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Abstract

The welfare effects of regulation are of crucial importance to policy makers. To this end, we present a model of n firms with differentiated costs competing in a linear market within the framework of spatial price discrimination. We prove that the Nash equilibrium locations of firms are always socially optimal irrespective of the number of competitors, the distribution of consumers, firms' cost heterogeneity, the level of privatization and the number and/or the varieties of the produced goods. We also provide an algorithm on how to find the unique Nash equilibrium in the case of uniformly distributed consumers.

Keywords: Mixed oligopoly; Social optimality; Spatial competition; Differentiated goods
JEL classification: L13; L32; L33; R32

1 Introduction

Individuals residing in different European countries buying the *Wall Street Journal Europe*, are witnessing a form of market segmentation due to discriminatory pricing dependent on geographical location. This pricing practice is called spatial price discrimination (Cabral, 2000). However, this is not the only market where this type of pricing is common. Spatial

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price discrimination manifests itself in markets in which firms are geographically differentiated[1] such as the markets of cement and steel or markets of customer-tailored goods. The wide application of this pricing strategy[2] together with the fact that it is forbidden by some countries when it cannot be justified on the grounds of transportation/delivery costs (e.g., Robinson-Patman Act, 1936 in the US),[3] makes the investigation of spatial price discrimination of great interest for both academics and policy makers.

The main goal of the current paper is to examine the welfare properties of the equilibrium in a market where operating firms exercise spatial price discrimination. To this purpose, we develop an integrated model giving new insight into the structure of customer-specific pricing markets. The existing literature adopts at least one of the following five assumptions: (i) the number of firms in the market does not exceed two (ii) all firms are privately owned (iii) only one homogeneous good is traded (iv) consumers are uniformly distributed and (v) firms have common marginal production costs. We relax all the above assumptions by assuming a market with an arbitrary number of heterogeneous competitors, an arbitrary distribution of consumers, an arbitrary level of privatization for each firm and an arbitrary number and/or varieties of traded goods. Firm heterogeneity is reflected by assuming different marginal costs of production. It should be emphasized that this heterogeneity is firm-specific and not product-specific (i.e., the marginal cost of production differs across firms but remains the same for all goods/varieties produced by the same firm). The effect of the above mentioned

[1] Greenhut (1981) provides evidence that spatial price discrimination is apparent in cases where transportation cost represents at least 5% of total costs.

[2] Such pricing strategy can also be apparent in cases where output is differentiated according to an ordered product characteristic (Schmalensee and Thisse, 1988) (e.g., the different departure time of airline flights between two destinations).

[3] For a review about the history of the enforcement of competition law against spatial discriminatory pricing, see Scherer and Ross (1980).

relaxation on the properties of the equilibrium is not clear. For example, Cremer et al. (1991) highlighted the importance of the number of competing firms on the welfare properties of the equilibrium.

The present paper contributes to the existing literature in many ways. We show that in a model of spatial price discrimination with linear transportation costs, where the produced goods have the same reservation price for the buyers, the market outcome will be socially optimal, and this result is independent of the number of firms in the market, the distribution of consumers, firm heterogeneity regarding marginal production costs, the level of privatization of each firm and the number and/or the varieties of the goods offered by each competitor. We further argue that when consumers are uniformly distributed, there is a unique Nash equilibrium which does not depend on the distribution of the marginal production costs. We also provide an algorithm for computing the Nash equilibrium locations in this case.

The driving force behind our welfare result is the same as in Lederer and Hurter (1986); a firm can increase its profit by opting for a production location so that the market is serviced with minimal total cost.[4] However, in Lederer and Hurter (1986) the discussion is restricted to only two exclusively privately owned firms offering the same good leaving untouched mixed markets with many competitors and multiple goods and the ensuing welfare questions. Moreover, our proof is completely different to the one found in Lederer and Hurter (1986) allowing for direct generalization.

Our findings about the characterization of the equilibrium are also in line with Vogel (2011). Vogel (2011) proves the existence of a unique equilibrium for an arbitrary number of privately owned heterogeneous firms located in a circular market (with uniformly distributed

[4]This implies the alignment of the social and private optima.

consumers) competing within the framework of spatial price discrimination bearing linear transportation costs, regardless of the distribution of their marginal production costs. We investigate the properties of the equilibrium, should it exist, when an arbitrary number of heterogeneous firms with various degrees of privatization offering multiple goods compete in a linear market under spatial price discrimination. In line with Vogel (2011), we prove the existence of a unique equilibrium when consumers are uniformly distributed. We succeed, however, in establishing the social optimality of the Nash equilibrium in all cases.

Moreover, the fact that our model imposes no constraints on the level of privatization extends our contribution to the theory of mixed oligopoly under spatial price discrimination. The studies which are closest to ours are those of Heywood and Ye (2009a) and Heywood and Ye (2009b). Heywood and Ye (2009a) assume a market with an arbitrary number of homogeneous firms having binary ownership status (private or public) and focus on the role of the public firm in the Stackelberg equilibrium where the leader is a private or a public firm. They extend their model accounting for the existence of foreign firms in Heywood and Ye (2009b). The aforementioned papers impose too many restrictions in their modeling structure. Specifically, the framework of Heywood and Ye (2009a) and Heywood and Ye (2009b) imposes restrictions on the distribution of consumers, the degree of privatization and the attributes/number of goods in the market. In addition, a fundamental restriction of the aforementioned papers lies in the assumption of common marginal production costs.

