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January 2020

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February 5, 2020

Abstract

We study three remarkable cost sharing rules in the context of shortest path problems, where agents have demands that can only be supplied by a source in a network. The demander rule requires each demander to pay the cost of their cheapest connection to the source. The supplier rule charges to each demander the cost of the second-cheapest connection and splits the excess payment equally between her access suppliers. The alexia rule averages out the lexicographic allocations, each of which allows suppliers to extract rent in some pre-specified order. We show that all three rules are anonymous and demand-additive core selections. Moreover, with three or more agents, the demander rule is characterized by core selection and a specific version of cost additivity. Finally, convex combinations of the demander rule and the supplier rule are axiomatized using core selection, a second version of cost additivity and two additional axioms that ensure the fair compensation of intermediaries.

JEL Classification: C71, D85.

Keywords: Shortest path, cost sharing, core selection, additivity.

*María Gómez-Rúa and Juan Vidal-Puga gratefully acknowledge financial support by the Spanish Ministerio de Economía, Industria y Competitividad (ECO2017-82241-R) and Xunta de Galicia (ED431B 2019/34).

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1 Introduction

We study shortest path problems, where agents must ship their demands of some commodity from a given source point to their respective geographic locations. Each agent can transport her demand directly from the source to her location, or indirectly (through intermediary nodes) if it turns out to be cheaper than the direct connection. The unit cost of shipping the commodity between any two nodes is constant but specific to the two nodes considered. We thus have a cost sharing problem where demanders have to determine the cheapest route (or shortest path) allowing to ship their demands, and the group has to decide how to reward intermediaries (who allow others to connect to the source at a lower cost).

Examples of applications include airline networks (Bryan and O'Kelly, 1999; Yang, 2009), distribution of power supply (Dutta and Kar, 2004), small package delivery (Sim et al., 2009), and biofuel supply chains (Roni et al., 2017). Dutta and Mishra (2012) provide further examples on multicast routing, irrigation systems from a dam, and information exchange. Most of the literature on shortest paths focuses on the construction of the optimal network. Our paper, along with a few others, examines the problem of splitting the total shipping cost between agents while satisfying some basic requirements of fairness and stability.

Every shortest path problem generates a cooperative game (with transferable cost) between the agents. A central and natural axiom is therefore the requirement that a solution for shortest path problems should be a core selector: no group of agents should jointly pay more than their stand-alone shipping cost. It is known from Rosenthal (2013) that the demander rule, which charges to each demander the cost of her shortest path, is a core selection. However, the demander rule does not reward intermediaries and produces an extreme allocation within the core of every shortest path problem: it is thus unfair towards access providers. Tijs et al. (2011) proposed a lexicographic rule that is core selector and that was studied by Bahel and Trudeau (2014) in the context of shortest path problems in order to compensate access providers. The current paper builds on these two works by proposing (i) a new family of cost sharing rules that allow to reward intermediaries (ii) additional axioms that are desirable in networks with linear costs; (iii) the first characterization results in the context of shortest path problems.

In addition to the requirement of Core Selection, we investigate other properties. The axiom of Additivity, whenever it applies, is a sensible and useful property in cooperative game theory and cost sharing problems. In the context of shortest path problems, it says that the cost shares should be additive in the *cost matrix* and the agents' *demands*.

Given that the shipping cost is linear on every arc linking two nodes, we show (in Lemma 3.1) that one can construct demand-additive rules (that are core selectors) by studying elementary problems (where a single agent demands one unit, and the others help her get that unit from the source). This approach allows us to define the family of *Anonymous and Demand-Additive Core Selections* (or ADACS). Anonymity simply says that the agents' labels should not be used in computing the cost shares. We prove that the demander rule and the average lexicographic rule (Tijs et al., 2011) are both ADACS (see Theorem 4.1 and Theorem 4.3). Moreover, we introduce the *supplier rule*, which charges to each demander the cost of her second-shortest path and splits the excess payment equally between her access providers. It is shown in Theorem 4.2 that the supplier rule is an ADACS.

However, it turns out (as pointed out in Section 5) that cost additivity is impossible to achieve. This impossibility is reminiscent of the one obtained in other types of cost sharing problems within networks —see for instance Bergantiños and Vidal-Puga (2009) for the case of minimum cost spanning trees or Bahel and Trudeau (2017) for the case of minimum cost arborescences. We thus propose two weaker versions of cost additivity that are compatible with Core Selection. The first version, *One-path Cost Additivity*, requires the cost shares in an elementary problem to be additive (in the cost matrix) only within families of cost matrices that exhibit a common shortest path to the demander. The second version, *Two-path Cost Additivity*, requires the cost shares to be additive only within families of cost matrices that exhibit both a common shortest-path and a common second-shortest path to the demander.

Two additional properties are studied in Section 6. *Supplier Equal Change* says that, whenever two cost matrices (c and c') have a common shortest path to j, the respective cost shares of all providers of j (on that common shortest path) should be affected in the same way when we move from c to c'. *Path Independence* says that a demander j should pay the same joint fee to her respective groups (G and G') of providers under c and c' whenever the cost savings generated by G are G' are identical.

Our results show that, with three agents or more, the demander rule is the only rule satisfying both Core Selection and One-path Cost Additivity (see Theorem 5.1). On the other hand, it turns out that Two-path Cost Additivity (which is obviously weaker than One-path Cost Additivity) does not preclude rewarding access providers. We show in Theorem 6.1 that an ADACS meets One-path Cost Additivity, Supplier Equal Change and Path Independence if and only if it is a convex combination of the demander rule and the supplier rule. To the best of our knowledge, the characterization results offered in Theorem 6.1 and Theorem 5.1 are the first axiomatizations of cost sharing rules within the literature on shortest path problems.

The paper is organized as follows. In Section 2, we define shortest path problems and describe the framework. In Section 3, we formally introduce Core Selection and other basic properties of cost sharing rules. In Section 4, we describe our three distinguished cost sharing rules and show that they are all ADACS. In Section 5, we present te respective versions of Cost Additivity, as well as the characterization result involving One-path Cost Additivity. In Section 6, we focus on the axiomatization of convex combinations of the demander rule and the supplier rule. Our concluding remarks are given in Section 7.

2 The model

Let $N = \{1, \ldots, n\}$ denote a set of $n \geq 2$ agents who need to ship units of some commodity from a fixed point $\mathbf{0}$ to their respective locations ($\mathbf{0}$ is called the *source*). We emphasize that this set of agents N is fixed and does not vary throughout the paper. A *Shortest Path Problem* (SPP) is a pair $P = (c, x)$, where:

- $c = \{c(i,j) : i \in N \cup \{\mathbf{0}\}, j \in N, i \neq j\}$ is a cost matrix of nonnegative numbers giving the unit cost of shipping demands through each arc (i, j).
- $x \in \mathbb{R}_+^N$ is the demand vector: each agent $i \in N$ has a demand $x_i \in \mathbb{R}_+$ (of the commodity) to ship from the source to her location.

Let us denote by $\mathbb{P}$ the set of shortest path problems (c, x), and by $\mathcal{C}$ the set of all cost matrices c. Note that: (a) the source $\mathbf{0}$ is not an agent, and (b) the unit costs $c(i, j)$ need not be symmetric —we may well have $c(i, j) \neq c(j, i)$ for some $i, j \in N$. If $c(i, j) = c(j, i)$ for any $i, j \in N$, $i \neq j$, we say that the SPP has *symmetric arcs*.

Definition 2.1 *Given $i \in N$, we call* ***path*** *(of* length *K)* ***to*** *i any sequence $p := (p_k)_{k=0,\ldots,K}$ such that:*

1. *$p_k \in N$, for $k = 1, 2, \ldots, K$;*
2. *$p_0 = \mathbf{0}$ and $p_K = i$;*
3. *$p_k \notin \{p_1, \ldots, p_{k-1}\}$ whenever $2 \leq k \leq K$.*

Note from Definition 2.1 that all paths p originate from the source $\mathbf{0}$ and cross any location p_k only once. Thus, the length of each path and the number of paths to any given $i \in N$ are both finite. We denote by $\mathcal{P}^i$ the set containing all paths to agent i. For any path p of length K, let $[p]$ refer to the set of players in the range of p, that is:

$$[p] := \{i \in N : p_k = i \text{ for some } k = 1, \ldots, K\}.$$

For any subset $M \subsetneq N$ and any path p (of length K) such that $M \subsetneq [p]$, we write $p \setminus M$ to refer to the unique path (of length $K - |M|$) where the agents of M have been excluded and the remaining agents (of $[p]$) appear in the same order as in p. To ease on notation, we often write i instead of $\{i\}$ and hence $p \setminus i$ instead of $p \setminus \{i\}$, for any $i \in [p]$.

Given $P = (c, x)$, one can extend the cost function c to paths as follows: for any path p (of length K) to i,

$$c(p) := \sum_{k=1}^{K} c(p_{k-1}, p_k).$$

In words, $c(p)$ stands for the cost of shipping one unit from the source to agent i via the path p. For any $i \in N$, we call *shortest path to* i any path $\bar{p}_c^i \in \mathcal{P}^i$ that solves the problem $\min_{p \in \mathcal{P}^i} c(p)$. In all cases where there is no possible confusion about the cost matrix c, we write $\bar{p}^i$ instead $\bar{p}_c^i$. Note that there exists a shortest path to any $i \in N$ — since the set $\mathcal{P}^i$ is nonempty and finite — but it need not be unique. Given a cost matrix c, we denote by $\overline{\mathcal{P}}^i(c)$ the set of shortest paths to each agent $i \in N$. The set of permutations of N is denoted by Π.

