node pays the cost of sending its flow and the cost of each hub is paid equally by the nodes sending more flow through this hub satisfies CS but not IIF.

In next proposition we prove that the Shapley rule satisfies all the properties introduced in Subsection 2.1 for the one-way flow case.

Proposition 3.2 *The Shapley rule satisfies Pos, ETE, CS, NF, FM, HM, CM, ETH, IIH and IIF.*

Proof. From Theorem 3.2, we deduce that $Sh\left(c^{of}\right)$ satisfies Pos, FM, HM, and CM. If i and j are equal in P, then it is not difficult to check that i and j are symmetric in c^{of}. Now, symmetry of the Shapley value implies that $Sh\left(c^{of}\right)$ satisfies ETE. Any $i \in N$ with $f_{ij} = 0$ for all $j \in N \setminus \{i\}$ is a dummy player in c^{of}. Hence, its Shapley value is zero, and so $Sh\left(c^{of}\right)$ satisfies NF. Let P, P' and k be given as in the definition of ETH and IIH. Given $i, j \in N$ such that $k \in H^{of}_i \cap H^{of}_j$, under Theorem 3.2,

$$Sh_i\left(c^{of}_P\right) - Sh_i\left(c^{of}_{P'}\right) = \frac{d_k - d'_k}{\left|\left\{l \in N : k \in H^{of}_l\right\}\right|} = Sh_j\left(c^{of}_P\right) - Sh_j\left(c^{of}_{P'}\right).$$

Given $i, j \in N$ such that $k \notin H^{of}_i \cup H^{of}_j$, by Theorem 3.2

$$Sh_i\left(c^{of}_P\right) - Sh_i\left(c^{of}_{P'}\right) = 0 = Sh_j\left(c^{of}_P\right) - Sh_j\left(c^{of}_{P'}\right).$$

Hence $Sh\left(c^{of}\right)$ satisfies ETH. Given $i \in N$ such that $k \notin H^{of}_i$, from Theorem 3.2 we know that $Sh_i\left(c^{of}\right)$ does not depend on d_k, and so $Sh_i\left(c^{of}_P\right) = Sh_i\left(c^{of}_{P'}\right)$ and hence $Sh\left(c^{of}\right)$ satisfies IIH. Let P, P' and i, j, k be given as in the definition of IIF. From Theorem 3.2 we have that $Sh_i\left(c^{of}\right)$ does not depend on f_{jk}. Hence, $Sh\left(c^{of}\right)$ satisfies IIF. From Proposition 3.1, it satisfies CS. ■

We now give two characterizations of the Shapley rule.

Theorem 3.3 (a) *The Shapley rule is the unique rule satisfying CS and ETH.*

(b) *The Shapley rule is the unique rule satisfying Pos, IIH, IIF, and ETH.*

Proof. (a) By Proposition 3.2 the Shapley rule satisfies these properties. We now prove the uniqueness. Let R be a rule satisfying CS and ETH. Let $P = (N, C, F, d, \alpha, h)$ be any hub network problem. For each $K \subseteq H$, let $P^K = \left(N, C, F, d^K, \alpha, h\right)$ with d^K defined as follows:

$$d_i^K = \begin{cases} 0 & \text{if } i \in H \setminus K \\ d_i & \text{otherwise.} \end{cases}$$

For all $k \in N$, let $N^{k,0} = \left\{i \in N : k \notin H_i^{of}\right\}$, $N^{k,1} = \left\{i \in N : k \in H_i^{of}\right\}$, $n^{k,0} = \left|N^{k,0}\right|$ and $n^{k,1} = \left|N^{k,1}\right|$ for all $k \in N$. ETH implies that, for each $k \in K$, there exist $x^{k,0}$ and $x^{k,1}$ such that for all $i \in N^{k,0}$,

$$R_i\left(P^K\right) - R_i\left(P^{K\setminus\{k\}}\right) = x^{k,0} \tag{4}$$

and for all $i \in N^{k,1}$

$$R_i\left(P^K\right) - R_i\left(P^{K\setminus\{k\}}\right) = x^{k,1}. \tag{5}$$

Since $N = N^{k,0} \cup N^{k,1}$ and

$$\sum_{i\in N} R_i\left(P^K\right) - \sum_{i\in N} R_i\left(P^{K\setminus\{k\}}\right) = d_k,$$

we have that for all $k \in K$,

$$n^{k,0}x^{k,0} + n^{k,1}x^{k,1} = d_k. \tag{6}$$

The equivalence relation in N defined as

$$i \sim j \Leftrightarrow \exists k \in K : i, j \in N^{k,1} \text{ or } i, j \in N^{k,0}$$

determines a partition $\mathcal{P}_K$ of N. It is straightforward to check that the cost game $c_{P^K}^{of}(N) = \sum_{S\in\mathcal{P}_K} c_{P^K}^{of}(S)$. So CS implies that $\sum_{i\in S} R_i\left(P^K\right) = c_{P^K}^{of}(S)$ for all $S \in \mathcal{P}_K$. Moreover, any $\mathcal{P}_L$ with $L \subset K$ is a refinement of $\mathcal{P}_K$, so $\sum_{i\in S} R_i\left(P^L\right) = c_{P^L}^{of}(S)$ for all $S \in \mathcal{P}_K$.

We now consider several cases.

Case 1. Assume that $\mathcal{P}_K$ has at least two components. Given $k \in K$, there exist $S, S' \in \mathcal{P}_K$ such that $k \in S \cap H$ and $S' \subseteq N^{k,0}$. Besides, $c_{P^{K\setminus\{k\}}}^{of}(S') = c_{P^K}^{of}(S')$. Thus,

$$\begin{aligned} c_{P^K}^{of}\left(S'\right) &= \sum_{i\in S'} R_i\left(P^K\right) \overset{(4)}{=} \sum_{i\in S'} R_i\left(P^{K\setminus\{k\}}\right) + \left|S'\right| x^{k,0} \\ &\overset{CS}{=} c_{P^{K\setminus\{k\}}}^{of}\left(S'\right) + \left|S'\right| x^{k,0} = c_{P^K}^{of}\left(S'\right) + \left|S'\right| x^{k,0} \end{aligned}$$

which implies that $x^{k,0} = 0$. Under (6), $x^{k,1} = \frac{d_k}{n^{k,1}}$ for all $k \in K$. Under (5), for each $i \in N$,

$$R_i\left(P^K\right) = R_i\left(P^{K\setminus\{k\}}\right) + \frac{d_k}{n^{k,1}}.$$

Repeating the same reasoning, we deduce that for each $i \in N$,

$$R_i\left(P^K\right) = R_i\left(P^{\emptyset}\right) + \sum_{k\in H_i^{of}} \frac{d_k}{n^{k,1}}.$$

Since R satisfies CS, under Theorem 3.1, $R_i\left(P^{\emptyset}\right) = \sum_{j\in N} \lambda_{ij}^h + y_i$ for all $i \in N$, where $0 \leq y_i \leq \sum_{j\in H_i^{of}} d_j^{\emptyset}$. By definition of $d^{\emptyset}$, we have $d_j^{\emptyset} = 0$ for all $j \in H$. Since $H_i^{of} \subseteq H$, we deduce $y_i = 0$ and so

$$R_i\left(P^{\emptyset}\right) = \sum_{j\in N} \lambda_{ij}^h.$$

By Theorem 3.2,

$$R_i\left(P^K\right) = \sum_{j\in N} \lambda_{ij}^h + \sum_{k\in H_i^{of}} \frac{d_k}{n^{k,1}} = Sh_i\left(P^K\right).$$

Case 2. Assume now $\mathcal{P}_K = \{N\}$. We consider several subcases.