Our paper is part of a wide literature on location analysis[5] which examines the welfare implications of spatial price discrimination; see, e.g., Greenhut and Ohta (1972), Holahan (1975), Thisse and Vives (1988), Hamilton et al. (1989), Hamilton et al. (1991), MacLeod et

[5]For a comprehensive review of the location analysis literature, see ReVelle and Eiselt (2005).

al. (1992), Claycombe (1996) and Braid (2008). Building on this literature, we make inroads into the theory of mixed oligopoly when firms have differentiated marginal production costs and the market is characterized by perfectly inelastic demand, as in Lederer and Hurter (1986).

The implications of our results can be summarized as follows: (i) A spatial price discriminatory market can serve as a typical example of how a 'laissez-faire' economy can lead to social optimality,[6] (ii) the social optimality of the equilibrium is independent of firm heterogeneity and (iii) the dependence of the equilibrium locations on the relative difference of the marginal production costs has important policy implications for government intervention. In the example of subsection 4.2.1 the government can (pre)determine the locations of firms through intervention on the marginal production costs (e.g., tax incentives and subsidies for firms located in low-populated areas etc.) without affecting social optimality.

Moreover, our results have important policy implications for the welfare effects of the recent privatization wave[7] and the recent tendency of states to buy private shares to prevent hostile bids. Specifically, the private outcome will be socially optimal irrespective of the degree of privatization in markets with inelastic demand (e.g., demand for gasoline in the short-run) where firms exercise spatial price discrimination. However, it should be noted that sequential decision making and vertically linked markets are not accounted for in our framework.

The rest of the paper is structured as follows. The next section presents the benchmark

[6] Anderson and Engers (1994) showed that in a spatial setting with inelastic demand there exists a unique price-taking equilibrium. However, even in this case social optimality can be achieved only if firm locations are regulated. In contrast, the social optimality of our results is independent from any regulatory intervention.

[7] See the Privatization Barometer Report 2014/2015 for more information about privatization trends (http://www.feem.it/userfiles/attach/2015112392244PB_Annual_Report_2014-2015.pdf).

model where an arbitrary number of privately owned firms offer a homogeneous good. The market is represented by a unit interval with the consumers arbitrary distributed along it. A three-stage game of complete information is played by firms and consumers. More specifically, in the first stage firms simultaneously choose their locations. In the second stage, after observing their competitors' locations, firms engage in Bertrand competition à la Hoover (1937) and Lerner and Singer (1937). In other words, firms set their prices simultaneously and are allowed to price discriminate by charging a different price for different locations. Finally, in stage three, consumers make their purchasing choices to clear the market. After presenting our theoretical construct, we solve the game and characterize the Nash equilibrium. The case of mixed oligopoly is presented in section 3. Section 4 generalizes the findings of section 3 for the case of multiple goods (or different varieties of the same good) and section 5 concludes.

2 Modelling the competition

We consider a market consisted of n private firms and a continuum of consumers distributed according to a continuous distribution density function g over the unit interval $[0, 1]$ representing a linear country. Let x_i, $i = 1, ..., n$, denote the location of firm i in the interval $[0, 1]$ with $0 \leq x_1 < x_2 < ... < x_n \leq 1$. All firms produce and sell the same homogeneous good. Each consumer has perfectly inelastic demand buying one unit of the good from the lowest price firm, providing that this price is lower or equal to her reservation price (i.e., the maximum price that the consumers are willing to pay for the good), $m > 0$. The marginal production cost of firm i is $c_i \geq 0$. Spatial price discrimination à la Hoover (1937) and Lerner and Singer (1937) is assumed. Specifically, the price charged for the good by the firm the consumer chooses to buy from, is equal to (or infinitesimally less than) the delivered cost of the remaining firms. Delivered costs are equal to the sum of transportation and production costs. Let $td(x_2, x_1) := t(|x_2 - x_1|)$ evaluate the transportation cost between points x_1 and x_2, where d denotes the shipped distance and $t > 0$ is the transport cost per unit of distance. Firms are located such that $td(x_{i+j}, x_i) > |c_{i+j} - c_i|$ (non-negative profit condition)[8] for any $j > 0$ with $i, i+j \in \{1, ..., n\}$ and c_{i+j}, c_i the corresponding marginal costs of firms $i+j$ and i located at points x_{i+j} and x_i respectively. Consumers and firms engage in a three-stage game of complete information. In stage one, firms simultaneously decide their location. Having observed the location of their competitors, firms simultaneously choose delivered price schedules in the second stage. In the final stage, consumers take their purchasing decisions.

Let $s_{i,i+j}$ denote the locations of the indifferent consumer with respect to firms i and $i+j$. The following Lemma essentially determines the location of the indifferent consumer relative to the locations of the firms; its proof is deferred for the Appendix.

Lemma 1. *(i) $x_i < s_{i,i+j} < x_{i+j}$ and (ii) if $j_1 < j_2$ then (a) $s_{i,i+j_1} < s_{i,i+j_2}$ and (b) provided $j_1 < j_2 < j$, $s_{i+j_1,i+j} < s_{i+j_2,i+j}$.*

The aggregate delivered cost for all locations z of consumers who buy from any of the n firms is equal to

$$T(x_1, ..., x_n) = \sum_{i=1}^{n} T_i(x_1, ..., x_n), \tag{1}$$

where

[8] In the opposite case if, $td(x_{i+j}, x_i) \leq |c_{i+j} - c_i|$, the total sales of either firm i or firm $i+j$ drop to zero.

