The cost of capital in markets with opaque intermediaries and the risk-structure of interest rates

Mierzejewski, Fernando

K.U.Leuven

31 July 2008

The Cost of Capital in Markets with Opaque Intermediaries and the Risk-Structure of Interest Rates

July 2008

Fernando Mierzejewski[†]

[†]*Faculty of Economy and Applied Economy, Katholieke Universiteit Leuven, Naamsestraat 69, 3000 Leuven, Belgium.*
Email: `Fernando.Mierzejewski@wis.kuleuven.be`, `femierze@lycos.com`

Abstract

The demand for cash balances of financial intermediaries that establish contractual liabilities with credit-sensitive customers is characterised. As stated by Merton, the success of the business activities of such firms crucially depends on their credit quality, and hence, they are obliged to rely on deposit insurance and capital cushions in order to assure that their promised payments are free of default. Unlike the Merton's approach, the optimal guaranteeing contract is formulated in actuarial terms in this paper, because in this way the model can be extended to consider the situation of firms that can only hedge up to a limited extent. Within this framework, the equilibrium in the market determines the rate at which a unit of capital is exchange by a unit of risk, or, in other words, it determines the market price of risk. Episodes of liquidity crises are meaningful in this theoretical setting.

Key words: The cost of capital; Liquidity preference; Monetary equilibrium; Financial instability; Liquidity crises.

JEL-Classification: E41, E44, G14, G15.

1   Introduction

Investors holding financial securities with random outcomes are exposed to unknown balances equal to the differences between the market values of outstanding assets and liabilities. When positive, such a surplus can be lent at the *overnight* interest rate in order to avoid keeping idle money, while, by contrast, short-term debt can be attracted when a net lost is suffered. Modigliani and Miller (1958) have noticed that, since in *perfect* capital markets such operations can be performed without restrictions, *rational* decision-makers — who are assumed to maximise value — should demand no equity, for it is costly and provides no benefit. However, frictions in capital markets can prevent this mechanism to fully operate in practice. As a matter of fact, premiums are normally charged on market prices depending

on *credit-ratings* and the size of loans, evidencing that the cost of restructuring debt can be incremented when the magnitude of losses increases up to certain levels. Those firms that cannot afford such extra costs are forced to default. Under such circumstances, the financial position and so the performance of companies are affected by the preferred surpluses.

Merton (1997) proposes a model to determine the price of insuring liability contracts established by *opaque* financial intermediaries with *credit-sensitive* and averse-to-risk customers. In this setting, *credit-sensitivity* characterises the attitude of economic agents who expect that the promised payments will be delivered with certainty, while *opacity* refers to the fact that the business activities of financial institutions are normally unknown to outsiders. Additionally, customers are supposed to lack the expertise to diversify and to hedge efficiently. Within this framework, the role of financial intermediaries is that of providing insured liability contracts to customers that cannot continuously modify their portfolios. The efficiency of such services crucially depends on the perceptions of customers about the credit quality of the intermediary. As long as the value of the guarantee and the value of the firm are interpreted as contingent claims on the market value of *net assets*, equal to the market values of assets minus liabilities, optimal arrangements can be determined by maximising the value of the firm and by minimising the cost of insolvency.

A fundamental assumption in the model of Merton is that financial intermediaries *can* hedge continuously and efficiently and that they can accordingly *sell* this service to their customers. But what is the situation in markets where even the most solvent institutions can only hedge *up to a limited extent*? After all, every financial company stops relying on trading at some limit and prefer to call instead for reinsurance and risk capital cushions. An actuarial approach is proposed in this paper to describe this situation, according to which (in a similar way to that adopted by Merton, 1974 and 1977), the optimal surpluses of shareholders and guarantors are respectively determined at the levels where the value of the firm is maximised and where the cost of bankruptcy is minimised. Then, as demonstrated in *Section* 2, the optimal levels of reserves are determined by the *quantile* functions of the series of capital returns of the intermediaries' portfolios.

The total cash balance demanded by the market is defined in *Section* 3 as the sum of the surpluses demanded by investors, customers and financial intermediaries. A more general framework can be characterised by assuming that creditors can control the total supply of capital in the market and the cost of capital is determined in order to satisfy the necessary condition that the total supply is equal to the total demand of equity. Consequently, as proved in *Section* 4, the monetary equilibrium determines a set of combinations of the supply of cash balances and the demand for money to spend on the transaction of securities, the rate of interest r, as well as the expected return and the volatility of the market portfolio.

On these grounds, the cost of capital in financial markets is defined by the probability that the market capital loss surpasses the ratio of the supply of cash balances to the volume of funds spent in the transactions of securities, or, in other words, it is equal to the probability that the cash-guarantee maintained at the aggregated level is insufficient to cover the total realised loss. Alternatively, it can be interpreted as the return to be paid for holding a given uncertain claim instead of investing on a non-risky zero-coupon bond with a predetermined maturity. Such a rate corresponds to the return to be obtained when investing one monetary unit on the underlying risk and can accordingly be regarded as the *Internal Rate of Return on Risk* ($IRRR$). Therefore, every pair of fixed levels of the equity supply and the transactions

2

volume determines a *Risk-Structure of Interest Rates* over the plane of expected returns and volatilities, providing the price to be asked for exchanging a given risk for equity. Such structure of interest rates is studied in *Section* 5.

The analysis of the *elasticity* of the demand for cash balances with respect to the interest rate and the shape of the *risk-structure* of interest rates in *Sections* 4 and 5 leads to a precise description of financial markets in terms of the capital flows and the risk-parameters of the market portfolio. More specifically, the derived expressions for the preference for liquidity and the internal rate of return on risk, which are dependent on observable variables, can be regarded as *measures* of the liquidity conditions in the market. On the grounds of a partial equilibrium approach, different states can be compared, thus allowing to distinguish the *more stable* and the *more vulnerable* to violent adjustments. The empirical analysis carried on in *Sections* 6, 7 and 8 illustrates the kind of representation that can be obtained from the model. In particular, the liquidity crises occurred in October 1987 and April 2000 (the *Black Monday* and the *dot-com bubble*) can be satisfactorily interpreted in terms of the paths followed by the empirical expected returns and volatilities of the series of capital returns of the corresponding indexes. *Section* 9 concludes the paper.

2   The Optimal Demand for Cash Balances

According to Merton (1997), the capital structure affects the value of firms that establish contractual liabilities with customers who are sensible to the default-risk of the involved payments. Such *customers* are characterised by their unwillingness to share the risk taken by the intermediary — even if they were offered a *fair* premium reduction — a situation that is aggravated due to the fact that the business activities of financial companies are usually unknown to outsiders, i.e. these institutions are *opaque* to outsiders. Additionally, customers are supposed to lack the expertise required to diversify or to hedge by themselves and to face higher transaction costs. By contrast, the *investors* in the stocks and bonds issued by financial companies agree to afford the cost of bankruptcy, provided they are compensated by the appropriate expected return. In this context, the role of financial intermediation is to attract funds from *risk-neutral* investors and from *averse-to-risk* customers to hold financial assets. The success of this kind of firms critically depends on their ability to assure to their customers that the promised payments will be made with certainty.

Three different strategies are available to provide protection against default-risk: hedging, insuring and risk-capital cushions. In the model of Merton (1997), intermediaries prefer *hedging*. Two different *option-based* contracts are then established, one involving the firm and stockholders and delivering the excess of the value of *net assets* (equal to assets minus liabilities) over the level of cash holdings, and another involving the firm and some external *guarantor*, who guarantees the cash flows promised to customers. The surplus S of the intermediary is then represented by the market value of net assets and is supposed to follow a *Brownian motion* with expected return μ and volatility σ:

$$dS = \mu dt + \sigma dz$$

where dz is a *Wiener* stochastic process. Then the value of the firm as well as the value of the guarantee are contingent claims on the value of net assets, which can be characterised

by imposing equilibrium-pricing formulas and providing adequate boundary conditions. The optimal contracts established with shareholders and guarantors can thus be respectively established in order to maximise the value of the firm and to minimise the cost of bankruptcy (see also Merton, 1974, 1977).

However (as already stated by Merton, 1997), the distinction between intermediaries, guarantors, customers and investors can be difficult in actual financial markets. As a matter of fact, some financial institutions acting as guarantors to some firms or individual investors, can be customers of or investors in other companies. Eventually, every financial institution will stop relying on hedging and will seek for reinsuring and risk capital to assure the solvency of its business activities. The capital structure of this kind of firms will be next characterise. As in the model of Merton, two contracts will be defined, one involving the maximisation of the value of the firm, and the other involving the minimisation of the cost of insolvency. But since these will be now supposed to be insuring contracts, an *actuarial* setting will be required.

In order to establish these contracts in formal terms, let us assume that every intermediary holds a *net-portfolio* combining capital and some *Gaussian* exposure X, characterised by a pair of expected return and volatility (μ, σ) and corresponded to the series of (random) capital *Profit and Losses (P&L)* of the value V of the underlying fund. Since each monetary unit invested on risk produces the percentage return X, the total $P\&L$ accrued by the underlying fund must be equal to:

$$L \cdot X = L \cdot \Delta V \quad with \quad X_t = \frac{V_t - V_{t-1}}{V_{t-1}} \quad \forall t$$

where the variable L denotes the amount of funds spent on transactions. The random variable X can be certainly related to the surplus S defined by Merton, but recall that X has been expressed in *percentage* terms — in such a way that the *absolute* return is also dependent on the transactions volume.

Let us additionally express the magnitude of the guarantee, in the following denoted by K, as a proportion k of the transactions volume, in such a way that $K = L \cdot k$. The agreements established by the financial intermediary are then defined in the following way. On the one hand, stockholders supply the balance $K = L \cdot k$ with the promise of receiving this amount back plus the *return in excess* $(X - k)$ if the firm is *solvent* and delivers a benefit over the level of capital, i.e. if $X > k$. On the other hand, the firm establishes a contract with some insurance company that promises to pay the *excess of loss* $(X + k)$ when bankruptcy is declared, i.e. when $X < -k$. Hence the total payments, respectively delivered to stockholders and received from the guarantor at the end of the investment period, are equal to $L \cdot (X - k)$ and $L \cdot (X + k)$. In other words, as long as the amount of capital K is invested in the market of cash balances to obtain the return r, the payments, per unit of investment, endured and received by the firm at the end of the investment period, are given by:

$$Y^S = (X - k) + r \cdot k \quad when \ X > k$$
$$Y^G = (X + k) + r \cdot k \quad when \ X < -k$$

which can be equivalently expressed as:

$$\begin{aligned} Y^S &= (X-k)_+ + r \cdot k = max(0, X-k) + r \cdot k \\ Y^G &= (X+k)_- + r \cdot k = -min(0, X+k) + r \cdot k \end{aligned} \tag{1}$$

Whenever $-k \leq X \leq k$, no capital gains are obtained and additional debt is required to pay the guarantee back, though the size of the losses is bounded by the level of reserves. Stockholders are expected to sell their shares or to call for restructuring in this case.

