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9 May 2019

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May 9, 2019

Abstract:
In this paper we propose the following conjecture: the equilibrium manifold $E(r) \subset \mathbb{R}^{LM-1}$, where L is the number of goods and M the number of consumers, is a minimal submanifold if and only if the price is unique for every economy. We show the validity of this conjecture for an arbitrary number of goods and two consumers and for an arbitrary number of consumers and two goods under the assumption that the normal vector field of $E(r)$ is constant outside a compact subset.

Keywords: Equilibrium manifold, uniqueness of equilibrium, minimal submanifold.

JEL Classification: D50, D51. **MSC Classification:** 53A10.

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*We would like to thank the participants at the SIE 59th Annual Conference, Bologna, for helpful discussions and critical comments. The first author was supported by Prin 2015 – Real and Complex Manifolds; Geometry, Topology and Harmonic Analysis – Italy, by INdAM. GNSAGA - Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni, by GESTA - Funded by Fondazione di Sardegna and Regione Autonoma della Sardegna and by KASBA- Funded by Regione Autonoma della Sardegna. The second author was supported by GROVID, funded by Fondazione di Sardegna and Regione Autonoma della Sardegna.

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1 Introduction

In this paper we explore the connection between minimality and uniqueness of equilibrium. We provide the following motivation. Let ω be an endowment with multiple equilibrium price vectors. One can assign a probability distribution to these set of prices, which play the role of events (which price vector will actually be the supporting price vector is the main issue of selection theory and goes behind the scope of this paper). Consider the value of the information content of a message saying that a price vector, e.g. p_1 with probability x_1, will be the equilibrium price vector. The information content of a message (see [14]) is a decreasing function of the probability that it occurs, $h(x_1)$, and is defined as $h(x_1) = -log(x_1)$. The expected information content of the distribution, or entropy, is

$$H(x) = \sum_{1=1}^{n} x_i h(x_i) = -\sum_{i=1}^{n} x_i log x_i$$

Entropy increases as the amount of uncertainty increases, and its maximum value is attained when all the events are equiprobable, i.e. $x_i = \frac{1}{n}$, $i = 1, \ldots, n$. Observe that an economy with a unique equilibrium price vector has entropy zero.

In a dynamic setting, one can link the length of a path from ω to ω' to its entropic measure. A minimal path is less entropic than a longer one, where the length of the path is measured taking into account the supporting equilibrium price vectors as we move from ω to ω'. Similarly, and more generally, one can establish a connection between the neighborhood of an economy and its entropy, where the volume is positively associated to the entropy. Since volume takes into account the equilibrium prices, it is natural to follow the equilibrium manifold approach, where the equilibrium manifold, $E(r)$, or Walras correspondence, is defined as the set of pairs of prices and endowments such that the aggregate excess demand function is equal to zero.

In this paper we investigate whether it is possible to have multiplicity of prices and minimal volume. This question leads us to the concept of minimal surfaces and, more generally, minimal submanifolds, a very active research area in differential geometry. Roughly speaking[1], a minimal submanifold $\mathcal{M}^n$ of a Riemannian manifold $\mathcal{N}^{n+k}$ minimizes volume locally, namely every point of $\mathcal{M}^n$ admits a neighborhood which minimizes volume among all submanifolds of $\mathcal{N}^{n+k}$ with the same boundary.

Even if the equilibrium manifold $E(r)$ is unknotted in its ambient space $\mathbb{R}^{LM-1}$, where L is the number of goods and M the number of consumers (see [4]), it can almost arbitrarily be twisted for an appropriate preference profile. Hence one could expect to find a configuration giving rise to multiplicity and minimality. We believe that exactly the opposite is true. Indeed we address the following:

[1]Throughout this paper we will content ourselves with this definition since in the proof of our main results we are not using the differential geometric machinery of the theory of minimal submanifolds. The interested reader is referred to [13] for more details and material on minimal submanifolds. The simplest examples of minimal submanifolds arise when $n = 1$: in this case they are simply geodesics of $\mathcal{N}^{k+1}$. In higher dimensions every totally geodesic submanifold is a minimal submanifold (cf. also [7] for some properties of geodesics and totally geodesic submanifolds of the equlibrium manifold). Nevertheless, there exist a lot of interesting minimal submanifolds (see [13] or Section 3 below).