$$T_i(x_1,...,x_n) = \begin{cases} \left(\begin{array}{c} \int_0^{x_1} [td(x_1,z)+c_1]g(z)dz \\ + \int_{x_1}^{s_{1,2}} [td(z,x_1)+c_1]g(z)dz \end{array} \right) & \text{for } i=1 \\ \\ \left(\begin{array}{c} \int_{s_{i-1,i}}^{x_i} [td(x_i,z)+c_i]g(z)dz \\ + \int_{x_i}^{s_{i,i+1}} [td(z,x_i)+c_i]g(z)dz \end{array} \right) & \text{for } 1<i<n \\ \\ \left(\begin{array}{c} \int_{s_{n-1,n}}^{x_n} [td(x_n,z)+c_n]g(z)dz \\ + \int_{x_n}^{1} [td(z,x_n)+c_n]g(z)dz \end{array} \right) & \text{for } i=n, \end{cases} \tag{2}$$

is the total delivered cost for those consumers buying from firm i. Using (2), we get

$$\begin{aligned} T(x_1,...,x_n) = & \int_0^{x_1} [td(x_1,z)+c_1]g(z)dz + \int_{x_1}^{s_{1,2}} [td(z,x_1)+c_1]g(z)dz + ... \\ & + \int_{s_{i-2,i-1}}^{x_{i-1}} [td(x_{i-1},z)+c_{i-1}]g(z)dz + \int_{x_{i-1}}^{s_{i-1,i}} [td(z,x_{i-1})+c_{i-1}]g(z)dz \\ & + \int_{s_{i-1,i}}^{x_i} [td(x_i,z)+c_i]g(z)dz + \int_{x_i}^{s_{i,i+1}} [td(z,x_i)+c_i]g(z)dz \\ & + \int_{s_{i,i+1}}^{x_{i+1}} [td(x_{i+1},z)+c_{i+1}]g(z)dz + \int_{x_{i+1}}^{s_{i+1,i+2}} [td(z,x_{i+1})+c_{i+1}]g(z)dz \\ & +... + \int_{s_{n-1,n}}^{x_n} [td(x_n,z)+c_n]g(z)dz + \int_{x_n}^{1} [td(z,x_n)+c_n]g(z)dz. \end{aligned} \tag{3}$$

As defined in Lederer and Hurter (1986) the social cost is the total supply cost when firms behave in a cooperative, cost minimizing manner. The socially optimal locations can be derived by minimizing the social cost with respect to each location x_i. In other words, social welfare is defined as the total consumer's willingness to pay less the aggregate transportation and production costs.

Firm i is selling its product at a price matching (or which is infinitesimally less than) the delivery cost of its direct competitor which is the firm nearest to its location. The indifferent consumer between firms i and $i+1$, according to Lemma 1, is located at $x_i < s_{i,i+1} < x_{i+1}$. Hence, the market share of firm i, $1 < i < n$, is $[s_{i-1,i}, s_{i,i+1}]$. The pricing strategy of firm i is determined by the following Lemma whose proof can be found in the Appendix.

Lemma 2. *(i)* $s_{i-1,i+1} \in (s_{i-1,i}, s_{i,i+1})$ *and (ii) the prices charged by firm* i*,* $1 < i < n$*, are* $td(z, x_{i-1}) + c_{i-1}$ *for* $z \in [s_{i-1,i}, s_{i-1,i+1}]$ *and* $td(x_{i+1}, z) + c_{i+1}$ *for* $z \in [s_{i-1,i+1}, s_{i,i+1}]$.

The pricing strategy of firms 1 and n can be derived along similar lines.

Thus, the profit function of firm i is

$$\Pi_i(x_1, .., x_n) = \begin{cases} \begin{array}{l} \int_0^{x_1} [td(x_2, z) + c_2 - td(x_1, z) - c_1] g(z) dz \\ + \int_{x_1}^{s_{1,2}} [td(x_2, z) + c_2 - td(z, x_1) - c_1] g(z) dz \end{array} & \text{if } i = 1 \\ \\ \begin{array}{l} \int_{s_{i-1,i}}^{s_{i-1,i+1}} [td(z, x_{i-1}) + c_{i-1} - td(x_i, z) - c_i] g(z) dz \\ + \int_{s_{i-1,i+1}}^{s_{i,i+1}} [td(x_{i+1}, z) + c_{i+1} - td(z, x_i) - c_i] g(z) dz \\ \text{if } 1 < i < n \end{array} & \\ \\ \begin{array}{l} \int_{s_{n-1,n}}^{x_n} [td(z, x_{n-1}) + c_{n-1} - td(x_n, z) - c_n] g(z) dz \\ + \int_{x_n}^{1} [td(z, x_{n-1}) + c_{n-1} - td(z, x_n) - c_n] g(z) dz \end{array} & \text{if } i = n. \end{cases} \tag{4}$$

The next result relates the marginal aggregate delivered cost with respect to location i to the the marginal profit of firm i and lies in the heart of what follows.

Proposition 1. *The marginal aggregate delivered cost with respect to the location of firm* i*,*

$i = 1, ..., n$, is opposite to the marginal profit of firm i, i.e.

$$\partial T(x_1, ..., x_n)/\partial x_i = -\partial \Pi_i(x_1, ..., x_n)/\partial x_i.$$

Proof. See Appendix $\square$

Following our discussion above, the socially optimal locations are derived by minimizing (3) with respect to each firm's location. Hence, the socially optimal locations satisfy the system:

$$\partial T(x_1, ..., x_n)/\partial x_i = 0,\ i = 1, ..., n. \tag{5}$$

whereas Nash equilibrium locations satisfy

$$\partial \Pi_i(x_1, ..., x_n)/\partial x_i = 0,\ i = 1, ..., n. \tag{6}$$

Proposition 1 ensures that the system of (5) and (6) are equivalent leading to

Proposition 2. *In models of spatial price discrimination where firms offer the same good to consumers having perfectly inelastic demands, the Nash equilibrium locations of firms are socially optimal.*

