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Abstract

This paper considers agglomeration economies. A new firm is planning to open a plant in a country divided into several regions. Each firm receives a positive externality if the new plant is located in its region. In a decentralized mechanism, the plant would be opened in the region where the new firm maximizes its individual benefit. Due to the externalities, it could be the case that the aggregated utility of all firms is maximized in a different region. Thus, the firms in the optimal region could transfer something to the new firm in order to incentivize it to open the plant in that region. We propose two rules that provide two different schemes for transfers between firms already located in the country and the newcomer. The first is based on cooperative game theory. This rule coincides with the nucleolus and the τ-value of the associated cooperative game. The second is defined directly. We provide axiomatic characterizations for both rules. We characterize the core of the cooperative game. We prove that both rules belong to the core.

Keywords: game theory, core, axiomatic characterization, agglomeration economies.

1. Introduction

There are many situations where the aggregate utility obtained by a group of agents is greater when they cooperate than when they make individual decisions. Some situations come from operations research, others from economics, and still others from other areas. For instance there are airport games (Littlechild and Owen, 1973), linear production problems (Owen, 1975), minimum cost spanning tree problems (Bird, 1976), bankruptcy problems (O'Neill, 1982), transportation situations (Sanchez-Soriano, 2006), and broadcasting sports events (Bergantiños and Moreno-Ternero, 2019). A major issue is how to divide the benefits of such cooperation. Our paper fits into this literature because we study how to divide the benefits of cooperation in an economic problem motivated by agglomeration economies.

Firms dedicated to similar activities tend to settle their plants in the same region because proximity between them provides mutual benefits. These situations are called agglomeration economies. They have

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been widely studied in economic literature. Marshall (1920) provides the first analysis on this theory. He states that benefits of firms from agglomeration come from three main sources: Input sharing, labor market pooling, and knowledge spillovers. There is also a large body of empirical literature measuring the relative importance of different agglomeration theories (see, for instance, Rosenthal and Strange, 2003 and Ellison et al., 2010).

In this paper, we consider that a firm is planning to open a new plant in a country divided into different regions where the plant could be opened. There are also a finite number of firms already located in these regions. The new plant generates agglomeration economies for all of them. This means that the firms in the region where the plant is opened receive a positive externality.

In a decentralized mechanism, the plant would be located in the most profitable region for the new firm, say k. However, if the new firm is located in a different region, say k^*, the aggregated utility of all firms in region k^* and the new firm could be greater than the aggregated utility of all firms in region k and the new firm. This is because the new firm could create more positive externalities in region k^* than in region k. Then, it makes sense for firms in region k^* to transfer something to the new firm in order to incentivize it to locate its plant in region k^* instead of region k. The question is what transfers should be made or, equivalently, how the aggregated utility generated when the new plant is located in region k^* should be divided up.

An indirect approach for answering this question is the following: First, associate a cooperative game with transferable utility to the situation. Second, compute a solution in the associated cooperative game. This indirect approach has been taken quite often in the literature. In airport games, Littlechild and Owen (1973) study the Shapley value and Littlechild (1974) studies the nucleolus. In linear production games Owen (1975) studies the core. In minimum cost spanning tree problems, Bird (1976) studies the core, Granot and Huberman (1984) study the nucleolus, and Kar (2002) studies the Shapley value. In bankruptcy problems O'Neill (1982) studies the Shapley value, Aumann and Maschler (1985) study the nucleolus, and Curiel et al. (1987) study the τ-value. In transportation situations Sanchez-Soriano (2006) studies the core. In broadcasting sport events Bergantiños and Moreno-Ternero (2019) study the core and the Shapley value.

Several reasonable ways can be found of associating a cooperative game with a given arbitrary problem. For instance, in minimum cost spanning tree problems, Bird (1976) gives one way and Bergantiños and Vidal-Puga (2007) give several options. In this paper we associate a cooperative game with each agglomeration problem as follows: In some countries, *e.g.* Spain, regional governments can incentivize firms to locate their plants in their territories. The usual way is to offer a subsidy to the firm for opening a new plant. Of course such subsidies come from the budget of the regional government, which in turn comes from the taxes paid by the economic agents in the region (which include other firms in the region). Thus, there is some transfer between the firms in the region and the firm opening the plant there. We think that the best way to model this situation is to consider that the coalitions that can share their benefits are those formed by the firm opening the plant and all firms in a given region.

We study several solution concepts for the cooperative game. The core is non-empty and can be described as follows: The new firm receives at least the aggregated utility (of all existing firms plus the new firm) when the new firm locates in the second best region. The difference between the aggregated utility of the best region and the second best region is divided among the new firm and the firms in the best region. Firms outside the best region receive 0.

The nucleolus and the τ-value coincide. We call this the egalitarian optimal location rule. The Shapley value also coincides with this rule in a subset of agglomeration problems. The egalitarian optimal location rule is the allocation of the core where the difference between the aggregated utility of the best region and that of the second best region is divided equally among the new firm and the firms of the best region. This rule has a problem because it allows some firms in the best region to receive transfers from other firms in the same region. We find this counterintuitive. Transfers should go from the firms in the best region to the new firm.

We also consider a rule, called the weighted optimal location rule, which is not defined through the associated cooperative game. This rule is defined as follows: The new firm receives the aggregated utility (of all existing firms plus the new firm) when the new firm locates in the second best region. The difference between the aggregated utility of the best and second best regions is divided among the firms located in the best region proportionally to the individual benefits generated for the firms by the location of the new firm. Firms outside the best region receive 0. In this case firms in the best region make transfers to the new firm and do not receive transfers from other firms.

Finally, we provide axiomatic characterizations of both rules. The egalitarian optimal location rule is characterized by core selection (the rule should select core allocations) and equal treatment inside optimal regions (if the aggregated utility of the best region increases, then all firms in that region and the new firm should improve by the same quantity).

We characterize the set of rules satisfying core selection and merging-splitting proofness (if one firm splits, the allocation to the rest of the firms does not change). This set of rules can be described as follows: The new firm receives the aggregated utility of the second best region for sure. Moreover, the new firm also receives an extra transfer from firms in the best region. The transfer from each firm located in the best region is proportional to the benefits of the firm. The weighted optimal location rule is a rule satisfying the two properties above that minimizes the transfer received by the new firm.

The rest of the paper is organized as follows. Section 2 formally introduces the problem and the rules. Section 3 associates a cooperative game with any agglomeration problem. Section 4 examines several cooperative solutions. Section 5 presents axiomatic characterizations of the rules, and Section 6 concludes.

2. The agglomeration problem

We introduce the formal model for studying the situations described in the introduction.

2.1. The model

An **agglomeration problem** (or simply, a "problem") is a tuple $\mathcal{A} = (0, N, P, b)$ where

- 0 is the firm which will open a plant in the country.
- $N = \{1, ..., n\}$ is the set of firms already located in the country. We denote by $N_0 = N \cup \{0\}$.
- $P = \{P_1, ..., P_r\}$ is a partition of N, where $R = \{1, ..., r\}$ is the set of regions in the country. P_k denotes the set of firms located in region k.
- $b = \left\{b_i^k : i \in N_0 \text{ and } k \in R\right\}$. $b_i^k \geq 0$ denotes the benefit obtained by firm i when 0 locates its plant in region k.

We assume that if an existing firm is outside the region where the new plant is located it does not obtain any (significant) benefit. Thus, for all $k \in R$ and all $i \notin P_k$, $b_i^k = 0$. There are no further assumptions about P. So there may be a region with no firms located in it, *i.e.*, $P_k = \emptyset$.

We now introduce some concepts and notation used throughout the paper.

For each $i \in N$, $k(i) \in R$ denotes the region where firm i is located, *i.e.*, $i \in P_{k(i)}$. For all $S \subseteq N_0$, $U(S)$ denotes the firms in regions contained in S. This is,

$$U(S) = \{i \in S : P_{k(i)} \subseteq S\}.$$

In a decentralized mechanism, firm 0 would locate its new plant in the region where the firm optimizes its individual benefit. Namely in

$$\arg\max_{k \in R} \left\{b_0^k\right\}.$$

Nevertheless, the aggregated benefit could be greater if the location was different, so it makes sense to locate the new plant maximizing the global benefit and then provide a compensation scheme. Thus, firm 0 gets more than in the decentralized mechanism and the other firms are not worse off.

Formally, we define the **global benefit** of any problem $\mathcal{A}$ as

$$g(\mathcal{A}) = \max_{k \in R} \left\{ \sum_{i \in N_0} b_i^k \right\}.$$

For each region $k \in R$, the benefit obtained by all firms in N_0 when the plant is located in region k is given by

$$\sum_{i \in N_0} b_i^k = \sum_{i \in P_k \cup \{0\}} b_i^k.$$

Given a problem $\mathcal{A}$ we say that k^* is an **optimal region** if locating firm 0 in region k^* results in the maximum global benefit. Namely for each $P_k \in P$,

$$\sum_{i \in P_{k^*} \cup \{0\}} b_i^{k^*} \geq \sum_{i \in P_k \cup \{0\}} b_i^k.$$

Obviously, k^* may be not unique and $g(\mathcal{A}) = \sum_{i \in P_{k^*} \cup \{0\}} b_i^{k^*}$ for each optimal region k^*.

For every problem $\mathcal{A}$, we now define $s(\mathcal{A})$ as the global benefit obtained by all firms in N_0 when the plant is located in the second best region. Formally, given an optimal region k^*

$$s(\mathcal{A}) = \max_{k \in R \setminus \{k^*\}} \left\{ \sum_{i \in P_k \cup \{0\}} b_i^k \right\}.$$

Although this definition could depend on the chosen k^*, it does not. When there are several optimal regions, $s(\mathcal{A}) = g(\mathcal{A})$. Otherwise, $s(\mathcal{A}) < g(\mathcal{A})$.

We now define $I_0(\mathcal{A})$ as the maximum between the individual benefit of firm 0 when it locates in an optimal region k^* and the benefit obtained by all firms when 0 locates in the second best region. Then,

$$I_0(\mathcal{A}) = \max_{k \in R} \left\{ \sum_{i \in N_0 \setminus P_{k^*}} b_i^k \right\} = \max \left\{ b_0^{k^*}, s(\mathcal{A}) \right\}.$$

Notice that $I_0(\mathcal{A})$ is the maximum utility that can be obtained without cooperating with firms in region k^*.