Case 2.1. Assume $K = \{k\}$. Since R satisfies CS, under Theorem 3.1,

$$\sum_{i\in N^{k,0}} R_i\left(P^{\{k\}}\right) = \sum_{i\in N^{k,0}} \sum_{j\in N} \lambda_{ij}^h + \sum_{j\in N^{k,0}} y_i$$

where $y \in \mathbb{R}_+^N$ and

$$0 \leq \sum_{i\in N^{k,0}} y_i \leq \sum_{j\in H_{N^{k,0}}^{of}} d_j^{\{k\}} = 0$$

which implies $\sum_{i\in N^{k,0}} y_i = 0$. Thus,

$$\sum_{i\in N^{k,0}} R_i\left(P^{\{k\}}\right) = \sum_{i\in N^{k,0}} \sum_{j\in N} \lambda_{ij}^h.$$

On the other hand,

$$\sum_{i\in N^{k,0}} R_i\left(P^{\{k\}}\right) \overset{(4)}{=} \sum_{i\in N^{k,0}} R_i\left(P^{\emptyset}\right) + n^{k,0}x^{k,0} = \sum_{i\in N^{k,0}} \sum_{j\in N} \lambda_{ij}^h + n^{k,0}x^{k,0}$$

which implies $x^{k,0} = 0$. So, for each $i \in N^{k,0}$,

$$R_i\left(P^{\{k\}}\right) = \sum_{j\in N} \lambda_{ij}^h = Sh_i\left(P^{\{k\}}\right).$$

Under (6), $x^{k,1} = \frac{d_k}{n^{k,1}}$. So, for each $i \in N^{k,1}$,

$$R_i\left(P^{\{k\}}\right) = R_i\left(P^{\emptyset}\right) + \frac{d_k}{n^{k,1}} = \sum_{j \in N} \lambda_{ij}^h + \frac{d_k}{n^{k,1}} = Sh_i\left(P^{\{k\}}\right).$$

Case 2.2. Assume now $|K| > 1$. We proceed by induction on $|K|$. Hence, we assume $R\left(P^{K'}\right) = Sh\left(P^{K'}\right)$ when $|K'| < |K|$. We have three cases:

Case 2.2.1. Assume first $n^{k,0} = 0$ for some $k \in K$. Under (5), for all $i \in N = N^{k,1}$,

$$R_i\left(P^K\right) = R_i\left(P^{K \setminus \{k\}}\right) + x^{k,1}.$$

Hence,

$$\sum_{i \in N} R_i\left(P^K\right) = \sum_{i \in N} R_i\left(P^{K \setminus \{k\}}\right) + nx^{k,1}$$

and thus

$$x^{k,1} = \frac{\sum_{i \in N} R_i\left(P^K\right) - \sum_{i \in N} R_i\left(P^{K \setminus \{k\}}\right)}{n} = \frac{d_k}{n^{k,1}}.$$

Now for all $i \in N = N^{k,1}$,

$$R_i\left(P^K\right) = R_i\left(P^{K \setminus \{k\}}\right) + \frac{d_k}{n^{k,1}}.$$

By induction hypothesis, for all $i \in N$

$$\begin{aligned} R_i\left(P^K\right) &= \sum_{j \in N} \lambda_{ij}^h + \sum_{j \in H_i^{of}} \frac{d_j^{K \setminus \{k\}}}{n^{k,1}} + \frac{d_k}{n^{k,1}} \\ &= \sum_{j \in N} \lambda_{ij}^h + \sum_{j \in H_i^{of}} \frac{d_j^K}{n^{k,1}} = Sh_i\left(P^K\right). \end{aligned}$$

Case 2.2.2. Assume now $n^{k,1} = 0$ for some $k \in K$. By (4), $R_i\left(P^K\right) = R_i\left(P^{K \setminus \{k\}}\right) + x^{k,0}$ for all $i \in N = N^{k,0}$. The rest of reasoning is analogous to the previous case and we omit it.

Case 2.2.3. Finally, assume $n^{k,0} > 0$ and $n^{k,1} > 0$ for all $k \in K$. We can assume w.l.o.g. $1, 2 \in K$. Let $i^1 \in N^{1,1}$ and $i^2 \in N^{1,0}$. Since $\mathcal{P}_K = \{N\}$, we know that there exists some $k \in K$ such that either $i^1, i^2 \in N^{k,1}$ or $i^1, i^2 \in N^{k,0}$. Assume w.l.o.g. that either $i^1, i^2 \in N^{2,1}$ or $i^1, i^2 \in N^{2,0}$. For each $k \in \{1, 2\}$ and each $l \in \{1, 2\}$, let $f^l(k) \in \{0, 1\}$ be defined such that $i^l \in N^{k, f^l(k)}$. Hence, we know that $f^1(1) = 1$ (because $i^1 \in N^{1,1}$), $f^2(1) = 0$ (because $i^2 \in N^{1,0}$), and $f^1(2) = f^2(2)$ (because either $i^1, i^2 \in N^{2,1}$ or $i^1, i^2 \in N^{2,0}$).

By induction hypothesis, for any $k \in \{1,2\}$ and any $l \in \{1,2\}$,

$$R_{i^l}\left(P^K\right) \overset{(4)(5)}{=} R_{i^l}\left(P^{K\setminus\{k\}}\right) + x^{k,f^l(k)} = Sh_{i^l}\left(P^{K\setminus\{k\}}\right) + x^{k,f^l(k)}$$
$$= \sum_{j\in N} \lambda^h_{i^l_j} + \sum_{j\in H^{of}_{i^l}} \frac{d_j^{K\setminus\{k\}}}{n^{j,1}} + x^{k,f^l(k)}.$$

Thus, for each $l \in \{1,2\}$,

$$x^{1,f^l(1)} - x^{2,f^l(2)} = \sum_{j\in H^{of}_{i^l}} \frac{d_j^{K\setminus\{1\}} - d_j^{K\setminus\{2\}}}{n^{j,1}} = \frac{d_1}{n^{1,1}} f^l(1) - \frac{d_2}{n^{2,1}} f^l(2).$$

In particular, taking $a = f^1(2) = f^2(2)$ and $l = 1$,

$$x^{1,1} - x^{2,a} = \frac{d_1}{n^{1,1}} - \frac{d_2}{n^{2,1}} a \tag{7}$$

and taking $l = 2$,

$$x^{1,0} - x^{2,a} = -\frac{d_2}{n^{2,1}} a. \tag{8}$$

Equations (6) for $k = 1,2$ and equations (7)-(8) can be written as a matrix equation as follows:

$$\begin{bmatrix} n^{1,1} & 0 & n^{1,0} & 0 \\ 0 & n^{2,1} & 0 & n^{2,0} \\ 1 & -a & 0 & a-1 \\ 0 & -a & 1 & a-1 \end{bmatrix} \cdot \begin{bmatrix} x^{1,1} \\ x^{2,1} \\ x^{1,0} \\ x^{2,0} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \frac{d_1}{n^{1,1}} - \frac{d_2}{n^{2,1}} a \\ -\frac{d_2}{n^{2,1}} a \end{bmatrix}.$$