On these grounds, two different kind of firms can be distinguished, depending on the criteria determining their capital structures. There are, in the first place, firms that seek to maintain their relationship with stockholders in the best terms and accordingly seek to maximise the return Y^S. In the second place, there are firms more concerned about their relationship with guarantors and customers that accordingly prefer to minimise the return Y^G. While those firms belonging to the first group can be regarded as *speculative* firms, as long as their focus on profits may lead them to implement *more aggressive* investment strategies, those in the second group can be regarded as *precautionary* firms, as long as their objective is to reduce their risk exposure to a minimum. Moreover, since the capital return X is represented by a random variable, both Y^S and Y^G are also represented by random variables, which means that the optimal surpluses should be determined based on the expected values of the residual exposures $E[(X-k)_+]$ and $E[(X+k)_-]$.

Recall that any random variable can be equivalently represented by its *cumulative probability* function F_X and its *tail probability* function T_X, functions that are related to each other through the fundamental identity:

$$T_X(k) = P[X > k] = 1 - P[X \leq k] = 1 - F_X(k)$$

Besides, in a Gaussian setting, both functions can be expressed in terms of the cumulative probability Φ of the *standard Gaussian*, whose mean return and volatility are respectively equal to zero and one:

$$T_X(k) = 1 - F_X(k) = 1 - \Phi\left(\frac{k-\mu}{\sigma}\right) \tag{2}$$

In fact, the Gaussian *cumulative* and *tail* probability function are explicitly defined as:

$$\begin{aligned} F_X(k) &= P[X \leq k] = \int_{-\infty}^{k} dF_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{k} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \\ \\ T_X(k) &= P[X > k] = \int_{k}^{+\infty} dF_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{k}^{+\infty} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \end{aligned} \tag{3}$$

Additionally, applying the mathematical definition of the *expectation* operator we obtain that:

$$E\left[(X-k)_+\right] = \int_{+k}^{+\infty} (x-k) \cdot dF_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{+k}^{+\infty} (x-k) \cdot \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \quad (4)$$

and also:

$$E\left[\,(X+k)_-\right] = -\int_{-\infty}^{-k}(x+k)\cdot dF_X(x) = -\frac{1}{\sigma\sqrt{2\pi}}\cdot\int_{-\infty}^{-k}(x+k)\cdot\exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]dx \quad (5)$$

Therefore, *speculative* firms (which maximise the expected excess return) determine their capital structures by solving the following optimisation problem:

$$\max_k\ E\left[(X-k)_+\right] + r\cdot k \quad (6)$$

Applying the Leibniz integral rule for the differentiation of a definite integral whose limits are functions of the differential variable,[1] we obtain from *Equation* 4 that:

$$\frac{\partial}{\partial k}\ E\left[\,(X-k)_+\right] = \int_{+k}^{+\infty}(-1)\cdot dF_X(x) = -T_X(k)$$

The solution to the maximisation problem of *Equation* 6 can then be obtained by applying *Lagrange optimisation*, which implies that the *optimal* cash balance k^S of speculative firms is characterised by the *first-order condition*:

$$\frac{\partial}{\partial k}\ E\left[\,(X-k^S)_+\right] + r\ =\ -T_X\left(k^S\right) + r = 0$$

The *optimal* surplus of speculative firms is thereby given by the inverse function of the tail probability:

$$k^S(r) = \sigma\cdot\Phi^{-1}(1-r) + \mu \quad (7)$$

According to this specification, the optimal balance of speculative firms is always a *non-increasing* and *continuous* function of the interest rate — for the cumulative function Φ is a *non-decreasing* and *continuous* function of its argument. Besides, the minimum and maximum cash balances are respectively maintained when $r \geq 1$ and $r \leq 0$.

In a similar way, the optimal balance of *precautionary* firms (which minimise the expected loss endured in case of default) is determined by the following minimisation problem:

$$\min_k\ E\left[(X+k)_-\right] + r\cdot k \quad (8)$$

Applying the Leibnitz's rule to the integral of *Equation* 5, we obtain that:

$$\frac{\partial}{\partial k}\ E\left[\,(X+k)_-\right] = -\int_{-\infty}^{-k}1\cdot dF_X(x) = -T_{-X}(k)$$

Then the first-order condition of the problem of *Equation* 8 leads to:

---

[1]According to this rule:

$$\frac{\partial}{\partial z}\int_{u(z)}^{v(z)}f(x,z)\ dx = \int_{u(z)}^{v(z)}\frac{\partial f(x,z)}{\partial z}\ dx + f\left(v(z),z\right)\cdot\frac{\partial v(z)}{\partial z} - f\left(u(z),z\right)\cdot\frac{\partial u(z)}{\partial z}$$

$$\frac{\partial}{\partial k} E\left[\left(X + k^G\right)_-\right] + r \;=\; -T_{-X}\left(k^G\right) + r = 0$$

and the *optimal* cash balance of precautionary firms is then equal to:

$$k^G(r) = \sigma \cdot \Phi^{-1}(1 - r) - \mu \tag{9}$$

Consequently, the *optimal* capital demand of precautionary firms also follows a non-increasing and continuous path and again the minimum and maximum levels of surplus are respectively demanded when $r \geq 1$ and $r \leq 0$. Discrepancies relative to preferred cash-balances can thus be explained on the basis of the underlying risks and the opportunity cost of capital.

From the actuarial point of view, the term $E\left[(X + k)_-\right]$ represents the *excess of loss* suffered by the issuer of some insuring policy covering the loss of the mutual fund X, where the *retention* or *deductible* is equal to k. Such a contract is known as a *layer* in the actuarial literature. The deductible is introduced in these contracts in order to assure that the holder of the policy will take every *reasonable care* to reduce the probability and the magnitude of losses. It then explicitly accounts for the *moral-hazard* inherent in the agreement — because insurers cannot directly observe the behaviour of policyholders. On these grounds, the *optimal* cash balance, as determined by *Equation* 7 or *Equation* 9, can be corresponded to the *optimal retention* or *optimal deductible* of the related deposit insurance contract. In the next section, the aggregated money demand of the market will be characterised by the sum of the optimal balances determined at the microscopic level.

3   The Aggregate Demand for Cash Balances

On the grounds of the *P&L* optimisation problem stated in the previous section, an *optimal* surplus exists that explicitly depends on the riskiness of the underlying fund, the return obtained for lending in the market of cash balances and the *type* of the firm. Indeed, from *Equations* 7 and 9, we obtain that the magnitudes of the optimal cash balance is always positively related to the volatility and negatively related to the rate of interest. However, while in the cases $\mu > 0$ and $\mu < 0$ *speculative* firms respectively increment and reduce the demand for capital as the magnitude of the mean return increases, when $\mu > 0$ and $\mu < 0$ *precautionary* firms respectively decrease and increase the level of equity with the magnitude of the mean return.

The amount of reserves demanded by the market will be characterised in the following by adding the cash balances maintained at different levels of aggregation, going from investors and small firms, to big companies, conglomerates, industries and economic sectors. Thus, notice in the first place that, although the capital decisions taken by speculative and precautionary firms are affected in quite the opposite way by the sign of the expected return, from the mathematical point of view this is only a matter of reference. In fact, if we assume that the individual exposures are distributed as Gaussians with means $\mu_1, \ldots, \mu_n$ and volatilities $\sigma_1, \ldots, \sigma_n$, then the optimal money demands can be simply written as:

$$k_i(r) \;=\; \sigma_i \cdot \Phi^{-1}(1 - r) - \mu_i \qquad \forall \, i = 1, \ldots, n \tag{10}$$

Under this specification, the *type* of firms is ultimately determined by the sign of the expected return, since the roles of *speculative* and *precautionary* firms are respectively corresponded to the cases $\mu_i < 0$ and $\mu_i > 0$. This convention is equivalent to assume that for every $i \in \{1, \ldots, n\}$ the mean return μ_i represents an average capital *loss*.

Summing up the balances maintained by all the firms in the market when their liquidity-preferences are given by the functions in *Equation* 10, we obtain that the total level of reserves K^M demanded by the market is equal to:

$$K^M = \sum_{i=1}^{n} L_i \cdot k_i(r) = \Phi^{-1}(1 - r) \cdot \sum_{i=1}^{n} L_i \, \sigma_i - \sum_{i=1}^{n} L_i \, \mu_i$$

where $L_1, \ldots, L_n$ denote the amounts of funds spent in each of the risks $X_1, \ldots, X_n$. Define $L = L_1 + \cdots + L_n$ as the total funds spent in the market, in such a way that:

$$K^M = L \cdot \left(\Phi^{-1}(1 - r) \cdot \sum_{i=1}^{n} \frac{L_i}{L} \, \sigma_i - \sum_{i=1}^{n} \frac{L_i}{L} \, \mu_i \right)$$

Hence the contributions of the individual exposures to the aggregate demand for cash balances are given by the coefficients $\omega_1, \ldots, \omega_n$, defined as:

$$\omega_i = \frac{L_i}{L_1 + \cdots + L_n} \quad \Leftrightarrow \quad L_i = \omega_i \cdot L \quad \forall \, i = 1, \ldots, n$$

Consequently, the liquidity-preference of the market $k^M = K^M/L$ must be defined as:

$$k^M(r) = k_{\mu,\sigma}(r) = \sigma \cdot \Phi^{-1}(1 - r) - \mu \tag{11}$$

with:

$$\mu = \sum_{i=1}^{n} \omega_i \cdot \mu_i \qquad and \qquad \sigma = \sum_{i=1}^{n} \omega_i \cdot \sigma_i$$

Note that assuming that the optimal surpluses are determined by *Equation* 10 does not necessarily imply that investors solve an optimisation problem before deciding their cash balances. However, as long as for every fund an optimal balance exists, there are incentives to accordingly adjust the capital structure, for big departures from this level can produce important losses and eventually lead to bankruptcy. Simple calibration and backtesting can be used to attain this objective. Thus, whatever the mechanism implemented to determine their levels of capital — going from a rule of thumb to complicated mathematical modeling — firms will be assumed to approach *as much as possible* to the optimal levels corresponded to their portfolios. On these grounds, a *weak* version of the *hypothesis of rational behaviour* is proposed, according to which the successive aggregate states will be characterised by adding the *risk-based* contributions of investors who combine equity and financial assets trying to build *optimal* portfolios.