For each optimal region k^* we have that

$$b_0^{k^*} \leq \sum_{i \in P_{k^*} \cup \{0\}} b_i^{k^*} = g(\mathcal{A}).$$

When there are several optimal regions, $s(\mathcal{A}) = g(\mathcal{A})$ and hence $I_0(\mathcal{A}) = g(\mathcal{A})$. Thus, $I_0(\mathcal{A})$ does not depend on the k^* chosen.

2.2. Rules

In this section we propose two rules: the egalitarian optimal location rule and the weighted optimal location rule.

A **rule** is a way of dividing the global benefit among the set of all firms, *i.e.*, a function f assigning to each problem $\mathcal{A}$ a vector in $\mathbb{R}^{N_0}$ that satisfies

$$\sum_{i \in N_0} f_i(\mathcal{A}) = g(\mathcal{A}).$$

For each problem $\mathcal{A}$, let k^* be an optimal region and let $|P_{k^*}|$ denote the number of firms in region k^*. The **egalitarian optimal location rule** (EOL), for each $i \in N_0$, is defined as

$$EOL_i(\mathcal{A}) = \begin{cases} I_0(\mathcal{A}) + \dfrac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}|+1}, & \text{if } i = 0 \\ \dfrac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}|+1}, & \text{if } i \in P_{k^*} \\ 0, & \text{otherwise.} \end{cases}$$

This rule has a nice interpretation. Firm 0 receives $I_0(\mathcal{A})$ for sure. Moreover, the surplus generated (with respect to $I_0(\mathcal{A})$) by P_{k^*} and 0 is divided equally among all firms generating that surplus. Firms outside the optimal region get zero. Below, we prove that the EOL rule can be obtained as a solution of a cooperative game.

Notice that when $\mathcal{A}$ has several optimal regions, $EOL_0(\mathcal{A}) = g(\mathcal{A})$ and $EOL_i(\mathcal{A}) = 0$ for all $i \in N$.

We think that this rule could assign "too much" to some firms. Consider the following example.

Example 1. Let $N = \{1,2,3\}$, $P = \{\{1\},\{2,3\}\}$, $b_0^1 = 6$, $b_1^1 = 2$, $b_0^2 = 5$, $b_2^2 = 1$ and $b_3^2 = 8$.

The optimal region is $P_2 = \{2,3\}$, $g(\mathcal{A}) = 14$ and $I_0(\mathcal{A}) = 8$. Now $EOL(\mathcal{A}) = (10,0,2,2)$.

The interpretation of this allocation is the following. To attract firm 0 to region 2, firms in that region must transfer something to firm 0. Thus it seems reasonable for firm 0 get 10. However, firm 2 gets 2, more than its individual benefit when firm 0 locates in region 2. Instead of transferring something to firm 0, firm 2 receives a transfer of 1 unit from firm 3. We do not find this very intuitive.

We now introduce a rule called the **weighted optimal location rule** (WOL), which avoids the problem mentioned for the EOL rule. The idea is simple: Firm 0 receives $I_0(\mathcal{A})$. If firm 0 receives $I_0(\mathcal{A})$ for locating in k^*, then it is known for sure that no region can offer firm 0 more than $I_0(\mathcal{A})$. The surplus generated when firm 0 locates in region k^*, $g(\mathcal{A}) - I_0(\mathcal{A})$, is divided among the firms in region k^* proportionally to b.

Formally, for each $i \in N_0$,

$$WOL_i(\mathcal{A}) = \begin{cases} I_0(\mathcal{A}), & \text{if } i = 0 \\ \dfrac{b_i^{k^*}}{\sum\limits_{j \in P_{k^*}} b_j^{k^*}} (g(\mathcal{A}) - I_0(\mathcal{A})), & \text{if } i \in P_{k^*} \\ 0, & \text{otherwise.} \end{cases}$$

Notice that there are two main differences between the egalitarian optimal location rule and the weighted optimal location rule. In EOL, $g(\mathcal{A}) - I_0(\mathcal{A})$ is divided equally among all firms in P_{k^*} and firm 0. While in WOL, $g(\mathcal{A}) - I_0(\mathcal{A})$ is divided only among firms in P_{k^*} and not equally but proportionally to b.

In Example 1 $WOL(\mathcal{A}) = (8, 0, 0.67, 5.33)$. In this case both firms in the optimal region transfer something to firm 0. We find this reasonable.

3. The cooperative game

In this section we approach the agglomeration problem as a cooperative game with transferable utility and study some properties of that game.

We first review some well-known definitions of cooperative games. Later we introduce a cooperative game modeling an agglomeration problem.

3.1. Basic notions of cooperative games

We now introduce cooperative games with transferable utility and some solutions such as the core, the nucleolus, the τ-value, and the Shapley value.

A **cooperative game with transferable utility** (TU game) is a pair (N, v) where $N \subset \mathbb{N}$ is the finite set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is the characteristic function. For any coalition $S \subseteq N$, $v(S)$ represents the amount that the members of coalition S can obtain if they cooperate. When no confusion arises, we refer to v as a game.

An allocation $x \in \mathbb{R}^N$ is an **imputation** in v if $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$, for all $i \in N$. The set of all imputations for a game v is denoted by $I(v)$.

The **core** of v is

$$C(v) = \left\{ x \in I(v) : \sum_{i \in S} x_i \geq v(S), \text{ for all } S \subseteq N \right\}.$$

We now present three single-value solutions of TU games: the nucleolus (Schmeidler, 1969), the τ-value (Tijs, 1981), and the Shapley value (Shapley, 1953).

The **excess** of $S \subseteq N$ with respect to any $x \in I(v)$ is defined as

$$e(S, x) = v(S) - \sum_{i \in S} x_i.$$

This number can be interpreted as the degree of dissatisfaction of coalition S when imputation x is realized.

For each $x \in I(v)$, let $\theta(x) \in R^{2^n}$ be the vector of all excesses $e(S, x)$ arranged in non-increasing order, *i.e.*, $\theta_i(x) \geq \theta_{i+1}(x)$ for all $i \in \{1, ..., 2^n - 1\}$.

For any $x, y \in I(v)$, x is more acceptable than y (and write $x \succ y$) if there is an integer $1 \leq j \leq 2^n$ such that $\theta_i(x) = \theta_i(y)$ if $1 \leq i < j$ and $\theta_j(x) < \theta_j(y)$.

The **nucleolus** of v is the set

$$\eta(v) = \{x \in I(v) : x \succeq y, \forall y \in I(v)\}.$$

The nucleolus consists of those imputations which are such that there is no more acceptable alternative. In other words, the nucleolus recursively minimizes the dissatisfaction of the worst treated coalitions.

It is well known that the nucleolus is always non-empty and it contains a unique allocation. Furthermore, if the game has a non-empty core, the nucleolus belongs to the core.

For any v and every $i \in N$, let $M_i(v)$ be player i's marginal contribution to the grand coalition, *i.e.*,

$$M_i(v) = v(N) - v(N \backslash \{i\}).$$

The vector $M(v) = (M_i(v))_{i \in N}$ is called the **utopia vector** of v.

The **minimum right vector** is $m(v) = (m_i(v))_{i \in N}$, where $m_i(v)$ is the greatest possible remainder for player i of $v(S)$ after every other player in the coalition obtains their utopia payoff. Formally, for all $i \in N$

$$m_i(v) = \max_{S \subset N : i \in S} \left\{ v(S) - \sum_{j \in S \backslash \{i\}} M_j(v) \right\}.$$

The **core cover** of v consists of the set of allocations that gives each player at least their minimum right and at most their utopia point. Namely,

$$CC(v) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N), m(v) \leq x \leq M(v) \right\}.$$

When the core cover is non-empty, the **τ-value** is defined as

$$\tau(v) = \alpha M(v) + (1 - \alpha) m(v)$$

with $\alpha \in [0, 1]$ such that $\sum_{i \in N} \tau_i(v) = v(N)$.

Let Π_N be the set of all permutations of the finite set $N \subset \mathbb{N}$. Given $\pi \in \Pi_N$, let $Pre(i, \pi)$ denote the set of elements of N which come before i in the order given by π, *i.e.* $Pre(i, \pi) = \{j \in N | \pi(j) < \pi(i)\}$.

The **Shapley value** of v is defined for all $i \in N$ as the average of the marginal contribution of agent i over the set of all permutations. Namely,

$$Sh_i(v) = \frac{1}{|N|!} \sum_{\pi \in \Pi_N} (v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi))).$$

Finally, we introduce some standard properties of TU games. We say that v is:

- **Monotone** if $v(S) \leq v(T)$ whenever $S \subseteq T$, for all $S, T \subseteq N$.
- **Superadditive** if for $S, T \subseteq N$ with $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$.
- **Convex** if $v(S \cup \{i\}) - (S) \geq v(T \cup \{i\}) - v(T)$, $\forall\ T \subseteq S \subseteq N \backslash \{i\}$, $\forall\ i \in N$.

Monotonicity states that the worth of a coalition increases as more players join it. Superadditivity says that it is more profitable for two disjoint coalitions to merge. In a convex game, the marginal contribution of a player is monotone with respect to the size of the coalition that they join.

3.2. The agglomeration game

We associate a cooperative game $v^{\mathcal{A}}$ with each agglomeration problem $\mathcal{A}$. We also study some basic properties of that cooperative game.

We now define the cooperative $v^{\mathcal{A}}$ under the assumption that benefits can only be shared between firm 0 and all firms in region k. Then, for any problem $\mathcal{A}$ and every $S \subseteq N_0$ the **agglomeration game** $v^{\mathcal{A}}$ can be defined as

$$v^{\mathcal{A}}(S) = \begin{cases} \max\limits_{k \in R} \left\{ b_0^k + \sum\limits_{i \in U(S)} b_i^k \right\}, & \text{if } 0 \in S \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $v^{\mathcal{A}}(N_0) = g(\mathcal{A})$. We compute $v^{\mathcal{A}}$ in Example 1.

Example 1 (*continuation*)**.** The table below shows the worth of coalitions $S \subseteq N_0$, with $0 \in S$ according to $v^{\mathcal{A}}$:

S	$\{0\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\{0,1,2\}$	$\{0,1,3\}$	$\{0,2,3\}$	N_0
$v^{\mathcal{A}}(S)$	6	8	6	6	8	8	14	14

Notice that, for example, the worth of coalition $\{0,3\}$ is 6, although the aggregated benefit is $b_0^2+b_3^2 = 13$. Since firm 2 is also located in region 2 and is not in the coalition, firm 0 would locate its new plant in region 1, the most profitable region for 0.