Example 2.1 *Consider the SPP (with symmetric arcs) given by $P = (c, x)$, where $N = \{1, 2, 3\}$, $x = (2, 0, 1)$ and the cost structure is depicted by Figure 1. Hence, we have $c(\mathbf{0}, 1) = 200$, $c(1, 3) = c(3, 1) = 10$, $c(1, 2) = c(2, 1) = 70$, and so on.*

One can see that there are 5 paths to agent 1, $(\mathbf{0}, 1), (\mathbf{0}, 2, 1)$, $(\mathbf{0}, 3, 1), (\mathbf{0}, 2, 3, 1)$, $(\mathbf{0}, 3, 2, 1)$; and the shortest path to 1 is $(\mathbf{0}, 2, 3, 1)$, with cost $c(\mathbf{0}, 2, 3, 1) = 60 + 20 + 10 = 90$. For agents 2 and 3, the costs of their respective shortest paths are $c(\mathbf{0}, 2) = 60$ and $c(\mathbf{0}, 2, 3) = 60 + 20 = 80$.

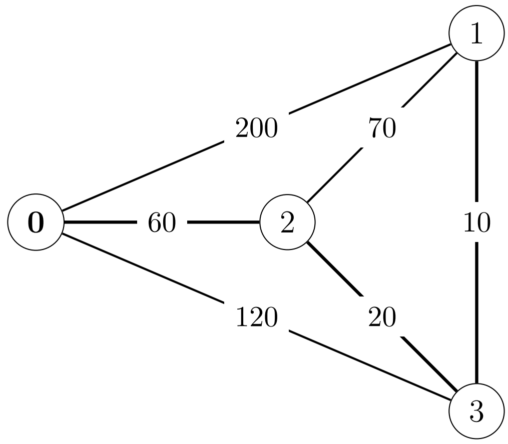

Figure 1: SPP with three agents.

For any vector $y \in \mathbb{R}^N$ and any subset $S \subseteq N$, we sometimes use the notation $y_S := \sum_{i \in S} y_i$. The cooperative game (with transferable cost) associated with P can be formulated as follows.

Define the cost of any nonempty coalition $S \subseteq N$ by:

$$C_P(S) := \min \left\{ \sum_{j \in S} x_j c(p^j) : p^j \in \mathcal{P}^j \text{ and } [p^j] \subseteq S, \forall j \in S \right\}. \tag{1}$$

Equation (1) gives the lowest possible cost of shipping (from the source) the respective demands of the members of S when using only the connections available in S. Note in particular that $C_P(S) = 0$ whenever $x_S = 0$ (there is no demand to ship). We also adopt the usual convention that $C_P(\emptyset) = 0$. As an illustration, for the problem P depicted in Example 2.1, note that $C_P(N) = 2 \cdot c(\mathbf{0}, 2, 3, 1) + 0 \cdot c(\mathbf{0}, 2) + 1 \cdot c(\mathbf{0}, 2, 3) = 180 + 80 = 260$.

Definition 2.2 *Given a shortest path problem $P = (c, x)$, an* ***allocation*** *is a profile of cost shares, $y \in \mathbb{R}^N$, such that $y_N = C_P(N)$. Let $\mathcal{A}(P)$ be the set containing all cost allocations.*

The above definition says that a cost allocation splits the (minimum) total cost of shipping the demands of all agents in N from the source to their respective locations. Remark that we allow for negative cost shares, which are desirable in particular if some agents have null demands while providing others with a cheaper access to the source.

Let us now define the solution concepts studied in this work.

Definition 2.3 *A* ***cost sharing rule*** *(CSR) is a mapping $y \colon \mathbb{P} \to \mathbb{R}^N$ that assigns to each $P \in \mathbb{P}$ a cost allocation $y(P) \in \mathbb{R}^N$ such that $y_N(P) := (y(P))_N = C_P(N)$.*

In words, a cost sharing rule is a mechanism which, for any given problem P, allows to divide between agents the total cost $C_P(N)$ of satisfying the respective demands (we refer to this property as *efficiency*). A classic example of CSR is the Shapley value (Shapley, 1953). In the case of the SPP given in Example 2.1, the Shapley value picks the allocation $(260, -30, 30)$. In the following sections we introduce and study some other specific CSR, as well as a number of desirable properties.

3 Core selection and other basic properties

The following definition provides the standard notion of stability: every coalition $S \subseteq N$ should jointly pay at most its stand-alone cost $C_P(S)$.

Definition 3.1 *Given a shortest path problem $P = (c, x)$, the* ***core*** *of P is the set*

$$Core(P) := \{y \in \mathcal{A}(P) : y_S \leq C_P(S), \forall S \subsetneq N\}.$$

An allocation y is called ***stable*** *if $y \in Core(P)$.*

In particular, the Shapley value does not always provide a stable allocation. In Example 2.1, $(260, -30, 30)$ does not belong to the core because $y_{\{1,3\}} = 260 + 30 > C_P(\{1, 3\})$.

In shortest path problems, there are no congestion externalities in the sense that shipping one unit to a given agent does not affect the minimum cost of shipping the next unit to any agent, and so on for the following units. Using this observation, we first study *elementary SPP*, which have the property that only one agent has a (unitary) demand.

For every $j \in N$, denote by $e^j \in \mathbb{R}^N$ the vector of demands characterized by $e^j_j = 1$ and $e^j_i = 0$, if $i \in N \setminus j$. Let $A, B \subseteq \mathbb{R}^N$ and $\alpha \in \mathbb{R}$. We use the following conventions: $A + B := \{a + b : a \in A, b \in B\}$; $\alpha \cdot A := \{\alpha \cdot a : a \in A\}$.

Lemma 3.1 *Given the problem $P = (c, x)$,*

$$\sum_{j \in N} x_j \cdot Core(P^j) \subseteq Core(P)$$

where $P^j := (c, e^j)$.

Proof. Fix a problem $P = (c, x)$ and suppose that $y^j \in Core(P^j)$ for every $j \in N$, where $P^j = (c, e^j)$. It then follows from (1) that

$$\sum_{i \in N} y^j_i = C_{P^j}(N) = c(\bar{p}^j) \tag{2}$$

for every $j \in N$, where $\bar{p}^j \in \overline{\mathcal{P}}^j(c)$ for all $j \in N$. Thus, defining $y := \sum_{j\in N} x_j y^j \in \mathbb{R}^N$, observe that y is a well-defined allocation for the problem P, since

$$\sum_{i\in N} y_i = \sum_{i\in N}\sum_{j\in N} x_j y_i^j = \sum_{j\in N} x_j \sum_{i\in N} y_i^j \overbrace{=}^{\text{by (2)}} \sum_{j\in N} x_j c(\bar{p}^j) \overbrace{=}^{\text{by (1)}} C_P(N).$$

Moreover, for any $S \subsetneq N$, given that $y^j \in Core(P^j)$ for all $j \in N$, we can write $\sum_{i\in S} y_i^j \leq C_{P^j}(S)$ for all $j \in N$. Therefore,

$$\sum_{i\in S} y_i = \sum_{i\in S}\sum_{j\in N} x_j y_i^j = \sum_{j\in N} x_j \sum_{i\in S} y_i^j \leq \sum_{j\in N} x_j C_{P^j}(S) \overbrace{=}^{\text{by (1)}} C_P(S).$$

Hence, $y \in Core(P)$; and we thus conclude that $\sum_{j\in N} x_j \cdot Core(P^j) \subseteq Core(P)$. ■

We now introduce a few basic requirements for cost sharing rules.

Given a bijection $\sigma : N \cup \{\mathbf{0}\} \to N \cup \{\mathbf{0}\}$ such that $\sigma(\mathbf{0}) = \mathbf{0}$, and given $P = (c, x), P' = (c', x') \in \mathbb{P}$, we say that P' is *σ-equivalent* to P if it holds that: (a) $x'_i = x_{\sigma(i)}$ for all $i \in N$; and (b) $c'(i, i') = c(\sigma(i), \sigma(i'))$ for all $i \in N \cup \{\mathbf{0}\}$, $i' \in N$ such that $i \neq i'$.

Definition 3.2 *A cost sharing rule y satisfies:*

1. ***Core Selection*** *if $y(P) \in Core(P)$ for all $P \in \mathbb{P}$.*

2. ***Demand Additivity*** *if $y(P) = \sum_{j\in N} x_j y(P^j)$ for all $P \in \mathbb{P}$.*

3. ***Anonymity*** *if, for all bijection $\sigma : N \cup \{\mathbf{0}\} \to N \cup \{\mathbf{0}\}$ with $\sigma(\mathbf{0}) = \mathbf{0}$, and all $P, P' \in \mathbb{P}$ such that P' is σ-equivalent to P, $y_i(P') = y_{\sigma(i)}(P)$ for all $i \in N$.*

The present work focuses on CSR that satisfy Core Selection, Demand Additivity and Anonymity. We call any CSR in this family an *Anonymous Demand-Additive Core Selection* (or $ADACS$, for short). We search for stable cost allocations in elementary SPP and extend these allocations (by Demand Additivity) to general SPP. By leveraging the decomposition result of Lemma 3.1, one can easily see that the CSR thus defined always satisfies Core Selection.

4 Some cost sharing rules

In this section we present three CSR that will be used throughout the paper.

4.1 The demander rule

A simple CSR obtains by requiring every agent to pay the cost of her shortest path for each unit demanded, with agents who demand zero paying nothing.

Definition 4.1 *The **demander rule** y^d is defined as follows: for all $(c, x) \in \mathbb{P}$ and $i \in N$,*

$$y_i^d(c, x) = x_i \min_{p \in \mathcal{P}^i} c(p).$$

Remark that this rule is favorable to demanders: they do not have to compensate any intermediaries who help them connect to the source at a lower cost. We show below that the demander rule is an $ADACS$.