The reader must be wondering at this point why should we consider the characterisation of the preference for liquidity in the terms presented so far, when the well-established utility-based framework is widely accepted in economic analysis — according to which the

8

demand for liquidity is represented by the surplus demanded by a *rational* decision-maker who maximises her or his expected utility. This is indeed the approach suggested by Tobin (1958) when characterising liquidity-preference *as behaviour towards risk* and also proposed in more recent contributions on the subject, such as Choi and Oh (2003) and Holmstrom and Tirole (2000). However, the following problems arise when implementing this model. Thus, in the first place, the expressions derived for the money demand from the first-order condition are not easy to work with and they cannot then be clearly interpreted in terms of *measurable* economic variables. Secondly, recall that a parametrical expression has to be determined for the utility function, as well as for the parameter representing the aversion-to-risk of the decision-maker. But this coefficient is not *observable*, a fact that leads the model to depend on *exogenous* variables that have to be determined on a subjective basis. As a consequence, every event of significance (involving adjustments beyond expected) has to be explained on the basis of exogenous events, like the sudden release of relevant information or the sudden change in the mood of investors — from risk-aversion to risk-loving or in the opposite direction. Therefore, and as a final point of criticism, the expected-utility framework cannot satisfactorily account for the whole of the recorded historical events — such that, in particular, it provides no satisfactory explanation for the liquidity crises occurred in the past.

By contrast, a great part of the criticism made for the utility approach can be avoided by adopting the characterisation of the preference for liquidity proposed in this paper. Indeed, a simple mathematical expression is provided in *Equations* 10 and 11 to describe the money demand — i.e. the quantile function of the probability distribution describing the underlying risk. Moreover, as established in the first-order condition, the optimal amount of capital is corresponded to an optimal exchange between a flow of probability and the return offered by a certain flow of cash. Such representation reinterprets the Tobin's (1958) characterisation of liquidity-preference as *behaviour towards risk*. Additionally, the market money demand explicitly depends on the mean return and the volatility of the series of capital returns of the market portfolio in the new model, which are certainly *observable* variables. The question remains, however, of whether the model can satisfactorily explains the evolution of the preference for liquidity in *normal* times as well as in times of turbulence. Before providing an answer to this question, I will ask the reader to wait until *Section* 6. First, in the next section, the cost of capital will be determined on the basis of the monetary equilibrium, when the money supply is controlled by creditors in deregulated markets.

4   The Cost of Capital as Determined by the Equilibrium in the Market of Cash Balances

According to the model developed in *Sections* 2 and 3, the determinants of the money demand in a *close* financial market are of two kinds. Firstly, since the optimal levels of reserves explicitly depends on risk and risks are represented by probability functions belonging to the class of Gaussian probability distributions, the optimal surplus are ultimately determined by the mean return and the volatility of the series of capital returns of the market portfolio. Besides, by imposing the market balance to be equal to the sum of the balances demanded by

investors, the mean return and the volatility of the market portfolio are expressed as the sum of the corresponding parameters representing the series of capital returns at the microscopic level (as stated in *Equations* 10 and 11). The other variable affecting the money demand is the *cost of capital*, defined as the return that allows investors to borrow or lend balances in the market. Equivalently, the cost of capital can be interpreted as the interest rate offered by a class of instruments which are regarded as *perfect* substitutes to money.[2]

Let us now introduce an alternative framework, where creditors are supposed to control the total supply of cash balances and the cost of capital is determined in order to satisfy the necessary equilibrium condition that the flows of funds provided to the market are equal to the total demanded balances. Accordingly, letting M denote the total amount of capital supplied by creditors and letting L (as in *Section* 2) denote the total amount of funds invested on securities, the following equation must hold:

$$M = K_{\mu,\sigma}(r) = L \cdot k_{\mu,\sigma}(r)$$

Replacing the liquidity-preference function $k_{\mu,\sigma}(r)$ according to *Equation* 11, we obtain that:

$$m := \frac{M}{L} = \sigma \cdot \Phi^{-1}(1-r) - \mu \tag{12}$$

where the ratio $m = M/L$ represents a *cash-to-risk* ratio, which can be also interpreted as a *supply-to-demand for transactions* ratio or a *relative money supply*. The monetary equilibrium can then be equivalently established in the following terms:

$$R := \frac{m+\mu}{\sigma} = \Phi^{-1}(1-r) \quad \Leftrightarrow \quad r = 1 - \Phi\left(\frac{m+\mu}{\sigma}\right) \tag{13}$$

In this context, both M and L are *control* variables of the market or, in other words, both lenders and borrowers can affect the monetary equilibrium. The expected return and the volatility of the market portfolio, on the other hand, can be regarded as *variables of state*, for they reflect the *impact* over the market of maintaining the current levels of supplied and demanded balances. Finally, the interest rate of equilibrium can be also considered as a *state* variable that measures the *opportunity* cost of capital, for it is related to the optimal exchange of money and risk in the market.

Within this setting, the ratio R represents the rate at which firms agree to exchange the sure flow $(m + \mu)$ for a unit of volatility and can accordingly be regarded as the *market price of risk*. It can then be regarded as an alternative measure of risk to the *Sharpe* ratio, which, as it is well-known, is equal to the slope of a straight line relating the expected return to the volatility of every financial security transacted in the market, in such a way that $\mu = r + S \cdot \sigma$ (see Sharpe, 1966a and 1966b). Unlike the Sharpe ratio, however, the measure R is respectively induced to *rise* and to *diminish* in response to *inflationary* and *deflationary* trends, respectively characterised by *increasing* and *decreasing* m. Such a difference is a consequence of the fact that, while in the model of equilibrium of William Sharpe firms can borrow or lend any amount of capital at the interest rate r, in the alternative equilibrium condition established in *Equation* 13 the liquidity conditions are determined by the relative

---

[2]The adopted theoretical framework can then be related to the model of debt managing with *free cash flows* suggested by Jensen (1986).

supply m. Accordingly, within any class of risks with the same market-price, different stocks of cash are necessarily corresponded to different risk-parameters.

The rate $r = T_X(m) = 1 - \Phi(R)$ thereby represents the return obtained when investing one monetary unit on the underlying risk and can be accordingly regarded as the *Internal Rate of Return on Risk* ($IRRR$). Thus, while the *Internal Rate of Return* (IRR) of a zero-coupon bond represents the opportunity cost of receiving a cash-flow at some future date instead of today (see, for example, Hull, 2000), the $IRRR$ can be interpreted as the return to be paid for holding a given uncertain claim instead of investing on a *non-risky* zero-coupon bond with a predetermined maturity.

Figure 1: Interest Rate Elasticity of the Optimal Demand for Cash Balances.
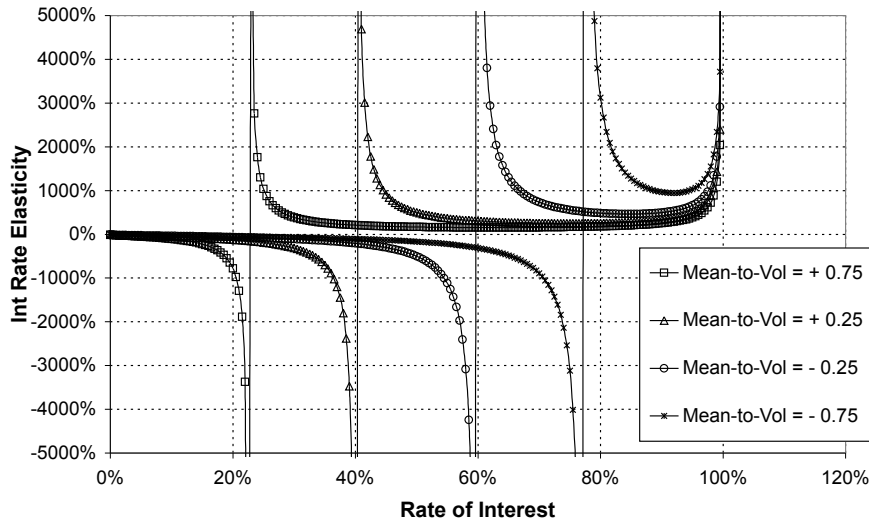
From the mathematical viewpoint, the $IRRR$ corresponds to the probability that the current stock of money will suffice to cover the expected imbalance of the market portfolio. In fact, the lower the $IRRR$, the lower the tail-probability that the excess of loss of the market portfolio was greater than the real money-supply. Such a situation is consistent with (overnight) markets where investors face less difficulties to adjust their end-of-day balances or equivalently, more *liquid* markets. By contrast, the tail-probability of the market portfolio increases with the $IRRR$ and so markets are more *illiquid* the greater the $IRRR$. Moreover, the higher the money spent in transactions with respect to the stock of capital in a given market, the higher its corresponding $IRRR$, since in this case the expansion of output is, to a greater extent, *nominal* in nature or, equivalently, more *inflationary*. Conversely, less inflated funds, characterised by greater cash-to-risk ratios, are assigned lower internal returns.

