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Dynamic Price Dispersion in a Bertrand-Edgeworth Model

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Abstract

This paper considers a dynamic model of price competition in which sellers are endowed with one unit of the good and compete by posting prices in every period. Buyers each demand one unit of the good and have a common reservation price. They have full information regarding the prices posted by each firm in the market; hence, search is costless. The number of buyers coming to the market in each period is random. We characterize the dynamics of market prices and show that price dispersion persists over time.

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1 Introduction

This paper captures the dynamic properties of price dispersion by introducing an infinite time horizon model of Bertrand-Edgeworth competition. Specifically, we consider a dynamic model of price competition in which sellers are endowed with one unit of the good and compete by posting prices in every period to maximize their expected profits with discounting. The number of buyers coming to the market in each period is random. Buyers each demand one unit of the good and have a common reservation price. They have full information regarding the prices posted by each firm in the market; hence, search is costless. We show that when excess demand occurs with positive probability, our model has a unique (symmetric) mixed-strategy equilibrium. In this equilibrium, sellers post prices according to non-degenerate distributions determined by the number of sellers, and the lowest possible market price, defined as the greatest lower bound of the support of the distribution played by sellers, is decreasing in the number of sellers. In other words, interfirm price dispersion not only exists in every period, but it also persists over time.

The used-textbook market on the internet is one example which fits our model well. At the end of each academic year, some students (sellers) try to sell their used textbooks by posting them on Amazon.com. Each seller has only one used textbook for sale, and we may assume that sellers adjust prices daily to maximize the expected revenue with discounting. Books are treated homogeneously by potential buyers if they have the same used-condition (e.g., like new). Clearly, buyers each demand one used textbook, and it is reasonable to assume that they have approximately the same reservation price. Furthermore, this market features demand uncertainty: the number of buyers coming to the market is uncertain in every period. Consequently this market satisfies the framework of our model, and, consistent with the prediction of our model, the phenomenon of price dispersion is prevalent in this market.[1]

Other applications of our model include certain labor markets. Consider the following environment. A fixed number of workers in the labor market wait for an opportunity to be hired by potential employers by posting their wages in each period. Workers have the same productivity; hence, they are homogeneous to potential employers. In every period, employers come to the market randomly. There is one position to be filled in each firm; thus each employer hires the worker with the lowest posted wage if it is below some reservation wage for the firm.

The model can also be applied to "scalping." At some universities, students are eligible to buy one ticket for most sports events. Some students offer to resell the ticket by posting a price on ebay.com. Interested buyers come to the market randomly. Tickets are homogeneous if they are located in the same section, and the buyer buys one ticket if the price is less than some reservation price. Students change the price daily (or hourly) to maximize their expected revenue with discounting.

Conventional wisdom says that the "law of one price" holds in markets of homogeneous goods, but substantial studies provide evidence that price dispersion is ubiquitous and persistent even on the internet, the so called frictionless market. Brynjolfsson and Smith (2000) investigate prices of books and CD's on the internet and at conventional sellers over a 15-month period and find that, although internet prices are lower than in conventional stores, price dispersion is more significant on the internet. Bailey (1998) compares the prices of 125 books, 108 musical titles, and 104 software titles sold in 1996 and 1997 through 52 internet and conventional outlets and finds that price dispersion among e-tailers is as least as great as that among conventional retailers.

[1]For example, when we checked the textbook "Probability and Measure" by Patrick Billingsley at Amazon.com, we got 29 sellers offering prices for new books and these prices range from $67 all the way up to $159.

Several previous studies have offered models to explain this puzzling empirical finding. Some authors attribute the occurance of price dispersion to heterogeneity of consumers or firms (e.g., Gatti (2000), Janssen and Moraga (2004), Milyo and Waldfogel (1999), Reinganum (1979), Rosenthal (1980), Salop and Stiglitz (1976), and Varian (1980)). Reinganum (1979) assumes that firms have different production costs. In Salop and Stiglitz (1976), consumers have different costs of information acquisition. In Wilde and Schwartz (1979), consumers have different propensities to search.

Some papers address the issue of price dispersion by introducing demand uncertainty or capacity constraints into the model (e.g., Arnold (2000), Dana (1999) and Prescott (1975)). Equilibrium price dispersion under perfect competition with demand uncertainty was first introduced by Prescott (1975) in his well-known "hotel model". Suppose there is a stochastic demand n for hotel rooms with distribution $F(n)$. All demanders are identical and purchase one unit if the price is below some reservation price $\overline{p}$. Owners of rooms incur cost c of providing one unit and set their price before the realization of demand, and entry is free. Under such a market structure, Prescott shows that in an equilibrium we have a distribution of prices rather than a single price.

Deneckere and Peck (2005) develop a dynamic version of Prescott's "hotel model." In an equilibrium, they show that there is price dispersion in the first period, followed by all firms learning the demand and charging the market-clearing price in the second period. The price dispersion comes from the trade-off between selling at the current price or selling at a higher price later if demand is strong or a lower price later if demand is revealed to be weak. Moreover, in Prescott's static model equilibrium is inefficient whenever consumers are heterogeneous. They demonstrate that efficiency can be restored by embedding the model in a dynamic framework.

Assuming bounded rationality, Baye and Morgan (2004) show that price dispersion occurs in epsilon and quantal response equilibria in a homogeneous product market where the uniform price Nash equilibrium is predicted in Bertrand competition. Statistical tests of two laboratory experiment data sets support the bounded rationality hypothesis and the pattern of price dispersion derived from it.

Burdett and Judd (1983), Gatti (2000) and Stigler (1961) characterize price dispersion equilibrium from the viewpoint of consumer search behavior. Burdett and Judd demonstrate that ex ante heterogeneity is not necessary to explain the appearance of price dispersion. In their model, firms maximize their expected profits given their beliefs about consumer search behavior, and consumers minimize their expected cost (i.e., buying price plus search cost) of purchasing one unit given their beliefs about the distribution of prices in the market. They construct a price dispersion equilibrium in which some consumers search once and others search twice. Price dispersion equilibria, however, cannot be sustained if search is costless. Thus the phonomenon of price dispersion on the internet cannot be explained from their model.

This paper proposes an alternative explanation of the existence of price dispersion in a homogeneous product market. Unlike other papers which assume static competition, our model allows sellers to compete in multiple periods. With discounting, each seller prefers to sell earlier rather than sell later if they can sell at the same price. We call this a discounting effect. On the other hand, the market is less competitive in the future since sellers with lower prices will be out of the market. So in order to sell earlier a seller must accept a lower price. We call this a competition effect. Consequently sellers maximize expected profit by balancing these two effects. In an equilibrium sellers play mixed-strategies because they are indifferent between posting a lower price and selling earlier and posting a higher price but selling later. This indifference contributes to the appearance of price dispersion.

The paper is organized as follows. Section 2 presents a simple binary demand model. Section 3 extends the model in section 2 by generalizing the underlying demand structure, and demonstrates that price dispersion occurs when excess demand occurs with positive probability. Section 4 concludes.