In the proposition below we discuss the properties satisfied by $v^{\mathcal{A}}$.

Proposition 1. *$v^{\mathcal{A}}$ is monotone and superadditive but not convex.*

Proof. Let $S \subseteq T \subseteq N_0$. Since $U(S) \subseteq U(T)$ it can be deduced that $v^{\mathcal{A}}(S) \leq v^{\mathcal{A}}(T)$ and hence $v^{\mathcal{A}}$ is monotone.

Let $S, T \subseteq N_0$ with $S \cap T = \emptyset$. Consider two cases. First, $0 \notin S \cup T$. Then $v^{\mathcal{A}}(S) = v^{\mathcal{A}}(T) = v^{\mathcal{A}}(S \cup T) = 0$ and $v^{\mathcal{A}}(S \cup T) \geq v^{\mathcal{A}}(S) + v^{\mathcal{A}}(T)$. Second, $0 \in S \cup T$. Assume that $0 \in S$ (the case $0 \in T$ is similar so we omit it). Since $v^{\mathcal{A}}$ is monotone, $v^{\mathcal{A}}(S) \leq v^{\mathcal{A}}(S \cup T)$. Since $0 \notin T$, $v^{\mathcal{A}}(T) = 0$. Then $v^{\mathcal{A}}(S \cup T) \geq v^{\mathcal{A}}(S) + v^{\mathcal{A}}(T)$. Hence $v^{\mathcal{A}}$ is superadditive.

We now prove that $v^{\mathcal{A}}$ may not be convex. Take $i = 1$, $S = \{0,2\}$ and $T = \{0,2,3\}$ in Example 1. Since $v^{\mathcal{A}}(S \cup \{i\}) - v^{\mathcal{A}}(S) = 8 - 6 = 2 > v^{\mathcal{A}}(T \cup \{i\}) - v^{\mathcal{A}}(T) = 14 - 14 = 0$, it can be deduced that $v^{\mathcal{A}}$ is not convex. □

In the next claim we state some obvious links between $I_0(\mathcal{A})$ and $v^{\mathcal{A}}$ that we then use in the proofs of our results.

Claim 1. *For any problem $\mathcal{A}$ and each $S \subseteq N_0$ with $0 \in S$, the following statements hold*

1. *$I_0(\mathcal{A}) \geq v^{\mathcal{A}}(S)$ when $P_{k^*} \nsubseteq S$.*

2. $I_0(\mathcal{A}) \leq v^{\mathcal{A}}(S)$ *when* $P_{k^*} \subseteq S$.

3. $I_0(\mathcal{A}) = v^{\mathcal{A}}(N_0 \backslash \{i\})$, *for any* $i \in P_{k^*}$.

4. $I_0(\mathcal{A}) = \max\limits_{k \in R \backslash \{k^*\}} \{v^{\mathcal{A}}(P_k \cup \{0\})\}$.

Now we discuss some links between $v^{\mathcal{A}}$ and other classes of games in the literature. Big boss games were introduced in Muto et al. (1988). A game v is a **big boss game** with a powerful player $i^* \in N$ if it satisfies the following three conditions: $(B1)$ v is monotone; $(B2)$ $v(S) = 0$ if $i^* \notin S$; and $(B3)$ $v(N) - v(S) \geq \sum\limits_{i \in N \backslash S} [v(N) - v(N \backslash \{i\})]$. Bahel (2016) extends the family of big boss games considering all games that satisfy $(B1)$ and $(B2)$ but not $(B3)$ and calls this family **generalized big boss games** or **veto games**.

$v^{\mathcal{A}}$ is not a big boss game but it is a generalized big boss game. It is also easy to see that for any problem $\mathcal{A}$ with $|P_{k^*}| = 1$, $v^{\mathcal{A}}$ is a big boss game.

4. Solutions of the agglomeration game

In this section we study the core, the nucleolus, the τ-value, and the Shapley value of the agglomeration game $v^{\mathcal{A}}$.

4.1. The core

We prove that the core of $v^{\mathcal{A}}$ is always non-empty. It can be described as follows: Firm 0 receives something between $I_0(\mathcal{A})$ and $g(\mathcal{A})$. Firms in the optimal region k^* receive something between zero and $g(\mathcal{A}) - I_0(\mathcal{A})$. Firms in other regions receive zero.

Theorem 1. *Given a problem* $\mathcal{A}$ *and an optimal region* k^**, the core of the game* $v^{\mathcal{A}}$ *is non-empty and is given by*

$$C(v^{\mathcal{A}}) = \left\{ x \in \mathbb{R}^{n+1} : \sum_{i \in N_0} x_i = g(\mathcal{A}), \begin{array}{l} I_0(\mathcal{A}) \leq x_0 \leq g(\mathcal{A}), \\ 0 \leq x_i \leq g(\mathcal{A}) - I_0(\mathcal{A}), \forall i \in P_{k^*}, \\ x_i = 0, \forall i \in N \backslash P_{k^*} \end{array} \right\}.$$

Proof. First, we prove "$\subseteq$". Let $x \in C(v^{\mathcal{A}})$. Then, $\sum\limits_{i \in N_0} x_i = v^{\mathcal{A}}(N_0) = g(\mathcal{A})$.

Take $i \in N$. Since $v^{\mathcal{A}}(\{i\}) = 0$, $x_i \geq 0$ holds. Moreover, since $\sum\limits_{j \in N_0 \backslash \{i\}} x_j \geq v^{\mathcal{A}}(N_0 \backslash \{i\})$,

$$x_i = v^{\mathcal{A}}(N_0) - \sum_{j \in N_0 \backslash \{i\}} x_j \leq g(\mathcal{A}) - v^{\mathcal{A}}(N_0 \backslash \{i\}).$$

Consider two cases:

- $i \notin P_{k^*}$. Notice that $b_j^{k^*} = 0$, for all $j \in P_{k(i)}$ and $U(N_0 \backslash \{i\}) = N \backslash P_{k(i)}$. Thus,

$$v^{\mathcal{A}}(N_0 \backslash \{i\}) = \max_{k \in R} \left\{ \sum_{j \in N_0 \backslash P_{k(i)}} b_j^k \right\} = \sum_{j \in N_0 \backslash P_{k(i)}} b_j^{k^*} = \sum_{j \in N_0} b_j^{k^*} = g(\mathcal{A}).$$

  Hence, $x_i = 0$.

- $i \in P_{k^*}$. By Claim 1.3, $v^{\mathcal{A}}(N_0 \backslash \{i\}) = I_0(\mathcal{A})$. Then, $0 \leq x_i \leq g(\mathcal{A}) - I_0(\mathcal{A})$.

We now prove that $x_0 \geq I_0(\mathcal{A})$. If $I_0(\mathcal{A}) = b_0^{k^*}$, $x_0 \geq v^{\mathcal{A}}(\{0\}) = b_0^{k^*} = I_0(\mathcal{A})$.
If $I_0(\mathcal{A}) = s(\mathcal{A})$, then there exists $\ell \in R \backslash \{k^*\}$ such that

$$x_0 = x_0 + \sum_{i \in P_\ell} x_i \geq v(\{0\} \cup P_\ell) = b_0^\ell + \sum_{i \in P_\ell} b_i^\ell = I_0(\mathcal{A}).$$

Finally, we prove that $x_0 \leq g(\mathcal{A})$.

$$x_0 = v^{\mathcal{A}}(N_0) - \sum_{i \in N} x_i \leq g(\mathcal{A}) - v^{\mathcal{A}}(N) = g(\mathcal{A}).$$

We now prove "$\supseteq$". It suffices to prove that $\sum_{i \in S} x_i \geq v^{\mathcal{A}}(S)$, for all $S \subseteq N_0$.
If $0 \notin S$, $\sum_{i \in S} x_i \geq 0 = v^{\mathcal{A}}(S)$. Now, assume that $0 \in S$. Since $x_i = 0$ when $i \notin P_{k^*}$, we have that

$$\sum_{i \in S} x_i = x_0 + \sum_{i \in S \cap P_{k^*}} x_i.$$

Again, we face two cases:

- $P_{k^*} \subseteq S$. Then,

$$\sum_{i \in S} x_i = v^{\mathcal{A}}(N_0) = v^{\mathcal{A}}(S).$$

- $P_{k^*} \nsubseteq S$. Then,

$$\sum_{i \in S} x_i \geq I_0(\mathcal{A}) + \sum_{i \in S \cap P_{k^*}} x_i \geq I_0(\mathcal{A}).$$

  By Claim 1.1 $I_0(\mathcal{A}) \geq v^{\mathcal{A}}(S)$. $\square$

As a consequence of this theorem if k^* is not unique, the core consists of a single element in which firm 0 gets the total worth of the grand coalition and the all other firms get zero. The same happens when $P_{k^*} = \emptyset$ or if $b_i^{k^*} = 0$ for all $i \in P_{k^*}$.

Taking into account the expression of the core, it is straightforward to check that both EOL and WOL belong to the core of $v^{\mathcal{A}}$.

4.2. The nucleolus

We now prove that the nucleolus of the game $v^{\mathcal{A}}$ coincides with the egalitarian optimal location rule.

Theorem 2. *For each problem $\mathcal{A}$, $\eta(v^{\mathcal{A}}) = EOL(\mathcal{A})$.*

Proof. Let k^* be an optimal region. Assume first that k^* is not unique. Then $EOL(\mathcal{A}) = (g(\mathcal{A}), 0, ..., 0)$. Since the core consist of a single element $(g(\mathcal{A}), 0, ..., 0)$, the nucleolus coincides with that element. Therefore, $\eta(v^{\mathcal{A}}) = EOL(\mathcal{A})$.