Finally, a word can be said about the *instability* of the market. In fact, as long as the market parameters are determined by *Equation* 12, changes in any of the control variables M and L must be followed by adjustments in μ, σ and the $IRRR$ in order to reestablish the monetary equilibrium. In the particular case when the mean return and the volatility of the market portfolio remain unchanged, variations in the preference for liquidity can be explicitly established in terms of the variations in the interest rate:

11

$$\frac{\Delta k_{\mu,\sigma}(r)}{k_{\mu,\sigma}(r)} = \epsilon(r) \cdot \frac{\Delta r}{r} \quad with \quad \epsilon(r) = \frac{r}{k_{\mu,\sigma}(r)} \cdot \frac{dk_{\mu,\sigma}(r)}{dr}$$

where the coefficient $\epsilon(r)$ represents the *elasticity* of the demand for cash balances with respect to the interest rate — equal to the percentage variation in the demanded cash balances in response to a percentage movement in the interest rate. From *Equation* 11, we obtain that the dependence on risk of the interest rate elasticity is in fact expressed as a dependence on the mean-to-volatility ratio μ/σ:

$$\epsilon\left(r, \frac{\mu}{\sigma}\right) = -r \sqrt{2\pi} \exp\left(\frac{[\Phi^{-1}(1-r)]^2}{2}\right) \cdot \left[\Phi^{-1}(1-r) - \frac{\mu}{\sigma}\right]^{-1} \tag{14}$$

The sign of the elasticity thereby depends on the state parameters, in such a way that:

$$\epsilon\left(r, \frac{\mu}{\sigma}\right) < 0 \quad \Leftrightarrow \quad \Phi^{-1}(1-r) - \frac{\mu}{\sigma} > 0 \quad \Leftrightarrow \quad r < 1 - \Phi\left(\frac{\mu}{\sigma}\right) \quad \Leftrightarrow \quad k_{\mu,\sigma}(r) > 0$$

$$\epsilon\left(r, \frac{\mu}{\sigma}\right) > 0 \quad \Leftrightarrow \quad \Phi^{-1}(1-r) - \frac{\mu}{\sigma} < 0 \quad \Leftrightarrow \quad r > 1 - \Phi\left(\frac{\mu}{\sigma}\right) \quad \Leftrightarrow \quad k_{\mu,\sigma}(r) < 0$$

When $r = 1 - \Phi\left(\frac{\mu}{\sigma}\right)$, the magnitude of the elasticity is *infinite*, meaning that at this point the whole adjustment is produced in quantities.

In this context, the level $r = 1 - \Phi\left(\frac{\mu}{\sigma}\right)$ represents a *turning-point* in capital markets that separates two different environments: the first one satisfying $k_{\mu,\sigma}(r) > 0$ and $\epsilon\left(r, \frac{\mu}{\sigma}\right) < 0$, and the second one, characterised by $k_{\mu,\sigma}(r) < 0$ and $\epsilon\left(r, \frac{\mu}{\sigma}\right) > 0$. As depicted in *Figure* 1, the interest rate elasticity is *negative* and *non-increasing* in r when $r < 1 - \Phi\left(\frac{\mu}{\sigma}\right)$, but it is *positive* and *U-shaped* when $r > 1 - \Phi\left(\frac{\mu}{\sigma}\right)$. An asymptote is produced at the level $r = 1 - \Phi\left(\frac{\mu}{\sigma}\right)$, which is located to the left of the axis $r = 0.5$ when $\mu > 0$ and to the right of the axis $r = 0.5$ when $\mu < 0$. Hence, the market is the *most unstable* when the internal return on risk approaches to the level $r = 1 - \Phi\left(\frac{\mu}{\sigma}\right)$. Moreover, the greater the magnitude of the mean-to-volatility ratio, the more the asymptote respectively approaches to the axis $r = 0$ and $r = 1$ when $\mu > 0$ and when $\mu < 0$.

A complete characterisation of the dynamic of the market would require to describe the variation of the preference for liquidity in terms of the variations in all of the state variables. Besides, since every cash-to-risk ratio m is corresponded to a set of parameters $(\mu, \sigma, IRRR)$, multiple equilibria are allowed and then additional conditions are required — determining an *optimal path* or a *most probable* state — in order to predict the future state of the market. The developing of such a model is beyond the scope of this paper. However, as long as the set of different combinations of the parameters which are compatible with the monetary equilibrium for a given relative stock m are determined by *Equation* 12, it is still possible to describe the possible states of the market on the basis of a *static* approach. As will be proved soon in *Section* 5, a *risk-structure* of interest rates can be defined in this way, providing the optimal return of exchange between risk and money in a market characterised by a given pair of expected return and volatility.

5 The Risk-Structure of Interest Rates

Let us consider in the following some capital market that follows a certain regime with $m = constant$. A *risk-structure* of interest rates on the plane of mean returns and volatilities is then determined by *Equation* 12, where each rate represents the return to be asked for receiving a *sure* cash-flow instead of some *uncertain* payment. Accordingly, as depicted in *Figure* 2, at any fixed level of volatility, the expected return and the internal rate of return on risk are inversely related. On the other hand, recall that the market price of risk is incremented when the $IRRR$ decreases and vice-versa (as established in *Equation* 13). Consequently, within a class of securities showing the same variability, risks offering different expected returns can be assigned the same $IRRR$ (or the same market price) only if more equity is supplied to the portfolio containing the fund with the lower expected return. This result ultimately provides a theoretical explanation of why investors show liquidity-preference.

Figure 2: The Risk-Structure of Interest Rates as a Function of the Expected Return.
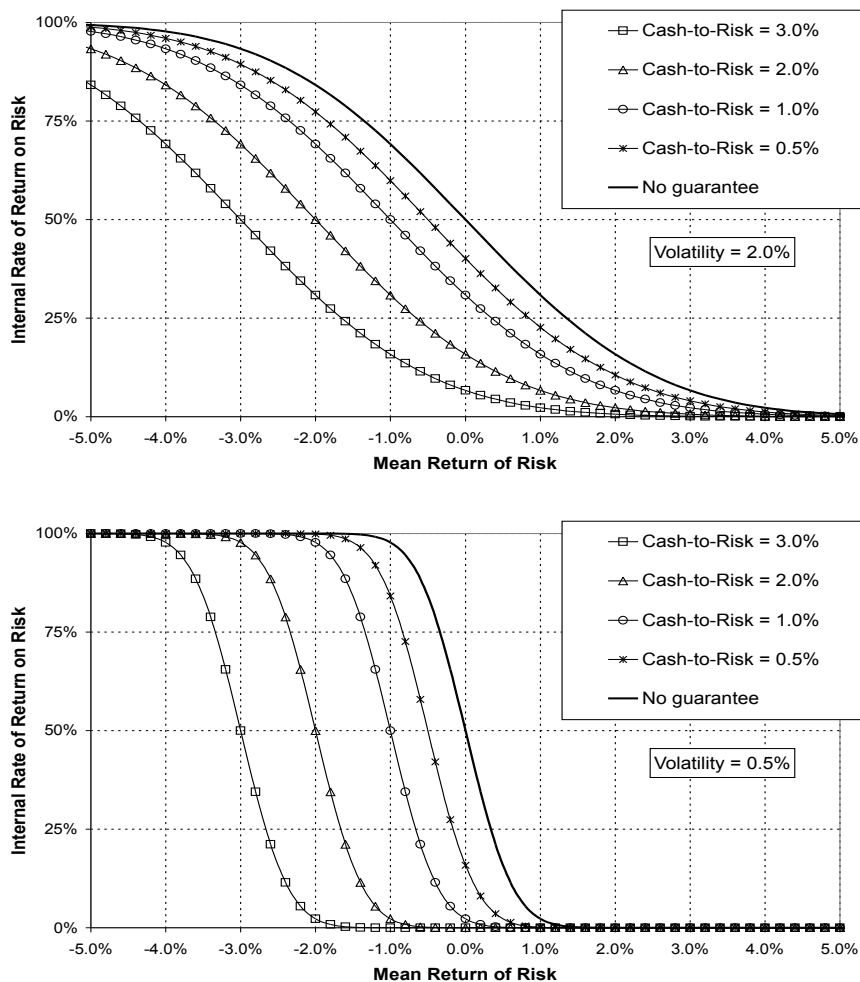
In *Figure* 3, the evolution of the internal return is shown as a function of the volatility,

given different fixed levels of the expected return. Thus, as depicted in the upper graph of $Figure$ 3, within a class of securities offering the same $positive$ expected return, the $IRRR$ always rises with variability, at the time that for every level of volatility, the $IRRR$ diminishes when the supply-to-demand ratio is increased. Consequently, the market price of risk can be inflated both by pumping liquidity and by reducing the variability of the market portfolio — relations that are compatible with economic intuition.

In the lower graph of $Figure$ 3, the risk-structure is shown for a class of funds offering $negative$ expected returns. Note that the relationship between internal returns and volatilities is inverted when the surplus is lower than the expected loss, in such a way that in this range higher internal returns (or lower market prices) are obtained for lower volatilities. Notice additionally, from $Equation$ 13, that:

$$IRRR > 0.5 \quad \Leftrightarrow \quad \tfrac{m+\mu}{\sigma} < \Phi^{-1}(0.5) = 0 \quad \Leftrightarrow \quad \mu < -m$$

$$IRRR < 0.5 \quad \Leftrightarrow \quad \tfrac{m+\mu}{\sigma} > \Phi^{-1}(0.5) = 0 \quad \Leftrightarrow \quad \mu > -m$$

Hence the case $IRRR > 0.5$ is corresponded to the situation when the average capital loss endured by the market portfolio is higher in magnitude to the relative stock m, while the case $IRRR > 0.5$ is related to the situation when the magnitude of the average capital loss is lower than the relative stock m. Accordingly, the cases $IRRR > 0.5$ and $IRRR < 0.5$ respectively characterises $insolvent$ and $solvent$ market portfolios. The asymptote separating both trends at the axis $r = 0.5$ corresponds to the case when the demanded and supplied levels of reserves are equal in magnitude. Besides, $\sigma \to +\infty$ when $IRRR \to 0.5$, no matter whether this level is approached from the left or the right of the axis $r = 0.5$. Indeed, $maximum\ uncertainty$ can be expected around the level $IRRR = 0.5$, for at this point the market portfolio is not $solvent$, as in the case $IRRR > 0.5$, nor $insolvent$, as in the case $IRRR < 0.5$, so that its credit quality is $not\ determined$.

Some general rules concerning the evolution of the market can thus be given by using a $partial$ equilibrium approach. Indeed, notice on the one hand, as shown in $Figure$ 2, that the equilibrium expected return tends to the cash-to-risk ratio (so that the market price of risk tends to zero) when the $IRRR$ tends to 0.5 (see $Equation$ 13). On the other hand, as depicted in $Figure$ 3, the internal return on risk tends to the value $r = 0.5$ when the volatility tends to infinite. Besides, the more the $IRRR$ approaches to 0.5, the greater the volatility adjustment that is required to account for a same absolute variation in the $IRRR$. Consequently, the model predicts the appearance of volatility $jumps$ in response to movements in the $IRRR$ if this variable is $close\ enough$ to the level $r = 0.5$. Moreover, since the concavity of the curve relating the $IRRR$ to σ increases as the risk-to-cash ratio is diminished and the curve eventually $degenerates$ when $m = 0$, more drastic volatility movements are expected for lower ratios.