2 The Binary Demand Model

Suppose there are N sellers in the market, and each of them has one unit of the good. Goods are non-perishable and homogeneous. The cost is normalized to 0 for each seller. Time is discrete and is indexed by $t = 0, 1, 2, ...$ in an infinite horizon. Denote p_{it} to be the price posted by seller i in period t. Each seller has to post a price before the demand is revealed in every period. In each period, the demand is either 0 or 1 with probability q and $1 - q$ respectively, where $0 < q < 1$. All potential buyers have the same reservation price $\overline{p}$. The buyer arriving in period t desires one unit of the good and can only buy the object in period t, so waiting is not allowed. Hence, the arriving buyer buys one unit of the good at the lowest price if and only if the price is no greater than $\overline{p}$. We assume that when there is a tie (the number of sellers at the lowest price is more than 1), each seller has an equal chance of making the sale.[2] Each seller i chooses price path $\{p_{it}\}$ to maximize the present discounted value of expected profits with a common discount factor δ, where $0 < \delta < 1$. Once the unit is sold, the seller is out of the market. Denote the current state variable S_t to be the number of units of goods remaining in the market at the beginning of period t. A selling game described above is given by $\Gamma(N, q)$. Since tacit collusion is not what we are concerned with, throughout this paper the equilibrium concept we use is subgame perfect Nash equilibrium(SPNE)

[2]This assumption can be dropped without changing any of our results.

without tacit collusion.

Definition 1 *Subgame perfect Nash equilibrium without tacit collusion: A SPNE in which each seller's strategy in every period depends on the current state variable S_t only. Hence, intertemporal punishments are not allowed in an equilibrium.*

First we notice that the environment is completely homogeneous. Goods are homogeneous, consumers have the same reservation price, and sellers have the same discount factor, capacity constraints and cost structure. Second, we see that there is no search cost. Consumers can observe the prices posted by each seller without paying any cost.

Next we offer some definitions. In the selling game $\Gamma(N, q)$, define $V(\overline{p}, N)$ to be the expected profit for a seller who simply posts a price $\overline{p}$ in every period and others post prices strictly less than $\overline{p}$. Then $V(\overline{p}, 1)$ is just the monopoly profit and can be calculated as

$$\begin{aligned} V(\overline{p}, 1) &= q\delta V(\overline{p}, 1) + (1-q)\overline{p} \\ &= \frac{1-q}{1-\delta q}\overline{p} \qquad (1) \end{aligned}$$

We have the following relation:

$$\begin{aligned} V(\overline{p}, N) &= q\delta V(\overline{p}, N) + (1-q)\delta V(\overline{p}, N-1) \\ &= \frac{(1-q)\delta V(\overline{p}, N-1)}{1-\delta q} \qquad (2) \end{aligned}$$

In the selling game $\Gamma(N, q)$, denote by p_N^* the price which gives a seller an expected profit equal to $V(\overline{p}, N)$ if he simply posts a price p_N^* in every period that is the lowest price in the market and there are no other sellers setting a price at p_N^*. Therefore p_N^* satisfies the following equation:

$$V(\overline{p}, N) = \frac{1-q}{1-\delta q}p_N^* \qquad (3)$$

From equations (1), (2) and (3) we can solve

$$V(\overline{p}, N) = \left[\frac{(1-q)\delta}{1-\delta q}\right]^{N-1} \frac{1-q}{1-\delta q}\overline{p} \tag{4}$$

$$p_N^* = \left[\frac{(1-q)\delta}{1-\delta q}\right]^{N-1} \overline{p} \tag{5}$$

We observe that $p_1^* = \overline{p}$ and $p_N^* = \delta V(\overline{p}, N-1)$.

Our first proposition characterizes an equilibrium in duopoly with the binary demand structure.

Proposition 1 *For the binary demand duopoly model, each seller adopts the strategy $p_t = p_n^*$ for all t s.t. $S_t = n$ constitutes a unique equilibrium in the selling game $\Gamma(2, q)$, and the expected profit for each seller is exactly $V(\overline{p}, 2)$. Hence, there is no price dispersion.*

Proof. See the Appendix.

Proposition 1 states that when there are two sellers in the market and in each period there is at most one consumer coming to the market with some positive probability $1-q$, a unique equilibrium must have both sellers set their prices at $p_2^* > 0$. Hence any asymmetric pricing behavior, pure or mixed, will never be observed. Therefore, price disperison cannot happen in duopoly with binary demand. We notice that the probability of excess demand is zero since there is at most one consumer coming to the market in every period. In section 3, we will show that price dispersion must happen when the probability of excess demand is positive.

Next we characterize the equilibrium in the selling game $\Gamma(N, q)$. Let $F_{it}(p)$ denote the equilibrium mixed strategy seller i plays in period t in the selling game $\Gamma(N, q)$. Define $l_{it} = \sup\{p : F_{it}(p) = 0\}$ and $u_{it} = \inf\{p : F_{it}(p) = 1\}$. Define $\Lambda_t(p)$ to be the number of sellers who set a price at p with probability one in period t.

Proposition 2 *For the binary demand model with N firms, any equilibirum in the selling game* $\Gamma(N,q)$ *has the following properties: (a)* $l_{it} \geq p_k^* \ \forall\ i$*, for all* t *s.t.* $S_t = k$*, (b)* $\Lambda_t(p_k^*) \geq 2$ *for all* t *s.t.* $S_t = k \geq 2$*, and (c) the expected profit for each seller is exactly* $V(\overline{p}, N)$*.*

Proof. Since most of the reasoning are parallel to the proof in Proposition 1, here we just give a sketch of the proof. When $N = 2$, the conclusion follows from Proposition 1. The proof proceeds by induction. Suppose the Proposition holds in the selling game $\Gamma(N-1,q)$. For a selling game $\Gamma(N,q)$, pick any period t with $S_t = N$ and WLOG let $l_{1t} \leq l_{2t} \leq ... \leq l_{Nt}$. Following the same line of reasoning in Claim 1 of Proposition 1, it is easy to show that $l_{1t} = l_{2t} \geq p_N^*$ and (a) is established. For part (b), let us define $V_i(t)$ to be the present discount value of expected profit to the seller i at period t, $V_i(t+1)$ to be the present discounted value of seller i's expected profit in period $t+1$ conditional on zero demand in period t, and $F_{-i}(p)$ to be the joint probability that all sellers except i set the price less or equal to p. Clearly $F_{-i}(p)$ is right continuous. Now we notice that $l_{1t} = l_{2t} > p_N^*$ cannot be sustained in any equilibrium. If it were indeed the case, then the expected profit for seller 1 from posting a price at l_{1t} would be

$$V_1(t) = q\delta V_1(t+1) + (1-q)[F_{-1}(l_{1t})\frac{1}{2}(\delta V(\overline{p}, N-1) + l_{1t}) + (1 - F_{-1}(l_{1t}))l_{1t}] \tag{6}$$