Conjecture: *The equilibrium manifold $E(r) \subset \mathbb{R}^{LM-1}$ is a minimal submanifold if and only if the equilibrium price is unique.*

In other words we believe that there is not an utility profile which minimizes volume by preserving multiplicity.

Notice that the "only if' part of our conjecture follows by Balasko's work (cf. Theorem 2.1 below). Here we are concerned with the other implication, namely we are conjecturing that minimality implies unicity.

In this paper we show the validity of this conjecture in the case of an arbitrary number of goods and two consumers (Theorem 3.1) and in the case of an arbitrary number of consumers and two goods (Theorem 4.1) under the additional assumption that the normal vector field of $E(r)$ is constant outside a compact subset of the ambient space. The proof of Theorem 3.1 strongly relies on the classification of ruled minimal submanifolds of the Euclidean space, on the bundle structure of the equilibrium manifold and on the positiveness of prices. On the other hand, the proof of Theorem 4.1 combines deep geometric results relating the topology of a minimal submanifold of the Euclidean space with the fact that $E(r)$ is globally diffeomorphic to an Euclidean space.

There is a relationship with a recent result [9], where it is shown that the zero curvature of the equilibrium manifold implies uniqueness of equilibrium. As in [9], we are concerned with uniqueness across the whole commodity space in a smooth exchange economy. But the issue, and the approach, is entirely different. Here we are concerned with the immersion of the manifold on its ambient space, while in [9] the explored geometric properties are intrinsic, i.e. they do not depend on the ambient space.

We believe that the connection between the broad literature on minimal submanifolds and uniqueness and stability issues (see [6, 11] for a survey) may deserve further investigation (cf. also Remark 4.4 at the end of the paper).

This paper is organized as follows. In Section 2 we recall the standard economic setup. Section 3 and Section 4 are dedicated to the proof of Theorem 3.1 and Theorem 4.1, respectively.

2 The economic setting

The economic setup is represented by a pure exchange smooth economy with L goods and M consumers under the standard smooth assumptions (see [1, Chapter 2]). The set of normalized prices is defined by

$$S = \{p = (p_1, \dots p_L) \in \mathbb{R}^L \mid p_l > 0, l = 1, \dots, L,\ p_L = 1\}$$

and the set $\Omega = (\mathbb{R}^L)^M$ denotes the space of endowments $\omega = (\omega_1, \dots, \omega_M)$, $\omega_i \in \mathbb{R}^L$. The *equilibrium manifold* E is the set of the pairs $(p, \omega) \in S \times \Omega$, which satisfy the equality:

$$\sum_{i=1}^{M} f_i(p, p \cdot \omega_i) = \sum_{i=1}^{M} \omega_i, \tag{1}$$

where $f_i(p, w_i)$ is consumer's i demand.
By [1, Lemma 3.2.1], E is a (closed) smooth submanifold of $S \times \Omega$, globally diffeomorphic to $S \times \mathbb{R}^M \times \mathbb{R}^{(L-1)(M-1)} = \mathbb{R}^{LM}$, i.e. $\phi_{|E} \cong \mathbb{R}^{LM}$, where the smooth mapping

$$\phi : S \times \Omega \to S \times \mathbb{R}^M \times \mathbb{R}^{(L-1)(M-1)}$$

is defined by

$$(p, \omega_1 \dots, \omega_M) \mapsto (p, p \cdot \omega_1, \dots, p \cdot \omega_M, \bar{\omega}_1, \dots, \bar{\omega}_{M-1}),$$

where $\bar{\omega}_i$ denotes the first $L-1$ components of ω_i, for $i = 1, \dots, M-1$.

We also introduce the following two subsets of E:

- the set of *no-trade equilibria* $T = \{(p, \omega) \in E | \, f_i(p, p \cdot \omega_i) = \omega_i, \, i = 1, \dots, M\}$;
- *the fiber* associated with $(p, w_1, \dots, w_M) \in S \times \mathbb{R}^M$, which is defined as the set of pairs $(p, \omega) \in S \times \Omega$ such that:
  - $p \cdot \omega_i = w_i$ for $i = 1, \dots, M$;
  - $\sum_i \omega_i = \sum_i f_i(p, w_i)$.