In conclusion, the mean return and the volatility of the market portfolio, as well as the internal return on risk, can be regarded as $physical$ properties of markets, which at equilibrium are connected to each other according to $Equation$ 12 for any given level of the relative stock of money $m = M/L$. Consequently, the equilibrium depends on the interactions between borrowers and lenders on the one hand, who respectively control the demand and the supply of capital, and the response of the market on the other, which can be regarded as the $involuntary$ output of the decisions simultaneously and independently

Figure 3: The Risk-Structure of Interest Rates as a Function of Volatility.
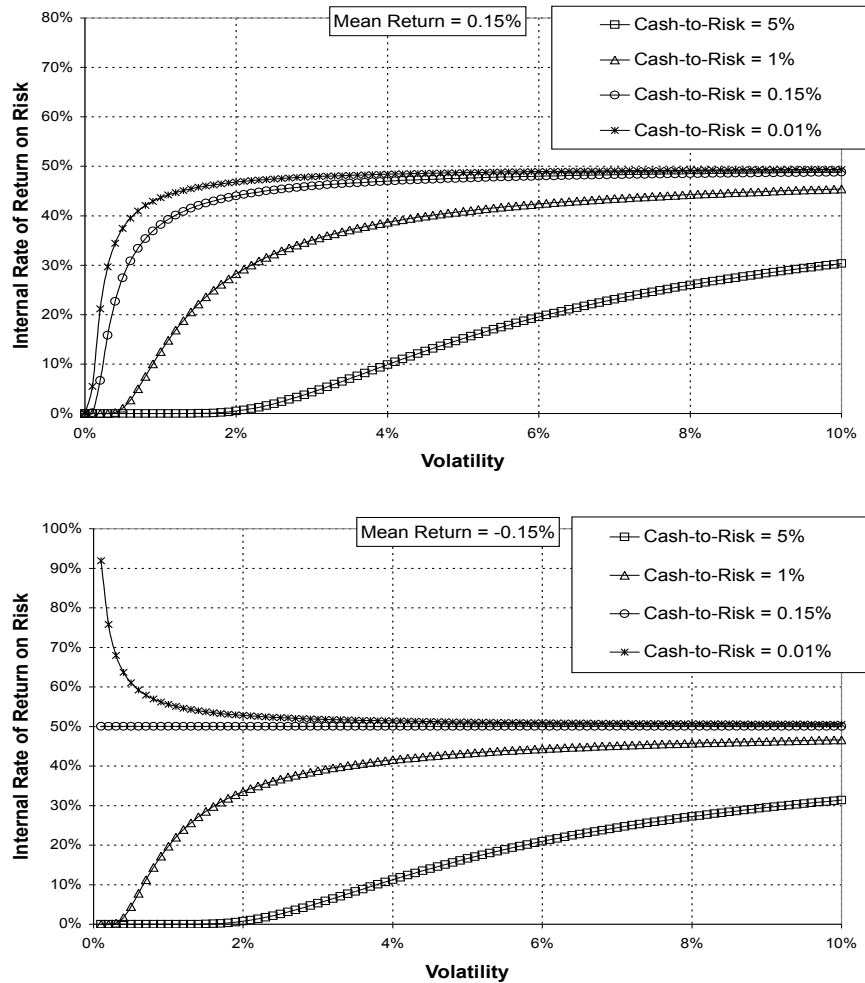

made by investors, intermediaries and monetary authorities, and which is corresponded to the probability distributions describing the processes of percentage returns. The fact that μ and σ are *observable* variables that can be *measured* on the base of the historical data, implies that the model can be empirically tested. This task will be performed in the coming sections.

6   Empirical Evidence

The theoretical approach presented in the previous sections can be applied to characterise the cost of capital in different economic contexts. Thus, for example, at the corporate level, it can be used for the design of a capital allocation mechanism in financial conglomerates (as proposed by Dhaene et al., 2003, Goovaerts et al., 2005, and Mierzejewski, 2006). In this case, the Gaussian random variables X_i represent the capital $P\&L$ of the portfolios maintained by the business units and the $IRRR$ represents the *internal* cost of capital, i.e. the return to be imposed on the transactions of cash balances *inside* the institution. Alternatively,

at the macroeconomic level, the model can be applied to deduce a (theoretically justified) expression for the money demand of the economy. In this situation, the aggregate exposure X should be related to the series of capital $P\&L$ of the level of nominal income.

Figure 4: Twenty Years of Evolution of the $DowJones$ and the $Nasdaq$ Composite Indexes and their Corresponding Daily Transacted Funds.
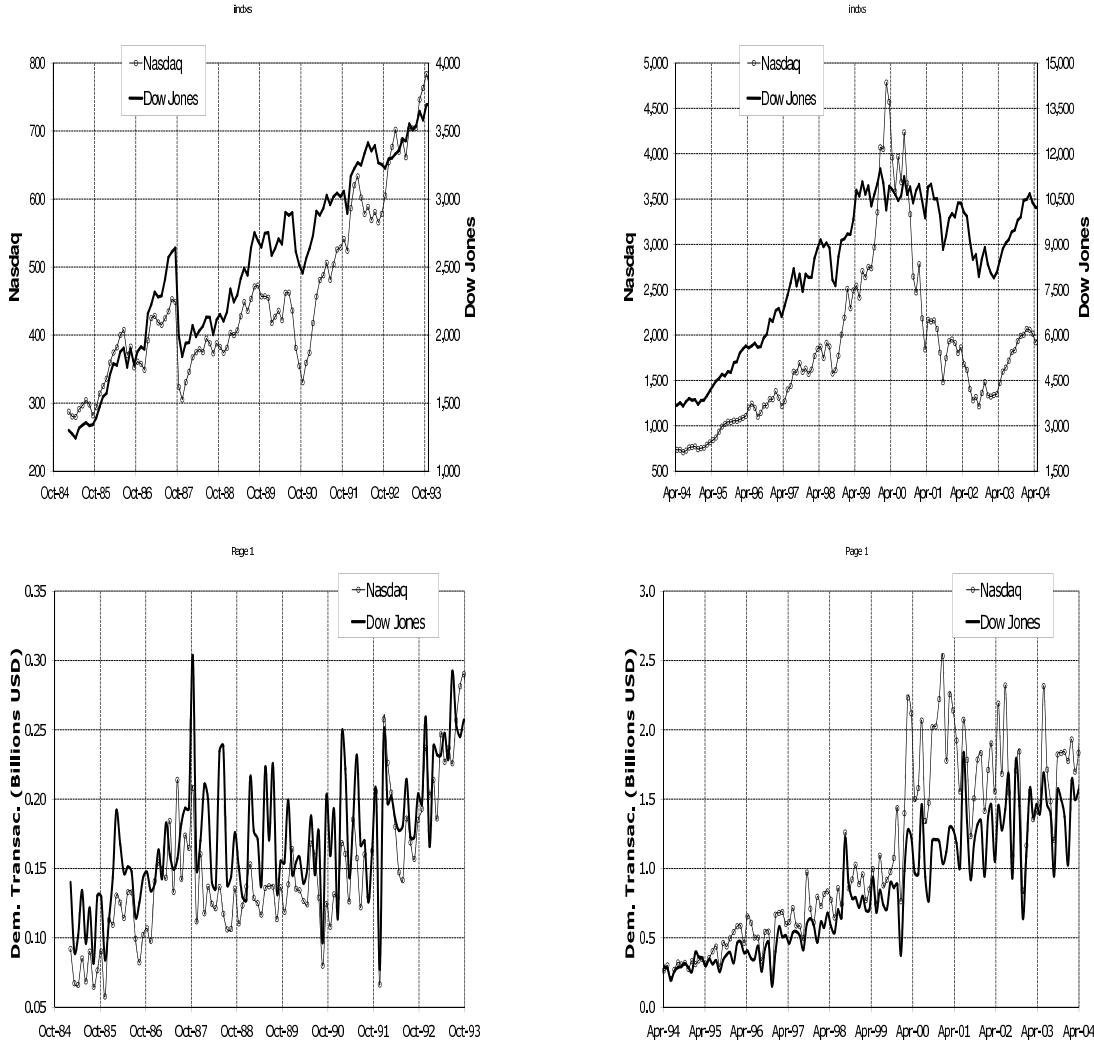
Finally, the model can be also used to describe the monetary equilibrium in a close financial market, where a total stock of cash M is supplied by creditors and stockholders. Financial intermediaries, on the other hand, spend the (nominal) quantities L_i on securities and demand equity up to the optimal levels $k_i(r)$ defined in $Equation$ 10, in such a way that the total demand for capital in the market is equal to $L \cdot k_{\mu,\sigma}$, where $k_{\mu,\sigma}$ denote the market liquidity-preference (defined in $Equation$ 11), μ and σ denote the risk-parameters of the market portfolio and L represents the total volume spent on the transactions of securities.

Assuming, in addition, that all the relevant information for investment decisions is contained in the values assigned in the stock exchange, the series of capital returns of the level of prices in this market should determine the riskiness of the market portfolio. This will be the framework adopted in the following for the empirical verification of the model. The

variables under analysis will be the *DowJones* and the *Nasdaq* composite indexes, which respectively represent the industrial and the *Information & Technology* (IT) sectors of the economy of United States. The daily valuations of these indexes are depicted in *Figure* 4, together with the amounts of funds daily spent on transactions in the corresponding sectors. The data have been obtained from $http://www.yahoo.com/finance$.

Two episodes of crisis can be clearly noticed in *Figure* 4, occurring in October 1987 and April 2000, and respectively known as the *Black Monday* and the *dot-com bubble* in the literature. In fact, on Monday October 19th, 1987, the *DowJones* lost more than 25% of its value to recover only about 5% the day after, thus accumulating in one week, in the two consecutive days, losses over 40% and 35% respectively. The *Nasdaq*, in the meantime, lost more than 10% of its value the same day and nearly 10% the day after, thus accounting for weekly losses around 20% and 30% on October 19th and 20th respectively. These are among the most severe declines observed during the whole past century. A striking feature of the episode is the short time that took the bubble to explode and its lack of fundamentals, for no major news or events appeared during the prior days to the crash.