Observing that $F_{-1}(l_{1t}) > 0$ cannot be the case, as it implies that

$$\begin{aligned} V_1(t) &\rightarrow q\delta V_1(t+1) + (1-q)[F_{-1}(l_{1t})\frac{1}{2}(\delta V(\overline{p}, N-1) + l_{1t}) + (1 - F_{-1}(l_{1t}))l_{1t}] \\ &< q\delta V_1(t+1) + (1-q)l_{1t} \text{ as } p \searrow l_{1t} \ , \end{aligned} \tag{7}$$

which implies that either a maximizer does not exist for seller 1 or $\sup\{p : F_{1t}(p) = 0\} > l_{it}$, a contradiction. But $F_{-1}(l_{1t})$ also cannot be zero. $F_{-1}(l_{1t}) = 0$ implies that conditional on demand being one in period t, the expected profit for every seller is at least l_{it}, which in turn implies that

$u_{1t} = u_{2t} = ... = u_{Nt}$ and $F_{it}(u_{it}-) = 1\ \forall i$. Also by the right continuity of $F_{-1}(p)$, F_{it} must be non-degenerate for some i. It is easy to see that it cannot be supported in any equilibrium. Consequently $l_{1t} = l_{2t} = p^*_N$ is the only feasible outcome. Suppose now that none of sellers plays p^*_N with probability 1, then $F_{-1}(p^*_N) < 1$ and right continuity again implies that seller 1 will set a price greater than p^*_N, violating the condition $l_{1t} = p^*_N$. Hence this cannot be the case also, and (b) is established. (c) follows from (a) and (b) immediately. Q.E.D.

Proposition 2 implies that if there are N>2 sellers in the market, in any equilibirum all sellers will post prices no less than p^*_N (the lowest market price), and at least two sellers will post prices at p^*_N with probability one. Moreover, all sellers have the same expected profit in any equilibrium, which is exactly equal to $V(\overline{p}, N)$, the expected profit for a monopoly when consumers' reservation price is p^*_N. Since we have "(N-2) degrees" of freedom in choosing prices between $[p^*_N, \overline{p}]$, one reasonable outcome is that (N-2) non-marginal sellers will set the price randomly. In this case, the model says that only the lowest market price can be predicted, and the degree of price dispersion is unpredictable! This result is not surprising at all. Since the probability of demand greater than one is zero, any non-marginal seller is indifferent between any two prices in the intervel $[p^*_N, \overline{p}]$. No one has an incentive to trigger a price war by cutting prices less than p^*_N, which makes the expected profit lower according to the definition of p^*_N. Therefore in every period, we may observe a distribution of prices rather than a single price. In terms of the actual transaction price, the lowest market price in this case, there is a unique effective price in the market, for at most one buyer will come to the market in every period. Under this definition we cannot say that it is a price dispersion equilibrium. However, in section 3 we will show that even under this more rigorous definition, equilibrium price dispersion still appears.

It has been shown that a pure strategy equilibrium does not exist in general when capacity constraints are introduced into a Bertrand competition model. Such result does not happen in our model. For example, each seller sets a price at $p_n^* \ \forall t$ s.t. $S_t = n$ constitutes a pure strategy equilibrium. Also in the standard model of Bertrand competition, a unique equilibrium occurs when sellers set prices equal to marginal cost and earn zero profits, but this equilibrium is not supported by empirical studies. After introducing demand uncertainty and capacity constraints, our model avoids such problems and every seller earns strictly positive profits.

Proposition 3 *For the binary demand oligopoly model, p_N^* is (a) linearly increasing in $\overline{p}$, (b) increasing and convex in δ, (c) decreasing and convex in N and (d) decreasing in q and convex in q if $N > \frac{2(1-\delta q)}{1-\delta}$.*

Proof. Part (a) is trivial. For part (b), taking the log of both sides of equation (5)

$$\ln p_N^* = (N-1)\left[\ln(1-q)\delta - \ln(1-\delta q)\right] + \ln \overline{p} \tag{8}$$

Taking a derivative w.r.t. δ, we get

$$\frac{1}{p_N^*}\frac{\partial p_N^*}{\partial \delta} = (N-1)\left[\frac{1}{\delta} + \frac{q}{1-\delta q}\right] \tag{9}$$

Therefore

$$\frac{\partial p_N^*}{\partial \delta} = (N-1)\left[\frac{1}{\delta} + \frac{q}{1-\delta q}\right] p_N^* > 0 \tag{10}$$

Taking a derivative w.r.t. δ again, we get

$$\begin{aligned}\frac{\partial^2 p_N^*}{\partial \delta^2} &= (N-1)\left[\frac{\partial p_N^*}{\partial \delta}(\frac{1}{\delta} + \frac{q}{1-\delta q}) + p_N^*(\frac{-1}{\delta^2} + \frac{q^2}{(1-\delta q)^2})\right] \\ &= (N-1)\left[(n-1)(\frac{1}{\delta} + \frac{q}{1-\delta q})^2 - \frac{1}{\delta^2} + \frac{q^2}{(1-\delta q)^2})\right] p_N^* > 0 \end{aligned} \tag{11}$$

For part (c), taking a derivative w.r.t. N in equation (5), we get

$$\begin{aligned}
\frac{\partial p_N^*}{\partial N} &= \left[\frac{1}{\delta} + \frac{q}{1-\delta q}\right]^{N-1} \overline{p} \ln[\frac{1}{\delta} + \frac{q}{1-\delta q}] < 0 \\
\frac{\partial^2 p_N^*}{\partial N^2} &= \left[\frac{1}{\delta} + \frac{q}{1-\delta q}\right]^{N-1} \overline{p} \left[\ln(\frac{1}{\delta} + \frac{q}{1-\delta q})\right]^2 > 0
\end{aligned} \tag{12}$$

For part (d), taking a derivative w.r.t. q in $\ln p_N^*$, we get

$$\begin{aligned}
\frac{\partial p_N^*}{\partial q} &= (N-1)\left[\frac{-1}{1-q} + \frac{\delta}{1-\delta q}\right] p_N^* \\
&= (N-1)\left[\frac{\delta - 1}{(1-q)(1-\delta q)}\right] p_N^* < 0
\end{aligned} \tag{13}$$

Taking a derivative w.r.t. q again, we get

$$\begin{aligned}
\frac{\partial^2 p_N^*}{\partial q^2} &= (N-1)\left[\frac{\partial p_N^*}{\partial q}(\frac{\delta - 1}{(1-q)(1-\delta q)}) + p_N^* \frac{(1-\delta)(-1-\delta+2\delta q)}{[(1-q)(1-\delta q)]^2}\right] \\
&= (N-1)(1-\delta) p_N^* \frac{(N-1)(1-\delta) - (1+\delta) + 2q\delta}{[(1-q)(1-\delta q)]^2}
\end{aligned} \tag{14}$$

Therefore,

$$\frac{\partial^2 p_N^*}{\partial q^2} > 0 \text{ iff } N > \frac{2(1-\delta q)}{1-\delta} \tag{15}$$

Q.E.D.