The determinant of the left matrix is $\left(an - n^{2,1}\right) n \neq 0$. Hence, the matrix equation has a unique solution given by $x^{k,1} = \frac{d_k}{n^{k,1}}$ and $x^{k,0} = 0$ for all $k \in \{1,2\}$. Thus,

$$R_{i^1}\left(P^K\right) \overset{(5)}{=} R_{i^1}\left(P^{K\setminus\{1\}}\right) + x^{1,1} = R_{i^1}\left(P^{K\setminus\{1\}}\right) + \frac{d_1}{n^{1,1}}$$
$$R_{i^2}\left(P^K\right) \overset{(4)}{=} R_{i^2}\left(P^{K\setminus\{1\}}\right) + x^{1,0} = R_{i^2}\left(P^{K\setminus\{1\}}\right).$$

By induction hypothesis,

$$R_{i^1}\left(P^K\right) = Sh_{i^1}\left(P^{K\setminus\{1\}}\right) + \frac{d_1}{n^{1,1}} \overset{Th.3.2}{=} \sum_{j\in N}\lambda^h_{i^1 j} + \sum_{k\in H^{of}_{i^1}} \frac{d_k^{K\setminus\{1\}}}{n^{k,1}} + \frac{d_1}{n^{1,1}}$$

$$= \sum_{j\in N}\lambda^h_{i^1 j} + \sum_{k\in H^{of}_{i^1}} \frac{d_k^{K}}{n^{k,1}} \overset{Th.3.2}{=} Sh_{i^1}\left(P^K\right)$$

$$R_{i^2}\left(P^K\right) = Sh_{i^2}\left(P^{K\setminus\{1\}}\right) \overset{Th.3.2}{=} \sum_{j\in N}\lambda^h_{i^2 j} + \sum_{k\in H^{of}_{i^2}} \frac{d_k^{K\setminus\{1\}}}{n^{k,1}}$$

$$= \sum_{j\in N}\lambda^h_{i^2 j} + \sum_{k\in H^{of}_{i^2}} \frac{d_k^{K}}{n^{k,1}} \overset{Th.3.2}{=} Sh_{i^2}\left(P^K\right).$$

Since i^1, i^2 were taken arbitrarily from $N^{1,1}$ and $N^{1,0}$, respectively, and these two sets form a partition of N, we conclude that $R_i\left(P^K\right) = Sh_i\left(P^K\right)$ for all $i \in N$.

(b) It follows from part (a) and Proposition 3.1. ■

Remark 3.2 *We now prove that the properties used in Theorem 3.3 are independent.*

- *Let R^0 be defined as $R^0(P) = x + Sh\left(c^{of}\right)$ for some $x \in \mathbb{R}^N$ with $\sum_{i\in N} x_i = 0$ and $x_i \neq 0$ for some $i \in N$. R^0 satisfies IIH, IIF and ETH, but fails CS and Pos.*

- *Let $\omega \in \mathbb{R}^N$ be such that $\omega_i > 0$ for all $i \in N$ and $\omega_i \neq \omega_j$ for some $i \neq j$. Let R^1 be defined for each P and each $i \in N$ as follows:*

$$R^1_i(P) = \sum_{j\in N}\lambda^h_{ij} + \sum_{j\in H^{of}_i} \frac{\omega_i}{\sum_{k\in N: j\in H^{of}_k}\omega_k} d_j.$$

  R^1 satisfies CS, Pos, IIH, and IIF, but fails ETH.

- *Let R^2 be defined for each P and $i \in N$ as follows:*

$$R^2_i(P) = \sum_{j\in N}\lambda^h_{ij} + \sum_{j\in H}\frac{d_j}{n}.$$

  R^2 satisfies Pos, IIF and ETH, but fails CS and IIH.

- *Let R^3 be defined for each P and $i \in N$ as follows:*

$$R^3_i(P) = \frac{\sum_{k\in N}\sum_{j\in N}\lambda^h_{kj}}{n} + \sum_{j\in H^{of}_i} \frac{d_j}{\left|\left\{k \in N : j \in H^{of}_k\right\}\right|}.$$

  R^3 satisfies Pos, IIH and ETH,but fails CS and IIF.

4 Two-way flow

In this section, we assume that nodes are interested in both outgoing and ingoing flow. Namely, the cost of a group of nodes depends on the outgoing and the ingoing flow of such nodes. We first associate to each hub network problem a cost game. Then, we study the core and the Shapley value of such game.

For each hub network problem P, we associate the cost game c_P^{tf} where for each $S \subseteq N$, $c_P^{tf}(S)$ is the cost of sending and receiving the flow of all nodes in S to and from all nodes through h. The cost game c_P^{tf} models situations where an (optimal) hub network h (with associated set of hubs H) has already been constructed. Thus, d can be considered as a vector of maintenance costs. We formally define this cost game as follows:

$$c_P^{tf}(S) = \sum_{(i,j)\notin(N\setminus S)\times(N\setminus S)} \lambda_{ij}^h + \sum_{i\in H_S^{tf}} d_i \tag{9}$$

for all $S \subseteq N$. When no confusion arises we write c^{tf} instead of c_P^{tf}.

4.1 The core

In next theorem we prove that in the core allocations of c^{tf} the cost of sending or receiving flow between two nodes is divided between them. Besides, the cost of any hub is divided among the nodes that use the hub for sending or receiving their flow. Before stating the theorem we need some notation.

Let $\Pi = \{\pi : N \longrightarrow N : \pi \text{ biyective}\}$ be the set of orderings of agents in N. Given $i \in N$ and $j \in H$, $\Pi_{ij} \subset \Pi$ is the set of orderings such that node i is the first that uses hub j, i.e. $\pi(l) = i$ implies $j \notin H_{\pi(l')}^{tf}$ for all $l' < l$.

Theorem 4.1 *For each hub network problem, c^{tf} is concave. Moreover, the core is nonempty and given by the convex hull of the following set of vectors:*

$$\left\{\left(\sum_{j\in N:\pi^{-1}(j)>\pi^{-1}(i)} \left(\lambda_{ij}^h + \lambda_{ji}^h\right) + \sum_{j\in H_i^{tf}:\pi\in\Pi_{ij}} d_j\right)_{i\in N}\right\}_{\pi\in\Pi}.$$

Proof. We first prove that $\left(N, c^{tf}\right)$ is concave. Let $l \in T \subset S \subseteq N$. Since for each $S' \subset N$, $H_{S'}^{tf} = \bigcup_{i\in S'} H_i^{tf}$, we have that

$$H_S^{tf} \setminus H_{S\setminus\{l\}}^{tf} \subset H_T^{tf} \setminus H_{T\setminus\{l\}}^{tf}. \tag{10}$$

Then,

$$c^{tf}(S') - c^{tf}\left(S' \setminus \{l\}\right) = \sum_{(i,j)\notin(N\setminus S')\times(N\setminus S')} \lambda_{ij}^{h} + \sum_{i\in H_{S'}} d_i$$
$$- \sum_{(i,j)\notin(N\setminus(S'\setminus\{l\}))\times(N\setminus(S'\setminus\{l\}))} \lambda_{ij}^{h} - \sum_{i\in H^{tf}_{S'\setminus\{l\}}} d_i$$
$$= \sum_{i\in N\setminus S'} \lambda_{il}^{h} + \sum_{j\in N\setminus S'} \lambda_{lj}^{h} + \sum_{i\in H^{tf}_{S'}\setminus H^{tf}_{S\setminus\{l\}}} d_i.$$

Since all terms are non-negative, $N \setminus S \subset N \setminus T$ and (10), we have that

$$c^{tf}(S) - c^{tf}(S \setminus \{l\}) \leq c^{tf}(T) - c^{tf}(T \setminus \{l\})$$

which proves that $\left(N, c^{tf}\right)$ is concave.