3 Mixed oligopoly

In our analysis so far, all firms are privately owned. Let us now assume that single firm l, $l = \{1, ..., n\}$ is partly privately owned and partly publicly owned in proportions a_l and

$1 - a_l$ (in other words a_l can be considered as the degree of privatization), respectively with $a_l \in [0,1]$. In such a case, firm l will decide about its optimal location by maximizing the weighted average of its own profits and social welfare with weights a_l and $1-a_l$, respectively. Social welfare is equal to the sum of the aggregate profits (the profit of all firms) and consumers' surplus. The consumers' surplus is given by

$$CS(x_1, ..., x_n) = \sum_{i=1}^{n} CS_i(x_1, ..., x_n),$$

where $CS_i(x_1, ..., x_n)$ is the consumer surplus generated for the consumers buying from firm i, therefore,

$$CS_i(x_1, ..., x_n) = \begin{cases} \begin{array}{c} \int_0^{x_1} [m - td(x_2, z) - c_2] g(z) dz \\ + \int_{x_1}^{s_{1,2}} [m - td(x_2, z) - c_2] g(z) dz \end{array} & \text{for } i = 1 \\ \\ \begin{array}{c} \int_{s_{i-1,i}}^{s_{i-1,i+1}} [m - td(z, x_{i-1}) - c_{i-1}] g(z) dz \\ + \int_{s_{i-1,i+1}}^{s_{i,i+1}} [m - td(x_{i+1}, z) - c_{i+1}] g(z) dz \end{array} & \text{for } 1 < i < n \\ \\ \begin{array}{c} \int_{s_{n-1,n}}^{x_n} [m - td(z, x_{n-1}) - c_{n-1}] g(z) dz \\ + \int_{x_n}^{1} [m - td(z, x_{n-1}) - c_{n-1}] g(z) dz \end{array} & \text{for } i = n. \end{cases}$$

Direct calculation proves

$$
\textbf{Lemma 3.}\ \Pi_i(x_1,...,x_n) + CS_i(x_1,...,x_n) = \begin{cases} \int_0^{s_{1,2}} mg(z)dz - T_1(x_1,...,x_n) \\ \qquad \text{for } i = 1 \\ \\ \int_{s_{i-1,i}}^{s_{i,i+1}} mg(z)dz - T_i(x_1,...,x_n) \\ \qquad \text{for } 1 < i < n \\ \\ \int_{s_{n-1,n}}^{1} mg(z)dz - T_n(x_1,...,x_n) \\ \qquad \text{for } i = n. \end{cases}
$$

Summing up over all firms one gets the following Proposition which could be viewed as the first main result of this section.

Proposition 3.

$$
\sum_{i=1}^{n} \Pi_i(x_1,...,x_n) + CS(x_1,...,x_n) = m - T(x_1,...,x_n)
$$

Proof. Straightforward calculations. □

The profit function of the partly publicly owned firm l when marginal costs of production are different will be

$$
\begin{aligned} \breve{\Pi}_l(x_1,...,x_n) &= \Pi_l(x_1,...,x_n) \\ +(1-a_l) &\left[\sum_{i\neq l} \Pi_i(x_1,...,x_n) + CS(x_1,...,x_n)\right], \end{aligned} \tag{7}
$$

where Π_l would be the profit function of firm l if it was fully privately owned.

Proposition 4. *Nash equilibria remain socially optima regardless of the degree of privatization of the individual firms $l, 1 \leq l \leq n$ and/or their marginal production costs.*

Proof. Fix a random $l, 1 \leq l \leq n$. Using Proposition 3 and (7), we get

$$\breve{\Pi}_l(x_1, ..., x_n) = \Pi_l(x_1, ..., x_n) + (1 - a_l) \left[m - T(x_1, ..., x_n) - \Pi_l(x_1, ..., x_n)\right].$$

From Proposition 1

$$\partial T / \partial x_l = -\partial \Pi_l / \partial x_l \Longleftrightarrow -\partial T / \partial x_l - \partial \Pi_l / \partial x_l = 0,$$

which implies that $\partial \breve{\Pi}_l / \partial x_l = \partial \Pi_l / \partial x_l$. Induction on i completes the proof. □

4 Invariance under multiple goods

4.1 Private firms

We now assume the existence of L different goods or different varieties of the same good or both. Let k_j denote the number of firms producing good j, $j = 1, ..., L$ with $1 \leq k_j \leq n$. Let T^j denote the aggregate transportation cost related to the provision of good j and Π_i^j the corresponding profit per consumer of firm i from selling good j with $\Pi_i^j := 0$ if good j is not produced by firm i. The fraction of consumers buying product j is now denoted by $h_j \in (0, 1]$ with $\sum_{j=1}^{L} h_j = 1$; hence, there will be buyers for all available products. In the case where good j is produced by only one firm, then this firm enjoys monopoly privileges and charges a price equal to, or infinitesimally smaller than, the reservation price m_j, i.e. the maximum price the consumer is willing to pay for good j. A fundamental assumption in this multi-good setting is that $m_1 = ... = m_L = m$ (i.e., the reservation price of all goods is identical).[9] Let $\tilde{T}$ denote the aggregate delivered cost for all products and $\tilde{\Pi}_i$ the total profit of firm i for all products it produces.

Proposition 5. *The marginal aggregate delivered cost with respect to the location of firm i is opposite to the marginal profit of firm i, namely* $\partial\tilde{T}/\partial x_i = -\ \partial\tilde{\Pi}_i/\partial x_i$.