By defining the two smooth maps

$$f : S \times \mathbb{R}^M \to S \times \mathbb{R}^{LM},$$

where $f(p, w_1, \dots, w_M) = (p, f_1(p, w_1), \dots, f_M(p, w_M))$, and

$$\phi_{Fiber} : E \to S \times \mathbb{R}^M,$$

where $\phi_{Fiber}(p, \omega_1, \dots, \omega_M) = (p, p \cdot \omega_1, \dots, p \cdot \omega_M)$, since $f(S \times \mathbb{R}^M) = T \subset E$ and $\phi_{Fiber} \circ f$ is the identity mapping, by applying [1, Lemma 3.2.1], Balasko shows [1, Proposition 3.3.2] that T is a smooth submanifold of E diffeomorphic to $S \times \mathbb{R}^M$.

By construction, every fiber associated with $(p, w_1, \dots, w_M)$ is a subset of E which is the inverse image of $(p, w_1, \dots, w_M)$ via the mapping ϕ_{Fiber}. It is intuitively clear that while holding $(p, w_1, \dots, w_M)$ fixed and letting ω varying along the fiber, there are not any nonlinearities which may arise from the aggregate demand. In fact the fiber is a linear submanifold of E of dimension $(L-1)(M-1)$ [1, Proposition 3.4.2].

Since every fiber contains only one no-trade equilibrium [1, Proposition 3.4.3], the equilibrium manifold E can be thought as a disjoint union of fibers parametrized by the no-trade equilibria T via the mapping $\phi_{|E} : E \to S \times \mathbb{R}^M \times \mathbb{R}^{(L-1)(M-1)}$: for a fixed $(p, w_1, \dots, w_M) \in S \times \mathbb{R}^M$, each fiber is parametrized by $\bar{\omega}_1, \dots, \bar{\omega}_{M-1}$. By letting $(p, w_1, \dots, w_M)$ varying in $S \times \mathbb{R}^M$, we obtain the *bundle structure* of the equilibrium manifold.

If total resources are fixed, the equilibrium manifold is defined as

$$E(r) = \{(p, \omega) \in S \times \Omega(r) \mid \sum_{i=1}^{M} f_i(p, p \cdot \omega_i) = r\}, \tag{2}$$

where $r \in \mathbb{R}^L$ is the vector that represents the total resources of the economy and $\Omega(r) = \{\omega \in \mathbb{R}^{LM} \mid \sum_{i=1}^{M} \omega_i = r\}$.
Let

$$B(r) = \{(p, w_1, \ldots, w_M) \in S \times \mathbb{R}^M \mid \sum_{i=1}^{M} f_i(p, w_i) = r\} \tag{3}$$

be the set of *price-income equilibria* (see [1, Definition 5.1.1]). $B(r)$ is a submanifold of $S \times \mathbb{R}^M$ diffeomorphic to $\mathbb{R}^{M-1}$ [1, Corollary 5.2.4] through the map $\theta : S \times \mathbb{R}^M \to \mathbb{R}^L \times \mathbb{R}^{M-1}$, defined by

$$(p, w) \mapsto (\sum_i f_i(p, w_i), u_1(f_1(p, w_1), \ldots, u_{M-1}(f_{M-1}(p, w_{M-1})). \tag{4}$$

The equilibrium manifold $E(r)$ is a submanifold of $S \times \Omega(r)$ diffeomorphic to $\mathbb{R}^{L(M-1)}$ [1, Corollary 5.2.5]

$$\phi(E(r)) = B(r) \times \mathbb{R}^{(L-1)(M-1)}. \tag{5}$$

Moreover we can define and $T(r) = T \cap S \times \Omega(r)$. By construction, even in a fixed total resource setting, the equilibrium manifold preserves its bundle structure property and, hence, $E(r)$ can be written as the disjoint union

$$E(r) = \sqcup_{x \in T(r)} F_x, \tag{6}$$

where F_x is an $(L-1)(M-1)$-affine subspace of $\mathbb{R}^{L(M-1)}$.