Proposition 3 says that p_N^*, the lowest market price when there are N sellers in the market, is higher when sellers become more patient, the reservation price of consumers is higher, or demand is stronger. Furthermore, it is a decreasing and convex function of the number of sellers.

3 Generalized Demand Structure

So far we were considering with an environment in which demand is either zero or one in every period. Sellers may charge different prices because posting any price above the lowest market price makes no difference to them, leading to price dispersion. Some may doubt whether price dispersion can happen if there is a positive probability of excess demand, i.e., a positive probability that the number of arriving buyers is larger than the number of units supplied. In this section, we show that price dispersion must appear when there is a positive probability of excess demand.

This section extends the model in section 2 by generalizing the underlying demand structure. We follow the same setup as in section 1 except that in each period the demand is i units with probability $q_i \geqq 0, i = 0, 1, 2, ...$ and $\sum_{i=0}^{\infty} q_i = 1$. We denote such selling game as $\Gamma(N, \{q_i\}_{i=0}^{\infty})$. Define $V(\overline{p}, N)$ to be the expected profit for a seller who simply posts a price $\overline{p}$ in every period and others post prices strictly less than $\overline{p}$. Then

$$V(\overline{p}, 1) = q_0 \delta V(\overline{p}, 1) + \sum_{i=1}^{\infty} q_i \overline{p} = \frac{1 - q_0}{1 - \delta q_0} \overline{p} \tag{16}$$

We have the following relation:

$$\begin{aligned} V(\overline{p}, N) &= q_0 \delta V(\overline{p}, N) + q_1 \delta V(\overline{p}, N-1) + ... + q_{N-1} \delta V(\overline{p}, 1) + \sum_{i=N}^{\infty} q_i \overline{p} \\ &= \frac{1}{1 - \delta q_0} [\sum_{i=1}^{N-1} q_i \delta V(\overline{p}, N-i) + \sum_{i=N}^{\infty} q_i \overline{p}] \end{aligned} \tag{17}$$

It is easy to see that $V(\overline{p}, 1) > V(\overline{p}, 2) > ... > V(\overline{p}, N)$. In $\Gamma(N, \{q_i\}_{i=0}^{\infty})$, let p_N^* be the price which yields a seller an expected profit $V(\overline{p}, N)$ if he simply posts a price p_N^* in every period which is the lowest price in the market and there are no other sellers setting a price at p_N^*. Therefore, p_N^* satisfies the following equation:

$$V(\overline{p}, N) = \frac{1 - q_0}{1 - \delta q_0} p_N^* \tag{18}$$

Define two distributions $W_i(\cdot) : [0, \infty) \to [0, 1]$ $i = 1, 2$ as:

$$W_1(p) = \begin{cases} 1 & , p \geq \overline{p} \\ 0 & , p < \overline{p} \end{cases} \tag{19}$$

$$W_2(p) = \begin{cases} 1 & , p > \overline{p} \\ 1 - \frac{\sum_2^\infty q_i}{q_1}[\frac{\overline{p}-\delta V(\overline{p},1)}{p-\delta V(\overline{p},1)} - 1] & , p_2^* \leq p \leq \overline{p} \\ 0 & , p < p_2^* \end{cases} \tag{20}$$

$W_1(p)$ is a degenerate distribution which puts probability one at $\overline{p}$. $W_2(p)$ is a distribution with $\sup\{x : W_2(x) = 0\} = p_2^*$ and $\inf\{x : W_2(x) = 1\} = \overline{p}$.

The following proposition characterizes a unique symmetric mixed-strategy equilibrium in duopoly with the generalized demand structure.

Proposition 4 *With the conditions $q_0 \neq 0, q_1 \neq 0$, and $\sum_{i=2}^\infty q_i > 0$, the duopoly model with a general demand structure has a unique symmetric mixed-strategy equilibrium which exhibits price dispersion. Specifically, each seller plays $W_n(p)$ for all t s.t. $S_t = n$ constitutes a unique symmetric mixed-strategy equilibrium in the selling game $\Gamma(2, \{q_i\}_{i=0}^\infty)$, and the expected profit for each seller is exactly $V(\overline{p}, 2)$.*

Proof. The case $S_t = 1$ is trivial. Pick any period $t \in \{0, 1, 2, ...\}$ s.t. $S_t = 2$. It is easy to show that a pure-strategy equilibrium does not exist in the selling game $\Gamma(2, \{q_i\}_{i=0}^\infty)$ when $q_0 \neq 0, q_1 \neq 0$, and $\sum_{i=2}^\infty q_i > 0$. Now we show that a symmetric mixed-strategy equilibrium exists. Let $F(p) : [0, \infty) \to [0, 1]$ denote the mixed-strategy played in period t in an equilbrium, and we define $l = \sup\{p : F(p) = 0\}$ and $u = \inf\{p : F(p) = 1\}$. Assume $F(\cdot)$ is differentiable. Also we define $V_i(t)$ to be the present discounted value of seller $i's$ expected profit in period t, and $V_i(t+1)$ to be the present discounted value of seller $i's$ expected profit in period $t + 1$ conditional on the demand in period t being zero. Apparently we must have $u = \overline{p}$. Suppose not and $u < \overline{p}$, then the expected profit for seller 1 from posting a price $\overline{p}$ is greater than that from posting a price u, i.e.,

$$\begin{aligned} V_1(t) &= q_0\delta V_1(t+1) + q_1\delta V(\overline{p},1) + \sum_{j=2}^{\infty} q_j\overline{p} \\ &> q_0\delta V_1(t+1) + q_1\delta V(\overline{p},1) + \sum_{j=2}^{\infty} q_j u \ , \end{aligned} \tag{21}$$

a contradiction. Since a seller is indifferent between posting a price at $u = \overline{p}$ and posting a price at l, we must have $l = p_2^*$ according to the definition of p_2^*. Then $\forall p \in [p_2^*, \overline{p}]$ the following relation holds:

$$(1-\delta q_0)V(\overline{p},2) = q_1[F(p)\delta V(\overline{p},1) + (1-F(p))p] + \sum_{i=2}^{\infty} q_i p \tag{22}$$

Therefore $\forall p \in [p_2^*, \overline{p}]$

$$\begin{aligned} & \frac{\partial}{\partial p} q_1[F(p)\delta V(\overline{p},1) + (1-F(p))p] + \sum_{i=2}^{\infty} q_i p] \\ = \ & q_1[\frac{\partial}{\partial p}F(p)\delta V(\overline{p},1) - p\frac{\partial}{\partial p}F(p) + 1 - F(p)] + \sum_{i=2}^{\infty} q_i = 0 \end{aligned} \tag{23}$$

which can be solved as:

$$F(p) = \begin{cases} 1 & ,p > \overline{p} \\ 1 - \frac{\sum_2^\infty q_i}{q_1}[\frac{\overline{p}-\delta V(\overline{p},1)}{p-\delta V(\overline{p},1)} - 1] & ,p_2^* \le p \le \overline{p} \\ 0 & ,p < p_2^* \end{cases} , \tag{24}$$

which is exactly $W_2(p)$. Q.E.D.