Unlike the stock crash of October 1987, the episode of April 2000 mainly affected the IT companies. In fact, while the *DowJones* attains a maximum at 11,722.98 points on January 14th, 2000, and then falls until reaching 9,811.24 points on March 14th, 2000, the *Nasdaq* attains a maximum at 5,048.62 points on March 10th, 2000, and then falls until reaching 3,400.91 points on May 31st, 2000, in this way respectively accounting for losses of 35% and 15% approximately. It can be also noted in *Figure* 4 that the funds demanded for transactions followed a growing path during the nineteen nineties, a fact that is consistent with the liberalisation of capital markets and the expansionary monetary policy followed by the Federal Reserve Bank in that period (see *Figure* 6). Moreover, although the total inflows of capital are shared in roughly equal parts most of the time, such a tendency is abandoned in the period starting shortly after April 1999 and extending until shortly before April 2003. It is well-known, indeed, that towards the mid-nineties, a sentiment of growing confidence about the future of the dot-com firms predominated. As a consequence, the credit conditions were facilitated to this industry, both through the reduction of premiums and controls, as through the publishing of new stock.

Let us then investigate whether the trends followed by the mean returns and the volatilities of the series of capital returns of the *DowJones* and the *Nasdaq*, respectively depicted in the upper and the lower rows of *Figure* 5, are consistent with the monetary equilibrium as described in *Equations* 12 and 13. The mean returns have been estimated as the average of the daily percentage variations of the indexes observed in the past ninety working days, while the volatilities have been calculated as the mean standard error:

$$\mu_t = \frac{1}{90} \cdot \sum_{i=t-89}^{t} X_i \quad and \quad \sigma_t = \sqrt{\frac{1}{90} \cdot \sum_{i=t-89}^{t} (X_i - \mu_t)^2}$$

Thus, in the first place, every peak in the volumes of funds spent on assets shown in the second row of *Figure* 4 should be corresponded to a pair of mean and volatility adjustments in *Figure* 5. By comparing both figures, we can verify that this is indeed the case. In particular, the peaks of trading activity occurred in October 1987 and April 2000 are accompanied by the major changes in the risk parameters. As a matter of fact, the volatilities

Figure 5: Expected Returns and Volatilities of the Series of Capital $P\&L$ of the $DowJones$ and the $Nasdaq$ Composite Indexes.
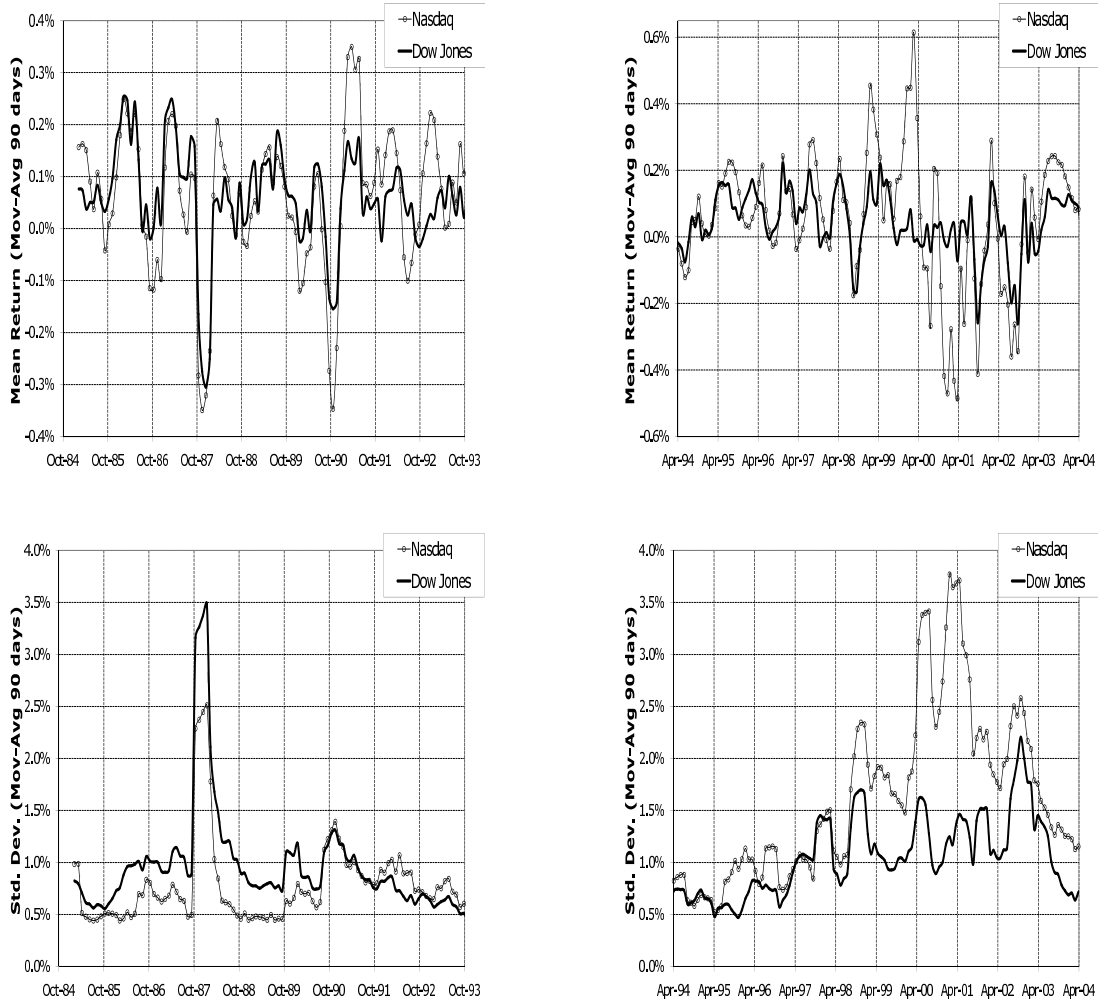
of the series of percentage returns of the $DowJones$ and the $Nasdaq$ respectively multiplied by more than three and more than four times in October 1987, while their mean returns suddenly felt from levels around $+0.1\%$ to levels around -0.3%. Regarding the episode of April 2000, there is no evidence of turbulence in the $DowJones$, for only small variations in the risk-parameters are observed — as it is expected indeed, since this index did not burst on that date. The volatility of the $Nasdaq$, in the meantime, was amplified 1.5 times approximately, although the major adjustment affected its expected return, which between February 2000 and May 2000 felt from a historical maximum over 0.6% to a level around -0.1%.

Other facts predicted by the model can be empirically verified. In fact, from $Equation$ 13, we obtain that if the $IRRR$ is rising (i.e. if the market price of risk is diminishing) while the relative supply $m = M/L$ is constant, either the expected return must be reduced or the volatility must augment. By contrast, if the $IRRR$ is diminishing (i.e. if the market price of risk is rising) while the relative supply $m = M/L$ is constant, either the expected return

Figure 6: Federal Funds Rate.
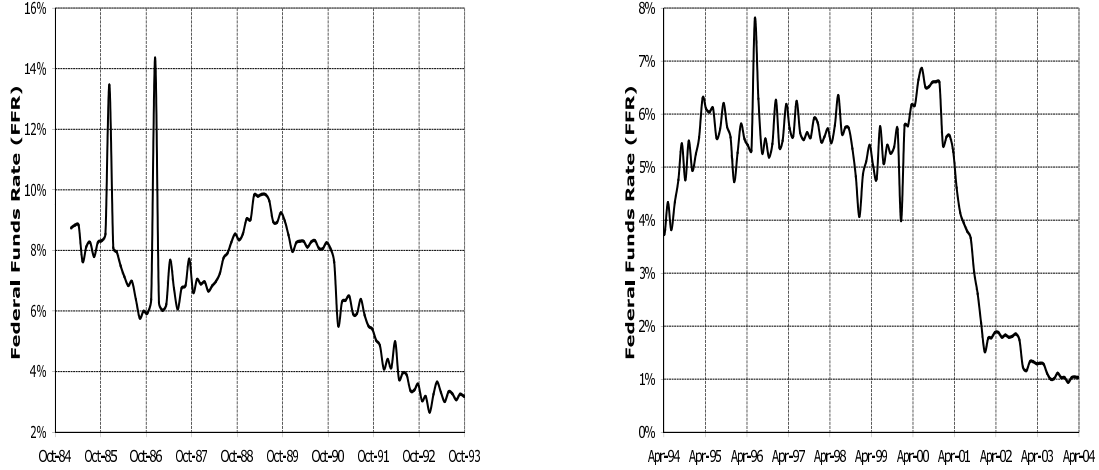
must augment or the volatility must be reduced.

On these grounds, we can say that under any of the combined regimes ($\mu \downarrow$, $\sigma \uparrow$) and ($\mu \uparrow$, $\sigma \downarrow$), the risk-parameters push the market in the same direction: i.e. the $IRRR$ is incremented and the market price of risk is reduced in the former case, while the $IRRR$ is reduced and the market price of risk is incremented in the later. In any different situation, i.e. either when ($\mu \downarrow$, $\sigma \downarrow$) and when ($\mu \uparrow$, $\sigma \uparrow$), the effects of the mean return and the volatility compensate each other. We then conclude that the most severe liquidity adjustments should be characterised by sudden movements of *opposite sign* affecting the risk-parameters. Several episodes occurred between October 1984 and April 2004 follow this pattern, like those observed in October 1990, April 1998 and April 2002, as well as in the stock crashes of October 1987 and April 2000 (see *Figures* 4 and 5).

To complete the description of the monetary equilibrium, still the paths followed by the liquidity-preference function and the internal rate of return on risk have to be analysed. This will be done in the next sections by separately considering two different settings. Thus, first the level of the rate of interest will be regarded *as given* in *Section* 7 (as in a competitive framework), so that the adjustments in markets will be exclusively produced in quantities, i.e. they will exclusively affect the variables M and L. Next in *Section* 8, the cash-to-risk ratio $m = M/L$ will be fixed, in such a way that the equilibrium will be reestablished by exclusively affecting the $IRRR$.