Proof. By definition $\tilde{T} = \sum_{j=1}^{L} h_j T^j$ and $\tilde{\Pi}_i = \sum_{j=1}^{L} h_j \Pi_i^j$. Applying Proposition 1 for every single traded product j we get

$$\partial\tilde{T}/\partial x_i = \sum_{j=1}^{L} h_j \partial T^j/\partial x_i = -\sum_{j=1}^{L} h_j \partial\Pi_i^j/\partial x_i = -\partial\tilde{\Pi}_i/\partial x_i.$$

□

Theorem 1. *In models of spatial price discrimination, where firms have different marginal production costs, produce different combination of goods, transportation costs are linear, consumers are arbitrary distributed along a linear city of unit length and have the same reservation price for all goods, the Nash equilibrium locations of firms are socially optimal.*

Proof. To derive the socially optimal locations we have to minimize $\tilde{T}$ with respect to each firm's location. Hence, the socially optimal locations satisfy the following system of equations:

$$\partial\tilde{T}/\partial x_i = 0,\ \ i = 1, ..., n. \tag{8}$$

[9] It should be noted that this assumption is more realistic in the case of the different varieties of the same good and less in the case of different goods.

On the other hand, the Nash equilibrium locations are given by the solution of the following system:

$$\partial\tilde{\Pi}_i/\partial x_i = 0,\ i = 1, ..., n. \tag{9}$$

Because of Proposition 5, systems (8) and (9) are equivalent and hence they have the same set of solutions. $\square$

4.2 Mixed oligopoly

Let's now turn to the case where some firm, say firm l is partly privately owned and partly publicly owned. In consistency with the notation developed in section 3 and subsection 4.1, let $\breve{\Pi}_l = \sum_{j=1}^{L} h_j \breve{\Pi}_l^j$ where $h_j \breve{\Pi}_l^j$ be the profit of the partially privatized firm l from selling good j and $h_j \in (0, 1]$ be the fraction of consumers buy product j. It is understood that $\breve{\Pi}_l^j = 0$ if good j is not produced by firm l.

Theorem 2. *The degree of privatization does not affect the socially optimal Nash equilibrium locations.*

Proof. From the proof of Proposition 4, we have that for every single product j

$$\partial\breve{\Pi}_l^j/\partial x_l = \partial\Pi_l^j/\partial x_l\,.$$

Therefore,

$$\partial\breve{\Pi}_l/\partial x_l = \sum_{j=1}^{L} h_j \partial\breve{\Pi}_l^j/\partial x_l = \sum_{j=1}^{L} \partial\Pi_l^j/\partial x_l = \partial\tilde{\Pi}_i/\partial x_i.$$

$\square$

The existence of the equilibrium is affected by the distribution of consumers. Specifically, we prove:

Theorem 3. *In a mixed oligopoly of n firms, with $n \geq 3$, producing different combinations of goods with differentiated marginal production costs, uniformly distributed consumers and linear transportation costs, there exists a unique socially optimal Nash equilibrium of locations for any $(c_1, ..., c_n)$ in the non bounded subset C, of the positive orthant $\mathbb{R}^n_+$, defined by the inequalities*

$$(n-2)c_i + (n-2)c_{i+1} - 2 \sum_{j=1, j \neq i, i+1}^{n} c_j < 1, \tag{10}$$

$$2(n-1)c_1 - 2\sum_{j=2}^{n} c_j < 1, \tag{11}$$

and

$$-2\sum_{j=1}^{n-1} c_j + 2(n-1)c_n < 1. \tag{12}$$

for $i = 1, ..., n$. Further any two marginal cost vectors $(c_1, ..., c_n)$ and $(c_1^{'}, ..., c_n^{'})$ in the subset C, such that $c_i' = c_i + u$ lead to the same equilibrium locations.

Proof. We prove Theorem 3 in the simplest possible setting that of private firms producing only one common good assuming the per distance transportation cost, t, equal to one. The general case for the mixed oligopoly with multiple goods and $t \neq 1$ can then be proved along similar to the analysis above lines.

According to Proposition 2, the optimal locations must satisfy the system

$$\begin{array}{lllllll}
3x_1 & -x_2 & & & = & -c_1 \;\; +c_2 & \\
-x_1 & +2x_2 & -x_3 & & = & -c_1 \qquad +c_3 & \\
& -x_2 & +2x_3 & -x_4 & = & \qquad -c_2 \qquad +c_4 & \\
& & \vdots & & & \qquad\qquad \vdots & \\
& & & -x_{n-1} \;\; +3x_n & = & & -c_{n-1}+c_n+2
\end{array}$$

It is straightforward to check that the above system is row equivalent to

$$\begin{array}{lllll}
3x_1 & -x_2 & & = & -c_1+c_2 \\
& \frac{5}{3}x_2 & -x_3 & = & -\frac{4}{3}c_1+\frac{1}{3}c_2+c_3 \\
& & \frac{7}{5}x_3 \;\; -x_4 & = & -\frac{4}{5}c_1-\frac{4}{5}c_2+\frac{3}{5}c_3+c_4 \\
& & \vdots & & \vdots \\
& & \frac{4n}{2n-1}x_n & = & -\frac{4}{2n-1}c_1-\frac{4}{2n-1}c_2-\ldots-\frac{4}{2n-1}c_{n-1}+\frac{4(n-1)}{2n-1}c_n+2
\end{array}$$

where the i-line $1 < i < n$ is given by

$$\frac{2i+1}{2i-1}x_i - x_{i+1} = -\frac{4}{2i-1}c_1 - \frac{4}{2i-1}c_2 - \ldots - \frac{4}{2i-1}c_{i-1} + \frac{2i-3}{2i-1}c_i + c_{i+1}\,.$$