Theorem 4.1 *The demander rule y^d is an $ADACS$.*

Proof. It is straightforward to see that y^d satisfies Anonymity and Demand Additivity. We prove Core Selection as follows. Fix $P = (c, x) \in \mathbb{P}$ and note that $\sum_{i \in N} y_i^d(P) = \sum_{i \in N} x_i \min_{p \in \mathcal{P}^i} c(p) = C_P(N)$. Moreover, for any coalition $S \subsetneq N$,

$$\sum_{i \in S} y_i^d(P) = \sum_{i \in S} x_i \min_{p \in \mathcal{P}^i} c(p) \leq \sum_{i \in S} x_i \min_{p \in \mathcal{P}^i : [p] \subseteq S} c(p) = C_P(S).$$

■

As an illustration note that, for the SPP depicted in Example 2.1, the demander rule yields the cost allocation $y^d = (180, 0, 100)$.

Although the demander rule is easy to compute, it does not reward nodes that provide a cheaper access to the source; and it is not difficult to see that the allocation produced by this CSR is extreme in the core of every elementary problem. In the remainder of the paper, we introduce and study other CSR that do reward access providers for their cooperation with demanders.

4.2 The supplier rule

In this subsection, we define a CSR that charges to every demander j the cost of her second-shortest path, and equally splits between all suppliers of j the excess payment thus collected. Let $\bar{p}^j$ denote (any of) the shortest path(s) to $j \in N$ under the cost matrix c. The supplier rule is formally defined as follows.

Definition 4.2 *The **supplier rule** y^s is the demand additive CSR defined as:*

$$y_i^s(P^j) = \begin{cases} \min_{p \in \mathcal{P}^j \setminus \{\bar{p}^j\}} c(p) & \text{if } i = j \\ \left(c(\bar{p}^j) - \min_{p \in \mathcal{P}^j \setminus \{\bar{p}^j\}} c(p)\right) / (|\bar{p}^j| - 1) & \text{if } i \in [\bar{p}^j] \setminus j \\ 0 & \text{if } i \notin [\bar{p}^j] \end{cases}$$

for each $P = (c, x) \in \mathbb{P}$ *and* $i, j \in N$.

From the above definition, note that (a) the demander j always pays the cost of her second-shortest path (which is at least the cost of her shortest path $\bar{p}^j$) per unit demanded; and (b) all agents i on the shortest path to j receive an equal compensation, which is the absolute value of their cost share given by $\left(c(\bar{p}^j) - \min_{p \in \mathcal{P}^j \setminus \{\bar{p}^j\}} c(p)\right) / (|\bar{p}^j| - 1) \leq 0$. Note from Definition 4.2 that, if there are multiple shortest paths to j, then every player $i \neq j$ pays a cost share of zero, that is, $y_i^s(P^j) = 0$. Therefore, the supplier rule is well defined, because it is independent of which particular shortest path $\bar{p}^j$ is picked. As stated in the following theorem, the supplier rule belongs to the $ADACS$ family.

Theorem 4.2 *The supplier rule* y^s *is an* $ADACS$.

Proof. The supplier rule is demand-additive by definition. It is also clear that it meets Anonymity. We show that it satisfies Core Selection. Fix $P = (c, x) \in \mathbb{P}$ and $j \in N$. From Lemma 3.1, it suffices to show that $y^s(P^j) \in Core(P^j)$ for all $j \in N$. To avoid triviality, we assume that $\{j\} \subsetneq [\bar{p}^j]$. Note from Definition 4.2 that $\sum_{i \in N} y_i^s(P^j) = c(\bar{p}^j) = C_{P^j}(N)$. Moreover, for any $S \subseteq N \setminus j$,

$$\sum_{i \in S} y_i^s(P^j) = \frac{c(\bar{p}^j) - \min_{p \in \mathcal{P}^j \setminus \{\bar{p}^j\}} c(p)}{|\bar{p}^j| - 1} |([\bar{p}^j] \setminus j) \cap S| \leq 0 = C_{P^j}(S).$$

If, instead, we have $j \in S$, then it follows that

$$\sum_{i \in S} y_i^s(P^j) = \min_{p \in \mathcal{P}^j \setminus \{\bar{p}^j\}} c(p) + \frac{c(\bar{p}^j) - \min_{p \in \mathcal{P}^j \setminus \{\bar{p}^j\}} c(p)}{|\bar{p}^j| - 1} |([\bar{p}^j] \setminus j) \cap S|. \tag{3}$$

We can then distinguish two cases.

Case 1. Suppose that $([\bar{p}^j] \setminus j) \cap S = S \setminus j$. Then, obviously $C_{P^j}(S) = c(\bar{p}^j)$ and it thus comes from (3) that $\sum_{i \in S} y_i^s(P^j) = C_{P^j}(S) = c(\bar{p}^j)$.

Case 2. Suppose instead that there exists $k \in ([\bar{p}^j] \setminus j) \setminus S$. Then, $\bar{p}^j \notin \left\{p \in \overline{\mathcal{P}}^j(c) : [p] \subseteq S\right\}$ and it thus follows from (1) that $C_{P^j}(S) \geq \min_{p \in \mathcal{P}^j \setminus \{\bar{p}^j\}} c(p)$. Combining this inequality with (3) finally yields: $\sum_{i \in S} y_i^s(P^j) \leq \min_{p \in \mathcal{P}^j \setminus \{\bar{p}^j\}} c(p) \leq C_{P^j}(S)$.

■

We illustrate the supplier rule by recalling Example 2.1. Note that the costs of the second shortest paths to 1 and 3 are respectively $c(\mathbf{0}, 2, 1) = 60 + 70 = 130$ and $c(\mathbf{0}, 3) = 100$. Also, there are two intermediaries (agents 2 and 3) who help agent 1 to connect to the source on the path $\bar{p}^1 = (\mathbf{0}, 2, 3, 1)$. We can therefore write: $y(P^1) = (130, -20, -20)$. As for agent 3, there is only one intermediary (agent 2) on the path $\bar{p}^3$; and hence $y(P^3) = (0, -40, 120)$. Thus, using demand additivity, the cost shares in the overall problem are $y(P) = 2 \cdot (130, -20, -20) + 1 \cdot (0, -40, 120) = (260, -80, 80)$.

4.3 The alexia rule for shortest-path games

In comparison with the two previous rules, our third distinguished CSR selects a more central cost allocation inside the core. We describe the computation procedure as follows.

Fix a permutation $\pi \in \Pi$. Focusing on an elementary problem $P^j = (c, e^j)$, let $\bar{p}^j$ be a shortest path to the demander j and let $m := |[\bar{p}^j] \setminus j|$. We define a particular cost allocation denoted by y^π. In the trivial cases where $m = 0$ or $\pi(j) = 1$, we have $y^\pi_j(P^j) = c(\bar{p}^j)$ and $y^\pi_i(P^j) = 0$ for all $i \in N \setminus j$. Otherwise, we write without loss of generality $\{1_\pi, \dots, m_\pi\} := \{i \in [\bar{p}^j] \setminus j : \pi(i) < \pi(j)\} \neq \emptyset$.

The procedure to compute y^π is formally described in Algorithm 1. Let us give here

Algorithm 1 Computation of $y^\pi(P^j)$

for all $i \in N$ **do**
    initialize $y_i \leftarrow 0$
choose $\bar{p}^j \in \arg\min_{p \in \mathcal{P}^j} c(p)$
define $m \leftarrow |\{i \in [\bar{p}^j] \setminus j : \pi(i) < \pi(j)\}|$
define $\alpha^0 \leftarrow c(\bar{p}^j)$
for all $k \in (N \setminus 1_\pi) \cup \{\mathbf{0}\}, l \in N \setminus 1_\pi$ (with $k \neq l$) **do**
    define $c_1(k,l) \leftarrow c(k,l)$
for all $t = 1, \dots, m$ **do**
    choose $p^j_t \in \arg\min \{c_t(p) : \ p \in \mathcal{P}^j \text{ s.t. } 1_\pi, \dots, t_\pi \notin [p]\}$
    define $\alpha^t \leftarrow c_t(p^j_t)$
    define $y^\pi_{t_\pi} \leftarrow \alpha^{t-1} - \alpha^t$
    for all $k \in (N \setminus \{1_\pi, \dots, t_\pi\}) \cup \{\mathbf{0}\}, l \in N \setminus \{1_\pi, \dots, t_\pi\}$ (with $k \neq l$) **do**
        define $c_{t+1}(k,l) \leftarrow \min\{c_t(k,l), c_t(k,t_\pi) + c_t(t_\pi, l) - y^\pi_{t_\pi}\}$
define $y^\pi_j \leftarrow c(\bar{p}^j) - y^\pi_{1_\pi} - \dots - y^\pi_{m_\pi}$
present $y^\pi \in \mathbb{R}^N$

a description of the steps of the algorithm. First, call p^j_1 (one of) the cheapest path(s) to j *among those that do not contain agent* 1_π; and then assign to player 1_π the cost share $y^\pi_{1_\pi}(P^j) = c(\bar{p}^j) - c(p^j_1) = \alpha^0 - \alpha^1 \leq 0$. Next, consider the reduced SPP $(N \setminus 1_\pi, c_1, e^j)$, where the cost matrix c_1 is defined by: for all $k \in (N \setminus 1_\pi) \cup \{\mathbf{0}\}, k \in (N \setminus 1_\pi)$ (with $k \neq l$), $c_1(k,l) = \min(c(k,l), c(k, 1_\pi) + c(1_\pi, l) - y^\pi_{1_\pi})$. In words, this means that any two agents of the reduced problem have the option to connect via agent 1_π (after paying to her the fee $|y^\pi_{1_\pi}|$) in case they find it beneficial. Then, mimicking the first step for this reduced problem, one can assign to agent 2_π the cost share $y^\pi_{2_\pi}(P^j) = c_1(\bar{p}^j_{c_1}) - c_1(p^j_2) =$

$\alpha^1 - \alpha^2$. We repeat the update of the cost matrix and compute the cost shares until all intermediaries $\{1_\pi, \ldots, m_\pi\}$ have been served. Finally, one must assign to the demander j a cost share that covers the cost of the shortest path and the fees paid to all intermediaries: $y_j^\pi(P^j) = c(\bar{p}^j) - y_{1_\pi}^\pi(P^j) - \cdots - y_{m_\pi}^\pi(P^j)$. Since the computed allocation corresponds to an arbitrary ordering of the players, a fairer and anonymous allocation rule obtains by averaging over all possible permutations of the player set:

Definition 4.3 *The **alexia rule** y^a is the demand additive CSR defined as:*

$$y^a(P) = \sum_{j \in N} x_j \underbrace{\frac{1}{n!} \cdot \sum_{\pi \in \Pi} y^\pi(P^j)}_{y^a(P^j)} = \frac{1}{n!} \cdot \sum_{j \in N} \sum_{\pi \in \Pi} x_j \cdot y^\pi(P^j) \tag{4}$$

for each $P = (c, x) \in \mathbb{P}$.