It is well-known that, when the cost game is concave, the core coincides with the Weber set. Thus, the core is the convex hull of the vectors of marginal contributions. Notice that the coordinate i of the vector of marginal contributions for $\pi \in \Pi$ is

$$\sum_{j\in N:\pi^{-1}(j)>\pi^{-1}(i)} \left(\lambda_{ij}^{h} + \lambda_{ji}^{h}\right) + \sum_{j\in H_i^{tf}:\pi\in\Pi_{ij}} d_j,$$

from where the result trivially holds. ■

Analogously to the one-way flow case, we consider an intermediate situation between a fixed hub network and a variable hub network. Assume that the optimal hub network is not unique. We can define the cost of a coalition as the minimum over all optimal hub networks. Namely, for each $S \subseteq N$,

$$c^{**}(S) = \min_{h\in\mathcal{H}, h \text{ is optimal}} \left\{ c^{tf}_{P(h)}(S) \right\}$$

where $P(h)$ is the hub network problem induced by the optimal hub network h. Next example shows that the core of c^{**} can be empty.

Example 4.1 *Let P be such that $N = \{1, 2, \ldots, 6\}$, $\alpha = 1$, $f_{12} = f_{34} = f_{56} = 1$, $f_{ij} = 0$ otherwise, $d_1 = d_2 = d_3 = 1$ and $d_i \geq 4$ otherwise. The cost matrix is given in the following table:*

c_{ij}	2	3	4	5	6
1	2	2	3	3	3
2		1	3	4	3
3			4	3	3
4				3	3
5					4

This hub problem is depicted in Figure 1.

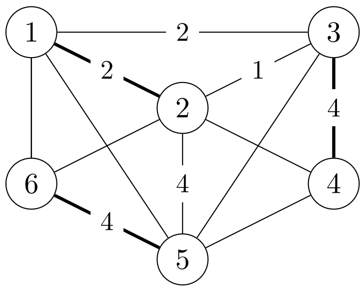

Figure 1: $c_{ij} = 3$ when no specified. Flow goes from 1 to 2, from 3 to 4, and from 5 to 6.

There exist three optimal hub networks h^1, h^2, *and* h^3, *corresponding to putting a single hub at either* 1, 2 *or* 3, *respectively. The cost of these networks is* 14 *each. Hence,* $c^{**}(N) = 14$. *Moreover, nodes* $1, 2, 3, 4$ *can cover their own flow at cost* 7 *when the hub is located at* 2. *Then,* $c^{**}(\{1, 2, 3, 4\}) = 7$. *Analogously, nodes* $1, 2, 5, 6$ *can cover their own flow at cost* 9 *when the hub is located at* 1, *so that* $c^{**}(\{1, 2, 5, 6\}) = 9$. *Analogously, nodes* $3, 4, 5, 6$ *can cover their own flow at cost* 11 *when the hub is located at* 3, *so that* $c^{**}(\{3, 4, 5, 6\}) = 11$. *Hence, a core allocation* y *should satisfy* $y_1 + y_2 + y_3 + y_4 \leq 7$, $y_1 + y_2 + y_5 + y_6 \leq 9$, *and* $y_3 + y_4 + y_5 + y_6 \leq 11$. *By adding these inequalities and dividing by* 2, *we deduce that* $\sum_{i \in N} y_i \leq 13.5$. *Since* $c^{**}(N) = 14$, *we deduce that the core of* c^{**} *is empty.*

4.2 The Shapley value

We now study the Shapley value of c^{tf}, which we also call the *Shapley rule*. In next theorem, we prove that in the Shapley rule the cost of sending flow between a pair of nodes (λ^h_{ij}) is divided equally between both nodes. Besides, the cost of any hub is divided equally among the nodes that use the hub for sending or receiving their flow.

Theorem 4.2 *For each hub network problem* P *and each* $i \in N$,

$$Sh_i\left(c^{tf}\right) = \sum_{j \in N} \frac{\lambda^h_{ij} + \lambda^h_{ji}}{2} + \sum_{j \in H^{tf}_i} \frac{d_j}{\left|\left\{k \in N : j \in H^{tf}_k\right\}\right|}. \tag{11}$$

Proof. It is well known that the Shapley value is the average of the vectors of marginal contributions. Thus,

$$Sh_i\left(c^{tf}\right) = \frac{1}{|\Pi|}\left(\sum_{\pi \in \Pi}\sum_{j \in N : \pi^{-1}(j) > \pi^{-1}(i)}\left(\lambda^h_{ij} + \lambda^h_{ji}\right) + \sum_{j \in H^{tf}_i}\sum_{\pi \in \Pi_{ij}} d_j\right).$$

Let $\Pi^{ij} = \left\{\pi \in \Pi : \pi^{-1}(j) > \pi^{-1}(i)\right\}$. Clearly, $|\Pi^{ij}| = \frac{|\Pi|}{2}$. Hence,

$$\begin{aligned}\frac{1}{|\Pi|}\sum_{\pi\in\Pi}\sum_{j\in N:\pi^{-1}(j)>\pi^{-1}(i)}\left(\lambda_{ij}^h+\lambda_{ji}^h\right) &= \frac{1}{|\Pi|}\sum_{j\in N}\sum_{\pi\in\Pi^{ij}}\left(\lambda_{ij}^h+\lambda_{ji}^h\right)\\ &= \frac{1}{|\Pi|}\sum_{j\in N}|\Pi^{ij}|\left(\lambda_{ij}^h+\lambda_{ji}^h\right)\\ &= \sum_{j\in N}\frac{\lambda_{ij}^h+\lambda_{ji}^h}{2}\end{aligned}$$

which is the first part of (11).

Let $T = \left\{k \in N : j \in H_k^{tf}\right\}$ and $t = |T|$. We still need to prove that $\frac{1}{|\Pi|}\sum_{j\in H_i^{tf}}\sum_{\pi\in\Pi_{ij}} d_j = \sum_{j\in H_i^{tf}} \frac{d_j}{t}$. Clearly, it is enough to prove that $\frac{|\Pi_{ij}|}{|\Pi|} = \frac{1}{t}$ for all $j \in H_i^{tf}$. Notice that Π_{ij} is the set of orderings in which the predecessors of i are not in T. In particular, $\Pi_{ij} = \bigcup_{s=1,\ldots,n-t+1}\Pi_{ij}^s$ where $\Pi_{ij}^s = \{\pi \in \Pi_{ij} : \pi(s) = i\}$. Hence, $\frac{|\Pi_{ij}|}{|\Pi|} = \sum_{s=1}^{n-t+1}\frac{\left|\Pi_{ij}^s\right|}{|\Pi|}$. Moreover, $\frac{\left|\Pi_{ij}^s\right|}{|\Pi|}$ is the probability of randomly picking up an order in Π satisfying that node i is in position s and it is preceded by $s-1$ nodes in $N\backslash T$. Let $|N \setminus T| = n-t$. Then,

$$\frac{\left|\Pi_{ij}^s\right|}{|\Pi|} = \frac{n-t}{n}\cdot\frac{n-t-1}{n-1}\cdot\dots\cdot\frac{n-t-s+2}{n-s+2}\cdot\frac{1}{n-s+1} = \frac{(n-s)!(n-t)!}{n!(n-t-s+1)!}$$