We now assume that k^* is unique. Take $S \subseteq N_0$. We compute $e(S, x)$ where $x = EOL(\mathcal{A})$. We consider several cases:

(i) $0 \notin S$ and $S \cap P_{k^*} = \emptyset$. Then,

$$e(S, x) = v^{\mathcal{A}}(S) - \sum_{i \in S} x_i = 0 - 0 = 0.$$

(ii) $0 \notin S$ and $S \cap P_{k^*} \neq \emptyset$. Then,

$$\begin{aligned} e(S, x) =& v^{\mathcal{A}}(S) - \sum_{i \in S} x_i = 0 - |S \cap P_{k^*}| \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1} \\ \leq& - \left(\frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1} \right). \end{aligned}$$

(iii) $0 \in S$ and $S \cap P_{k^*} = P_{k^*}$. Then,

$$e(S, x) = v^{\mathcal{A}}(S) - \sum_{i \in S} x_i = g(\mathcal{A}) - x_0 - \sum_{i \in P_{k^*}} x_i = g(\mathcal{A}) - g(\mathcal{A}) = 0.$$

(iv) $0 \in S$ and $S \cap P_{k^*} \neq P_{k^*}$. By Claim 1.1, $v^{\mathcal{A}}(S) \leq I_0(\mathcal{A})$. Then,

$$\begin{aligned} e(S, x) =& v^{\mathcal{A}}(S) - \sum_{i \in S} x_i = v^{\mathcal{A}}(S) - x_0 - \sum_{i \in S \cap P_{k^*}} x_i \\ \leq& I_0(\mathcal{A}) - \left(I_0(\mathcal{A}) + \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1} \right) - |S \cap P_{k^*}| \left(\frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1} \right) \\ =& - (|S \cap P_{k^*}| + 1) \left(\frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1} \right) \leq - \left(\frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1} \right). \end{aligned}$$

Thus, $\theta(x)$ can be expressed as

$$\theta(x) = (0, ..., 0, e(S^1, x), e(S^2, x), ...)$$

where the $0, ..., 0$ corresponds to cases (i) and (iii) and $e(S^1, x), e(S^2, x), ...$ corresponds to cases (ii) or (iv). It has already been shown above that for all S^h,

$$e(S^h, x) \leq - \left(\frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1} \right).$$

In order to prove that x is the nucleolus it suffices to prove that x is more acceptable than any other element in the core.

Let $y \in C(v^{\mathcal{A}})$. It is easy to see that for cases (i) and (iii), $e(S, y) = 0$. Thus, $\theta(y)$ can be expressed as

$$\theta(y) = (0, ..., 0, e(T^1, y), e(T^2, y), ...)$$

where the $0, ..., 0$ corresponds to cases (i) and (iii) and $e(T^1, y), e(T^2, y), ...$ corresponds to cases (ii) or (iv). Now, it suffices to prove that there is S in cases (ii) or (iv) satisfying that

$$e(S, y) > -\left(\frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1}\right).$$

Consider two cases:

- $y_0 < x_0$. Let $S = N_0 \backslash P_{k^*}$,

$$e(S, y) = v^{\mathcal{A}}(N_0 \backslash P_{k^*}) - \sum_{i \in N_0 \backslash P_{k^*}} y_i = I_0(\mathcal{A}) - y_0 > I_0(\mathcal{A}) - x_0 = -\left(\frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1}\right).$$

- $y_0 \geq x_0$. Then, there is $i \in P_{k^*}$ such that $y_i < x_i$. If we take $S = \{i\}$,

$$e(S, y) = v^{\mathcal{A}}(\{i\}) - y_i > 0 - x_i = -\left(\frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1}\right). \quad \square$$

In general, the computation of the nucleolus is NP hard. As a consequence of Theorem 2, in agglomeration games η can be computed in polynomial time.

4.3. The τ-value

We now prove that the τ-value of the game $v^{\mathcal{A}}$ coincides with the egalitarian optimal location rule.

Theorem 3. *For each problem $\mathcal{A}$, $\tau(v^{\mathcal{A}}) = EOL(\mathcal{A})$.*

Proof. Let k^* be an optimal region. Assume first that k^* is not unique. Thus, $v^{\mathcal{A}}(N_0 \backslash \{0\}) = 0$ and for all $i \in N$, $v^{\mathcal{A}}(N_0 \backslash \{i\}) = g(\mathcal{A})$. Hence, $\tau(v^{\mathcal{A}}) = EOL(\mathcal{A})$.

We now consider the case when k^* is unique. Since for all $S \supseteq P_{k^*} \cup \{0\}$, $v^{\mathcal{A}}(S) = v^{\mathcal{A}}(N_0)$, $v^{\mathcal{A}}(N_0 \backslash \{0\}) = 0$, and for all $i \in P_{k^*}$ $v^{\mathcal{A}}(N_0 \backslash \{i\}) = I_0(\mathcal{A})$, it can be deduced that

$$M_i(v^{\mathcal{A}}) = \begin{cases} g(\mathcal{A}), & \text{if } i = 0 \\ g(\mathcal{A}) - I_0(\mathcal{A}), & \text{if } i \in P_{k^*} \\ 0, & \text{otherwise.} \end{cases}$$

Let $i \in N$ and $S \subseteq N_0$ with $i \in S$. If $0 \notin S$, then $v^{\mathcal{A}}(S) = 0$. Since $M_j(v^{\mathcal{A}}) \geq 0$ for all $j \in N_0$,

$$v^{\mathcal{A}}(S) - \sum_{j \in S \backslash \{i\}} M_i(v^{\mathcal{A}}) \leq 0.$$

Assume that $0 \in S$. Since $M_0(v^{\mathcal{A}}) = g(\mathcal{A})$ and $g(\mathcal{A}) = v^{\mathcal{A}}(N_0) \geq v^{\mathcal{A}}(S)$,

$$v^{\mathcal{A}}(S) - \sum_{j \in S \setminus \{i\}} M_i(v^{\mathcal{A}}) \leq g(\mathcal{A}) - g(\mathcal{A}) - \sum_{j \in S \setminus \{i,0\}} M_i(v^{\mathcal{A}}) \leq 0.$$

Moreover, for $S = \{i\}$,

$$v^{\mathcal{A}}(S) - \sum_{j \in S \setminus \{i\}} M_i(v^{\mathcal{A}}) = 0.$$

Therefore, $m_i(v^{\mathcal{A}}) = 0$, for all $i \in N$.

Let $S \subseteq N_0$ with $0 \in S$. If $P_{k^*} \nsubseteq S$, by Claim 1.1,

$$v^{\mathcal{A}}(S) - \sum_{j \in S \setminus \{0\}} M_j(v^{\mathcal{A}}) \leq v^{\mathcal{A}}(S) \leq I_0(\mathcal{A}).$$

Now, assume that $P_{k^*} \subseteq S$. Thus,

$$\begin{aligned} v^{\mathcal{A}}(S) - \sum_{j \in S \setminus \{0\}} M_j(v^{\mathcal{A}}) =& v^{\mathcal{A}}(N_0) - \sum_{j \in P_{k^*}} M_j(v^{\mathcal{A}}) \\ =& g(\mathcal{A}) - |P_{k^*}|(g(\mathcal{A}) - I_0(\mathcal{A})) \\ \leq& g(\mathcal{A}) - (g(\mathcal{A}) - I_0(\mathcal{A})) = I_0(\mathcal{A}). \end{aligned}$$

If $I_0(\mathcal{A}) = b_0^{k^*}$, take $S = \{0\}$. Then,

$$v^{\mathcal{A}}(S) - \sum_{j \in S \setminus \{0\}} M_j(v^{\mathcal{A}}) = b_0^{k^*} = I_0(\mathcal{A}).$$

If $I_0(\mathcal{A}) \neq b_0^{k^*}$, take $S = P_\ell \cup \{0\}$ where $\ell \in R \setminus \{k^*\}$ is such that $I_0(\mathcal{A}) = v^{\mathcal{A}}(P_\ell \cup \{0\})$. Then,

$$v^{\mathcal{A}}(S) - \sum_{j \in S \setminus \{0\}} M_j(v^{\mathcal{A}}) = v^{\mathcal{A}}(P_\ell \cup \{0\}) = I_0(\mathcal{A}).$$

Then, $m_0(v^{\mathcal{A}}) = I_0(\mathcal{A})$.

We know that $\tau(v^{\mathcal{A}}) = \alpha M(v^{\mathcal{A}}) + (1-\alpha) m(v^{\mathcal{A}})$ where $\alpha \in [0,1]$ and $\sum_{i \in N_0} \tau_i(v^{\mathcal{A}}) = v^{\mathcal{A}}(N_0) = g(\mathcal{A})$. Thus,

$$\begin{aligned} g(\mathcal{A}) =& \alpha \sum_{i \in N_0} M_i(v^{\mathcal{A}}) + (1-\alpha) \sum_{i \in N_0} m_i(v^{\mathcal{A}}) \\ =& \alpha(g(\mathcal{A}) + |P_{k^*}|(g(\mathcal{A}) - I_0(\mathcal{A}))) + (1-\alpha) I_0(\mathcal{A}) \\ =& \alpha(g(\mathcal{A}) + |P_{k^*}| g(\mathcal{A}) - |P_{k^*}| I_0(\mathcal{A}) - I_0(\mathcal{A})) + I_0(\mathcal{A}) \\ =& \alpha(|P_{k^*}| + 1)(g(\mathcal{A}) - I_0(\mathcal{A})) + I_0(\mathcal{A}). \end{aligned}$$

Therefore,

$$\alpha(|P_{k^*}| + 1)(g(\mathcal{A}) - I_0(\mathcal{A})) = g(\mathcal{A}) - I_0(\mathcal{A}) \Rightarrow \alpha = \frac{1}{|P_{k^*}| + 1}.$$

Let $i \in N$. If $i \notin P_{k^*}$, $m_i(v^{\mathcal{A}}) = M_i(v^{\mathcal{A}}) = 0$ and $\tau_i(v^{\mathcal{A}}) = 0$. If $i \in P_k$,

$$\tau_i(v^{\mathcal{A}}) = \alpha M_i(v^{\mathcal{A}}) + (1-\alpha) m_i(v^{\mathcal{A}}) = \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}|+1}.$$

Moreover,

$$\begin{aligned}\tau_0(v^{\mathcal{A}}) =& \alpha M_0(v^{\mathcal{A}}) + (1-\alpha) m_0(v^{\mathcal{A}}) = \frac{g(\mathcal{A})}{|P_{k^*}|+1} + \frac{|P_{k^*}|}{|P_{k^*}|+1} I_0(\mathcal{A}) \\ =& \frac{|P_{k^*}| I_0(\mathcal{A}) + I_0(\mathcal{A}) + g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}|+1} = I_0(\mathcal{A}) + \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}|+1}.\end{aligned}$$

Thus, $\tau(v^{\mathcal{A}}) = EOL(\mathcal{A})$. □

In general, the computation of the τ-value is NP hard. As a consequence of Theorem 3, in agglomeration games τ can be computed in polynomial time.

4.4. The Shapley value

The Shapley value is the most popular single value solution for cooperative games. It is well known that, in non-convex games the Shapley value can lie outside the core. Moreover, the computation of this allocation is NP hard.