The following result states that the alexia rule belongs to the $ADACS$ family.

Theorem 4.3 *The alexia rule y^a is an $ADACS$.*

Proof. The alexia rule is demand-additive by definition. It is also clear that it meets Anonymity, since it is computed by averaging over all possible permutations of the set N. Finally, Bahel and Trudeau (2014) prove that, for every $P \in \mathbb{P}$, $y^\pi(P) \in Core(P)$ and hence $y^a(P) \in Core(P)$, since the core is a convex set. ■

The following example illustrates the alexia rule and Algorithm 1.

Example 4.1 *Recall the SPP of Example 2.1, where $x = (2, 0, 1)$ and $\bar{p}^1 = (\mathbf{0}, 2, 3, 1)$ is the shortest path to agent 1. Fixing the permutation $\pi = \overline{321}$ and the agent $j = 1$, note that Algorithm 1 yields $m = 2$, $1_\pi = 3$, and $\alpha^0 = c(\bar{p}^1) = 60 + 20 + 10 = 90$. Thus, it comes that $y_3^\pi = 90 - 130 = -40$ —remark that the lowest cost of serving agent 1 while excluding agent 3 is $c_1(\mathbf{0}, 2, 1) = 60 + 70 = 130 = \alpha^1$. The procedure then continues as follows (for $t = 2$): $2_\pi = 2$, $c_2(0, 1) = \min(200, 120 + 10 - (-40)) = 170 = \alpha^2$; and hence $y_2^\pi = \alpha^1 - \alpha^2 = 130 - 170 = -40$. Finally, we get $y_1^\pi = 90 - (-40) - (-40) = 170$, that is to say, $y^\pi = (170, -40, -40)$.*

Proceeding as explained above, we obtain the cost shares y^π described by Table 1, for each permutation $\pi \in \Pi$ and each elementary problem P^j. Using the definition given by Equation (4), it is then not difficult to check that $y^a(P^1) = (130, -20, -20), y^a(P^3) = (0, -20, 100)$. Hence, $y^a(P) = x_1 y^a(P^1) + x_3 y^a(P^3) = 2y^a(P^1) + y^a(P^3) = (260, -60, 60)$.

It follows that our three distinguished rules are all $ADACS$. In order to differentiate them, we introduce and study some additional properties in the next sections.

Order π	$y^\pi(P^1)$	$y^\pi(P^2)$	$y^\pi(P^3)$
$\overline{123}$	$(90,0,0)$	$(0,60,0)$	$(0,-40,120)$
$\overline{132}$	$(90,0,0)$	$(0,60,0)$	$(0,0,80)$
$\overline{213}$	$(130,-40,0)$	$(0,60,0)$	$(0,-40,120)$
$\overline{231}$	$(170,-40,-40)$	$(0,60,0)$	$(0,-40,120)$
$\overline{312}$	$(130,0,-40)$	$(0,60,0)$	$(0,0,10)$
$\overline{321}$	$(170,-40,-40)$	$(0,60,0)$	$(0,0,10)$
Average	$(130,-20,-20)$	$(0,60,0)$	$(0,-20,100)$

Table 1: Allocations $y^\pi(P^j)$ obtained from Algorithm 1.

5 Cost additivity: weak versions and a result

This section shows that it is not possible to require that the cost shares be additive in the cost matrix c. As a consequence, we propose two weakened versions of cost additivity that turn out to be compatible with our main axiom of Core Selection.

Definition 5.1 *A CSR y satisfies **Cost Additivity** if $y(c+c', e^j) = y(c, e^j) + y(c', e^j)$ for any two elementary problems $(c, e^j), (c', e^j) \in \mathbb{P}$.*

Cost Additivity says that cost shares should be additive in the cost matrix. However, no CSR satisfies this property. Take for example $N = \{1, 2\}$ and $c, c' \in \mathcal{C}$ given by $c(\mathbf{0}, 1) = c(1, 2) = c(2, 1) = 0, c(\mathbf{0}, 2) = 1$, and $c'(\mathbf{0}, 2) = c'(1, 2) = c'(2, 1) = 0, c'(\mathbf{0}, 1) = 1$. Then, for any $j \in N$, $C_{(c,e^j)}(N) = C_{(c',e^j)}(N) = 0$ whereas $C_{(c+c',e^j)}(N) = 1$ and hence this property is incompatible with efficiency.

Definition 5.2 *A CSR y satisfies **One-path Cost Additivity** if whenever two elementary problems $(c, e^j), (c', e^j) \in \mathbb{P}$ have a common shortest path to j, it holds that: $y(c + c', e^j) = y(c, e^j) + y(c', e^j)$.*

One-path Cost Additivity is a weaker version of Cost Additivity; and it is compatible with efficiency, as our next results show.

Theorem 5.1 *If $n > 2$, then the demander rule y^d is the unique CSR satisfying Core Selection and One-path Cost Additivity.*
If instead $n = 2$ (say, $N = \{1, 2\}$), then a CSR y satisfies Core Selection and One-path

Cost Additivity if and only if there exists a function $\alpha : \mathbb{R}_+^{\{1,2\}} \to [0,1]^2$ *such that, for all* $(c,x) \in \mathbb{P}$,

$$y(c,x) = \begin{cases} \alpha_1(x) \cdot y^s(c,x) + (1-\alpha_1(x)) \cdot y^d(c,x), & \textit{if } c(\mathbf{0},2) > c(\mathbf{0},1) + c(1,2); \\ \alpha_2(x) \cdot y^s(c,x) + (1-\alpha_2(x)) \cdot y^d(c,x), & \textit{otherwise.} \end{cases}$$

Proof. Suppose first that $n > 2$. Recall that the demander rule satisfies Core Selection (as implied by Theorem 4.1). It is also easy to check that the demander rule satisfies One-path Cost Additivity.

Conversely, consider now a CSR y that satisfies Core Selection and One-path Cost Additivity. We must show that it coincides with the demander rule, that is, $y = y^d$. Fix an arbitrary problem $P = (c,x) \in \mathbb{P}$; and let $\bar{p}^j = (\bar{p}_0^j = \mathbf{0}, \bar{p}_1^j, \ldots, \bar{p}_{K_j}^j = j) \in \bar{\mathcal{P}}^j(c)$ be a shortest path to every player $j \in N$. We construct a new cost matrix c^a as follows: for all distinct $i \in N \cup \mathbf{0}, j \in N$,

$$c^a(i,j) = \begin{cases} c(\bar{p}^j), & \text{if } i = \mathbf{0}; \\ c(i,\bar{p}_{k+1}^j) + \ldots + c(\bar{p}_{K_j-1}^j, j), & \text{if } i = \bar{p}_k^j \text{ for some } k \in \{1,\ldots,K_j-1\}; \\ c(i,j), & \text{otherwise.} \end{cases} \tag{5}$$

In words, if i belongs to the set of suppliers of j under the shortest path $\bar{p}^j$, then $c^a(i,j)$ gives the sum of the costs for the sequence of consecutive arcs leading to j from i. Otherwise, we simply have $c^a(i,j) = c(i,j)$.

It is not difficult to see from (5) that

$$c^a(\mathbf{0},j) = c^a(\bar{p}^j) = c(\bar{p}^j) \leq c^a(p^j), \text{ for all } i \in N \text{ and } p^j \in \mathcal{P}^j. \tag{6}$$

That is to say, for any $j \in N$, both $\bar{p}^j$ and $(\mathbf{0}, j)$ are shortest paths to j under c^a;

Moreover, observe from (5) and the assumption $\bar{p}^j \in \bar{\mathcal{P}}^j(c)$ that me must have $c^a(i,j) \leq c(i,j)$, for all $c^a(i,j) = c(i,j)$. Thus, it comes that $c^b := c - c^a \in \mathcal{C}$, since $c^b(i,j) = c^a(i,j) - c(i,j) \geq 0$, for all distinct $i \in N \cup \mathbf{0}, j \in N$. Next, define the cost matrices c^0 and c^k (for all $k \in N$) as follows: for all distinct $i \in N \cup \mathbf{0}, j \in N$,

$$c^0(i,j) = \begin{cases} 0, & \text{if } i = \mathbf{0} \\ c^b(i,j) & \text{otherwise;} \end{cases} \qquad c^k(i,j) = \begin{cases} c^b(0,j), & \text{if } i = \mathbf{0} \text{ and } k = j \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

It is straightforward to see that $c^b = c^0 + c^1 + c^2 + \ldots + c^n$ and $c^0(\bar{p}^j) = c^k(\bar{p}^j) = 0$, for all $j,k \in N$. Hence, $\bar{p}^j$ is a shortest path to every $j \in N$ for any of the cost matrices $c^a, c^0, c^1, \ldots, c^n$. Given that $c = c^a + c^0 + c^1 + \ldots + c^n$, One-path Cost Additivity therefore yields

$$y(c,x) = y(c^a,x) + y(c^0,x) + \ldots + y(c^n,x). \tag{8}$$

But note that the cooperative game induced by $P^a = (c^a, x)$ is additive, since its characteristic cost function C_{P^a} satisfies

$$C_{P^a}(S) = \sum_{j\in S} x_j c^a(0,j) = \sum_{j\in S} x_j c(\bar{p}^j) = \sum_{j\in S} y_j^d(c,x),\ \forall S \in 2^N \setminus \{\emptyset\}.$$