So

$$\begin{aligned}\frac{|\Pi_{ij}|}{|\Pi|} &= \frac{(n-t)!}{n!}\sum_{s=1}^{n-t+1}\frac{(n-s)!}{(n-s-t+1)!}\\ &= \frac{(n-t)!t!}{n!}\sum_{s=1}^{n-t+1}\frac{(n-s)!}{(n-s-t+1)!(t-1)!}\cdot\frac{1}{t}\\ &= \frac{1}{\binom{n}{t}}\sum_{s=1}^{n-t+1}\binom{n-s}{t-1}\frac{1}{t}.\end{aligned}$$

Then, it is enough to prove that $\binom{n}{t} = \sum_{s=1}^{n-t+1}\binom{n-s}{t-1}$. This is trivially true when $n = 1$. By induction hypothesis on n, and using Stidel formula:

$$\begin{aligned}\binom{n}{t} &= \binom{n-1}{t-1}+\binom{n-1}{t} = \binom{n-1}{t-1}+\sum_{s=1}^{n-t}\binom{n-1-s}{t-1}\\ &= \binom{n-1}{t-1}+\sum_{s=2}^{n-t+1}\binom{n-s}{t-1} = \sum_{s=1}^{n-t+1}\binom{n-s}{t-1}.\end{aligned}$$

■

We now define the property of core selection in the two-way flow case.

Core Selection (CS) For any hub network problem P, we have that

$$R(P) \in Core\left(c_P^{tf}\right).$$

The analogous results for Proposition 3.1 also hold in the two-way flow case.

Proposition 4.1 *(a) CS implies Pos.*
(b) Pos, IIH and IIF imply CS.

Proof. It is analogous to the proof of Proposition 3.1 and we omit it. ■

CS does not imply neither IIH nor IIF. The rule in which each node pays half the cost of sending and receiving her flow and the cost of each hub is paid equally by the nodes that use the most expensive hubs among those that use that hub satisfies CS but not IIH. The rule in which each node pays half the cost of sending and receiving her flow and the cost of each hub is paid equally by the nodes sending more flow through this hub satisfies CS but not IIF.

In next proposition we prove that the Shapley rule satisfies all the properties we have defined in Subsection 2.1 for the two-way flow case.

Proposition 4.2 *The Shapley rule satisfies Pos, ETE, CS, NF, FM, HM, CM, ETH, IIH, IIF and ETF.*

Proof. The proof for Pos, ETE, CS, NF, FM, HM, CM, ETH, IIH and IIF is analogous to that of Proposition 3.2 (using Theorem 4.2 instead of Theorem 3.2 and Proposition 4.1 instead of Proposition 3.1) and we omit it.

Let P, P' be given as in the definition of ETF. We consider two cases:

1. $\{i,j\} = \{k,l\}$. Let $\lambda^{h'}$ the λ^h associated with P'. By Theorem 4.2,

$$Sh_i\left(c_P^{tf}\right) - Sh_i\left(c_{P'}^{tf}\right) = \frac{\lambda_{ij}^h - \lambda_{ij}^{h'}}{2} = Sh_j\left(c_P^{tf}\right) - Sh_j\left(c_{P'}^{tf}\right),$$

   and hence $Sh\left(c^{tf}\right)$ satisfies ETF.

2. $\{i,j\} \cap \{k,l\} = \emptyset$. Under Theorem 4.2,

$$Sh_i\left(c_P^{tf}\right) - Sh_i\left(c_{P'}^{tf}\right) = 0 = Sh_j\left(c_P^{tf}\right) - Sh_j\left(c_{P'}^{tf}\right).$$

■

Similarly to Theorem 3.3, we give two characterizations of the Shapley rule.

Theorem 4.3 (a) *The Shapley rule is the unique rule satisfying* CS*,* ETH *and* ETF*.*

(b) *The Shapley rule is the unique rule satisfying Pos,* IIH*,* IIF*,* ETH*, and* ETF*.*

Proof. (a) By Proposition 4.2 the Shapley rule satisfies these properties. We now prove the uniqueness. Let R be a rule satisfying CS, ETH and ETF.

Let $P = (N, C, F, d, \alpha, h)$ be a hub network problem. We assume $d_i = 0$ for all $i \in H$; the extension to positive hub costs is analogous to the proof of Theorem 3.3 and we omit it.

Let $E = \{(i,j) : f_{ij} > 0\}$ and, for each $i \in N$ and $e \in E$, let $a^i(e) = 1$ when node i is adjacent to e, and $a^i(e) = 0$ otherwise. Denote $E = \{e_1, \ldots, e_\gamma\}$. We assume, w.l.o.g., $e_1 = (1,2)$. We also assume, w.l.o.g., $e_2 = (2,1)$ in case $(2,1) \in E$.

For each $\varepsilon > 0$, let $P^\varepsilon = (N, C, F^\varepsilon, d, \alpha, h)$ defined by $f^\varepsilon_{ij} = \varepsilon$ for all (i,j) with $f_{ij} > 0$, and $f^\varepsilon_{ij} = f_{ij} = 0$ otherwise.

Let $a(P)$ de defined as in the proof of Proposition 3.1. Suppose that, for ε small enough, there exists $x^P \in \mathbb{R}^N$ with $-7^{|E|}a(P)\varepsilon \leq x^P_i \leq 7^{|E|}a(P)\varepsilon$ for all $i \in N$ such that

$$R_i(P) = \sum_{e \in E} \frac{\lambda^h_e}{2} a^i(e) + x^P_i \text{ for all } i \in N. \tag{12}$$

Since $R_i(P)$ does not depend on ε, we deduce that for all $i \in N$

$$R_i(P) = \sum_{e \in E} \frac{\lambda^h_e}{2} a^i(e) = \sum_{j \in N} \frac{\lambda^h_{ij} + \lambda^h_{ji}}{2} = Sh_i(P).$$

Hence, we just need to prove that (12) holds.

For each $e_k \in E$, we define $P^{-k} = \left(N, C, F^{-k}, d, \alpha, h\right)$ with $f^{-k}_{e_k} = \varepsilon$ and $f^{-k}_{ij} = f_{ij}$ otherwise. For notational convenience, we write λ^h_k, f^{-k}_{ij} and $a^i(k)$ instead of $\lambda^h_{e_k}$, $f^{-e_k}_{ij}$ and $a^i(e_k)$, respectively.

We proceed by induction on $|E|$. Case $E = \emptyset$ is not possible because H is nonempty and for each $k \in H$ we assume that there exist $i, j \in N$ with $f_{ij} > 0$ and $k \in \{h(i), h(j)\}$.

Assume then $E = \{e_1\}$. In this case, $P^{-1} = P^\varepsilon$. Let $x^P_i = 0$ if $i \notin \{1,2\}$ and $x^P_i = R_i(P) - \frac{\lambda^h_1}{2}$ if $i \in \{1,2\}$. We prove that for all $i \in N$, x^P_i lies on the interval $[-7a(P)\varepsilon, 7a(P)\varepsilon]$.

Let $i \notin \{1,2\}$. By CS, $R_i(P) \leq c^{tf}_P(\{i\}) = 0$. Since $a^i(e_1) = 0$ and $\lambda^h_e = 0$ when $e \neq e_1$, (12) holds trivially.

We now prove it for $i = 1$ (the case $i = 2$ is analogous). By ETF, there exists $y^{1,1} \in \mathbb{R}$ such that $R_1(P) - R_1\left(P^{-1}\right) = R_2(P) - R_2\left(P^{-1}\right) = y^{1,1}$.