Our feeling is that the allocation obtained through the Shapley value does not work very well in agglomeration games. In Example 1 the Shapley value is $(9.17, 0.5, 2.17, 2.17)$. Note that this allocation is outside the core because firm 1, which is not located in the optimal region, gets a positive amount. Moreover, firm 2 receives more than b_2^2. As argued above for EOL, we do not find this very reasonable.

Since the Shapley value can lie outside the core, firms in the optimal region can transfer money to firms outside the optimal region. This is quite difficult to imagine in the situation that we are considering. Furthermore, we so not have a closed expression for the Shapley value for any agglomeration problem. This makes it impossible to compute Sh in polynomial time.

Nevertheless, in agglomeration problems where $I_0(\mathcal{A}) = b_0^{k^*}$, the Shapley value coincides with the egalitarian optimal location rule. The next theorem formally states this result.

Theorem 4. *For any problem $\mathcal{A}$ such that $I_0(\mathcal{A}) = b_0^{k^*}$, $Sh(v^{\mathcal{A}}) = EOL(\mathcal{A})$.*

Proof. Let k^* be an optimal region. Assume that k^* is not unique. It is known that $EOL(\mathcal{A}) = (g(\mathcal{A}), 0, ..., 0)$.

Since $I_0(\mathcal{A}) = b_0^{k^*}$ and k^* is not unique it can be deduced that $I_0(\mathcal{A}) = g(\mathcal{A})$. Then, for all $S \subseteq N_0$

$$v^{\mathcal{A}}(S) = \begin{cases} g(\mathcal{A}), & \text{if } 0 \in S \\ 0, & \text{otherwise.} \end{cases}$$

For any $S \subseteq N_0$ such that $0 \notin S$, $v^{\mathcal{A}}(S \cup \{0\}) - v^{\mathcal{A}}(S) = g(\mathcal{A})$. Then, $Sh_0(v^{\mathcal{A}}) = g(\mathcal{A})$.

For any $i \in N$ and any $S \subseteq N_0$ such that $i \notin S$, $v^{\mathcal{A}}(S \cup \{i\}) = v^{\mathcal{A}}(S)$. Then, $Sh_i(v^{\mathcal{A}}) = 0$ for any $i \in N$. Therefore, $Sh(v^{\mathcal{A}}) = EOL(\mathcal{A})$.

We now consider that k^* is unique. Then, for all $S \subseteq N_0$

$$v^{\mathcal{A}}(S) = \begin{cases} g(\mathcal{A}), & \text{if } 0 \in S \text{ and } P_{k^*} \subseteq S \\ I_0(\mathcal{A}), & \text{if } 0 \in S \text{ and } P_{k^*} \not\subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

Take $i \in N \backslash P_{k^*}$ and $S \subseteq N_0$ with $i \notin S$. Then $v^{\mathcal{A}}(S \cup \{i\}) = v^{\mathcal{A}}(S)$. Therefore, $Sh_i(v^{\mathcal{A}}) = 0$, $\forall i \in N \backslash P_{k^*}$.

Take $i \in P_{k^*}$ and let $S \subseteq N_0$ with $i \notin S$.

$$v^{\mathcal{A}}(S \cup \{i\}) - v^{\mathcal{A}}(S) = \begin{cases} g(\mathcal{A}) - I_0(\mathcal{A}), & \text{if } P_{k^*} \cup \{0\} \subseteq S \cup \{i\} \\ 0, & \text{otherwise.} \end{cases}$$

Let Π' be the subset of Π_{N_0} given by the permutations π where i is the last element of $P_{k^*} \cup \{0\}$ in π. Thus,

$$\begin{aligned} Sh_i(v) =& \frac{1}{|N_0|!} \sum_{\pi \in \Pi_{N_0}} (v(Pre(i,\pi) \cup \{i\}) - v(Pre(i,\pi))) \\ =& \frac{1}{|N_0|!} \sum_{\pi \in \Pi'} (g(\mathcal{A}) - I_0(\mathcal{A})) \\ =& \frac{|\Pi'|}{|N_0|!} (g(\mathcal{A}) - I_0(\mathcal{A})). \end{aligned}$$

Since 1 in $|P_{k^*}| + 1$ permutations in Π_{N_0} belongs to Π' it can be deduced that

$$Sh_i(v) = \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1}, \quad \forall i \in P_{k^*}.$$

Finally,

$$\begin{aligned} Sh_0(v^{\mathcal{A}}) =& g(\mathcal{A}) - \sum_{i \in N} Sh_i(v^{\mathcal{A}}) \\ =& g(\mathcal{A}) - \frac{|P_{k^*}|(g(\mathcal{A}) - I_0(\mathcal{A}))}{|P_{k^*}| + 1} \\ =& I_0(\mathcal{A}) + \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1}. \end{aligned}$$

Therefore, $Sh(v^{\mathcal{A}}) = EOL(\mathcal{A})$. □

5. Axiomatic characterizations

In this section we introduce some properties of rules. We analyze which of those properties are fulfilled by the egalitarian optimal location rule and the weighted optimal location rule. Finally, we present axiomatic characterizations for both rules.

Core selection says that the rule should select core allocations.

Core selection (CS): For any problem $\mathcal{A}$, $f(\mathcal{A}) \in C(v^{\mathcal{A}})$.

Monotonicity says that if the individual benefit of a firm increases (and the rest of the problem remains the same), then that firm should not end up worse off.

Monotonicity (M): For any two problems $\mathcal{A} = (0, N, P, b)$ and $\mathcal{A}' = (0, N, P, b')$ such that $b_i'^k > b_i^k$ for some $i \in N_0$ and $k \in R$ and $b_j'^\ell = b_j^\ell$ otherwise, $f_i(\mathcal{A}') \geq f_i(\mathcal{A})$.

Consider two firms belonging to the same region that obtain the same benefit when 0 opens the new plant in their region. Symmetry says that they should receive the same amount.

Symmetry (SYM): For any problem $\mathcal{A}$ and each pair of firms $i, j \in P_k$ such that $b_i^k = b_j^k$, $f_i(\mathcal{A}) = f_j(\mathcal{A})$.

Assume that firm 0 locates in region k^*. Thus it would be desirable to have rules where firms in P_{k^*} transfer something to firm 0 in order to incentivize firm 0 to locate in region k^*.

Let k^* be an optimal region. For each $i \in N_0$, define $t^f(\mathcal{A})$, the **transfer vector** associated to rule f, as $t_i^f(\mathcal{A}) = b_i^{k^*} - f_i(\mathcal{A})$.

The next property says that firms in P_{k^*} should transfer something, but they cannot receive transfers.

No transfer to local firms ($NTLF$): For any problem $\mathcal{A}$, each optimal region k^* and each $i \in P_{k^*}$ we have that $t_i^f(\mathcal{A}) \geq 0$.

Equal treatment inside optimal regions says that if the value of an optimal region increases (and the rest of the problem remains the same), then all firms in that region and firm 0 are affected in the same amount.

Equal treatment inside optimal regions ($ETOR$): Let $\mathcal{A} = (0, N, P, b)$ and $\mathcal{A}' = (0, N, P, b')$ be two problems such that $\sum\limits_{i \in P_{k^*}} b_i'^{k^*} > \sum\limits_{i \in P_{k^*}} b_i^{k^*}$ and $b_j'^\ell = b_j^\ell$ otherwise. Then, for each $i, j \in P_{k^*} \cup \{0\}$,

$$f_i(\mathcal{A}') - f_i(\mathcal{A}) = f_j(\mathcal{A}') - f_j(\mathcal{A}).$$

Notice that although this property considers that the individual benefit of firm 0 remain the same, the aggregated benefits of the rest of the firms in the optimal region increase. This increase in benefit is generated by the positive externalities of firm 0 in that region. Thus, it seems reasonable for the increase in benefit to affect firm 0 in the same way as the rest of the firms in the optimal region.

Consider a situation in which a firm in N splits (for instance, each of its plants comes to be considered as an independent firm). The next property says that the amount obtained by each of the other pre-existing firms does not change.

Merging-splitting proofness (MSP): Let $\mathcal{A} = (0, N, P, b)$ and $\mathcal{A}' = (0, N', P', b')$ be two problems such that, for some $i \in N$

- $N' = (N \backslash \{i\}) \cup \{i^1, ..., i^m\}$.
- $P' = \{P_1', ..., P_r'\}$ where $P_k' = P_k$ for all $k \neq k(i)$ and $P_{k(i)}' = (P_{k(i)} \backslash \{i\}) \cup \{i^1, ..., i^m\}$.

- $b'_j = b^k_j$ for all $j \in N_0 \backslash \{i\}$ and $k \in R$ and $b^{k(i)}_i = \sum_{\ell=1}^{m} b'^{k(i)}_{i^\ell}$.

Then, $f_j(\mathcal{A}) = f_j(\mathcal{A}')$ for all $j \in N_0 \backslash \{i\}$.

Alternatively, this property can also be motivated by saying that a subset of firms merges into a unique firm.

In the propositions below we discuss what properties are satisfied by each rule.

Proposition 2. (a) *The egalitarian optimal location rule satisfies core selection, monotonicity, symmetry, and equal treatment inside optimal regions.*

(b) *The egalitarian optimal location rule does not satisfy no transfer to local firms or merging-splitting proofness.*

Proof. (a) By Theorem 2, EOL coincides with the nucleolus of $v^{\mathcal{A}}$ and the nucleolus is always in the core of $v^{\mathcal{A}}$. Thus, EOL satisfies CS. It is straightforward to check that EOL satisfies SYM.

We prove that EOL satisfies M. Let $\mathcal{A}' = (0, N, P, b')$ be a problem such that $b'^k_i > b^k_i$ for some $i \in N_0$ and $k \in R$.

Assume that $i \in N$. Notice that necessarily $k = k(i)$. There are three possibilities for $k(i)$:

- $k(i)$ is an optimal region for $\mathcal{A}$. Then, $k(i)$ is the unique optimal region for $\mathcal{A}'$. Therefore, $g(\mathcal{A}') > g(\mathcal{A})$ and $I_0(\mathcal{A}') = I_0(\mathcal{A})$. Then, $EOL_i(\mathcal{A}') > EOL_i(\mathcal{A})$.
- $k(i)$ is not an optimal region for either $\mathcal{A}$ or $\mathcal{A}'$. Then, $EOL_i(\mathcal{A}') = 0 = EOL_i(\mathcal{A})$.
- $k(i)$ is not an optimal region for $\mathcal{A}$ but it is for $\mathcal{A}'$. Then, $EOL_i(\mathcal{A}') \geq 0 = EOL_i(\mathcal{A})$.