The problem $P^a = (c^a, x)$ thus has a unique core allocation, $(x^j c(\bar{p}^j))_{j\in N} = y^d(c,x)$; and given that y is a Core Selection, we must have

$$y(c^a, x) = y^d(c,x). \tag{9}$$

Moreover, since $c^0(\mathbf{0}, j) = 0$ for all $j \in N$, it is obvious that the Core Selection y should pick $y(c^0, x) = (0, \ldots, 0) \in \mathbb{R}^N$. Finally, given that $n > 2$, remark from (7) that, for any distinct $i, j, l \in N$, we have (a) $c^k(\mathbf{0}, j) = 0$ if $j \neq k$ and (b) $c^k(\mathbf{0}, i, j) = 0 = c^k(\mathbf{0}, l, j)$ if $j = k$. Since y meets Core Selection, it thus follows from (a)-(b) above that $y_j(c^k, x) \leq 0$ (for all $j, k \in N$); and efficiency then implies $y(c^k, x) = y(c^0, x) = (0, \ldots, 0) \in \mathbb{R}^N$. Substituting these equalities in (8) and recalling (9), one gets the desired result:

$$y(c,x) = y(c^a, x) = y^d(c,x).$$

Suppose now that $n = 2$, that is, $N = \{1, 2\}$. It is not difficult to see that y satisfies Core Selection and One-path Cost Additivity whenever there exists $\alpha : \mathbb{R}_+^{\{1,2\}} \to [0,1]^2$ satisfying the properties described in the statement of Theorem 5.1. Conversely, we must show that such a function α exists for any CSR y that meets our two axioms.

Fix then a CSR y that satisfies Core Selection and One-path Cost Additivity; and define the set of cost matrices

$$\begin{aligned} \mathcal{C}^1 &= \{c \in \mathcal{C} : c(\mathbf{0},2) > c(\mathbf{0},1) + c(1,2)\}\,; \\ \mathcal{C}^2 &= \{c \in \mathcal{C} : c(\mathbf{0},2) \leq c(\mathbf{0},1) + c(1,2)\}\,; \\ \mathcal{C}_0^1 &= \{c \in \mathcal{C} : c(\mathbf{0},2) > 0 = c(\mathbf{0},1) = c(1,2) = c(2,1)\}\,. \end{aligned} \tag{10}$$

Note that we have $\mathcal{C} = \mathcal{C}^1 = \mathcal{C}^1 \cup \mathcal{C}^2$ and $\mathcal{C}_0^1 \subsetneq \mathcal{C}^1$. Let $\tilde{c} \in \mathcal{C}_0^1$ be defined by $\tilde{c}(\mathbf{0}, 2) = 1$ and $\tilde{c}(\mathbf{0}, 1) = \tilde{c}(1,2) = \tilde{c}(2,1) = 0$. For any $x \in \mathbb{R}^N$, define

$$\alpha_1(x) = \begin{cases} \frac{y_2(\tilde{c},x)}{x_2}, & \text{if } x_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

We can now prove the following claims.

<u>*Claim 1*</u>: $y_2(c,x) = \alpha_1(x) c(\mathbf{0},2) x_2$, for all $c \in \mathcal{C}_0^1$ and $x \in \mathbb{R}^N$.

Fix $c \in \mathcal{C}_0^1$. The claim trivially holds (by Core Selection) whenever $x_2 = 0$. Suppose then that $x_2 > 0$ and notice first that Claim 1 holds by One-path Cost Additivity whenever $c(\mathbf{0}, 2) = \frac{a}{b} \in \mathbb{Q}$ (with $a, b \in \mathbb{N} \setminus \{0\}$). Indeed, since $\tilde{c}$, $c = \frac{a}{b}\tilde{c}$ and $\frac{1}{b}\tilde{c}$ all have the same shortest paths to 1 and 2, applying One-path Cost Additivity repeatedly gives

$$y_2(c,x) = y_2(a(\frac{1}{b}\tilde{c}), x) = a \cdot y_2(\frac{1}{b}\tilde{c}, x) = a[\frac{1}{b} \cdot y_2(\tilde{c}, x)] = \frac{a}{b} \cdot y_2(\tilde{c}, x) = c(\mathbf{0}, 2)\alpha_1(x)x_2.$$

Consider now that $\theta := c(\mathbf{0}, 2) \notin \mathbb{Q}$ and, for any $t \in \{1, 2, \ldots\}$, write $\theta = q_t \frac{1}{2^t} + r_t$ (with $q_t \in \mathbb{N}$ and $0 \leq r_t < \frac{1}{2^t}$) as the result of the Euclidean division of θ by the rational number $\frac{1}{2^t}$ (remark that $\lim_{t \to \infty} r_t = 0$ and $\lim_{t \to \infty} \frac{q_t}{2^t} = \theta$).

Since $c = \theta\tilde{c}$, it follows from One-path Cost Additivity that

$$y_2(c,x) = y_2(\frac{q_t}{2^t}\tilde{c} + r_t\tilde{c}, x) = y_2(\frac{q_t}{2^t}\tilde{c}, x) + y_2(r_t\tilde{c}, x) = \frac{q_t}{2^t} y(\tilde{c}, x) + y_2(r_t\tilde{c}, x). \tag{11}$$

But we must have $0 \leq y_2(r_t\tilde{c}, x) \leq \underbrace{r_t x_2}_{\to 0}$ by Core Selection; and Substituting these two inequalities in (11) thus yields at the limit:

$$\lim_{t \to \infty} \frac{q_t}{2^t} y_2(\tilde{c}, x) - 0 \leq \lim_{t \to \infty} y_2(\theta\tilde{c}, x) \leq \lim_{t \to \infty} \frac{q_t}{2^t} y_2(\tilde{c}, x).$$

That is to say, $y_2(c,x) = \lim_{t \to \infty} y_2(\theta\tilde{c}, x) = \lim_{t \to \infty} \underbrace{\frac{q_t}{2^t}}_{\to \theta} y_2(\tilde{c}, x) = \theta\alpha_1(x)x_2 = c(\mathbf{0}, 2)\alpha_1(x)x_2$.

Claim 2: $y(c,x) = \alpha_1(x)y^s(c,x) + (1 - \alpha_1^d(x))y(c,x)$, for all $c \in \mathcal{C}^1$ and $x \in \mathbb{R}^N$.

Fix $x \in \mathbb{R}^N$ and $c \in \mathcal{C}^1$. Again, assume that $x_2 > 0$ (the claim trivially holds by Core Selection if $x_2 = 0$). First, notice that $c = c_1 + c_2$, where $c_1 = [c(\mathbf{0}, 2) - c(\mathbf{0}, 1) - c(1, 2)] \cdot \tilde{c} \in \mathcal{C}_0^1$ and $c_2 = c - c_1 \in \mathcal{C}$. Second, remark that c_1 and c_2 have a common path to player 1 [which is $(\mathbf{0}, 1)$] and a common path to player 2 [which is $(\mathbf{0}, 1, 2)$]. Thus, letting $\theta = c(\mathbf{0}, 2) - c(\mathbf{0}, 1) - c(1, 2)$, one can use One-path Cost Additivity to write

$$y_2(c,x) = y_2(c_1, x) + y_2(c_2, x) = y_2(\theta\tilde{c}, x) + y_2(c_2, x) \underbrace{=}_{\text{by Claim 1}} \alpha_1(x)\theta x_2 + y_2(c_2, x). \tag{12}$$

But Core Selection requires that $y_2(c_2, x) = c_2(\mathbf{0}, 2)x_2 = [c(\mathbf{0}, 1) + c(1, 2)]x_2$: this is because, under c_2, the direct connections $(\mathbf{0}, 1)$ and $(\mathbf{0}, 2)$ are both shortest paths to the respective players 1 and 2 — the core of $P_2 = (c_2, x)$ is thus a singleton. Substituting the value of $y_2(c_2)$ in (12) thus gives

$$\begin{aligned} y_2(c,x) =& y_2(c_2, x) + \alpha_1(x)\theta x_2 \\ =& \underbrace{[c(\mathbf{0}, 1) + c(1, 2)]x_2}_{y_2^d(c,x)} + \alpha_1(x) \underbrace{x_2[c(\mathbf{0}, 2) - c(\mathbf{0}, 1) - c(1, 2)]}_{y_2^s(c,x) - y_2^d(c,x)} \\ =& \alpha_1 y_2^s(c,x) + (1 - \alpha)y_2^d(c,x). \end{aligned} \tag{13}$$

Using (13) and efficiency —that is, $y_1(c, x) + y_2(c, x) = x_1 c(\mathbf{0}, 1) + x_2[c(\mathbf{0}, 1) + c(1, 2)]$, one can write as well $y_1(c, x) = \alpha_1 y_1^s(c, x) + (1 - \alpha) y_1^d(c, x)$; and Claim 2 is proved.

Let now $\hat{c} \in \mathcal{C}^2$ be defined by $\hat{c}(\mathbf{0}, 1) = 1$ and $\hat{c}(\mathbf{0}, 2) = \hat{c}(1, 2) = \hat{c}(2, 1) = 0$. For any $x \in \mathbb{R}^N$, define

$$\alpha_2(x) = \begin{cases} \frac{y_1(\hat{c}, x)}{x_1}, & \text{if } x_1 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

<u>*Claim 3*</u>: $y(c, x) = \alpha_2(x) y^s(c, x) + (1 - \alpha_2(x)) y^d(c, x)$, for all $c \in \mathcal{C}^2$ and $x \in \mathbb{R}^N$.

The proof of Claim 3 is omitted (it is similar to that of Claim 2).