From above we see that it is not hard to find an equilibrium in the selling game $\Gamma(2, \{q_i\}_{i=0}^{\infty})$, but finding an equilibrium in the selling game $\Gamma(N, \{q_i\}_{i=0}^{\infty})$ becomes very complicated, and there is no explicit solution for an equilibrium strategy when N is greater than five. We give some definitions first. $\forall t$ s.t. $S_t = n$ let $F_n(p) : [0,\infty) \to [0,1]$ denote the mixed-strategy played in an equlibrium, and define $l_n = \sup\{p : F_n(p) = 0\}$ and $u_n = \inf\{p : F_n(p) = 1\}$. Also define

$$Z_{k,n}(x) = \sum_{i=0}^{k} \binom{n-1}{i} (1-x)^{n-1-i} x^i \tag{25}$$

If there are n sellers in the market and one seller sets a price at p and other $n-1$ sellers play mixed strategies according to the distribution function $F_n(p)$,then $Z_{k,n}(F_n(p))$ is just the probability that there are at most k prices less than p.

Now we guess that an equilibrium has the following properties: in the selling game $\Gamma(N, \{q_i\}_{i=0}^{\infty})$, each seller plays a mixed-strategy $F_n(\cdot)$ $\forall t$ s.t. $S_t = n$, and the expected profit for each is $V(\overline{p}, N)$. Then the expected profit for a seller posting any price $p \in [l_N, u_N]$ should be equal to $V(\overline{p}, N)$, i.e.,

$$\begin{aligned} V(p, F_N) &= q_0 \delta V(p, F_N) + q_1[Z_{0,N}(F_N(p))p + (1 - Z_{0,N}(F_N(p)))\delta V(\overline{p}, N-1)] \\ &\quad +... + q_{N-1}[Z_{N-2,N}(F_N(p))p + (1 - Z_{N-2,N}(F_N(p)))\delta V(\overline{p}, 1)] + \sum_{i=N}^{\infty} q_i p \\ &= \frac{1}{1-\delta q_0}[\sum_{i=1}^{N-1} q_i[Z_{i-1,N}(F_N(p))p + (1 - Z_{i-1,N}(F_N(p)))\delta V(\overline{p}, N-i)] + \sum_{i=N}^{\infty} q_i p] \\ &= V(\overline{p}, N) \end{aligned} \tag{26}$$

For any fixed $p \in [l_N, u_N]$, if we treat $F_N(p)$ as a variable x, then the equation (23) is a polynomial of degree$N-1$. Showing that there is a unique symmetric mixed-strategy equilibrium in the selling game $\Gamma(N, \{q_i\}_{i=0}^{\infty})$ is equivalent to showing that there is a unique real root between [0,1] for any $p \in [l_N, u_N]$, and it must be increasing in p. According to Abel's impossibility theorem, there exist no explicit solutions for a polynomial of degree greater than four.[3] However, we still can show that a unique solution exists in this problem.

[3]Since we have N+2 parameters in this equation (q_0, q_1,..., q_{N-1}, $\sum_{i=N}^{\infty} q_i$, δ, $\overline{p}$), it becomes very complicated for us to determine the number of roots between [0,1] in this equation. Some standard root-finding approaches such as Fourier-Budan and Sturm are intractable in this problem.

Define $G_n(x) : [0, 1] \rightarrow R$ to be

$$G_n(x) = \frac{\sum_{i=1}^{n-1} q_i Z_{i-1,n}(x) \delta V(\overline{p}, n-i) + \sum_{i=n}^{\infty} q_i \overline{p}}{\sum_{i=1}^{n-1} q_i Z_{i-1,n}(x) + \sum_{i=n}^{\infty} q_i} \tag{27}$$

and

$$W_n(p) = \begin{cases} 1 & , p \geq \overline{p} \\ G_n^{-1}(p) & , p_n^* \leq p \leq \overline{p} \\ 0 & , p \leq p_n^* \end{cases} \tag{28}$$

With the definitions in place, we can state our main result.

Proposition 5 *When $q_i > 0$, $i = 0, 1, \ldots N-1$, and $\sum_{i=N}^{\infty} q_i \neq 0$, the oligopoly model with a general demand structure has a unique symmetric mixed-strategy equilibrium which exhibits price dispersion. Specifically, each seller plays $W_n(p)$ for all t s.t. $S_t = n$ constitutes a unique symmetric mixed-strategy equilibrium in the selling game $\Gamma(N, \{q_i\}_{i=0}^{\infty})$, and the expected profit for each seller is exactly $V(\overline{p}, N)$.*

Proof. See the Appendix.

Proposition 5 says that when there is a positive probability of excess demand (i.e., $\sum_{i=N}^{\infty} q_i \neq 0$ when there are N sellers in the market), each seller must play a non-degenerate mixed-strategy in a unique equilbrium. Moreover, the greatest lower bound of the support of the distribution played by sellers increases as the number of sellers decreases. However, it is still possible for us to observe that the lowest market price is lower when N decreases. More competition in the market (higher N) does not imply lower market prices.

4 Discussion and Conclusion

In this paper, we construct a simple, dynamic model to explain the phenomenon of price dispersion in an environment in which firms are capacity constrained, demand is uncertain, and search is costless for consumers. First, we show that when in every period there is at most one consumer coming to the market with some positive probability, price dispersion happens except the case of duopoly, and the degree of price dispersion is unpredictable. Moreover, the lowest market price is higher when sellers become more patient, the reservation price of consumers is higher, or demand is stronger. Furthermore, this lowest market price is a decreasing and convex function of the number of sellers. Secondly, with a generalized demand structure, when there is a positive probability of excess demand, there exists a unique mixed-strategy symmetric equilibrium. In this equilibrium, sellers set prices according to non-degenerate distributions, and the lowest possible market price is decreasing in the number of sellers. Again, uniform price equilibria cannot appear in this case.