If $i = 0$, the following cases need to be analyzed:

- k is an optimal region for $\mathcal{A}$. Then k is the unique optimal region for $\mathcal{A}'$, $g(\mathcal{A}') > g(\mathcal{A})$ and $I_0(\mathcal{A}') > I_0(\mathcal{A})$. Then,

$$\begin{aligned}
EOL_0(\mathcal{A}') =& I_0(\mathcal{A}') + \frac{g(\mathcal{A}') - I_0(\mathcal{A}')}{|P_k| + 1} \\
=& I_0(\mathcal{A}) + (I_0(\mathcal{A}') - I_0(\mathcal{A})) + \frac{g(\mathcal{A}') - g(\mathcal{A})}{|P_k| + 1} + \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_k| + 1} - \frac{I_0(\mathcal{A}') - I_0(\mathcal{A})}{|P_k| + 1} \\
=& EOL_0(\mathcal{A}) + \frac{|P_k|(I_0(\mathcal{A}') - I_0(\mathcal{A}))}{|P_k| + 1} + \frac{g(\mathcal{A}') - g(\mathcal{A})}{|P_k| + 1} \\
>& EOL_0(\mathcal{A}).
\end{aligned}$$

- k is not an optimal region for either $\mathcal{A}$ or $\mathcal{A}'$. Then $g(\mathcal{A}') = g(\mathcal{A})$ and $I_0(\mathcal{A}') \geq I_0(\mathcal{A})$. Thus,

$$\begin{aligned}
EOL_0(\mathcal{A}') =& I_0(\mathcal{A}') + \frac{g(\mathcal{A}') - I_0(\mathcal{A}')}{|P_k| + 1} \\
=& I_0(\mathcal{A}) + (I_0(\mathcal{A}') - I_0(\mathcal{A})) + \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_k| + 1} - \frac{I_0(\mathcal{A}') - I_0(\mathcal{A})}{|P_k| + 1} \\
=& EOL_0(\mathcal{A}) + \frac{|P_k|(I_0(\mathcal{A}') - I_0(\mathcal{A}))}{|P_k| + 1} \\
\geq& EOL_0(\mathcal{A}).
\end{aligned}$$

- k is not an optimal region for $\mathcal{A}$ but it is for $\mathcal{A}'$. Then, $s(\mathcal{A}') = g(\mathcal{A})$. Hence, $I_0(\mathcal{A}') \geq g(\mathcal{A})$. Now,

$$EOL_0(\mathcal{A}') = I_0(\mathcal{A}') + \frac{g(\mathcal{A}') - I_0(\mathcal{A}')}{|P_k| + 1} \geq I_0(\mathcal{A}') \geq g(\mathcal{A}) \geq EOL_0(\mathcal{A}).$$

Thus, EOL satisfies M.

Finally, we prove that EOL satisfies $ETOR$. Let $\mathcal{A}$ and $\mathcal{A}'$ be two problems fulfilling the conditions in $ETOR$. Then k^* is also an optimal region for $\mathcal{A}'$, $g(\mathcal{A}') > g(\mathcal{A})$ and $I_0(\mathcal{A}') = I_0(\mathcal{A})$. Therefore, for all $i \in P_{k^*} \cup \{0\}$,

$$EOL_i(\mathcal{A}') - EOL_i(\mathcal{A}) = \frac{g(\mathcal{A}') - g(\mathcal{A})}{|P_{k^*}| + 1}.$$

Thus, EOL satisfies $ETOR$.

(b) Consider Example 1. Clearly, EOL does not fulfill $NTLF$ since

$$t_2^{EOL}(\mathcal{A}) = b_2^2 - EOL_2(\mathcal{A}) = 1 - 2 = -1 < 0.$$

Let $\mathcal{A}$ be as in Example 1. Let $\mathcal{A}' = (0, N', P', b')$ be a problem such that $N' = \{1, 2, 3^1, 3^2\}$, $P = \{\{1\}, \{2, 3^1, 3^2\}\}$, $b'^2_{3^1} = b'^2_{3^2} = 4$, and $b'^k_i = b^k_i$, otherwise. However,

$$EOL_2(\mathcal{A}') = \frac{14 - 8}{4} = 1.5 < 2 = EOL_2(\mathcal{A}).$$

Thus, EOL does not satisfy MSP. □

Proposition 3. (a) *The weighted optimal location rule satisfies core selection, monotonicity, symmetry, no transfer to local firms, and merging-splitting proofness.*

(b) *The weighted optimal location rule does not satisfy equal treatment inside optimal regions.*

Proof. (a) WOL satisfies CS by Theorem 1. It is straightforward to prove that WOL also satisfies SYM.

We now prove that WOL satisfies M. Let $\mathcal{A}' = (0, N, P, b')$ be a problem such that $b'^k_i > b^k_i$ for some $i \in N_0$ and $k \in R$.

Let $i \in N$. Notice that necessarily $k = k(i)$. There are three possibilities for $k(i)$:

- $k(i)$ is an optimal region for $\mathcal{A}$. Then, $k(i)$ is the unique optimal region for $\mathcal{A}'$. Therefore, $g(\mathcal{A}') > g(\mathcal{A})$ and $I_0(\mathcal{A}') = I_0(\mathcal{A})$. Then,

$$WOL_i(\mathcal{A}') = \frac{b_i'^{k^*}}{\sum\limits_{j\in P_{k^*}} b_j'^{k^*}}(g(\mathcal{A}') - I_0(\mathcal{A}')) \geq \frac{b_i^{k^*}}{\sum\limits_{j\in P_{k^*}} b_j^{k^*}}(g(\mathcal{A}) - I_0(\mathcal{A})) = WOL_i(\mathcal{A}).$$

- $k(i)$ is not an optimal region for either $\mathcal{A}$ or $\mathcal{A}'$. Then, $WOL_i(\mathcal{A}') = 0 = WOL_i(\mathcal{A})$.

- $k(i)$ is not an optimal region for $\mathcal{A}$ but it is for $\mathcal{A}'$. Then, $WOL_i(\mathcal{A}') \geq 0 = WOL_i(\mathcal{A})$.

If $i = 0$, $WOL_0(\mathcal{A}') = I_0(\mathcal{A}') \geq I_0(\mathcal{A}) = WOL_0(\mathcal{A})$. Therefore, WOL satisfies M.

We prove that WOL satisfies $NTLF$. Let k^* be an optimal region. If k^* is not unique, then $WOL_i(\mathcal{A}) = 0$, for all $i \in N$. Hence $t_i^{WOL}(\mathcal{A}) = b_i^{k^*} \geq 0$ for all $i \in N$.

Now consider that k^* is unique. For all $i \in P_{k^*}$,

$$t_i^{WOL}(\mathcal{A}) = b_i^{k^*} - \frac{b_i^{k^*}}{\sum\limits_{j\in P_{k^*}} b_j^{k^*}}(g(\mathcal{A}) - I_0(\mathcal{A})) = b_i^{k^*}\left[1 - \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{\sum\limits_{j\in P_{k^*}} b_j^{k^*}}\right].$$

Let $i \in P_{k^*}$. If $b_i^{k^*} = 0$, then $t_i^{WOL}(\mathcal{A}) = 0$. Assume that $b_i^{k^*} > 0$. Thus,

$$\begin{aligned} t_i^{WOL}(\mathcal{A}) \geq 0 \Leftrightarrow & 1 - \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{\sum\limits_{j\in P_{k^*}} b_j^{k^*}} \geq 0 \\ \Leftrightarrow & \sum_{j\in P_{k^*}} b_j^{k^*} \geq g(\mathcal{A}) - I_0(\mathcal{A}) \\ \Leftrightarrow & I_0(\mathcal{A}) \geq g(\mathcal{A}) - \sum_{j\in P_{k^*}} b_j^{k^*} = b_0^{k^*}, \end{aligned}$$

which always holds. Thus, WOL satisfies $NTLF$.

We now prove that WOL satisfies MSP. Let $i \in N$ and $\mathcal{A}'$ fulfilling the conditions in MSP. Let k^* be an optimal region for $\mathcal{A}$. If k^* is not unique, $WOL_j(\mathcal{A}') = 0 = WOL_j(\mathcal{A})$, for all $j \in N_0\backslash\{i\}$.

Now assume that k^* is unique. Notice that k^* is an optimal region for $\mathcal{A}'$. Moreover, $g(\mathcal{A}') = g(\mathcal{A})$ and $I_0(\mathcal{A}) = I_0(\mathcal{A}')$. Then, for all $j \in P_{k^*}\backslash\{i\}$,

$$WOL_j(\mathcal{A}') = \frac{b_j'^{k^*}}{\sum\limits_{\ell\in P'_{k^*}} b_\ell'^{k^*}}(g(\mathcal{A}') - I_0(\mathcal{A}')) = \frac{b_j^{k^*}}{\sum\limits_{\ell\in P_{k^*}} b_\ell^{k^*}}(g(\mathcal{A}) - I_0(\mathcal{A})) = WOL_j(\mathcal{A}).$$

Since $WOL_j(\mathcal{A}') = 0 = WOL_j(\mathcal{A})$, for all $j \in N_0\backslash P_{k^*}$, WOL satisfies MSP.

(b) Again, consider the problem introduced in Example 1. Now let $\mathcal{A}' = (0, N, P, b')$ be a problem such that $b_2'^2 = 4$. Note that $\mathcal{A}$ and $\mathcal{A}'$ fulfill the conditions of $ETOR$. Since $WOL(\mathcal{A}') = (8, 0, 3, 6)$ it can be deduced that WOL does not satisfy $ETOR$. $\square$

We now provide an axiomatic characterization for the EOL rule with core selection and equal treatment inside optimal regions.

Theorem 5. *The egalitarian optimal location rule is the only rule that satisfies core selection and equal treatment inside optimal regions.*

Proof. By Proposition 2, EOL satisfies CS and $ETOR$.

We now prove uniqueness. Let f be a rule satisfying both properties. Assume that $\mathcal{A}$ has several optimal regions. Since the core has a single element and f satisfies CS it is obvious that f coincides with EOL.

Assume that $\mathcal{A}$ has a unique optimal region k^*. We consider two cases.