Together, Claim 2 and Claim 3 mean that there exists $\alpha : \mathbb{R}_+^{\{1,2\}} \to [0, 1]^2$ such that, for all $(c, x) \in \mathbb{P}$,

$$y(c, x) = \begin{cases} \alpha_1(x) \cdot y^s(c, x) + (1 - \alpha_1(x)) \cdot y^d(c, x), & \text{if } c(\mathbf{0}, 2) > c(\mathbf{0}, 1) + c(1, 2); \\ \alpha_2(x) \cdot y^s(c, x) + (1 - \alpha_2(x)) \cdot y^d(c, x), & \text{otherwise.} \end{cases}$$

■

The two properties used in Theorem 5.1 are independent: The egalitarian rule, defined as $e_i(P) = \frac{C_P(N)}{n}$ for all $i \in N$ and $P \in \mathbb{P}$, satisfies One-path Cost Additivity but fails Core Selection. On the other hand, the alexia rule satisfies Core Selection but fails One-path Cost Additivity. In order to check that, note in Example 2.1 that the allocation provided by the alexia rule (Example 4.1) differs from that proposed by the demander rule.

Corollary 5.1 *If $n > 2$, then the demander y^d is unique ADACS satisfying One-path Cost Additivity.*

*If instead $n = 2$, then an ADACS y satisfies One-path Cost Additivity **if and only if** there exists $\alpha \in [0, 1]$ such that $y(P) = \alpha \cdot y^s(P) + (1 - \alpha) \cdot y^d(P), \forall P \in \mathbb{P}$.*

Proof. Given Theorem 5.1, it suffices to check that, if $n = 2$, then $\alpha_1(x) = \alpha_2(x')$, for all $x, x' \in \mathbb{R}_+^N$. But this property easily follows from Anonymity and Demand Additivity.

■

Hence, if $n > 2$, our results yield the demander rule as the unique ADACS satisfying One-path Cost Additivity. In the case where $n = 2$, the ADACS satisfying One-path Cost Additivity are the convex combinations of the demander rule and the supplier rule.

We now provide an alternative weakening of the axiom Cost Additivity.

Definition 5.3 *A CSR y satisfies **Two-path Cost Additivity** if whenever two elementary problems $(c, e^j), (c', e^j) \in \mathbb{P}$ have a common shortest path and a common second-shortest path to j, it holds that: $y(c + c', e^j) = y(c, e^j) + y(c', e^j)$.*

Two-path Cost Additivity is another weak version of Cost Additivity and it also weakens One-path Cost Additivity. Indeed, note from Definition 5.3 that Two-path Cost Additivity requires the additivity of the cost shares only when the summand matrices have a common shortest path and a common second-shortest path to the demander j. The axiom does not impose any restriction at all when these two conditions are not met. In Section 6, we characterize a remarkable family of ADACS using Two-path Cost Additivity and two new axioms.

6 A family of rules containing the demander rule and the supplier rule

This section introduces some new properties that allow to characterize a distinguished family of ADACS.

Definition 6.1 *A CSR y satisfies:*

1. ***Supplier Equal Change*** *if whenever two elementary problems $P^j = (c, e^j)$, $P'^j = (c', e^j) \in \mathbb{P}$ have a common shortest path (to j) $\bar{p}^j_c = \bar{p}^j_{c'}$, then it holds that:*

$$y_i(P^j) - y_i(P'^j) = y_k(P^j) - y_k(P'^j)$$

for all $i, k \in [\bar{p}^j_c] \setminus j$.

2. ***Path Independence*** *if whenever two elementary problems $P^j = (c, e^j)$, $P'^j = (c', e^j) \in \mathbb{P}$ satisfy $c(\bar{p}^j_c) = c'(\bar{p}^j_{c'})$ and $\min_{p\in\mathcal{P}^j\setminus\bar{p}^j_c} c(p) = \min_{p\in\mathcal{P}^j\setminus\bar{p}^j_{c'}} c'(p)$, we have:*

$$y_j(P^j) = y_j(P'^j)$$

for all $j \in N$.

If the shortest path to j remains the same from the cost matrix c to the cost matrix c', then the axiom Supplier Equal Change says that all suppliers of j should see their cost shares change in the same way. Since the same agents contribute to shipping j's demand under both matrices, it is natural to require that no supplier of j be affected more than the others by the passage from c to c'.

Path Independence says that a demander j should pay the same subsidy to her respective groups of suppliers (under c and c') if these groups have the same added-value.

In other words, what should determine the amount paid to the suppliers is the reduction in the demander's shipping cost rather than the size or composition of the group of suppliers.

The next theorem characterizes the family of *ADACS* that satisfy these three properties.

Theorem 6.1 *An ADACS y satisfies Supplier Equal Change, Path Independence, and Two-path Cost Additivity **if and only if** it is a convex combination of the supplier rule and the demander rule, that is to say, if and only if $y = \alpha \cdot y^s + (1-\alpha) \cdot y^d$ for some $\alpha \in [0,1]$.*

Proof. It is not difficult to check that both y^d and y^s (and hence their convex combinations) are ADACS that satisfy Supplier Equal Change, Path Independence and Two-path Cost Additivity. Fix an ADACS y that meets Supplier Equal Change, Path Independence and Two-path Cost Additivity. By Demand Additivity, it suffices to show that there exists $\alpha \in [0,1]$ such that $y(c, e^j) = \alpha \cdot y^s(c, e^j) + (1-\alpha) \cdot y^d(c, e^j)$, for all $j \in N$ and all $c \in \mathcal{C}$. Consider an arbitrary $j \in N$ and, for all $c \in \mathcal{C}$, denote by $\beta_j(c) = \min_{p \in \overline{\mathcal{P}}^j(c)} c(p)$ the cost of every shortest path (to j) under c. Moreover, picking any $\bar{p}^j$ such that $c(\bar{p}^j) = \beta_j(c)$, write $\gamma_j(c) = \min_{p \in \mathcal{P}^j \setminus \{\bar{p}^j\}} c(p)$ to denote the cost of every second-shortest path to j.[1] Let us then define the following sets of cost matrices:

$$\mathcal{C}_1^j = \{c \in \mathcal{C} : c(\mathbf{0}, j) = \beta_j(c) \text{ or } \gamma_j(c) = \beta_j(c)\}\,; \tag{14}$$

$$\mathcal{C}_2^j = \left\{c \in \mathcal{C} : c(\mathbf{0}, j) = \gamma_j(c) = \min_{p \in \mathcal{P}^j:\ c(p) > \beta_j(c)} c(p)\right\}; \tag{15}$$

$$\mathcal{C}_3^j = \left\{c \in \mathcal{C} : c(\mathbf{0}, j) > \gamma_j(c) = \min_{p \in \mathcal{P}^j:\ c(p) > \beta_j(c)} c(p)\right\}. \tag{16}$$

Remark that $\mathcal{C} = \mathcal{C}_1^j \cup \mathcal{C}_2^j \cup \mathcal{C}_3^j$. For any $c \in \mathcal{C}$, construct $\tilde{c} \in \mathcal{C}$ as follows:

$$\tilde{c}(k,l) = \begin{cases} c(k,l) & \text{if } (k,l) \neq (\mathbf{0}, j); \\ c(\bar{p}^j) & \text{if } (k,l) = (\mathbf{0}, j). \end{cases}$$

Note that $\bar{p}^j$ (a shortest path for c) is also by construction a shortest path for $\tilde{c}$, with $\tilde{c}(\bar{p}^j) = c(\bar{p}^j)$. Moreover, applying Core Selection gives $\begin{cases} y_j(\tilde{c}, e^j) \leq \tilde{c}(\mathbf{0}, j) = c(\bar{p}^j) \\ y_j(\tilde{c}, e^j) \leq 0, \forall i \neq j. \end{cases}$ Since efficiency requires $y_j(\tilde{c}, e^j) + y_{N\setminus j}(\tilde{c}, e^j) = c(\bar{p}^j)$, it thus follows that $y_j(\tilde{c}, e^j) = c(\bar{p}^j)$

[1] In case there exist *multiple* shortest paths to j, note that $\gamma_j(c)$ is independent of which one is picked. Indeed, observe in this case that $\gamma_j(c) = \min_{p \in \overline{\mathcal{P}}^j(c) \setminus \{\bar{p}^j\}} c(p) = \beta_j(c)$, for *any* $\bar{p}^j$ such that $c(\bar{p}^j) = \beta_j(c)$.

and $y_i(\tilde{c}, e^j) = 0$ for all $j \neq i$. Hence, Supplier Equal Change gives $y_i(c, e^j) - 0 = y_k(c, e^j) - 0$, that is,

$$y_i(c, e^j) = y_k(c, e^j) \quad \forall i, k \in [\bar{p}^j] \setminus j. \tag{17}$$

From this point on, we will follow six steps to complete the proof.

Step 1. For all $c \in \mathcal{C}_1^j$, we have $y_i(c, e^j) = 0 = y_i^d(c, e^j) = y_i^s(c, e^j)$, $\forall i \in N \setminus j$.