Solving for x_i we get

$$\begin{aligned}
x_i \;\; = \;\; & \frac{2i-1}{2i+1}[x_{i+1} + \frac{4}{2i-1}(c_i - c_1) + \ldots + \frac{4}{2i-1}(c_i - c_\rho) \\
& + \frac{\upsilon}{2i-1}(c_i - c_{\rho+1}) + \frac{4-\upsilon}{2i-1}(c_{i+1} - c_{\rho+1}) \\
& + \frac{4}{2i-1}(c_{i+1} - c_{\rho+2}) + \ldots + \frac{4}{2i-1}(c_{i+1} - c_{i-1})],
\end{aligned}$$

where $2i-3=4\rho+\upsilon$, $0<\upsilon<4$.

Inherent to the discussion leading to Proposition 2 was the assumption that $x_1 < x_2 < \ldots < x_n$. It is a straightforward, albeit tedious, calculation to show that

$$x_i < x_{i+1} \Longleftrightarrow (n-2)c_i + (n-2)c_{i+1} - 2 \sum_{j=1, j\neq i,i+1}^{n} c_j < 1\,.$$

Further, we get

$$0 < x_1 \Longleftrightarrow 2(n-1)c_1 - 2\sum_{j=2}^{n} c_j < 1$$

and

$$x_n < 1 \Longleftrightarrow -2\sum_{j=1}^{n-1} c_j + 2(n-1)c_n < 1\,.$$

To prove that the domain, C, defined by the above set of inequalities is not bounded it suffices to consider all n-tuples $(c_1,\ldots,c_n)$ with $c_1=\ldots=c_n$ (homogeneous case) on the positive part of the main diagonal of $\mathbb{R}^n_+$. For $n=2$, the situation is considerably simpler and is treated thoroughly in subsection 4.2.1. $\square$

4.2.1 Policy implications

To highlight the policy implications of our findings in subsection 4.2, we present an application for a duopoly with linear transportation costs. Let c_i denote the marginal production cost of firm i. There are three varieties of a differentiated product offered to consumers, U and W from firm 1 and V and W from firm 2. Let also the fraction of consumers buying only good U equal the fraction of consumers buying good V, with both set equal to e. Product W is bought by a fraction b of consumers. Transportation costs are linear and equal to td, where t is a positive scalar and d is the distance shipped. The locations of firm 1 and 2 over the interval $[0,1]$ are x_1 and x_2, respectively (without loss of generality $x_1 < x_2$). Keeping the structure of the game and the rest of the notation as above, the profit functions of firms 1 and 2 when both are privately owned are:

$$\tilde{\Pi}_1 = \left(e(m-c_1) - \tfrac{et}{2}\left[x_1^2 + (1-x_1)^2\right]\right) + \left(\begin{array}{c} \int_0^{x_1} b[t(x_2-x_1)+c_2-c_1]dz \\ + \int_{x_1}^{\left(\frac{x_1+x_2}{2}+\frac{c_2-c_1}{2t}\right)} b[t(x_1+x_2-2z)+c_2-c_1]dz \end{array}\right), \tag{13}$$

$$\tilde{\Pi}_2 = \left(\begin{array}{c} e(m-c_2) \\ -\frac{et}{2}\left[x_2^2+(1-x_2)^2\right] \end{array}\right) + \left(\begin{array}{c} \int_{\left(\frac{x_1+x_2}{2}+\frac{c_2-c_1}{2t}\right)}^{x_2} b[t(2z-x_1-x_2)+c_1-c_2]dz \\ + \int_{x_2}^{1} b[t(x_2-x_1)+c_1-c_2]dz \end{array}\right), \tag{14}$$

with $\frac{c_2-c_1}{2t} \leq \frac{x_2-x_1}{2}$.[10] The location s of the indifferent consumer for good W is determined by equating the two delivered costs in regard to the common good W: $t(x_2-s)+c_2 = t(s-x_1)+c_1 \Rightarrow s = \frac{x_1+x_2}{2} + \frac{c_2-c_1}{2t}$.

Having evaluated the integrals, (13) and (14) become

$$\begin{aligned} \tilde{\Pi}_1 = e(m-c_1) - \tfrac{et}{2}\left[x_1^2+(1-x_1)^2\right] \\ + bx_1[t(x_2-x_1)+c_2-c_1] \\ + \tfrac{b}{4t}[t(x_2-x_1)+c_2-c_1]^2, \end{aligned} \tag{13b}$$

[10] If $\frac{c_2-c_1}{2t} > \frac{x_2-x_1}{2}$, both firms are reduced to spatial-price discriminating monopolists where the common good W is now provided only by firm 1. We consider this case trivial and focus only on the case where $\frac{c_2-c_1}{2t} \leq \frac{x_2-x_1}{2}$.

$$
\begin{aligned}
\tilde{\Pi}_2 &= e(m-c_2) - \tfrac{et}{2}\left[x_2^2 + (1-x_2)^2\right] \\
&\quad + b(1-x_2)[t(x_2-x_1)+c_1-c_2] \\
&\quad + \tfrac{b}{4t}[t(x_2-x_1)+c_1-c_2]^2.
\end{aligned} \tag{14b}
$$

Firm 1 chooses x_1 to maximize (13b), and firm 2 chooses x_2 to maximize (14b), leading to the following Nash equilibrium locations

$$
(x_1, x_2) = \left(\frac{1}{2} - A + \omega, \frac{1}{2} + A + \omega\right), \tag{15}
$$

where $\omega = \frac{b(c_2-c_1)}{2t(b+2e)}$ and $A = \frac{b}{4(b+e)}$.