- $I_0(\mathcal{A}) = s(\mathcal{A})$. There exists $\ell \in R\backslash\{k^*\}$ such that $I_0(\mathcal{A}) = \sum_{i \in P_\ell \cup \{0\}} b_i^\ell$. Define $\mathcal{A}^1 = (0, N, P, b^1)$ such that $\sum_{i \in P_{k^*}} b_i^{1k^*} = s(\mathcal{A}) - b_0^{k^*}$ and $b_j^{1k} = b_j^k$, otherwise.

  In $\mathcal{A}^1$ there are at least two optimal regions: k^* and ℓ. Moreover, $g(\mathcal{A}^1) = I_0(\mathcal{A})$. By Theorem 1, the core of $v^{\mathcal{A}^1}$ has a single element $(I_0(\mathcal{A}), 0, ..., 0)$. By CS $f(\mathcal{A}^1) = (I_0(\mathcal{A}), 0, ..., 0)$.

  Since $\mathcal{A}$ and $\mathcal{A}^1$ fulfill the conditions of $ETOR$, we have that for each $i, j \in P_{k^*} \cup \{0\}$,

  $$f_i(\mathcal{A}) - f_i(\mathcal{A}^1) = f_j(\mathcal{A}) - f_j(\mathcal{A}^1).$$

  Fix $i \in P_{k^*} \cup \{0\}$, then

  $$\begin{aligned}(|P_{k^*}| + 1)(f_i(\mathcal{A}) - f_i(\mathcal{A}^1)) = &\sum_{j \in P_{k^*} \cup \{0\}} (f_j(\mathcal{A}) - f_j(\mathcal{A}^1)) \\ =& g(\mathcal{A}) - g(\mathcal{A}^1) \\ =& g(\mathcal{A}) - I_0(\mathcal{A})\end{aligned}$$

  $$\Rightarrow f_i(\mathcal{A}) = f_i(\mathcal{A}^1) + \frac{g(\mathcal{A}) - I_0(\mathcal{A})}{|P_{k^*}| + 1}.$$

  Then f coincides with EOL on $P_{k^*} \cup \{0\}$. Since f satisfies CS and Theorem 1 it follows that $f_i(\mathcal{A}) = 0$ for all $i \in N\backslash P_{k^*}$. Hence f coincides with EOL on $N\backslash P_{k^*}$.

- $I_0(\mathcal{A}) = b_0^{k^*}$. Let $\mathcal{A}^2 = (0, N, P, b^2)$ be a problem such that $b_i^{2k^*} = 0$, for all $i \in P_{k^*}$ and $b_j^{2k} = b_j^k$, otherwise. Note that k^* is also the unique optimal region for $\mathcal{A}^2$, $g(\mathcal{A}^2) = I_0(\mathcal{A})$ and $b_i^{2k^*} = 0$, for all $i \in P_{k^*}$. By Theorem 1 and CS $f(\mathcal{A}^2) = (I_0(\mathcal{A}), 0, ..., 0)$.

  Since $\mathcal{A}$ and $\mathcal{A}^2$ also fulfill the conditions of $ETOR$, it can be concluded using arguments similar to those used in the previous case that f coincides with EOL. □

Remark 1. *The properties used in Theorem 5 are independent.*

The WOL rule satisfies CS but not $ETOR$.

The rule f given by $f_i(\mathcal{A}) = \dfrac{g(\mathcal{A})}{n+1}$, $\forall i \in N_0$ *satisfies $ETOR$ but not CS.*

In the next theorem we characterize the rules that satisfy core selection and merging-splitting proofness. These rules have a nice interpretation. Firm 0 receives $I_0(\mathcal{A})$ for sure. Moreover firm 0 receives an extra transfer $x_0(\mathcal{A})$ from firms in the optimal region k^*. The transfer of each firm located in region k^* is proportional to b.

Theorem 6. *A rule f satisfies core selection and merging-splitting proofness if and only if for each $\mathcal{A}$ there is $x_0(\mathcal{A}) \in [0, g(\mathcal{A}) - I_0(\mathcal{A})]$ such that*

$$f_i(\mathcal{A}) = \begin{cases} I_0(\mathcal{A}) + x_0(\mathcal{A}), & \text{if } i = 0 \\ \dfrac{b_i^{k^*}}{\sum\limits_{j \in P_{k^*}} b_j^{k^*}} (g(\mathcal{A}) - I_0(\mathcal{A}) - x_0(\mathcal{A})), & \text{if } i \in P_{k^*} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It is straightforward to prove that any rule such as in the statement of this theorem satisfies the two properties.

Let f be a rule satisfying CS and MSP. We prove that f also satisfies SYM. Let $i, j \in P_k$ as in the definition of SYM. Consider the following problems:

- $\mathcal{A}^{ip}$ obtained from $\mathcal{A}$ by splitting firm i into two, p_1 and p_2, with benefit $b_{p_1}^k = b_{p_2}^k = \dfrac{b_i^k}{2}$.
- $\mathcal{A}^{ip,jq}$ obtained from $\mathcal{A}^{ip}$ by splitting firm j into two, q_1 and q_2 with benefit $b_{q_1}^k = b_{q_2}^k = \dfrac{b_i^k}{2}$.
- $\mathcal{A}^{jp}$ obtained from $\mathcal{A}$ by splitting firm j into two, p_1 and p_2, with benefit $b_{p_1}^k = b_{p_2}^k = \dfrac{b_i^k}{2}$.
- $\mathcal{A}^{jp,iq}$ obtained from $\mathcal{A}^{jp}$ by splitting firm i into two, q_1 and q_2 with benefit $b_{q_1}^k = b_{q_2}^k = \dfrac{b_i^k}{2}$.

Since the sequence of problems $\mathcal{A}$, $\mathcal{A}^{ip}$ and $\mathcal{A}^{ip,jq}$ is under the hypothesis of MSP, it follows that

$$f_i(\mathcal{A}) = f_{p_1}(\mathcal{A}^{ip}) + f_{p_2}(\mathcal{A}^{ip}) = f_{p_1}(\mathcal{A}^{ip,jq}) + f_{p_2}(\mathcal{A}^{ip,jq}).$$

Since the sequence of problems $\mathcal{A}$, $\mathcal{A}^{jp}$, and $\mathcal{A}^{jp,iq}$ is under the hypothesis of MSP it follows that

$$f_j(\mathcal{A}) = f_{p_1}(\mathcal{A}^{jp}) + f_{p_2}(\mathcal{A}^{jp}) = f_{p_1}(\mathcal{A}^{jp,iq}) + f_{p_2}(\mathcal{A}^{jp,iq}).$$

Since the problems $\mathcal{A}^{ip,jq}$ and $\mathcal{A}^{jp,iq}$ coincide, it follows that $f_i(\mathcal{A}) = f_j(\mathcal{A})$. Hence, f satisfies SYM.

Assume that $\mathcal{A}$ has several optimal regions. Since f satisfies CS and Theorem 1 it can be deduced that $f(\mathcal{A}) = (g(\mathcal{A}), 0, ..., 0)$. Thus, $x_0(\mathcal{A}) = g(\mathcal{A}) - I_0(\mathcal{A})$.

Now assume that $\mathcal{A}$ has a unique optimal region k^*. Also assume, without loss of generality, that $P_{k^*} = \{1, ..., p\}$.

Since f satisfies CS, by Theorem 1 it can be deduced that

$$f(\mathcal{A}) = (I_0(\mathcal{A}) + x_0, x_1, ..., x_p, 0, ..., 0)$$

where $0 \leq x_i \leq g(\mathcal{A}) - I_0(\mathcal{A})$ for each $i = 0, 1, ..., p$.

Define $x_0(\mathcal{A}) = x_0$. For any $\varepsilon > 0$, it is possible to find $\{n_i^\varepsilon\}_{i \in P_{k^*}}$, b^ε, and $\{b_i^\varepsilon\}_{i \in P_{k^*}}$ such that

- $b^\varepsilon \in \mathbb{R}$ and $b^\varepsilon \leq \varepsilon$.
- For each $i \in P_{k^*}$, $n_i^\varepsilon \in \mathbb{N}$, $0 \leq b_i^\varepsilon \leq b^\varepsilon$ and $b_i^{k^*} = n_i^\varepsilon b^\varepsilon + b_i^\varepsilon$.

For every $h = 1, ..., p$, let $\mathcal{A}^h$ be the problem obtained from $\mathcal{A}$ by splitting each firm $i = 1, ..., h$ into $n_i^\varepsilon + 1$ firms. The first n_i^ε firms with $b_{i^\ell}^{hk^*} = b^\varepsilon$ and firm $n_i^\varepsilon + 1$ with $b_{i^{n_i^\varepsilon+1}}^{hk^*} = b_i^\varepsilon$.

Formally, for each $h = 1, ..., p$, let $\mathcal{A}^h = (0, N^h, P^h, b^h)$ be a problem such that

- $N^h = (N \backslash \{1, ..., h\}) \cup \left(\cup_{i=1}^h \{i^1, ..., i^{n_i^\varepsilon+1}\}\right)$.
- $P^h = \{P_1^h, ..., P_r^h\}$ where $P_{k^*}^h = (P_{k^*} \backslash \{1, ..., h\}) \cup \left(\cup_{i=1}^h \{i^1, ..., i^{n_i^\varepsilon+1}\}\right)$ and $P_k^h = P_k$ for all $k \neq k^*$.
- $b_j^{hk} = b_j^k$, for all $j \in N \backslash \{1, ..., h\}$ and $k \in R$. Given $i \in \{1, ..., h\}$ and $\ell = 1, ..., n_i^\varepsilon + 1$, $b_{i^\ell}^{hk} = 0$ if $k \neq k^*$, $b_{i^\ell}^{hk^*} = b^\varepsilon$ if $\ell = 1, ..., n_i^\varepsilon$ and $b_{i^\ell}^{hk^*} = b_i^\varepsilon$ if $\ell = n_i^\varepsilon + 1$.

Notice that in the sequence of problem $\mathcal{A} \to \mathcal{A}^1 \to \cdots \to \mathcal{A}^p$ it is possible to apply MSP to each pair of consecutive problems.