Fix $c \in \mathcal{C}_1^j$. Suppose first that c satisfies $c(\mathbf{0}, j) = \beta_j(c)$. Then Core Selection requires

$$\begin{cases} y_j(\tilde{c}, e^j) \leq \tilde{c}(\mathbf{0}, j) = \beta_j(c) \\ y_j(\tilde{c}, e^j) \leq 0, \forall i \neq j \\ y_j(\tilde{c}, e^j) + y_{N \setminus j}(\tilde{c}, e^j) = \beta_j(c), \end{cases}$$

which implies that $y_i(c, e^j) = 0 = y_i^d(c, e^j) = y_i^s(c, e^j)$ for all $j \neq i$. Suppose next that $\gamma_j(c) = \beta_j(c)$. Then there exists another shortest path $\hat{p}^j \in \overline{\mathcal{P}}^j(c)$, that is, $c(\hat{p}^j) = c(\bar{p}^j) = \min_{p \in \mathcal{P}^j} c(p)$, and $k \notin [\hat{p}^j]$ for some $k \in [\bar{p}^j]$. Note that we have $y_k(c, e^j) = 0$ by Core Selection. Indeed, we must have $y_k(c, e^j) \leq C_{P_j}(k) = 0$; and assuming $y_k(c, e^j) < 0$ implies $y_{N \setminus k}(c, e^j) = c(\hat{p}^j) - y_k(c, e^j) > c(\hat{p}^j) = C_{P_j}(N \setminus k)$, which contradicts Core Selection. Substituting $y_k(c, e^j) = 0$ in (17) then gives $y_i(c, e^j) = 0$ for all $i \in \bar{p}^j$ and it easily follows that $y_i(c, e^j) = 0 = y_i^d(c, e^j) = y_i^d(c, e^j)$ for any $i \neq j$.

Step 2. For all $c \in \mathcal{C}_2^j \cup \mathcal{C}_3^j$, there exists $\alpha_j(c) \geq 0$ s.t. $y_i(c, e^j) = -\alpha_j(c) \frac{\gamma_j(c) - \beta_j(c)}{|\bar{p}^j| - 1}$, $\forall i \in [\bar{p}^j] \setminus j$.

For any $c \in \mathcal{C}_2^j \cup \mathcal{C}_3^j$, note that $\gamma_j(c) - \beta_j(c) > 0$ and then define

$$\alpha_j(c) = -\frac{y_{N \setminus j}(c, e^j)}{\gamma_j(c) - \beta_j(c)}. \tag{18}$$

Remark that $\alpha_j(c) \geq 0$ by Core Selection. Recalling (17) yields the desired result: $y_i(c, e^j) = -\frac{\alpha_j(c)}{|\bar{p}^j| - 1}$.

The following steps will show that $\alpha_j(c)$ is in fact independent of c and j.

Step 3. We have $\alpha_j(c + \hat{c}) = \alpha_j(c)$, for all $c \in \mathcal{C}_2^j \cup \mathcal{C}_3^j$ and all $\hat{c} \in \mathcal{C}_1^j$ that have a *common shortest path* and a *common second-shortest path*.

Fix $c \in \mathcal{C}_2^j \cup \mathcal{C}_3^j$, $\hat{c} \in \mathcal{C}_1^j$; and suppose that c and $\hat{c}$ have a common shortest path and a common second-shortest path. It is easy to see that $\beta_j(c + \hat{c}) = \beta_j(c) + \beta_j(\hat{c})$, $\gamma_j(c + \hat{c}) = \gamma_j(c) + \gamma_j(\hat{c}) = \gamma_j(c) + \beta_j(\hat{c})$; and hence $\gamma_j(c + \hat{c}) - \beta_j(c + \hat{c}) = \gamma_j(c) - \beta_j(c)$. Moreover, Two-path Cost Additivity yields $y_i(c + \hat{c}, e^j) = y_i(c, e^j) + y_i(\hat{c}, e^j)$. Substituting the last two equalities in (18) thus gives

$$\alpha_j(c + \hat{c}) = -\frac{y_{N \setminus j}(c, e^j) + \overbrace{y_{N \setminus j}(\hat{c}, e^j)}^{=0 \text{ by Step 1}}}{\gamma_j(c) - \beta_j(c)} = -\frac{y_{N \setminus j}(c, e^j)}{\gamma_j(c) - \beta_j(c)} = \alpha_j(c).$$

Step 4. For all $c \in \mathcal{C}_2^j \cup \mathcal{C}_3^j$ and all $\theta > 0$, we have $\alpha_j(\theta c) = \alpha_j(c)$.

Fix $c \in \mathcal{C}_2^j \cup \mathcal{C}_3^j$ and $\theta > 0$. Note first that we have $\beta_j(\theta c) = \theta \beta_j(c)$ and $\gamma_j(\theta c) = \theta \gamma_j(c)$, since c and λc have the same shortest path(s) and second-shortest path(s) to j (for all $\lambda > 0$).

Second, remark that the statement of Step 4 holds if $\theta = \frac{a}{b} \in \mathbb{Q}$ (with $a, b \in \mathbb{N} \setminus \{0\}$). Indeed, since c, $\frac{a}{b}c$ and $\frac{1}{b}c$ all have the same shortest path(s) and second-shortest path(s) to j, applying Two-path Cost Additivity repeatedly gives

$$y(\theta c, e^j) = y(a(\frac{1}{b}c), e^j) = a \cdot y(\frac{1}{b}c, e^j) = a[\frac{1}{b} \cdot y(c, e^j)] = \frac{a}{b} \cdot y(c, e^j) = \theta \cdot y(c, e^j).$$

Combining this equality and $\gamma_j(\theta c) = \theta \gamma_j(c)$ in (18) gives $\alpha_j(\frac{a}{b} \cdot c) = \alpha_j(c)$.

Suppose now that $\theta \notin \mathbb{Q}$ and, for any $t \in \{1, 2, \ldots\}$, write $\theta = q_t \frac{1}{2^t} + r_t$ (with $q_t \in \mathbb{N}$ and $0 \leq r_t < \frac{1}{2^t}$) as the result of the Euclidean division of θ by the rational number $\frac{1}{2^t}$. This means in particular that $\lim_{t \to \infty} r_t = 0$ and $\lim_{t \to \infty} \frac{q_t}{2^t} = \theta$.

It follows from Two-path Cost Additivity that

$$\begin{aligned} y_{N \setminus j}(\theta c, e^j) =& y_{N \setminus j}(\frac{q_t}{2^t} c + r_t c, e^j) \\ =& y_{N \setminus j}(\frac{q_t}{2^t} c, e^j) + y_{N \setminus j}(r_t c, e^j) \\ =& \frac{q_t}{2^t} y_{N \setminus j}(c, e^j) + y_{N \setminus j}(r_t c, e^j) \quad \text{since } \frac{q^t}{2^t} \in \mathbb{Q}. \end{aligned} \tag{19}$$

By Core Selection, we must have $-\underbrace{r_t(c(\mathbf{0}, j) - \alpha_j(c))}_{\to 0} \leq y_{N \setminus j}(r_t c, e^j) \leq 0$. Substituting these two inequalities in (19) and taking the limit thus gives

$$\lim_{t \to \infty} \frac{q_t}{2^t} y_{N \setminus j}(c, e^j) - 0 \leq \lim_{t \to \infty} y_{N \setminus j}(\theta c, e^j) \leq \lim_{t \to \infty} \frac{q_t}{2^t} y_{N \setminus j}(c, e^j).$$

That is, $y_{N \setminus j}(\theta c, e^j) = \lim_{t \to \infty} y_{N \setminus j}(\theta c, e^j) = \lim_{t \to \infty} \underbrace{\frac{q_t}{2^t}}_{\to \theta} y_{N \setminus j}(c, e^j) = \theta y_{N \setminus j}(c, e^j)$. Finally, using $y_{N \setminus j}(\theta c, e^j) = \theta y_{N \setminus j}(c, e^j)$ in (18) [and recalling that $\beta_j(\theta c) = \theta \beta_j(c)$, $\gamma_j(\theta c) = \theta \gamma_j(c)$] gives the desired result, $\alpha_j(c) = \alpha_j(\theta c)$.

Step 5. We have $\alpha_j(c) = \alpha_j(c') \leq 1$, for all $c, c' \in \mathcal{C}_2^j$.

Let $c, c' \in \mathcal{C}_2^j$. We will distinguish two cases.

Substep 5.1. Suppose first that $c(\bar{p}_c^j) = c'(\bar{p}_{c'}^j)$ and $\overbrace{\min_{p \in \mathcal{P}^j \setminus \bar{p}^j} c(p)}^{\beta_j(c)} = \overbrace{\min_{p \in \mathcal{P}^j \setminus \bar{p}_{c'}^j} c'(p)}^{\beta_j(c')}$.

Then it follows from Path Independence that $y_j(c, e^j) = y_j(c', e^j)$, that is to say, $y_{N \setminus j}(c, e^j) = c(\bar{p}^j) - y_j(c, e^j) = c'(\bar{p}_{c'}^j) - y_j(c', e^j) = y_{N \setminus j}(c', e^j)$. Substituting in (18) hence gives $\alpha_j(c) = \alpha_j(c')$.

Substep 5.2. Suppose now that $c(\bar{p}^j) \neq c'(\bar{p}^j_{c'})$ or $\beta_j(c) \neq \beta_j(c')$.