5 Appendix

Proof of Proposition 1. Pick any period $t \in \{0, 1, 2, ...\}$. If in period t there is only one seller remaining in the market, then the seller will simply set a price $p = \overline{p} = p_1^*$ in every period. Suppose there are two sellers in the market in period t, and let $F_i(p) : [0, \overline{p}] \to [0, 1]$ denote the strategy played by seller i in period t ($F_i(p)$ can be degenerate) in an equilbrium. Define $l_i = \sup\{p : F_i(p) = 0\}$ and $u_i = \inf\{p : F_i(p) = 1\}$. Also define $V_i(t)$ to be the present discounted value of seller $i's$ expected profit in period t, and $V_i(t + 1)$ to be the present discounted value of seller $i's$ expected profit to at period t+1 conditional on demand in period t is zero. We prove the proposition by establishing the following claims:

Claim 1. $l_1 = l_2 \geq p_2^*$: Suppose $l_1 \neq l_2$ and WLOG let $l_1 < l_2$. Seller 1 can increase his profit by putting all probability between $[l_1, l_2)$ to $l_2 - \varepsilon$. Hence in any equilibrium we must have $l_1 = l_2$. Suppose now we have $l_1 = l_2 < p_2^*$ and $F_2(l_2) < 1$. Since a distribution is right continuous,[4] $\exists$ $\eta \in (l_2, p_2^*)$ such that $F_2(l_2) \leq F(p) < 1 \ \forall \ p \in (l_2, \eta)$. Given seller 2's equilibrium strategy, seller 1's expected profit from setting any price $p \in (l_2, \eta)$ is

$$\begin{aligned}
V_1(t) &= q\delta V_1(t+1) + (1-q)[F_2(p-)\delta V(\overline{p}, 1) \\
&\quad + (F_2(p) - F_2(p-))(\frac{1}{2}\delta V(\overline{p}, 1) + \frac{1}{2}p) + (1 - F_2(p))p] \\
&< q\delta V_1(t+1) + (1-q)[F_2(p-)\delta V(\overline{p}, 1) \\
&\quad + (F_2(p) - F_2(p-))\delta V(\overline{p}, 1) + (1 - F_2(p))\delta V(\overline{p}, 1)] \\
&= q\delta V_1(t+1) + (1-q)[F_2(p_2^*-)\delta V(\overline{p}, 1) \\
&\quad + (F_2(p_2^*) - F_2(p_2^*-))\frac{1}{2}(\delta V(\overline{p}, 1) + p_2^*) + (1 - F_2(p_2^*))p_2^*] \qquad (29)
\end{aligned}$$

Consequently we must have $l_1 = \sup\{p : F_1(p) = 0\} \geq \eta$, a contradiction! Also we notice that $l_1 = l_2 < p_2^*$ and $F_2(l_2) = 1$ cannot hold in any equilibrium because seller 2's expected profit in this case is

$$\begin{aligned}
V_2(t) &= q\delta V_2(t+1) + (1-q)[F_1(l_2)[\frac{1}{2}\delta V(\overline{p}, 1) + \frac{1}{2}l_2] + (1 - F_1(l_2))l_2] \\
&< q\delta V_2(t+1) + (1-q)\delta V(\overline{p}, 1) \\
&= q\delta V_2(t+1) + (1-q)[F_1(p_2^*-)\delta V(\overline{p}, 1) \\
&\quad + (F_1(p_2^*) - F_1(p_2^*-))\frac{1}{2}(\delta V(\overline{p}, 1) + p_2^*) + (1 - F_1(p_2^*))p_2^*] \qquad (30)
\end{aligned}$$

Thus in an equilibrium we must have $l_1 = l_2 \geq p_2^*$.

[4] Here we use standard definiton of distribution function $F(p) = \mu((-\infty, p])$, where μ is a finite Borel measure. The right continuity of the distribution follows from the fact that a finite measure is continuous from above.

Claim 2. $u_1 = u_2 = p_2^*$: Suppose $u_1 \neq u_2$ and WLOG let $u_1 < u_2$. From Claim 1, we have $p_2^* \leq u_1 < u_2$. First, we observe that $p_2^* < u_1 < u_2$ cannot be supported by any equilibrium, for if it is indeed the case then seller 2's expected profit from posting any $p \in (p_2^*, u_1)$ is greater than that from posting any price $p > u_1$, *i.e.*, $\forall\ p \in (p_2^*, u_1)$

$$\begin{aligned} V_2(t) &= q\delta V_2(t+1) + (1-q)[F_1(p-)\delta V(\overline{p}, 1) \\ &\quad +(F_1(p) - F_1(p-))(\frac{1}{2}\delta V(\overline{p}, 1) + \frac{1}{2}p) + (1 - F_1(p))p] \\ &> q\delta V_2(t+1) + (1-q)[F_1(p-)\delta V(\overline{p}, 1) \\ &\quad +(F_1(p) - F_1(p-))\delta V(\overline{p}, 1) + (1 - F_1(p))\delta V(\overline{p}, 1)] \\ &= q\delta V_2(t+1) + (1-q)\delta V(\overline{p}, 1), \end{aligned} \tag{31}$$

which implies $\inf\{p : F_2(p) = 1\} \leq u_1$, and we get a contradiction. On the other hand, $p_2^* = u_1 < u_2$ also cannot be sustained in any equilibrium, for seller 1's expected profit from posting any $p \in (p_2^*, u_2)$ is greater than that from posting any price $p \leq p_2^*$, *i.e.*, $\forall\ p \in (p_2^*, u_2)$

$$\begin{aligned} V_1(t) &= q\delta V_1(t+1) + (1-q)[F_2(p-)\delta V(\overline{p}, 1) \\ &\quad +(F_2(p) - F_2(p-))(\frac{1}{2}\delta V(\overline{p}, 1) + \frac{1}{2}p) + (1 - F_2(p))p] \\ &> q\delta V_1(t+1) + (1-q)\delta V(\overline{p}, 1), \end{aligned} \tag{32}$$

which implies that $F_1(p_2^*) = 0$, violating our premise that $u_1 = \inf\{p : F_1(p) = 1\}$. Therefore we have shown that $u_1 = u_2$ in any equilibrium. Now suppose $u_1 = u_2 > p_2^*$. Combining the result from Claim 1, there are two possible cases: either $u_1 = u_2 > l_1 = l_2 = p_2^*$ or $u_1 = u_2 \geq l_1 = l_2 > p_2^*$. If $u_1 = u_2 > l_1 = l_2 = p_2^*$, define

$$\xi = \sup_{p \in [0, \overline{p}]} [F_2(p-)\delta V(\overline{p}, 1) + (F_2(p) - F_2(p-))(\frac{1}{2}\delta V(\overline{p}, 1) + \frac{1}{2}p) + (1 - F_2(p))p], \tag{33}$$

then ξ is the supremum of seller 1's expected profit conditional on demand being one in period t, and it is easy to see that $p_2^* < \xi < u_2$. Seller 1's expected profit from posting any $p \in [p_2^*, \xi)$ is

$$\begin{aligned}
V_1(t) &= q\delta V_1(t+1) + (1-q)[F_2(p-)\delta V(\overline{p},1) \\
&\quad +(F_2(p) - F_2(p-))(\frac{1}{2}\delta V(\overline{p},1) + \frac{1}{2}p) + (1-F_2(p))p] \\
&< q\delta V_1(t+1) + (1-q)[F_2(p-)\delta V(\overline{p},1) \\
&\quad +(F_2(p) - F_2(p-))(\frac{1}{2}\delta V(\overline{p},1) + \frac{1}{2}\xi) + (1-F_2(p))\xi] \\
&\leq q\delta V_1(t+1) + (1-q)[F_2(p-)\delta V(\overline{p},1) + (1-F_2(p-))\xi] \\
&\leq q\delta V_1(t+1) + (1-q)\ \xi, \qquad (34)
\end{aligned}$$