# 7   Maintaining the Monetary Equilibrium with Adjustments in Quantities

Let us suppose in the following that firms are allowed to exchange overnight balances without restrictions at the free-of-risk interest rate r_0, which will be represented by the *Federal Funds Rate* (FFR) and whose evolution is depicted in *Figure* 6. This is equivalent to assume that lenders are *price-takers* and that no transaction costs nor credit premiums are charged to

the price of debt. The optimal demand for money is then given by the inverse of the quantile function of the series of capital $P\&L$ of the market portfolio, as in $Equation$ 11, and the level of reserves $K_{\mu,\sigma}$ demanded by the market is given by:

$$K_{\mu,\sigma}(r_0) \; = \; L \cdot k_{\mu,\sigma}(r_0) \; = \; L \cdot \big[\; \sigma \cdot \Phi^{-1}(1 - r_0) - \mu \; \big] \tag{15}$$

Let us then analise the empirical evidence provided by the series of capital $P\&L$ of the $DowJones$ and the $Nasdaq$ during the nineteen eighties and the nineteen nineties. We would like to know whether the preference for liquidity remains more or less stable — as assumed in standard macroeconomic analysis — and whether the episodes of liquidity crises already described can be corresponded to adjustments in the liquidity-preference and the demanded reserves functions.

Figure 7: Preference for Liquidity and Demanded Reserves.
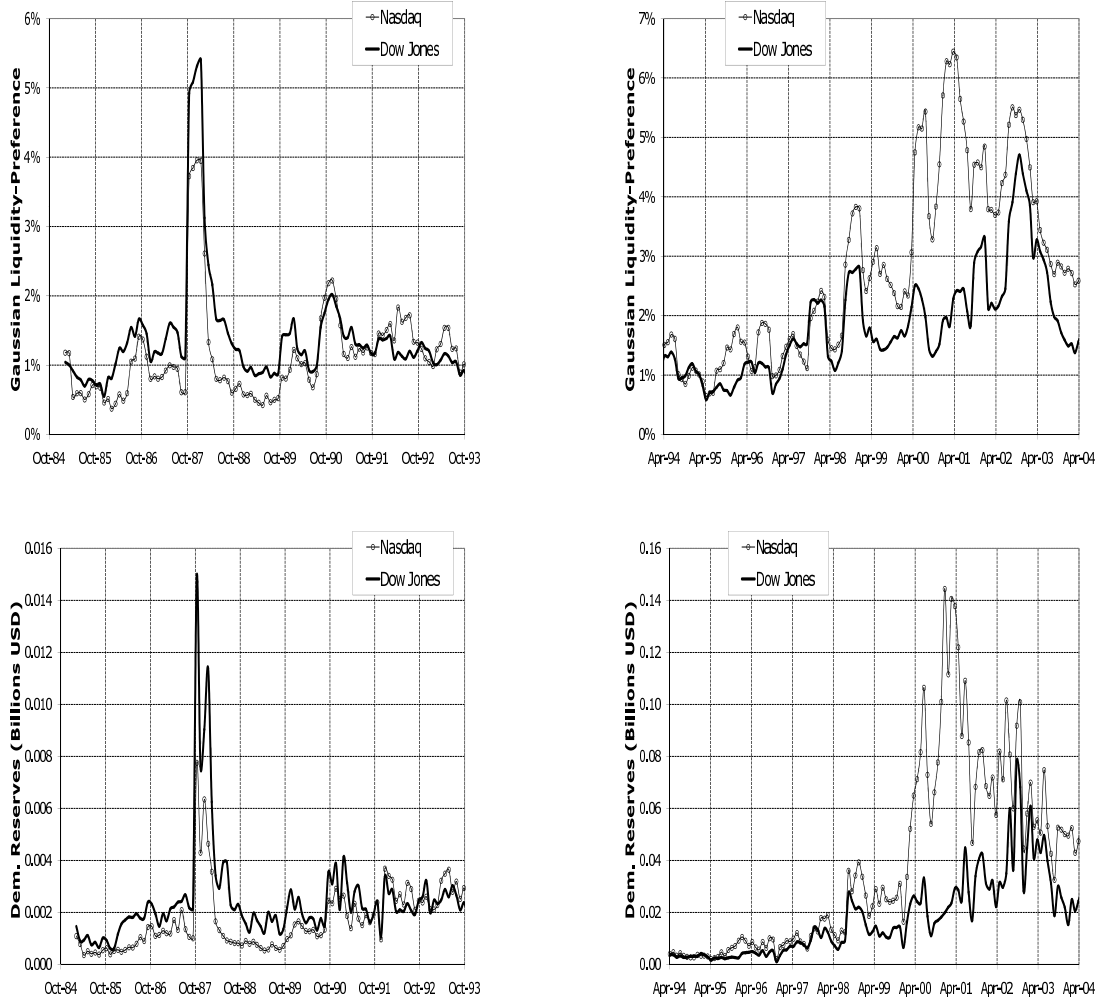
As depicted in the first column of $Figure$ 7, although the preference for liquidity and the demand for cash balances are indeed fluctuating variables, they show a *stable* growing tendency during nearly the whole period going from October 1984 to October 1993, the

only exception being the sharp increments shown in October 1987. In fact, in the case of the *DowJones*, the level of demanded reserves was multiplied by eight around October 1987 (going from about USD 2 MM to about USD 16 MM), a result that is compatible with the fact that the liquidity demand was multiplied by five around October 87 (rising from 1% to 5%), while the level of demanded cash-balances L was multiplied by 1.5 (rising from less than USD 200 MM to about USD 300 MM). Regarding the *Nasdaq*, the level of demanded reserves and the preference for liquidity were respectively amplified about 8 and 6 times, while the demand for cash balances was multiplied by 1.3 approximately. Notice also that the effects of the crisis were endured in both industries, although the magnitude of the required adjustment was greater for the *DowJones* composite index. Finally, the *short-lasting* nature of the event is capture by the liquidity-preference function as well, for the levels that predominated before the crisis had already been attained towards October 1988.

Let us now analise the episode of April 2000. As shown in the second column of *Figure* 7, although the liquidity-preference of both indexes rose steadily from roughly the first quarter of 1997 until the last quarter of 2002, most of the time the magnitude of this coefficient is greater for the *Nasdaq*, specially after April 2000. The crisis signaled at this point must then be regarded as one particular adjustment occurred at the middle of a more extended period of turbulence primarily affecting the *IT* industry. Inspecting the graph located to the lower right corner of *Figure* 7, we can additionally notice that the increment in the preference for liquidity is partly compensated by a reduction in the funds demanded for transactions L (see also *Figure* 4), in such a way that only from the beginnings of the year 2000 until the ends of 2003 the balances demanded for precautionary motives surpassed the levels considered as *normal* up to that moment. However, since the lift of the liquidity-preference of the *Nasdaq* is sharpest on April 2000, the burst of the bubble can be established at this point, in accordance to the analysis of the previous section.

In conclusion, the two episodes of crisis so far analised can be fully described by means of the money demand functions defined in *Equations* 11 and 15, in the sense that not only the moments when the bubbles bursted are precisely determined, also the main characteristics of the crises are satisfactorily explained.

8 Maintaining the Monetary Equilibrium with Adjustments in the Cost of Capital

Let us now assume that firms are obliged to fit their reserves in order to peg their cash-to-risk ratios to a fixed proportion α, i.e. $m = K/L = \alpha$. Such could be a restriction established by a regulatory authority, but it could be also imposed as a tacit requirement in markets with informational asymmetries. As a matter of fact, when intermediaries cannot fully observe the portfolios held by their customers, the leverage ratios provide a signal of the capacity to pay liabilities back and thus explicitly affect the price of debt. The level of the internal return on risk that is compatible with the monetary equilibrium under such circumstances is obtained by replacing $m = \alpha$ in *Equation* 13:

Figure 8: Internal Rate of Return on Risk (IRRR) and Interest Rate Elasticity of the Liquidity Demand with Constant Relative Supply.
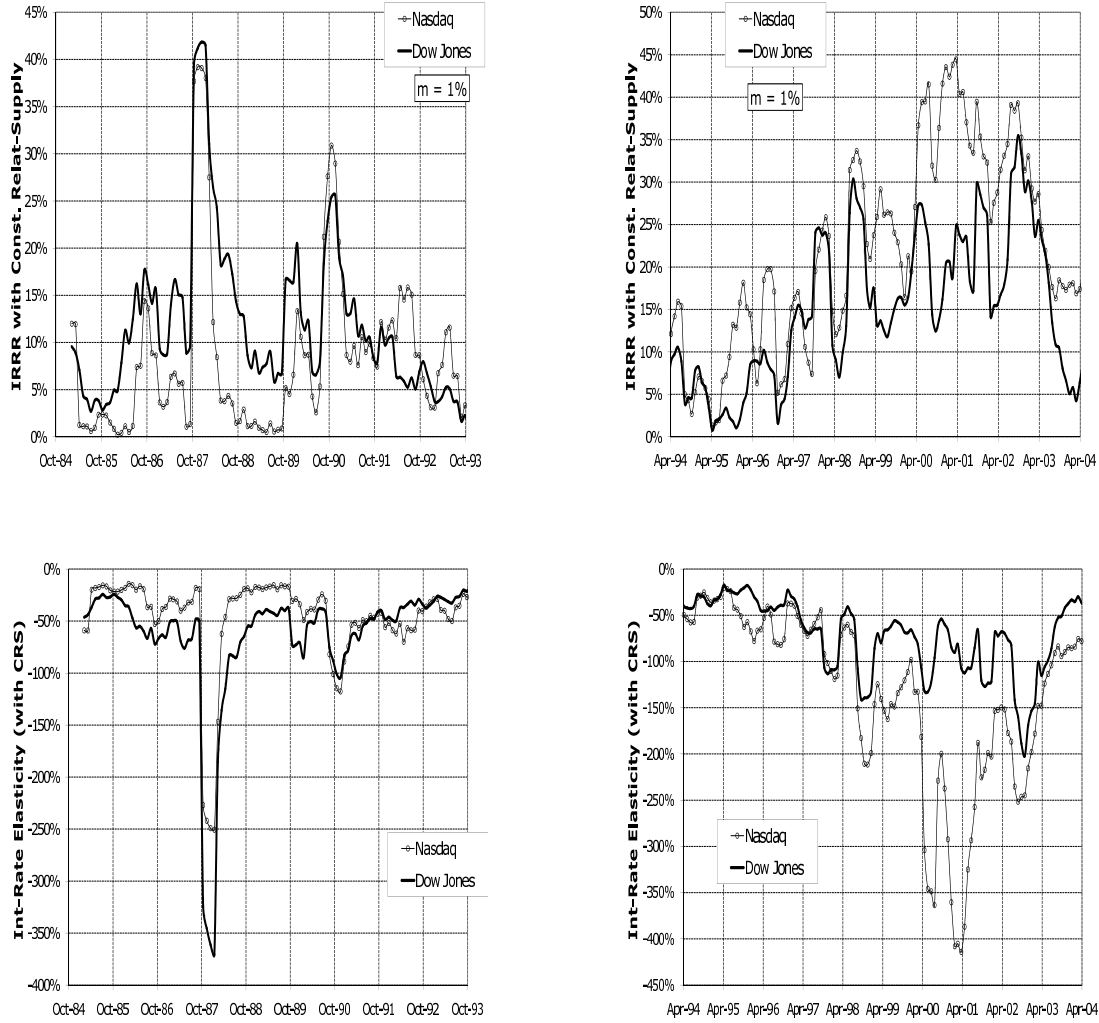
$$r_\alpha = 1 - \Phi\left(\frac{\mu + \alpha}{\sigma}\right) \tag{16}$$