Since $c, c' \in \mathcal{C}_2^j$, notice that $\beta_j(c) - \alpha_j(c) > 0$ and $\beta_j(c') - \alpha_j(c') > 0$. Letting then $\theta = \frac{\beta_j(c)-\alpha_j(c)}{\beta_j(c')-\alpha_j(c')} > 0$, define the cost matrix $\tilde{c}' \equiv \theta c' \in \mathcal{C}_2^j$. It comes from Step 4 that

$$\alpha_j(\tilde{c}') = \alpha_j(\tilde{c}). \tag{20}$$

Also note that we have

$$\beta_j(\tilde{c}') - \alpha_j(\tilde{c}') = \theta(\beta_j(c') - \alpha_j(c')) = \beta_j(c) - \alpha_j(c). \tag{21}$$

Assuming without loss of generality that $\delta \equiv \alpha_j(\tilde{c}') - \alpha_j(c) > 0$ and $\bar{p}^j = (\mathbf{0}, i_1, \ldots, i_{K-1}, j)$, define the cost matrix $\hat{c}$ as follows:

$$\hat{c}(k,l) = \begin{cases} \delta & \text{if } (k,l) = (\mathbf{0}, j), (\mathbf{0}, i_1); \\ 0 & \text{if } (k,l) = (i_t, i_{t+1}) \text{ for some } t = 1, \ldots, K-1; \\ \delta + 1 & \text{otherwise.} \end{cases}$$

Note that **(i)** $\bar{p}^j = (\mathbf{0}, i_1, \ldots, i_{K-1}, j)$ is a shortest path for $\hat{c}$ (as well as c); and **(ii)** $(\mathbf{0}, j)$ is a second-shortest path for $\hat{c}$ and (as well as c), with $\hat{c}(\bar{p}^j) = \hat{c}(\mathbf{0}, j) = \delta$ (i.e, $\hat{c} \in \mathcal{C}_1^j$). Therefore, letting $\tilde{c} = c + \hat{c}$, it comes from Step 3 that

$$\alpha_j(\tilde{c}) = \alpha_j(c + \hat{c}) = \alpha_j(c). \tag{22}$$

Furthermore, (i)-(ii) above mean that

$$\alpha_j(\tilde{c}) = \delta + \alpha_j(c) = \alpha_j(\tilde{c}'); \; \beta_j(\tilde{c}) = \delta + \beta_j(c) = \alpha_j(\tilde{c}') - \alpha_j(c) + \beta_j(c) \overbrace{=}^{\text{by (21)}} \beta_j(\tilde{c}'). \tag{23}$$

Since $\tilde{c}, \tilde{c}' \in \mathcal{C}_2^j$, it comes from (23) and Substep 5.1 above that $\alpha_j(\tilde{c}) = \alpha_j(\tilde{c}')$. Combining this equality with (22) and (20) hence gives $\alpha_j(c) = \alpha_j(c')$, which is the desired result.

To conclude Step 5, remark that writing $\alpha_j(c) > 1$ for some $c \in \mathcal{C}_2^j$ would mean in Equation (18) that $y_{N\setminus j}(c, e^j) < -(\gamma_j(c) - \beta_j(c))$, that is to say,

$$y_j(c, e^j) = \beta_j(c) - y_{N\setminus j}(c, e^j) > \beta_j(c) + (\gamma_j(c) - \beta_j(c)) = \gamma_j(c) = c(\mathbf{0}, j);$$

and this would violate Core Selection.

Step 6. For all $c, c' \in \mathcal{C}_2 \cup \mathcal{C}_3$, we have $\alpha_j(c) = \alpha_j(c')$.

For any $c \in \mathcal{C}_2 \cup \mathcal{C}_3$, perform the decomposition $c = \tilde{c} + \hat{c}$, where

$$\tilde{c}(k,l) = \begin{cases} \gamma_j(c) & \text{if } (k,l) = (\mathbf{0}, j) \\ c(k,l) & \text{otherwise} \end{cases} \tag{24}$$

$$\hat{c}(k,l) = \begin{cases} c(0,j) - \gamma_j(c) & \text{if } (k,l) = (\mathbf{0}, j) \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Since $c(\mathbf{0}, j) \geq \gamma_j(c) > c(\bar{p}^j)$ (because $\hat{c} \in \mathcal{C}_2 \cup \mathcal{C}_3$), observe from (24)-(25) that (**i**) $\bar{p}^j$ is a common shortest path for $\tilde{c}$ and $\hat{c}$; (**ii**) $(\mathbf{0}, j)$ is a common second-shortest path for $\tilde{c}$ and $\hat{c}$; (**iii**) $\tilde{c} \in \mathcal{C}_2^j$ and $\hat{c} \in \mathcal{C}_1^j$. Hence, by Step 3, we have

$$\alpha_j(c) = \alpha_j(\tilde{c}), \ \forall c \in \mathcal{C}_2 \cup \mathcal{C}_3. \tag{26}$$

Pick then any $c, c' \in \mathcal{C}_2 \cup \mathcal{C}_3$. It comes from (26) that $\alpha_j(c) = \alpha_j(\tilde{c})$ and $\alpha_j(c') = \alpha_j(\tilde{c}')$. But given that $\tilde{c}, \tilde{c}' \in \mathcal{C}_2^j$, we have $\alpha_j(\tilde{c}) = \alpha_j(\tilde{c}')$ from Step 5; and hence $\alpha_j(c) = \alpha_j(c')$.

We are now ready to conclude the proof of Theorem 6.1. The six steps above allow to claim that, for all $j \in N$, there exists $\alpha_j \in [0,1]$ such that

$$y(c, e^j) = \alpha_j \underbrace{y^s(c, e^j)}_{\gamma_j(c)} + (1 - \alpha_j) \underbrace{y^d(c, e^j)}_{\beta_j(c)}, \quad \forall c \in \mathcal{C}. \tag{27}$$

It now remains to show that $\alpha_j = \alpha_{j'}$ for any $j, j' \in N$.

Fix distinct $j, j' \in N$ and pick any $c \in \mathcal{C}$ such that $\gamma_j(c) > \beta_j(c)$. Recalling (27) gives

$$y_j(c, e^j) = \beta_j(c) + \alpha_j(\gamma_j(c) - \beta_j(c)). \tag{28}$$

Defining $c' \in \mathcal{C}$ by $c'(k,l) = c(\sigma_{jj'}(k), \sigma_{jj'}(l))$ for all (k,l), remark that $\gamma_{j'}(c') = \gamma^j(c)$ and $\beta_{j'}(c') = \beta_j(c)$. It thus comes from (27) that

$$y_{j'}(c', e^{j'}) = \beta_{j'}(c') + \alpha_{j'}(\gamma_{j'}(c') - \beta_{j'}(c')) = \beta_j(c) + \alpha_{j'}(\gamma_j(c) - \beta_j(c)). \tag{29}$$

Since c and c' are jj'-symmetric, observe that Anonymity gives $y_{j'}(c', e^{j'}) = y_j(c, e^j)$. Combining this equality with (28) and (29) finally gives $\alpha_j = \alpha_{j'}$. ∎

Theorem 6.1 says that, within the set of $ADACS$, we must pick a convex combination of y^s and y^d if one requires the cost sharing mechanism to satisfy the three requirements of Supplier Equal Change, Path Independence, and Two-path Cost Additivity. Within this family, the demander rule y^d is the most advantageous to the demander j and the supplier rule y^s is the most advantageous to j's suppliers. A natural compromise is obtained by taking the average of these two extremes: $y = \frac{1}{2}y^s + \frac{1}{2}y^d$.

We now argue that the three axioms used in the characterization are independent. First, define the ADACS $\tilde{y}$: for all $(c, e^j) \in \mathbb{P}$ and $i \in N$,

$$\tilde{y}_i(c, e^j) = \begin{cases} \frac{\beta(c) - \gamma(c)}{|\bar{p}^j|} & \text{if } i \in [\bar{p}^j] \setminus j; \\ \gamma(c) + \frac{\beta(c) - \gamma(c)}{|\bar{p}^j|} & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\tilde{y}$ satisfies Supplier Equal Change and Two-Path Additivity, but it violates Path Independence because the cost share paid by a demander, $\tilde{y}_j(c, e^j) = \gamma(c) + \frac{\beta(c)-\gamma(c)}{|\bar{p}^j|}$, depends on the length of the shortest path to j. Indeed, if we take two matrices c and c' such that $\beta(c) = \beta(c')$ and $\gamma(c) = \gamma(c')$ but $|\bar{p}^j_c| > |\bar{p}^j_{c'}|$, then we will have $y_j(c, e^j) > y_j(c, e^j)$, a violation of Path Independence.

Second, define the ADACS $\hat{y}$: for all (c, e^j) and $i \in N$,

$$\hat{y}_i(c, e^j) = \begin{cases} \beta(c) - \gamma(c) & \text{if } i = \min_{k \in [\bar{p}^j] \setminus j} k; \\ \gamma(c) & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Remark that $\tilde{y}$ satisfies Two-Path Additivity and Path Independence, but it does not meet Supplier Equal Change since only the lowest-label supplier of j sees her cost share decrease when we move from a matrix c to a matrix c' such that $\bar{p}^j \in \overline{\mathcal{P}}^j(c) \cap \overline{\mathcal{P}}^j(c')$ and $\gamma(c) = 0 < \beta(c') - \gamma(c')$. This is a violation of Supplier Equal Change.

Finally, recalling the demander rule y^d and the supplier rule y^s, consider the ADACS $\breve{y}$ defined as follows: for all (c, e^j) and $i \in N$,

$$\breve{y}(c, e^j) = \begin{cases} y^s(c, e^j) & \text{if } \beta(c) > 100 \\ y^d(c, e^j) & \text{otherwise.} \end{cases}$$

It is easy to see that $\breve{y}$ satisfies Supplier Equal Change and Path Independence. However, note that $\breve{y}$ violates Two-Path Additivity since $\breve{y}(\lambda \cdot c, e^j) \neq \lambda \cdot \breve{y}(c, e^j)$ for any $c \in \mathcal{C}$ such that $\beta(c) = 60$ and any $\lambda \geq 2$, even though the two matrices c and $\lambda \cdot c$ have identical shortest paths and identical second-shortest paths.

7 Conclusion

The paper has introduced the family of Anonymous and Demand-Additive Core Selections (or ADACS) for shortest path problems, which are network problems where the shipping cost on every arc (linking two nodes) is linear in the flow crossing it.

We have identified three remarkable rules that belong to the family of ADACS: the demander rule, the supplier rule, and the alexia rule.

Besides the standard axiom of Core Selection, we have introduced and studied many properties that are natural for shortest path problems. In particular, it has been shown that only restricted versions of Cost Additivity are possible. With three players or more, the property of One-path Cost Additivity (combined with Core Selection) characterizes the demander rule. On the other hand, we have shown that the combination of Two-Path Cost Additivity, Supplier Equal Change and Path Independence characterizes the convex combinations of the demander rule and the supplier rule (within the family of ADACS).

These results provide the first axiomatizations of cost sharing rules in the context of shortest path problems. Future research proposing additional rules, axioms, or characterization results would certainly contribute to this literature.

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