which implies that, given seller 2's strategy $F_2(p)$, seller 1 will never choose any $p < \xi$ in period t Therefore $\sup\{p : F_1(p) = 0\} \geq \xi > p_2^*$. So we get a contradiction, and $u_1 = u_2 > l_1 = l_2 = p_2^*$ cannot be the case in any equilibrium. On the other hand, $u_1 = u_2 \geq l_1 = l_2 > p_2^*$ also cannot be supported in any equilibrium. Suppose $u_1 = u_2 \geq l_1 = l_2 > p_2^*$ in some equilibrium, and define

$$\theta = \sup_{p\in[0,\overline{p}]} [F_2(p-)\delta V(\overline{p},1) + (F_2(p) - F_2(p-))(\frac{1}{2}\delta V(\overline{p},1) + \frac{1}{2}p) + (1-F_2(p))p], \qquad (35)$$

then we have $\theta \geq l_1 > p_2^*$. If $F_2(u_2-) = 1$, then $\exists\ 0 < \vartheta < u_2$ such that $F_2(p) > 1-\varepsilon\ \forall p \in (\vartheta, u_2)$, where $\varepsilon = (l_1 - p_2^*)/(\delta V(\overline{p},1)/2 + 3u_2/2)$. Therefore,

$$\begin{aligned}
V_1(t) &= q\delta V_1(t+1) + (1-q)[F_2(p-)\delta V(\overline{p},1) \\
&\quad +(F_2(p) - F_2(p-))\frac{1}{2}(\delta V(\overline{p},1) + p) + (1-F_2(p))p] \\
&< q\delta V_1(t+1) + (1-q)[F_2(p-)\delta V(\overline{p},1) + \frac{\varepsilon}{2}(\delta V(\overline{p},1) + p) + \varepsilon p] \\
&\leq q\delta V_1(t+1) + (1-q)[\delta V(\overline{p},1) + \varepsilon(\frac{1}{2}\delta V(\overline{p},1) + \frac{3}{2}p)] \\
&< q\delta V_1(t+1) + (1-q)[\delta V(\overline{p},1) + \varepsilon(\frac{1}{2}\delta V(\overline{p},1) + \frac{3}{2}u_2)]
\end{aligned}$$

$$= \quad q\delta V_1(t+1) + (1-q)l_1, \tag{36}$$

which implies that $\inf\{p : F_1(p) = 1\} \le \vartheta < u_2$, a contradiction! Hence we must have $F_2(u_2-) < 1$ in an equilibrium. Since $u_2 = \inf\{p : F_2(p) = 1\}$, $F_2(u_2) - F_2(u_2-) > 0$. It is easy to see that seller 1 will never set his price at u_1 in period t (he can always get higher profit by cutting down the price). Since $u_1 = \inf\{p : F_1(p) = 1\}$, we must have $F_1(u_1-) = 1$, but this also cannot be sustained in any equilibrium from our previous argument. Thus the necessary condition in an equilibrium is that $u_1 = u_2 = l_1 = l_2 = p_2^*$, and it is easy to verify that it is indeed an equilibrium. Q.E.D.

Proof of Proposition 5. The statement is true for N=2 from Proposition 4. Suppose the proposition is true for the selling game $\Gamma(N-1, \{q_i\}_{i=0}^{\infty})$. We prove it by establishing the following lemmas.

Lemma 1 *Let $\Phi_n(x) : R \to R$ be of the form*

$$\Phi_n(x) = \frac{\alpha_1(x)C_1 + \alpha_2(x)C_2 + ... + \alpha_n(x)C_n}{\alpha_1(x) + \alpha_2(x) + ... + \alpha_n(x)},$$

where $C_1 < C_2 < ... < C_n$, $\alpha_i(x) \ge 0$ is differentiable for i=1,2,...,n, and $\alpha_n(x) \ne 0$ $\forall x \in R$. Then $\frac{\partial}{\partial x}W(x) > 0$ if $\alpha_j(x)\frac{\partial}{\partial x}[\sum_{i=j+1}^{n}\alpha_i(x)] - \sum_{i=j+1}^{n}\alpha_i(x)\frac{\partial}{\partial x}\alpha_j(x) \ge 0$ for j=1,...,n-1, and this holds with strict inequality for some j.

Proof. The result is trivially true for $n = 2$. When $n = 3$, we can rewrite it as

$$\Phi_3(x) = \frac{\alpha_1(x)C_1 + \alpha_2(x)C_2 + \alpha_3(x)C_3}{\alpha_1(x) + \alpha_2(x) + \alpha_3(x)} = \frac{\alpha_1(x)C_1 + [\alpha_2(x) + \alpha_3(x)]\{\frac{\alpha_2(x)C_2+\alpha_3(x)C_3}{\alpha_2(x)+\alpha_3(x)}\}}{\alpha_1(x) + [\alpha_2(x) + \alpha_3(x)]} \tag{37}$$

Fixing $\frac{\alpha_2(x)C_2+\alpha_3(x)C_3}{\alpha_2(x)+\alpha_3(x)}$, $\frac{\partial}{\partial x}\Phi_3(x) > 0$ if $\alpha_1(x)\frac{\partial}{\partial x}[\alpha_2(x) + \alpha_3(x)] - [\alpha_2(x) + \alpha_3(x)]\frac{\partial}{\partial x}\alpha_1(x) > 0$ (here we use the fact that $C_1 < \frac{\alpha_2(x)C_2+\alpha_3(x)C_3}{\alpha_2(x)+\alpha_3(x)}$). Furthermore $\frac{\partial}{\partial x}\frac{\alpha_2(x)C_2+\alpha_3(x)C_3}{\alpha_2(x)+\alpha_3(x)} > 0$ given that

$\alpha_2(x)\frac{\partial}{\partial x}\alpha_3(x) - \alpha_3(x)\frac{\partial}{\partial x}\alpha_2(x) > 0$. Hence we prove the case for $n = 3$. Applying a similar argument for any n, we get the conditions stated above. Q.E.D.

Lemma 2 $\frac{\partial}{\partial x}Z_{k,n}(x) = -(k+1)\binom{n-1}{k+1}(1-x)^{n-2-k}x^k < 0 \;\forall\; x \in (0,1)$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial x}Z_{k,n}(x) &= \frac{\partial}{\partial x}\sum_{i=0}^{k}\binom{n-1}{i}(1-x)^{n-1-i}x^i \\
&= \sum_{i=0}^{k}[-(n-1-i)\binom{n-1}{i}(1-x)^{n-2-i}x^i + i\binom{n-1}{i}(1-x)^{n-1-i}x^{i-1}] \\
&= \sum_{i=0}^{k}[-(i+1)\binom{n-1}{i+1}(1-x)^{n-2-i}x^i + i\binom{n-1}{i}(1-x)^{n-1-i}x^{i-1}] \\
&= -(k+1)\binom{n-1}{k+1}(1-x)^{n-2-k}x^k
\end{aligned} \tag{38}$$

Q.E.D.