For $\mathcal{A}$ and $\mathcal{A}^1$,

$f_i(\mathcal{A}) = f_i(\mathcal{A}^1)$, $\forall i \in N_0 \backslash \{1\}$ and

$$f_1(\mathcal{A}) = \sum_{\ell=1}^{n_1^\varepsilon+1} f_{1^\ell}(\mathcal{A}^1).$$

For $\mathcal{A}^1$ and $\mathcal{A}^2$,

$f_i(\mathcal{A}^1) = f_i(\mathcal{A}^2)$, $\forall i \in N_0 \backslash \{1, 2\}$,

$f_{1^\ell}(\mathcal{A}^1) = f_{1^\ell}(\mathcal{A}^2)$, $\forall \ell = 1, ..., n_1^\varepsilon + 1$ and

$$f_2(\mathcal{A}^1) = \sum_{\ell=1}^{n_2^\varepsilon+1} f_{2^\ell}(\mathcal{A}^2).$$

Iterating the previous argument, the following is obtained for $\mathcal{A}^{p-1}$ and $\mathcal{A}^p$,

$f_i(\mathcal{A}^{p-1}) = f_i(\mathcal{A}^p)$, $\forall i \in N_0 \backslash P_{k^*}$,

$f_{i^\ell}(\mathcal{A}^{p-1}) = f_{i^\ell}(\mathcal{A}^p)$, $\forall i = 1, ..., p-1$, $\ell = 1, ..., n_i^\varepsilon + 1$ and

$$f_p(\mathcal{A}^{p-1}) = \sum_{\ell=1}^{n_p^\varepsilon+1} f_{p^\ell}(\mathcal{A}^p).$$

Therefore,

$$f_1(\mathcal{A}) = \sum_{\ell=1}^{n_1^\varepsilon+1} f_{1^\ell}(\mathcal{A}^1) = \sum_{\ell=1}^{n_1^\varepsilon+1} f_{1^\ell}(\mathcal{A}^2) = \cdots = \sum_{\ell=1}^{n_1^\varepsilon+1} f_{1^\ell}(\mathcal{A}^p),$$

$$f_2(\mathcal{A}) = f_2(\mathcal{A}^1) = \sum_{\ell=1}^{n_2^\varepsilon+1} f_{2^\ell}(\mathcal{A}^2) = \cdots = \sum_{\ell=1}^{n_2^\varepsilon+1} f_{2^\ell}(\mathcal{A}^p),$$

$$\vdots$$

$$f_p(\mathcal{A}) = f_p(\mathcal{A}^1) = \cdots = f_p(\mathcal{A}^{p-1}) = \sum_{\ell=1}^{n_p^\varepsilon+1} f_{p^\ell}(\mathcal{A}^p).$$

From these equations and SYM, it can be concluded that, for any $i \in P_{k^*}$,

$$f_i(\mathcal{A}) = \sum_{\ell=1}^{n_i^\varepsilon+1} f_{i^\ell}(\mathcal{A}^p) = \sum_{\ell=1}^{n_i^\varepsilon} f_{i^\ell}(\mathcal{A}^p) + f_{i^{n_i^\varepsilon+1}}(\mathcal{A}^p) = n_i^\varepsilon f_{i^1}(\mathcal{A}^p) + f_{i^{n_i^\varepsilon+1}}(\mathcal{A}^p).$$

Notice also that $f_{i^1}(\mathcal{A}^p) = f_{j^1}(\mathcal{A}^p)$, for all $i, j \in P_{k^*}$. Then denote $x^\varepsilon = f_{i^1}(\mathcal{A}^p)$ and $x_i^\varepsilon = f_{i^{n_i^\varepsilon+1}}(\mathcal{A}^p)$. Thus, $\forall i \in P_k^*$,

$$f_i(\mathcal{A}) = n_i^\varepsilon x^\varepsilon + x_i^\varepsilon.$$

Fix $i \in P_{k^*}$. Let $\mathcal{A}^\alpha$ be the problem obtained from $\mathcal{A}^p$ by splitting firm i^1 into two firms: α^1 with $b_{\alpha^1}^{\alpha k^*} = b_i^\varepsilon$ and α^2 with $b_{\alpha^2}^{\alpha k^*} = b^\varepsilon - b_i^\varepsilon$. By MSP and because f is non-negative (by CS),

$$f_{\alpha^1}(\mathcal{A}^\alpha) \leq f_{\alpha^1}(\mathcal{A}^\alpha) + f_{\alpha^2}(\mathcal{A}^\alpha) = f_{i^1}(\mathcal{A}^p) = x^\varepsilon.$$

Moreover, $x_i^\varepsilon = f_{i^{n_1^\varepsilon+1}}(\mathcal{A}^p) = f_{i^{n_1^\varepsilon+1}}(\mathcal{A}^\alpha) = f_{\alpha^1}(\mathcal{A}^\alpha)$. Hence, $0 \leq x_i^\varepsilon \leq x^\varepsilon$, for all $i \in P_{k^*}$.

Since

$$g(\mathcal{A}) = f_0(\mathcal{A}) + \sum_{j \in P_{k^*}} f_j(\mathcal{A}) = I_0(\mathcal{A}) + x_0 + \sum_{j \in P_{k^*}} n_j^\varepsilon x^\varepsilon + \sum_{j \in P_{k^*}} x_j^\varepsilon$$

and $0 \leq x_j^\varepsilon \leq x^\varepsilon$ for all $j \in P_{k^*}$, it can be deduced that

$$g(\mathcal{A}) \geq I_0(\mathcal{A}) + x_0 + \sum_{j \in P_{k^*}} n_j^\varepsilon x^\varepsilon \tag{1}$$

and

$$g(\mathcal{A}) \leq I_0(\mathcal{A}) + x_0 + \sum_{j \in P_{k^*}} (n_j^\varepsilon + 1) x^\varepsilon. \tag{2}$$

By (1),

$$x^\varepsilon \leq \frac{g(\mathcal{A}) - I_0(\mathcal{A}) - x_0}{\sum_{j \in P_{k^*}} n_j^\varepsilon} = \frac{(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0) b^\varepsilon}{\sum_{j \in P_{k^*}} (b_j^{k^*} - b_j^\varepsilon)}.$$

Therefore, for all $i \in P_{k^*}$,

$$\begin{aligned} f_i(\mathcal{A}) =& n_i^\varepsilon x^\varepsilon + x_i^\varepsilon \leq (n_i^\varepsilon + 1)x^\varepsilon \\ \leq& (n_i^\varepsilon + 1)\left(\frac{(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0)b^\varepsilon}{\sum\limits_{j \in P_{k^*}} (b_j^{k^*} - b_j^\varepsilon)}\right) \\ =& \frac{b_i^{k^*} - b_i^\varepsilon + b^\varepsilon}{\sum\limits_{j \in P_{k^*}} (b_j^{k^*} - b_j^\varepsilon)}(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0). \end{aligned}$$

By (2),

$$x^\varepsilon \geq \frac{g(\mathcal{A}) - I_0(\mathcal{A}) - x_0}{\sum\limits_{j \in P_{k^*}} (n_i^\varepsilon + 1)} = \frac{(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0)b^\varepsilon}{pb^\varepsilon + \sum\limits_{j \in P_{k^*}} (b_j^{k^*} - b_j^\varepsilon)}.$$

Therefore, for all $i \in P_{k^*}$,

$$\begin{aligned} f_i(\mathcal{A}) =& n_i^\varepsilon x^\varepsilon + x_i^\varepsilon \geq n_i^\varepsilon x^\varepsilon \\ \geq& n_i^\varepsilon \left(\frac{(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0)b^\varepsilon}{pb^\varepsilon + \sum\limits_{j \in P_{k^*}} (b_j^{k^*} - b_j^\varepsilon)}\right) \\ =& \frac{b_i^{k^*} - b_i^\varepsilon}{pb^\varepsilon + \sum\limits_{j \in P_{k^*}} (b_j^{k^*} - b_j^\varepsilon)}(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0). \end{aligned}$$

Then, for all $i \in P_{k^*}$,

$$\frac{b_i^{k^*} - b_i^\varepsilon}{pb^\varepsilon + \sum\limits_{j \in P_{k^*}} (b_j^{k^*} - b_j^\varepsilon)}(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0) \leq f_i(\mathcal{A}) \leq \frac{b_i^{k^*} - b_i^\varepsilon + b^\varepsilon}{\sum\limits_{j \in P_{k^*}} (b_j^{k^*} - b_j^\varepsilon)}(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0).$$

Taking the limit when ε tends to zero in the previous inequality, it is obtained that for all $i \in P_{k^*}$,

$$\frac{b_i^{k^*}}{\sum\limits_{j \in P_{k^*}} (b_j^{k^*})}(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0) \leq f_i(\mathcal{A}) \leq \frac{b_i^{k^*}}{\sum\limits_{j \in P_{k^*}} (b_j^{k^*})}(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0).$$

Hence,

$$f_i(\mathcal{A}) = \frac{b_i^{k^*}}{\sum\limits_{j \in P_{k^*}} b_j^{k^*}}(g(\mathcal{A}) - I_0(\mathcal{A}) - x_0)$$

and the desired expression is obtained. □

Remark 2. *The properties used in Theorem 6 are independent.*

The EOL rule satisfies CS but not MSP.

The rule f given by

$$f_i(\mathcal{A}) = \begin{cases} I_0(\mathcal{A}), & \text{if } i = 0 \\ \dfrac{b_i^{k(i)}}{\sum\limits_{j \in N} b_j^{k(j)}}(g(\mathcal{A}) - I_0(\mathcal{A})), & \text{if } i \in N \end{cases}$$

satisfies MSP but not CS.

The next corollary of Theorem 6 provides a characterization of the WOL rule.

Corollary 1. *Of all the rules satisfying core selection and merging-splitting proofness, the weighted optimal location rule is the one that minimizes transfers to firm* 0.

The proof of this corollary is obvious because WOL coincides with the rule characterized in Theorem 6 when $x_0(\mathcal{A}) = 0$ for all $\mathcal{A}$.

6. Concluding remarks

We introduce a new type of location problem that considers a widely studied economic phenomenon: Agglomeration economies. Once the optimal region has been determined, the main issue is to provide an appropriate compensation scheme for the firms involved in the problem. We analyze this problem using cooperative game theory. We first prove that the core is non empty and characterize all the allocations in the core. We also consider a rule, called the egalitarian optimal location rule, which always selects an element in the core. We also prove that this rule can be obtained as the nucleolus or the τ-value of the cooperative game. We provide an axiomatic characterization for this rule.

We argue that the egalitarian optimal location rule has some shortcomings (some firms already located in the optimal region could receive transfers). Thus we consider a new rule called the weighted optimal location rule. That rule is also characterized.

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