Let $t = (t_1, \ldots, t_{l-1})$, $\bar{\omega}_j = (\omega_1^1, \ldots, \omega_1^{l-1})$ and $p(t) = (p_1(t), \ldots, p_{l-1}(t)$. Following [1] and [9], we can parametrize $B(r)$ via the map:

$$\varphi : \mathbb{R}^{M-1} \to B(r),\ t \to (p(t), w_1(t) \ldots, w_{m-1}(t)) \tag{7}$$

and $E(r)$ via the map:

$$\Phi : \mathbb{R}^{L(M-1)} \to E(r), \tag{8}$$

$$(t, \omega_1^1, \ldots, \omega_{M-1}^1, \ldots, \omega_1^1, \ldots, \omega_{M-1}^{L-1}) \mapsto (p(t), \bar{\omega}_1, w_1(t) - p(t) \cdot \bar{\omega}_1, \ldots, w_{M-1}(t) - p(t) \cdot \bar{\omega}_{M-1})$$

We end this section with a theorem due to Balasko which is related with the issue raised in this paper.

Theorem 2.1 [1, p. 188 Theorem 7.3.9 part (2)] *If for every $\omega \in \Omega(r)$ there is uniqueness of equilibrium, the equilibrium correspondence is constant: The equilibrium price vector p associated with ω does not depend on ω.*

Remark 2.2 This theorem will be used to prove the "only if" part of Theorem 3.1 and Theorem 4.1. In this paper we are concerned with the other implication: does minimality implies uniqueness? (cf. the conjecture in the introduction).

3 The case $M = 2$

In this section we prove the following:

Theorem 3.1 *Let $M = 2$. Then $E(r)$ is a minimal submanifold of $\mathbb{R}^{2L-1}$ if and only if the price is unique.*

Before proving the theorem we need some definitions.

- a submanifold $\mathcal{M}^n \subset \mathbb{R}^{n+k}$ is said to be ruled if $\mathcal{M}^n$ is foliated by affine subspaces of dimension $n-1$ in $\mathbb{R}^{n+k}$.
- a *generalized helicoid* is the ruled submanifold $\mathcal{M}^n(a_1, \dots, a_k, b) \subset \mathbb{R}^{n+k}$, $k \leq n$, admitting the following parametrization:

$$(s, t_1, \dots, t_{n-1}) \mapsto (t_1 \cos(a_1 s), t_1 \sin(a_1 s), \dots, t_k \cos(a_k s), t_k \sin(a_k s), t_{k+1}, \dots, t_{n-1}, bs)),$$

where a_j, $j = 1, \dots, k$, and b are real numbers (we are not escluding that one of these coefficients could vanish and the generalized helicoid becomes an affine subspace).

The key ingredient in the proof of Theorem 3.1 is the following classification result on ruled minimal submanifolds of the Euclidean space. We refer the reader to [5, Section 1] and references therein (in particular [10] for a proof).

Theorem 3.2 ([10]) *A minimal ruled submanifold $\mathcal{M}^n \subset \mathbb{R}^{n+k}$ is, up to rigid motions*[2] *of $\mathbb{R}^{n+k}$, a generalized helicoid.*

We need also the following simple but fundamental fact:

Lemma 3.3 *Let $\mathcal{M}^n(a_1, \dots, a_k, b) \subset \mathbb{R}^{n+k}$ be a generalized helicoid such that $b \cdot \prod_{i=1}^k a_i \neq 0$. Then $\mathcal{M}^n$ intersects any affine hyperplane of $\mathbb{R}^{n+k}$.*

Proof: In cartesian coordinates $x_1, y_1, \dots, x_k, y_k, x_{k+1}, \dots, x_{n-1}, x_n$ an hyperplane of $\mathbb{R}^{n+k}$ has equation:

$$\alpha_1 x_1 + \beta_1 y_1 + \dots + \alpha_k x_k + \beta_k y_k + \alpha_{k+1} x_{k+1} + \dots + \alpha_{n-1} x_{n-1} + \alpha_n x_n = \delta,$$

where $\alpha_i, \beta_i, i = 1, \dots k$, $\alpha_j, j = k+1, \dots n$ and δ are real numbers such that

$$\sum_{i=1}^{k} (\alpha_i^2 + \beta_i^2) + \sum_{j=1}^{n} \alpha_j^2 \neq 0.$$

[2]A rigid motion of the Euclidean space $\mathbb{R}^l$ is an isometry of $\mathbb{R}^l$ given by the composition of an orthogonal $l \times l$ matrix and a translation by some vector $v \in \mathbb{R}^l$.