Lemma 3 *Given* $q_i \neq 0$ *for* $i = 0, ..., n-1$ *and* $\sum_{i=n}^{\infty} q_i > 0$, $\frac{\partial}{\partial x}G_n(x) > 0 \;\forall x \in (0,1)$

Proof. We observe that $\delta V(\overline{p}, n-1) < \delta V(\overline{p}, n-2) < ... < \delta V(\overline{p}, 1) < \overline{p}$, $\{q_i Z_{i-1,n}(x)\}_{i=1}^{n-1} > 0$ and $\sum_{i=n}^{\infty} q_i > 0$. Hence, it satisfies all of the conditions of Lemma 1. By lemma 2, we have

$$\begin{aligned}
& q_{n-1}Z_{n-2,n}(x)\frac{\partial}{\partial x}[\sum_{i=n}^{\infty} q_i] - [\sum_{i=n}^{\infty} q_i]\frac{\partial}{\partial x}[q_{n-1}Z_{n-2,n}(x)] \\
= \;& [\sum_{i=n}^{\infty} q_i]q_{n-1}(n-1)x^{n-2} > 0 \quad \forall x \in (0,1)
\end{aligned} \tag{39}$$

We can apply Lemma 1 if the following conditions hold: $\forall x \in (0,1), \;\; j = 3, ..., n$

$$\Psi_j(x) = \begin{array}{c} q_{n-j+1}Z_{n-j,n}(x)\frac{\partial}{\partial x}[\sum_{i=1}^{j-2} q_{n-j+1+i}Z_{n-j+i,n}(x) + \sum_{i=n}^{\infty} q_i] - \\ [\sum_{i=1}^{j-2} q_{n-j+1+i}Z_{n-j+i,n}(x) + \sum_{i=n}^{\infty} q_i]\frac{\partial}{\partial x}[q_{n-j+1}Z_{n-j,n}(x)] \end{array} \geq 0 \tag{40}$$

$\forall x \in (0,1), \;\; j = 3, ..., n$

$$\begin{aligned}
\Psi_j(x) &= q_{n-j+1}Z_{n-j,n}(x)\frac{\partial}{\partial x}[\sum_{i=1}^{j-2} q_{n-j+1+i}Z_{n-j+i,n}(x) + \sum_{i=n}^{\infty} q_i] - \\
&\quad [\sum_{i=1}^{j-2} q_{n-j+1+i}Z_{n-j+i,n}(x) + \sum_{i=n}^{\infty} q_i]\frac{\partial}{\partial x}[q_{n-j+1}Z_{n-j,n}(x)] \\
&= q_{n-j+1}Z_{n-j,n}(x)\frac{\partial}{\partial x}[\sum_{i=1}^{j-2} q_{n-j+1+i}Z_{n-j+i,n}(x)] - \\
&\quad [\sum_{i=1}^{j-2} q_{n-j+1+i}Z_{n-j+i,n}(x) + \sum_{i=n}^{\infty} q_i]\frac{\partial}{\partial x}[q_{n-j+1}Z_{n-j,n}(x)] \\
&> q_{n-j+1}Z_{n-j,n}(x)\frac{\partial}{\partial x}[\sum_{i=1}^{j-2} q_{n-j+1+i}Z_{n-j+i,n}(x)] - \\
&\quad [\sum_{i=1}^{j-2} q_{n-j+1+i}Z_{n-j+i,n}(x)]\frac{\partial}{\partial x}[q_{n-j+1}Z_{n-j,n}(x)]
\end{aligned} \tag{41}$$

After simplification, we can get:

$$\begin{aligned}
& q_{n-j+1}Z_{n-j,n}(x)\frac{\partial}{\partial x}[\sum_{i=1}^{j-2} q_{n-j+1+i}Z_{n-j+i,n}(x)] - [\sum_{i=1}^{j-2} q_{n-j+1+i}Z_{n-j+i,n}(x)]\frac{\partial}{\partial x}[q_{n-j+1}Z_{n-j,n}(x)] \\
= \;& q_{n-j+1}\sum_{i=1}^{j-2}\sum_{k=n-j+1}^{n-j+i}\sum_{l=0}^{n-j} q_{n-j+1+i}\binom{n-1}{k}\binom{n-1}{l}[(1-x)^{2n-2-l-k}x^{l+k}](k-l)\frac{\partial}{\partial x}\log\frac{x}{1-x} > 0
\end{aligned}$$

Hence, the lemma is established. Q.E.D.

Lemma 4 *The $W_n(p)$ defined before is indeed a distribution with $W_n(\overline{p}) = 1$ and $W_n(p_n^*) = 0$.*

It is easy to see that $W_n(p_n^*) = G_n^{-1}(p_n^*) = 0$ and $W_n(\overline{p}) = G_n^{-1}(\overline{p}) = 1$. Since $G_n(p)$ is strictly increasing in [0,1], $W_n(p) = G_n^{-1}(p)$ is well defined and increasing in $[p_n^*, \overline{p}]$. *Q.E.D.*

Now the only thing that needs to be shown is that no one will deviate if the other N-1 sellers play mixed-strategies according to $W_N(p)$ when there are N sellers in the market. It is equivalent to showing that setting any price between $[p_N^*, \overline{p}]$ gives a seller the same expected profits, *i.e.*, $\forall p \in [p_N^*, \overline{p}]$

$$
\begin{aligned}
V(\overline{p}, N) &= \frac{1}{1-\delta q_0}[\sum_{i=1}^{N-1} q_i \delta V(\overline{p}, N-i) + \sum_{i=N}^{\infty} q_i \overline{p}] \qquad (42) \\
&= \frac{1}{1-\delta q_0}\left\{\sum_{i=1}^{N-1} q_i [Z_{i-1}(W_N(p))p + (1 - Z_{i-1}(W_N(p)))\delta V(\overline{p}, N-i)] + \sum_{i=N}^{\infty} q_i p\right\}
\end{aligned}
$$

or

$$
\sum_{i=N}^{\infty} q_i \overline{p} = \sum_{i=1}^{N-1} q_i Z_{i-1}(W_N(p))[p - \delta V(\overline{p}, N-i)] + \sum_{i=N}^{\infty} q_i p \qquad (43)
$$

Therefore,

$$
p = \frac{\sum_{i=1}^{N-1} q_i Z_{i-1}(W_N(p)) \delta V(\overline{p}, N-i) + \sum_{i=N}^{\infty} q_i \overline{p}}{\sum_{i=1}^{N-1} q_i Z_{i-1}(W_N(p)) + \sum_{i=N}^{\infty} q_i} \qquad (44)
$$

which is exactly the definition of $W_N(p)$. Q.E.D.

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