Hence,
$$y^{1,1} = \frac{R_1(P) + R_2(P) - R_1(P^{-1}) - R_2(P^{-1})}{2}.$$
By CS, $0 \leq R_1(P) + R_2(P) = \lambda_1^h$ and $0 \leq R_1(P^{-1}) + R_2(P^{-1}) \leq a(P)\varepsilon$. Hence,
$$y^{1,1} \in \left[\frac{\lambda_1^h}{2} - a(P)\varepsilon, \frac{\lambda_1^h}{2} + a(P)\varepsilon\right].$$

By induction hypothesis, $x_1^{P^{-1}} \in [-a(P)\varepsilon, a(P)\varepsilon]$. Thus,
$$R_1(P) = R_1\left(P^{-1}\right) + y^{1,1} = x_1^{P^{-1}} + y^{1,1} \in \left[\frac{\lambda_1^h}{2} - 2a(P)\varepsilon, \frac{\lambda_1^h}{2} + 2a(P)\varepsilon\right].$$
and so (12) holds with $x_1^P = R_1(P) - \frac{\lambda_1^h}{2}$.

Assume now (12) holds when $|E| < \gamma$ and suppose $|E| = \gamma$. We consider several cases:

Case 1. $\gamma = 2$ and $e_2 = (2,1)$, so that $E = \{e_1, e_2\}$. Then, we proceed as above defining $y^{1,1}$ in the same way.

Case 2. Either $\gamma > 2$ or $e_2 \neq (2,1)$. Notice that this implies $n > 2$. Fix $i \in N$. We consider two cases.

Case 2.1. $a^i(e) = 0$ for all $e \in E$. We take $x_i^P = 0$. Then, by CS, (12) holds because $R_i(P) = 0$.

Case 2.2. There exists $k \in E$ such that $a^i(k) = 1$. Fix also $e_l \in E \setminus \{e_k\}$ with different adjacent nodes than e_k. We can find such e_l because either $\gamma > 2$ or $e_2 \neq (2,1)$.

ETF implies that there exist $y^{k,0}, y^{k,1}, y^{l,0}$ and $y^{l,1}$ such that $R_j(P) - R_j\left(P^{-k}\right) = y^{k,a^j(k)}$ and $R_j(P) - R_j\left(P^{-l}\right) = y^{l,a^j(l)}$ for all $j \in N$.Since
$$\sum_{j \in N} R_j(P) - \sum_{j \in N} R_j\left(P^{-k}\right) = \lambda_k^h - \frac{\lambda_k^h}{f_k}\varepsilon,$$
we have
$$2y^{k,1} + (n-2)y^{k,0} = \lambda_k^h + z^{k,1} \tag{13}$$
where $z^{k,1} = -\frac{\lambda_k^h}{f_k}\varepsilon \in [-a(P)\varepsilon, 0]$. Analogously,
$$2y^{l,1} + (n-2)y^{l,0} = \lambda_l^h + z^{l,1} \tag{14}$$
with $z^{l,1} \in [-a(P)\varepsilon, 0]$.

On the other hand, $R_i(P) = R_i(P^{-k}) + y^{k,1} = R_i(P^{-l}) + y^{l,a^i(l)}$. Hence,
$$y^{k,1} - y^{l,a^i(l)} = R_i(P^{-l}) - R_i(P^{-k})$$
by induction hypothesis,
$$= \frac{\lambda_k^h}{2} - \frac{\lambda_l^h}{2}a^i(l) + x_i^{P^{-l}} - x_i^{P^{-k}}$$

with $x_i^{P^{-l}}, x_i^{P^{-k}} \in [-7^{\gamma-1}a(P)\varepsilon, 7^{\gamma-1}a(P)\varepsilon]$.

We define $z^{k,0} = x_i^{P^{-l}} - x_i^{P^{-k}} \in [-2\cdot 7^{\gamma-1}a(P)\varepsilon, 2\cdot 7^{\gamma-1}a(P)\varepsilon]$ so that

$$y^{k,1} - y^{l,a^i(l)} = \frac{\lambda_k^h}{2} - \frac{\lambda_l^h}{2}a^i(l) + z^{k,0}. \tag{15}$$

We repeat the reasoning for some $j \in N$ adjacent to e_l (i.e. $a^j(l) = 1$) but not to e_k (i.e. $a^j(k) = 0$). We can find such j because e_l has different adjacent nodes than e_k. Then, we get

$$y^{l,1} - y^{k,0} = \frac{\lambda_l^h}{2} + z^{l,0} \tag{16}$$

with $z^{l,0} \in [-2\cdot 7^{\gamma-1}a(P)\varepsilon, 2\cdot 7^{\gamma-1}a(P)\varepsilon]$.

Equations (13)-(14)-(15)-(16) form a system of linear equations given by

$$\begin{bmatrix} 2 & 0 & n-2 & 0 \\ 0 & 2 & 0 & n-2 \\ 1 & -a^i(l) & 0 & a^i(l)-1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y^{k,1} \\ y^{l,1} \\ y^{k,0} \\ y^{l,0} \end{bmatrix} = \begin{bmatrix} \lambda_k^h + z^{k,1} \\ \lambda_l^h + z^{l,1} \\ \frac{\lambda_k^h}{2} - \frac{\lambda_l^h}{2}a^i(l) + z^{k,0} \\ \frac{\lambda_k^h}{2} + z^{l,0} \end{bmatrix}.$$

The determinant of the first matrix is $(n + 2a^i(l) - 4)n$. We consider several cases.

Case 2.2.1. $a^i(l) = 1$. Then, $(n + 2a^i(l) - 4)n \neq 0$. Thus, the previous system of linear equations have a unique solution which is given for $y^{k,1}$ by

$$y^{k,1} = \frac{\lambda_k^h}{2} + \frac{z}{(n-2)n}$$

where $z = (n-2)z^{k,1} - nz^{l,1} + (n^2 - 4n + 4)z^{k,0} + (n^2 - 4n + 4)z^{l,0}$.

Since $z^{k,1}, z^{l,1}, z^{k,0}, z^{l,0} \in [-2\cdot 7^{\gamma-1}a(P)\varepsilon, 2\cdot 7^{\gamma-1}a(P)\varepsilon]$, we deduce that $z \in [-2(2n^2 - 6n + 6)\cdot 7^{\gamma-1}a(P)\varepsilon, 2(2n^2 - 6n + 6)\cdot 7^{\gamma-1}a(P)\varepsilon]$.

For $n > 2$, we have $\frac{2(2n^2-6n+6)}{(n-2)n} \leq 6$ and hence

$$\frac{z}{(n-2)n} \in [-6\cdot 7^{\gamma-1}a(P)\varepsilon, 6\cdot 7^{\gamma-1}a(P)\varepsilon]. \tag{17}$$

By induction hypothesis,

$$R_i(P) = R_i(P^{-k}) + y^{k,1} = \sum_{e\in E} \frac{\lambda_e^h}{2}a^i(e) + x_i^{P^{-l}} + \frac{z}{(n-2)n}.$$

Let us define $x_i^P = x_i^{P^{-l}} + \frac{z}{(n-2)n}$. By (17) and $x_i^{P^{-l}} \in [7^{\gamma-1}a(P)\varepsilon, 7^{\gamma-1}a(P)\varepsilon]$, we deduce that $x_i^P \in [-7^{\gamma}a(P)\varepsilon, 7^{\gamma}a(P)\varepsilon]$.