On the other hand the following equation represents the condition to be satisfied for a point of the hyperplane to belong to the generalized helicoid:

$$\sum_{i=1}^{k} t_i(\alpha_i \cos(a_i s) + \beta_i \sin(a_i s)) + \sum_{j=k+1}^{n-1} \alpha_j t_j + \alpha_n bs = \delta.$$

Since one can always find a pair (s_0, t_0) satisfying the previous equation, the lemma is proved. $\square$

Proof of Theorem 3.1: Since $M = 2$, by the bundle structure property (see (6)), $E(r)$ is a ruled submanifold in $\mathbb{R}^{2L-1}$. By Theorem 3.2, $E(r)$ is (up to rigid motions) a generalized helicoid. If some a_i or b are equal to zero then $E(r)$ is an hyperplane and, by Theorem 2.1, the price is unique. Otherwise if $b \cdot \prod_i a_i \neq 0$, by combining Lemma 3.3 with the fact that $E(r)$ is contained in the open set of $\mathbb{R}^{2L-1}$ consisting of those points with $p > 0$ (p being the price) one deduces that $E(r)$ is an affine hyperplane and so the price is unique. The "only if" part follows by Theorem 2.1 (see Remark 2.2). $\square$

Remark 3.4 In the previous theorem we assumed the minimality of $E(r)$. We can prove the same result by only assuming that the no-trade equilibria $T(r)$ (which is one dimensional for $M = 2$) is a minimal submanifold of $E(r)$, namely it is a geodesic. Indeed, by using the diffeomorphism between $T(r)$ and $B(r)$, and the parametrization Φ of $E(r)$ (see (8)), $T(r)$ can be parametrized through Φ by letting $v = 0$:

$$\Phi(t, 0) = \gamma(t).$$

Hence, if $T(r)$ is a geodesic in $E(r)$, its acceleration $\gamma''(t)$ is parallel, for every t, to the unit normal vector $N(t)_{|v=0}$ of $E(r)$ or, equivalently, $\gamma''(t) \wedge N(t)_{|v=0} = 0$. We have $\gamma''(t) = \beta''(t) = (\ddot{p}, 0, \ddot{w})$ and, since $v = 0$, $\Phi_t \wedge \Phi_v = \dot{\beta} \wedge \delta = (-\dot{w}, p\dot{p}, \dot{p})$. Hence $\gamma''(t) \wedge N(t)_{|v=0} = \beta'' \wedge (\beta' \wedge \delta) = 0$ if and only if

$$(-p\dot{p}\ddot{w}, p\ddot{p} + \dot{w}\ddot{w}, p\dot{p}\ddot{p}) = (0, 0, 0).$$

This implies that, for every t, $\dot{p}\ddot{p} = 0$, i.e. $(\dot{p}\dot{p})' = 0$, hence p is (constant and) unique.

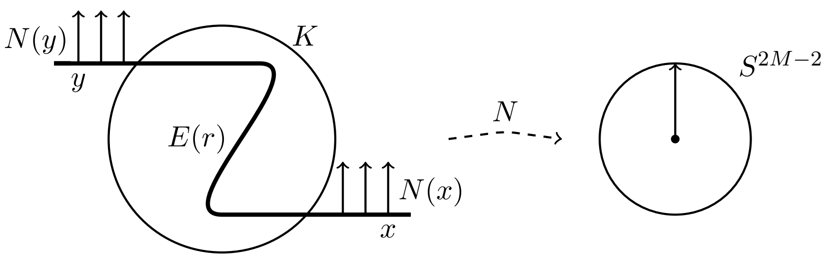
4 The case $L = 2$

In this section we consider an economy with two goods and an arbitrary number of consumers. In this case the equilibrium manifold is a hypersurface. More precisely, the equilibrium manifold $E(r)$ has dimension $\mathbb{R}^{2M-2}$ and the ambient space has dimension $\mathbb{R}^{2M-1}$. So it makes sense to consider the normal vector field N along $E(r)$, namely for each $x \in E(r)$ we consider a unit vector $N(x)$ parallel to one dimensional affine subspace $T_x E(r)^{\perp} \subset \mathbb{R}^{2M-1}$ normal to the tangent space $T_x E(r)$ of $E(r)$ at x. The smooth map $N : E(r) \to S^{2M-2}$ which takes x to the point $N(x)$ of the unit sphere $S^{2M-2} \subset \mathbb{R}^{2M-1}$

is called the *Gauss map*. Obviously, if the Gauss map is constant then the price is constant and hence $E(r)$ is an affine hyperplane in $\mathbb{R}^{2M-1}$. In the following theorem, which represents the second main result of the paper, we show that the minimality assumption together with the constancy of the Gauss map outside a compact set imply uniqueness of the equilibrium price.