The total delivered cost will be equal to

$$
\begin{aligned}
\tilde{T} &= \tfrac{et}{2}\left[x_1^2 + (1-x_1)^2\right] + \tfrac{et}{2}\left[x_2^2 + (1-x_2)^2\right] + ec_1 + ec_2 \\
&+ \left(\int_0^{x_1} b[t(x_1-z)+c_1]dz + \int_{x_1}^{\left(\frac{x_1+x_2}{2}+\frac{c_2-c_1}{2t}\right)} b[t(z-x_1)+c_1]dz\right) \\
&+ \left(\begin{array}{c} \int_{\left(\frac{x_1+x_2}{2}+\frac{c_2-c_1}{2t}\right)}^{x_2} b[t(x_2-z)+c_2]dz \\ + \int_{x_2}^{1} b[t(z-x_2)+c_2]dz \end{array}\right).
\end{aligned} \tag{16}
$$

Maximizing (16) with respect to x_1 and x_2 gives the socially optimal locations

$$
\left(\frac{1}{2} - A + \omega, \frac{1}{2} + A + \omega\right). \tag{17}
$$

We now turn to the case where firm 2 is partly privately owned and partly publicly owned in proportions a_2 and $1-a_2$, respectively with $a_2 \in [0,1]$. In this case, the profits of firm 2 will be

$$
\begin{aligned}
\breve{\Pi}_2 = e(m-c_2) - \tfrac{et}{2}\left[x_2^2 + (1-x_2)^2\right] \\
+b(1-x_2)[t(x_2-x_1)+c_1-c_2] \\
+\tfrac{b}{4t}[t(x_2-x_1)+c_1-c_2]^2 + (1-a_2)v(x_1,x_2),
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
v(x_1,x_2) &= \left(e(m-c_1) - \tfrac{et}{2}\left[x_1^2+(1-x_1)^2\right]\right) \\
&+ \left(\begin{array}{c} \int_0^{x_1} b[t(x_2-x_1)+c_2-c_1]dz \\ + \int_{x_1}^{\left(\frac{x_1+x_2}{2}+\frac{c_2-c_1}{2t}\right)} b[t(x_1+x_2-2z)+c_2-c_1]dz \end{array}\right) \\
&+ \int_0^{\left(\frac{x_1+x_2}{2}+\frac{c_2-c_1}{2t}\right)} b[m-t(x_2-z)-c_2]dz \\
&+ \int_{\left(\frac{x_1+x_2}{2}+\frac{c_2-c_1}{2t}\right)}^1 b[m-t(z-x_1)-c_1]dz \\
&= \tfrac{(b+e)}{2}\left[2tx_1(1-x_1)+2m-t-2c_1\right].
\end{aligned}
\tag{19}
$$

It is straightforward to show that $\partial v(x_1,x_2)/\partial x_2 = 0$ showing that the equilibrium remains intact irrespective of the degree, a_2, of privatization.

Furthermore, the distance between the optimal locations, x_1 and x_2, is independent of marginal production costs and t and equals $2A$. It follows that anybody who wishes to influence the location x_1 of either firm 1, with $x_1 \in (0,1/2)$,[11] or the location of firm 2, x_2, with $x_2 \in (1/2,1)$, can do so by intervening on the marginal cost relative difference, $c_2 - c_1$. For example, given an a priori $X \in (0,1/2)$, it suffices to choose $c_1 < c_2$ in such a way that $c_2 - c_1 = \frac{2t(b+2e)}{b}(X - \frac{1}{2} + A)$ for firm 1 to locate optimaly on the given X.

[11] $x_1 \in (0,1/2)$ is implied by the fact that $\frac{c_2-c_1}{2t} \leq \frac{x_2-x_1}{2}$ (non-negative profit condition) and $x_2 - x_1 = 2A$.

5 Conclusion

We have proved that when firms exercise spatial price discrimination, the equilibrium outcome is socially optimal and independent of the underlying assumptions on the number of firms, firm heterogeneity, the distribution of consumers, the number or the varieties of the provided goods and the degree of privatization. To the best of our knowledge, our analysis is the first attempt to present an 'holistic' view of models of spatial price discrimination. Moreover, our findings verify the robustness of the 'laissez-faire' doctrine and can be easily applied to the case of vertically related markets (see Eleftheriou and Michelacakis, 2016). It is also not hard to deduce the validity of our findings for 'discontinued' markets where specific locations are ruled out. Possible extensions could investigate sequential decision making, strategic delegation effects and spatial two dimensional markets.

Appendix

Proof of Lemma 1

By definition of $s_{i,i+j}$,

$$td(s_{i,i+j}, x_i) + c_i = td(x_{i+j}, s_{i,i+j}) + c_{i+j}.$$

If $s_{i,i+j} \in [0, x_i] \cup [x_{i+j}, 1]$ then

$$td(x_{i+j}, x_i) = |c_{i+j} - c_i|,$$

a contradiction, thus, $x_i < s_{i,i+j} < x_{i+j}$ which proves *(i)*.