Hence $r_\alpha \to 0.5$ when $\sigma \to \infty$ (see *Figure* 3). In fact, as already stated in *Section* 5, under regimes that maintain a constant relative stock of money, decreasing expected return and increasing volatility are both markets trends that push the induced internal return r_α to the axis $r = 0.5$ (or push the market price of risk to zero) and consequently push the market to more *unstable* states.

From *Equations* 14 and 16, the following expression is obtained for the elasticity:

$$\epsilon_\alpha = -r_\alpha \frac{\sigma\sqrt{2\pi}}{\alpha} \cdot \exp\left[\frac{(\mu + \alpha)^2}{2\sigma^2}\right] \tag{17}$$

Therefore, the sign of the interest rate elasticity is always *negative*, which means that increments of the cost of capital are always followed by reductions in the amount of demanded

cash balances and vice versa. Besides, the magnitude of the elasticity converges to *infinite* when $\sigma \to \infty$, i.e. $|\epsilon_\alpha| \to \infty$ when $\sigma \to \infty$. This result reinforces the claim stated in the previous paragraph, that the market becomes *more unstable* as $\sigma \to \infty$ and $r_\alpha \to 0.5$.

The internal returns on risk and the elasticities of the demand for cash balances, computed according to *Equations* 16 and 17, are respectively depicted in the first and the second rows of *Figure* 8, when the cash-to-risk ratio is pegged to the level $\alpha = 1\%$. Notice that jumps in r_α and $\epsilon_r(r_\alpha)$ are produced around the episodes of crisis — in the same way that this events were distinguished by jumps in the preference for liquidity when the cost of capital was peg to the *Federal Funds Rate* in the previous section. In fact, around October 1987, r_α drastically rose to a historical maximum, while $\epsilon_r(r_\alpha)$ suddenly felt to a historical minimum. Besides, the signal appeared in both indexes and its influence was short-lasting, in accordance to the empirical facts characterising this crisis. Regarding the episode of April 2000, the times of turbulence are located on a broader interval, with a clear predominance of the signals related to the *Nasdaq*, in consistence with the fact that only this index crashed on that date, and the most unexpected movements are produced precisely in April 2000.

We can then conclude that many of the facts observed in stock markets can be satisfactorily explained within the theoretical framework proposed in this paper. Besides, both the configurations of the risk-parameters μ and σ, as well as the values of the liquidity-preference function and the $IRRR$, can be corresponded to specific states of the market, in such a way that, in particular, episodes of crisis can be precisely described. However, although in this context $k_{\mu,\sigma}$ and the $IRRR$ can be regarded as *signals* of liquidity adjustments, such signals only appear *after* the corresponding events have been produced and hence they cannot be used to make predictions. This is a consequence of the fact that only a *partial* equilibrium approach has been followed in this paper and hence the equilibrium ultimately depends on variables whose evolution is unknown (as it is the case with L, μ and σ) and have to be treated in practice as *exogenous* variables. Although this is certainly an important limitation, the model still provides a precise definition of *liquidity* and of *instability*, capable of accounting for the behaviour of capital markets both under *normal* circumstances and in times of turbulence — as illustrated with the empirical analysis so far developed.

9   Conclusions

As long as in *perfect* capital markets funds can be lend and borrow without restrictions at a single and exogenously determined interest rate, financial institutions can always adjust their surpluses and accordingly, their market values do not depend on the implemented capital structures (Modigliani and Miller, 1958). However, this mechanism does not work efficiently in markets of contractual liabilities established by opaque intermediaries with default sensitive customers (Merton, 1974, 1977 and 1997). Within this framework, the role of financial institutions is that of providing the services of hedging and diversification to customers that cannot trade by themselves. Optimal option-based arrangements, contingent on the outstanding surplus (equal to difference between the outstanding assets and liabilities) at the maturity date, can then be issued to assure that the value of the firm is maximised and the cost of insolvency is minimised.

An alternative approach is proposed in this paper for the characterisation of markets

where intermediaries can only hedge and diversify *up to a limited extent* and are accordingly obliged to rely on deposit insurance and capital cushions. The optimal contracts are established in actuarial terms, in such a way that the expected insured capital gain of the intermediary's portfolio is maximised and the cost of bankruptcy is minimised (see *Equations* 6 and 8), while the optimal liquidity demands are expressed as the *quantile* function of the probability distributions describing risks (as in *Equations* 7, 9 and 10).

The level of reserves demanded by the market is thereby characterised by the sum of the surpluses demanded by investors, intermediaries and customers. As a result, the market expected returns and volatilities are expressed as the weighted averages of the mean returns and volatilities of the individual portfolios (see *Equation* 11). The role of investors when deciding their balances in deregulated markets can then be related to that of subsidiaries in *decentralised* organisations, where an internal cost of capital is given and the total surplus is equal to the sum of the divisional contributions (see Merton and Perold, 1993, and also Mierzejewski, 2006).

More generally, when the supply of reserves is controlled by creditors, the cost of capital is endogenously determined in order to satisfy the equilibrium in the market of cash balances (as in *Equations* 12 and 13). In this context, the market interest rate represents the return obtained when investing one monetary unit on risk and can be accordingly regarded as the *Internal Rate of Return on Risk* (*IRRR*). Given any fixed pair of liquidity supply and balances demanded for transactions, a risk-structure of interest rates is determined in the plane of expected returns and volatilities (see *Figures* 2 and 3).

In conclusion, variations in the rate of growth of the funds spent on transactions or in the stock of money must be necessarily followed by market adjustments reflected in the relative prices of securities. Up to some extent, such movements will be compensated by the internal return on risk, but also the mean return and the volatility of the market portfolio are expected to be affected, for these parameters completely characterise risks in a Gaussian setting. Different scenarios are thus corresponded to different combinations of the involved parameters, some of which can be related to more stable and others to more vulnerable states. In this way, both *normal* times, when credit is plentiful, as well as times of financial turmoil, are possible in the model, depending on the flows of capital and the paths followed by the risk parameters. The equilibrium mechanism of capital markets is thus essentially *physical* in nature — and not only affected by *psychological* and informational shocks. The predictions of the model can be verified by analysing the historical data.

References

[1] Choi, W.G., and Oh, S., 2003. A Money Demand Function with Output Uncertainty, Monetary Uncertainty, and Financial Innovations. *Journal of Money, Credit and Banking* **35** (5), pp. 685-709.

[2] Dhaene, J., Goovaerts, M. and Kaas, R., 2003. Economic Capital Allocation Derived from Risk Measures. *North American Actuarial Journal* **7** (2), pp.44-59.

[3] Friedman, M., 1970. A Theoretical Framework for Monetary Analysis. *The Journal of Political Economy* **78** (2), pp. 193-238.

[4] Goovaerts M.J., Van den Borre E. and Laeven R., 2005. Managing economic and virtual economic capital within financial conglomerates. *North American Actuarial Journal* **9** (3), pp.77-89.

[5] Holmstrom, B., and Tirole, J., 2000. Liquidity and Risk Management. *Journal of Money, Credit and Banking* **32** (3), pp.295-319.

[6] Howells, P. and Bain, K., 2005. *The Economics of Money, Bankig and Finance.* Prentice Hall.

[7] Hull, J.C., 2000. *Options, Futures and Other Derivatives.* Fourth Edition. Prentice Hall.

[8] Jensen, M.C., 1986. Agency Costs of Free Cash Flow, Corporate Finance, and Takeovers. *The American Economic Review* **76** (2), pp. 323-329.

[9] Merton, R.C., 1974. On the Pricing of Corporate Debt: The Risk-Structure of Interest Rates. *Journal of Finance* **29** (2), pp. 449-470.

[10] Merton, R.C., 1977. An Analytic Derivation of the Cost of Deposit Insurance and Loan Guarantees. *Journal of Banking and Finance* **1**, pp.3-11.

[11] Merton, R.C., 1997. A Model of Contract Guarantees for Credit-Sensitive, Opaque Financial Intermediaries. *European Finance Review* **1** (1), pp.1-13.

[12] Merton, R.C. and Perold, A.F. (1993). Theory of Risk Capital in Financial Firms. *Journal of Applied Corporate Finance* **5**, pp. 16-32.

[13] Mierzejewski, F., 2006. Optimal Capital Allocation Confronting Bankruptcy and Agency Costs. *Bank- en Financiewezen* **2006/2**, pp. 72-77.

[14] Modigliani, F. and Miller, M.H., 1958. The Cost of Capital, Corporation Finance and the Theory of Investment. *The American Economic Review* **48** (3), pp. 261-297.

[15] Sharpe, W.F., 1966a. Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *The Journal of Finance* **19** (3), pp. 425-442.

[16] Sharpe, W.F., 1966b. Mutual Fund Performance. *The Journal of Business* **39** (1), pp. 119-138.

[17] Tobin, J., 1958. Liquidity Preference as Behavior Towards Risk. *The Review of Economic Studies* **25** (2), pp. 65-86.