Case 2.2.2. $a^i(l) = 0$ and $n \neq 4$. Then, $(n + 2a^i(l) - 4)n \neq 0$. Thus, the previous system of linear equations have a unique solution which is given for $y^{l,0}$ by

$$y^{l,0} = \frac{z}{(n-4)n}$$

where $z = -2z^{k,1} + (n-2)z^{l,1} + 4z^{k,0} + (-2n+4)z^{l,0}$.

Since $z^{k,1}, z^{l,1}, z^{k,0}, z^{l,0} \in [-2 \cdot 7^{\gamma-1}a(P)\varepsilon, 2 \cdot 7^{\gamma-1}a(P)\varepsilon]$, we deduce that

$$z \in \left[-3n \cdot 7^{\gamma-1}a(P)\varepsilon, 3n \cdot 7^{\gamma-1}a(P)\varepsilon\right].$$

For $n \geq 3$, $n \neq 4$ we have $\frac{6n}{(n-4)n} \leq 6$ and hence

$$y^{l,0} \in [-6 \cdot 7^{\gamma-1}a(P)\varepsilon, 6 \cdot 7^{\gamma-1}a(P)\varepsilon]. \tag{18}$$

By induction hypothesis,

$$R_i(P) = R_i(P^{-l}) + y^{l,0} = \sum_{e \in E} \frac{\lambda_e^h}{2} a^i(e) + x_i^{P^{-l}} + y^{l,0}.$$

Let us define $x_i^P = x_i^{P^{-l}} + y^{l,0}$. By (18) and $x_i^{P^{-l}} \in [-7^{\gamma-1}a(P)\varepsilon, 7^{\gamma-1}a(P)\varepsilon]$, we deduce that $x_i^P \in [-7^{\gamma}a(P)\varepsilon, 7^{\gamma}a(P)\varepsilon]$.

Case 2.2.3. $a^i(l) = 0$ and $n = 4$. Then, $(n + 2a^i(l) - 4)n = 0$. In this case we replace equation (16) by either $y^{k,0} - y^{l,0} = z^{lk,0}$ or $y^{l,1} - y^{k,1} = \frac{\lambda_l^h}{2} - \frac{\lambda_k^h}{2} + z^{lk,1}$, with $z^{lk,\cdot} \in [-2 \cdot 7^{\gamma-1}\varepsilon, 2 \cdot 7^{\gamma-1}\varepsilon]$. Now the resulting determinant is non zero. The rest of the proof is similar and we omit further details.

(b) It follows from (a), Proposition 4.2, and Proposition 4.1. ■

Remark 4.1 *We now prove that the properties used in Theorem 4.3 are independent.*

- *Let R^0 be defined as $R^0(P) = x + Sh\left(c^{tf}\right)$ for some $x \in \mathbb{R}^N$ with $\sum_{i \in N} x_i = 0$ and $x_i \neq 0$ for some $i \in N$. R^0 satisfies IIH, IIF, ETH and ETF, but fails CS and Pos.*

- *Let $\omega \in \mathbb{R}^N$ be such that $\omega_i > 0$ for all $i \in N$ and $\omega_i \neq \omega_j$ for some $i \neq j$. Let R^1 be defined for each P and each $i \in N$ as follows:*

$$R_i^1(P) = \sum_{j \in N} \frac{\lambda_{ij}^h + \lambda_{ji}^h}{2} + \sum_{j \in H_i^{tf}} \frac{\omega_i}{\sum_{k \in N: j \in H_k^{tf}} \omega_k} d_j.$$

R^1 satisfies CS, Pos, IIH, IIF and ETF, but fails ETH.

- *Let R^2 be defined for each P and $i \in N$ as follows:*

$$R_i^2(P) = \sum_{j \in N} \frac{\lambda_{ij}^h + \lambda_{ji}^h}{2} + \sum_{j \in H} \frac{d_j}{n}.$$

R^2 satisfies Pos, IIF, ETH and ETF, but fails CS and IIH.

- *Let R^3 be defined for each P and $i \in N$ as follows:*

$$R_i^3(P) = \frac{\sum_{k \in N} \sum_{j \in N} \lambda_{kj}^h}{n} + \sum_{j \in H_i^{tf}} \frac{d_j}{\left| \left\{ k \in N : j \in H_k^{tf} \right\} \right|}.$$

R^3 satisfies Pos, ETH, IIH and ETF, but fails CS and IIF.

- *Let R^4 be defined for each P and $i \in N$ as follows:*

$$R_i^4(P) = \sum_{j \in N} \lambda_{ij}^h + \sum_{j \in H_i^{tf}} \frac{d_j}{\left| \left\{ k \in N : j \in H_k^{tf} \right\} \right|}.$$

R^4 satisfies CS, Pos, IIH, IIF and ETH but fails ETF.

5 Some examples

We now present some examples where we compute the Shapley rule and compare it with the rule considered in Matsubayashi et al., 2005.

We first provide an example of a hub problem in which the proportional rule proposed by (Matsubayashi et al., 2005, Theorem 3.1) does not belong to the core, whereas the one-way Shapley value does.

Example 5.1 *Let P be such that $N = \{1, 2, 3\}$, $\alpha = 0.1$, $d_i = 3$ for all $i \in N$, $c_{12} = 1$, $c_{13} = c_{23} = 10$, $f_{13} = f_{23} = 1$ and $f_{ij} = 10$ otherwise. There is no congestion cost. In this problem, there exists a unique optimal hub network. It has three hubs, one at each node, and the total cost is 33. The proportional rule gives the cost allocation $u^* = (8.643, 8.643, 15.714)$, which does not belong to the core because $c^{of}(\{1, 2\}) = 13 < 2 \cdot 8.643 = u_1^* + u_2^*$. As opposed, the Shapley rule gives the cost allocation $(5, 5, 23)$, which does belong to the core.*

We now apply our results to Example 4.3 in Matsubayashi et al. (2005).

Example 5.2 (Example 4.3 in Matsubayashi et al. (2005)) *Let P be such that $N = \{1, 2, \ldots, 7\}$, $\alpha = 0.2$, $d_i = 65$ for all $i \in N$. The flow and cost matrices are both symmetric and given in the following respective tables[3]:*

f_{ij}	2	3	4	5	6	7
1	0.1	0.1	1.5	1.5	1	1
2		0.1	1.5	1.5	1	1
3			1.5	1.5	1	1
4				0.1	1	1
5					1	1
6						f_{67}

[3]These tables correct some irrelevant typos presented in Matsubayashi et al. (2005).

c_{ij}	2	3	4	5	6	7	7
1	1.5	1.5	59	58	96	93	87
2		1.5	60	59	96	93	86
3			60	59	97	94	87
4				2	106	110	116
5					107	111	117
6						4	10

The cost matrix assigns two columns to node 7. *The first one applies when node* 7 *represents Osaka. The second one applies when node* 7 *represents Seoul. The other nodes represent London (node* 1*), Brussels (node* 2*), Paris (node* 3*), Washington (node* 4*), Ottawa (node* 5*), and Tokyo (node* 6*). There is also a congestion factor that increases the cost at each node by 0.1% of the flow that goes through it.*

When $f_{67} = 0.1$ *or* $f_{67} = 1$*, and node* 7 *represents Osaka, the optimal hub network is depicted in Figure 2, with hubs at nodes* 1*,* 5 *and* 7*.*

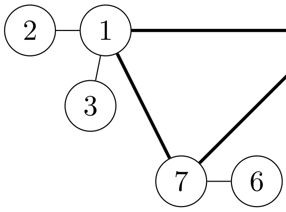

Figure 2: Optimal hub network which arises in Example 4.3(a) and Example 4.3(b) in Matsubayashi et al. (2005).