Theorem 4.1 *Let $L = 2$ and assume that the Gauss map is constant outside a compact subset of $E(r)$. Then $E(r)$ is a minimal submanifold of $\mathbb{R}^{2M-1}$ if and only if the price is unique.*

This theorem can be intepreted by saying that if the equilibrium manifold is minimal and there exists a compact subset K of $\mathbb{R}^{2M-1}$ such that $(\mathbb{R}^{2M-1} \setminus K) \cap E(r)$ is the union of open subsets of hyperplanes each parallel to the hyperplane $p = const$, then $E(r)$ is indeed an hyperplane. As a consequence, the usual one-dimensional representation of the equilibrium manifold cannot be minimal (see figure below).



The proof of Theorem 4.1 relies on the following theorem obtained in turn by suitably combining some deep results obtained by Anderson in [2].

Theorem 4.2 *Let $\mathcal{M}^n \subset \mathbb{R}^{n+1}$, $n > 2$, be a minimal hypersurface such that the following conditions are satisfied:*

1. *$\mathcal{M}^n$ has one end;*
2. *$\mathcal{M}^n$ is a C^1-diffeomorphic to a compact manifold $\bar{\mathcal{M}}^n$ punctured at a finite number of points $\{p_i\}$.*
3. *the Gauss map $N : \mathcal{M}^n \to S^n$ extends to a C^1-map of $\bar{\mathcal{M}}^n$.*

Then $\mathcal{M}^n$ is an affine n-plane.

Remark 4.3 The number of ends of a smooth manifold is a topological invariant which, roughly speaking, measures the number of connected components "at infinity". The reader is referred to [2] for a rigorous definition. What we are going to use in the proof of Theorem 4.1 is that for $n > 1$, the Euclidean space $\mathbb{R}^n$ has only one end. This is because $\mathbb{R}^n \setminus K$ has only one unbounded component for any compact set K.

Proof of Theorem 4.1: If $M = 2$ we can apply Theorem 3.1. We can then assume $M > 2$ and so $\dim E(r) = 2M - 2 > 2$. Hence, in order to prove the "if" part it is enough to verify that the three conditions of Theorem 4.2 are satisfied for $E(r) \subset \mathbb{R}^{2M-1}$. Conditon 1 follows by the previous remark, since $E(r)$ is globally diffeomorphic to $\mathbb{R}^{2M-2}$. Notice that the unit sphere S^{2M-2} is the Alexandroff compactification of $E(r) \cong \mathbb{R}^{2M-2}$, namely it can be obtained by adding one point, called ∞, to $E(r)$. In other words $E(r)$ is diffeomorphic to the sphere S^{2M-2} punctured to ∞ and so also condition 2 holds true. The assumption that the Gauss map $N : E(r) \to S^{2M-2}$ is constant outside a compact set K means that $N(x) = N_0$, where N_0 is a fixed vector in S^{2M-2}, for all $x \in E(r) \setminus K$. Therefore, one can extend N to a C^∞-map $\hat{N} : S^{2M-2} \to S^{2M-2}$ by simply defining $\hat{N}(\infty) = N_0$, and so also condition 3 is satisfied. The "only if" part follows by Theorem 2.1 (see Remark 2.2). $\square$

Remark 4.4 Given a submanifold $\mathcal{M}^n$ of a Riemannian manifold $\mathcal{N}^{n+k}$, one can express the minimality condition in terms of the vanishing of the mean curvature H. If $k = 1$, namely when $\mathcal{M}^n$ is an hypersurface, the minimality condition, namely $H = 0$, is equivalent to the vanishing of the trace of the differential of the Gauss map (see e.g. [3]). Thus, for $L = 2$ one could try to show that minimality of $E(r)$ implies uniqueness of the equilibrium price without imposing the extra condition on the constancy of the Gauss map outside a compact set (as in Theorem 4.1), by computing the Gauss map through the parametrisation (8) above and imposing the vanishing of the trace of its differential. This gives rise to a complicated PDE equation, which also when $M = 3$ the authors were not able to handle.

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