To prove *(ii) (a)* we argue by contradiction. From the proof of (i), we get that

$$td(s_{i,i+j_1}, x_i) + c_i = td(x_{i+j_1}, s_{i,i+j_1}) + c_{i+j_1} \tag{A.1}$$

$$td(s_{i,i+j_2}, x_i) + c_i = td(x_{i+j_2}, s_{i,i+j_2}) + c_{i+j2}. \tag{A.2}$$

If $s_{i,i+j_1} \geq s_{i,i+j_2}$, then

$$td(s_{i,i+j_2}, x_i) < td(s_{i,i+j_1}, x_i), \tag{A.3}$$

and

$$td(x_{i+j_2}, s_{i,i+j_1}) \ < td(x_{i+j_2}, s_{i,i+j_2}). \tag{A.4}$$

From (A.3)

$$\begin{aligned} td(s_{i,i+j_2}, x_i) + c_i \quad &< \quad td(s_{i,i+j_1}, x_i) + c_i \overset{(A.1),\,(A.2)}{\Rightarrow} \\ td(x_{i+j_2}, s_{i,i+j_2}) + c_{i+j_2} \quad &< \quad td(x_{i+j_1}, s_{i,i+j_1}) + c_{i+j_1}. \end{aligned} \tag{A.5}$$

Hence

$$td(x_{i+j_2}, x_{i+j_1}) + td(x_{i+j_1}, s_{i,i+j_1}) = td(x_{i+j_2}, s_{i,i+j_1}) \overset{(A.4)}{<} td(x_{i+j_2}, s_{i,i+j_2}).$$

Thus

$$td(x_{i+j_2}, x_{i+j_1}) < td(x_{i+j_2}, s_{i,i+j_2}) - td(x_{i+j_1}, s_{i,i+j_1}) \overset{(A.5)}{<} c_{i+j_1} - c_{i+j_2},$$

a contradiction. The proof of *(ii) (b)* follows similar lines.

Proof of Lemma 2

From Lemma 1 $s_{i-1,i} < s_{i-1,i+1}$ and $s_{i-1,i+1} < s_{i,i+1}$ which proves *(i)*. For *(ii)* distinguish between two cases; either $s_{i-1,j_1} < .... < s_{i-1,j_l} \in (s_{i-1,i}, s_{i,i+1})$ with $i+1 = j_1 < .... < j_l$ or $s_{i-1,j_1} < .... < s_{i-1,j_l} \in (s_{i-1,i}, s_{i,i+1})$ with $j_1 < .... < j_l = i+1$. We do the first case. The second case is done similarly. If the consumer's location $z \in [s_{i-1,i}, s_{i-1,i+1}]$, the consumer will buy from firm i, because $s_{i-1,i} < z$. Since $z < s_{i-1,i+1}$, firm i will charge a price equal to the delivered cost of firm $i-1$, i.e. $td(z, x_{i-1}) + c_{i-1}$. If the consumer's location $z \in [s_{i-1,i+1}, s_{i,i+1}]$ then $z \in [s_{i-1,j_k}, s_{i-1,j_k+1}]$ for some $1 < k < l$. $z > s_{i-1,j_k}$ means that the consumer buys cheaper from firm j_k than from firm $i-1$ ($z < s_{i-1,j_k+1}$ means that the consumer buys cheaper from firm $i-1$ than from firm j_k+1). Since $s_{i-1,i+1} < s_{i-1,j_k} < z$ the consumer buys cheaper from firm $i+1$ than from firm $i-1$. Therefore, the price firm i will charge is $td(x_{i+1}, z) + c_{i+1}$.

Proof of Proposition 1

We prove the Proposition for i, $1 < i < n$; the border cases, for $i = 1$ and $i = n$, being very similar.

$$\begin{aligned} \frac{\partial \Pi_i(x_1,...,x_n)}{\partial x_i} &= \frac{\partial}{\partial x_i}\left[\int_{s_{i-1,i}}^{s_{i-1,i+1}} [td(z, x_{i-1}) + c_{i-1} - td(x_i, z) - c_i] g(z) dz\right] \\ &+ \frac{\partial}{\partial x_i}\left[\int_{s_{i-1,i+1}}^{s_{i,i+1}} [td(x_{i+1}, z) + c_{i+1} - td(z, x_i) - c_i] g(z) dz\right]. \end{aligned} \tag{A.6}$$

On the other hand

$$\frac{\partial T(x_1,\ldots,x_n)}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\int_{x_{i-1}}^{s_{i-1,i}}[td(z,x_{i-1})+c_{i-1}]g(z)dz\right] + \frac{\partial}{\partial x_i}\left[\int_{s_{i,i+1}}^{x_{i+1}}[td(x_{i+1},z)+c_{i+1}]g(z)dz\right]$$
$$+\frac{\partial}{\partial x_i}\left[\int_{s_{i-1,i}}^{s_{i-1,i+1}}[td(x_i,z)+c_i]g(z)dz + \int_{s_{i-1,i+1}}^{s_{i,i+1}}[td(z,x_i)+c_i]g(z)dz\right]. \tag{A.7}$$

However, $\frac{\partial}{\partial x_i}\left[\int_{s_{i-1,i}}^{s_{i-1,i+1}}[td(z,x_{i-1})+c_{i-1}]g(z)dz\right] = -\frac{\partial}{\partial x_i}\left[\int_{x_{i-1}}^{s_{i-1,i}}[td(z,x_{i-1})+c_{i-1}]g(z)dz\right]$ and $\frac{\partial}{\partial x_i}\left[\int_{s_{i-1,i+1}}^{s_{i,i+1}}[td(x_{i+1},z)+c_{i+1}]g(z)dz\right] = -\frac{\partial}{\partial x_i}\left[\int_{s_{i,i+1}}^{x_{i+1}}[td(x_{i+1},z)+c_{i+1}]g(z)dz\right]$.

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