When $f_{67} = 25$ *and node* 7 *represents Osaka, or* $f_{67} = 10$ *and node* 7 *represents Seoul, the optimal hub network is depicted in Figure 3, with hubs at nodes* 1*,* 5*,* 6 *and* 7*.*

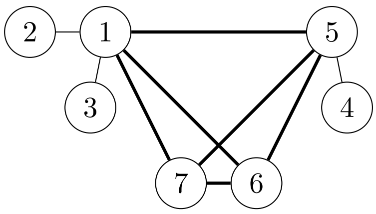

Figure 3: Optimal hub network which arises in Example 4.3(c) and Example 4.3(d) in Matsubayashi et al. (2005).

We now compare the rule suggested by (Matsubayashi et al., 2005, Theorem 3.2), also denoted as u^**, and the one-way Shapley rule.*[4] *The allocation*

[4]Due to the symmetry of the matrices, the fact that all flows are positive, and the low congestion rate, both the one-way case and two-way case give very similar values for the Shapley rule.

for Osaka is represented in next table (in parenthesis, the value of f_{67} for each column):

i	*Osaka* ($f_{67} = 0.1$)		*Osaka* ($f_{67} = 1$)		*Osaka* ($f_{67} = 25$)	
	u_i^*	Sh_i	u_i^*	Sh_i	u_i^*	Sh_i
1	109.6	107.5	109.4	107.5	102.0	113.4
2	109.6	115.2	109.4	115.2	102.0	121.1
3	109.6	115.2	109.4	115.2	102.0	121.1
4	144.7	146.6	144.5	146.6	135.6	151.2
5	144.7	133.6	144.5	133.6	135.6	138.1
6	143.6	153.8	147.8	157.4	199.9	166.0
7	143.6	133.7	147.8	137.4	199.7	165.8

For each $i \in N$, let $f_{i\cdot} = \sum_j f_{ij}$ denote the flow that leaves node i. Next table shows the cost allocation per unit of flow ($y_i/f_{i\cdot}$):

i	*Osaka* ($f_{67} = 0.1$)		*Osaka* ($f_{67} = 1$)		*Osaka* ($f_{67} = 25$)	
	$u_i^*/f_{i\cdot}$	$Sh_i/f_{i\cdot}$	$u_i^*/f_{i\cdot}$	$Sh_i/f_{i\cdot}$	$u_i^*/f_{i\cdot}$	$Sh_i/f_{i\cdot}$
1	21.08	20.67	21.05	20.67	19.61	21.81
2	21.08	22.15	21.05	22.15	19.61	23.29
3	21.08	22.15	21.05	22.15	19.61	23.29
4	21.92	22.22	21.89	22.22	20.55	22.91
5	21.92	20.24	21.89	20.24	20.55	20.92
6	28.16	30.15	24.63	26.24	6.662	5.534
7	28.16	26.22	24.63	22.90	6.656	5.527

The main differences between the one-way Shapley rule and u^* are the following:

1. *Under the Shapley rule, it is preferable to be a hub because hub nodes pay less than the other nodes connected to it. Under u^*, nodes connected to the same hub pay the same. For instance, when $f_{67} = 1$, node 1 is a hub and nodes 2 and 3 are connected to node 1. Under the Shapley rule, node 1 pays less than nodes 2 and 3. Under u^*, node 1 pays the same as nodes 2 and 3.*

2. *When f_{67} increases from 0.1 to 1 and the other flows do not change, the optimal hub network does not change. Under the Shapley rule, the cost increase is charged to the nodes responsible for it (nodes 6 and 7), leaving the rest unaffected. Under, u^*, nodes 6 and 7 pay more whereas the rest of nodes pay less.*

3. *When f_{67} increases from 1 to 25 and the other flows do not change, the optimal hub network also changes. Under the Shapley rule, all nodes pay more (specially nodes 6 and 7). Under u^*, nodes 6 and 7 pay more whereas the rest of nodes pay less.*

Similar comments apply to the case of Seoul. Both allocations for Seoul are represented in next table (in parenthesis, the value of f_{67}):

	Seoul ($f_{67} = 10$)			
i	u_i^*	Sh_i	$u_i^*/f_{i\cdot}$	$Sh_i/f_{i\cdot}$
1	102.9	112.2	19.79	21.57
2	102.9	119.9	19.79	23.05
3	102.9	119.9	19.79	23.05
4	139.0	152.3	21.05	23.08
5	139.0	139.2	21.05	23.08
6	191.9	163.5	12.79	10.90
7	190.5	162.1	12.70	10.80

6 Concluding remarks

We have considered two cost sharing problem associated with hub network problems. We have defined two respective cooperative cost games and we have proved that their cores are non-empty. Besides, we have computed the Shapley value of such cost games. We have proved that the Shapley value belongs to the core and we have provided some axiomatic characterizations.

We now compare our results with the three papers that study other cost sharing problems associated with hub networks. We consider several issues:

- About the class of hub network problems.

  Skorin-Kapov (1998, 2001) considers that the number of hubs to be located is fixed. Matsubayashi et al. (2005) and this paper consider that it is a variable.

  Skorin-Kapov (1998, 2001) assumes that locating a hub at some node has no cost. Matsubayashi et al. (2005) and this paper consider that there may be a cost.

  In Skorin-Kapov (2001), direct connection between non-hubs nodes are possible. In Skorin-Kapov (1998), Matsubayashi et al. (2005), and this paper, a non-hub node is only directly connected to a hub node.

  In Skorin-Kapov (1998, 2001) and this paper, there is no congestion cost. In Matsubayashi et al. (2005), there is a congestion cost.

- About the cooperative game. When the cost of a coalition S of nodes is computed:

  In Skorin-Kapov (1998, 2001) and Matsubayashi et al. (2005), the cost only depends on the outgoing flow of nodes in S. In this paper, we consider two cases. In the one-way flow case, the cost only depends on the outgoing flow of nodes in S. In the two-way flow case, the cost depends on the outgoing and the ingoing flow of nodes in S.

In some games in Skorin-Kapov (1998, 2001) and this paper, nodes in S can use only the hubs of the constructed network h. In some games in Skorin-Kapov (1998, 2001); Matsubayashi et al. (2005) and this paper, nodes in S can construct the hub they use for sending the flow optimally.

In the games in Skorin-Kapov (1998, 2001) and this paper, nodes in S can decide only on the traffic generated by the nodes in S. In Matsubayashi et al. (2005), nodes in S can decide on the traffic of all nodes.

- About the core of the cooperative game.

  In some games in Skorin-Kapov (1998, 2001) and this paper, the core is always nonempty. In Matsubayashi et al. (2005) and some games in Skorin-Kapov (1998, 2001) and this paper, the core can be empty.

- About the rule considered.

  Skorin-Kapov (1998, 2001) does not study any specific rule. In this paper, we study two rules based on the Shapley value. In Matsubayashi et al. (2005), they study the rule that divides the cost proportionally to the traffic generated by each node.

  The rules considered in this paper are characterized axiomatically. The rule in Matsubayashi et al. (2005) is not.

The main differences between this paper and the other papers are two. First, we consider the two-way case, which has not been studied before. Second, we give an explicit formula for the Shapley rule and we provide axiomatic characterizations of it. Notice that no characterization of other rules has been provided in the other papers.

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