A Note on Socially Insufficient Advertising in Tirole's Duopoly Model

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ABSTRACT. In his textbook Tirole (1988, pp. 291-294) presents a model of advertising with Hotelling duopolists. A condition for there to be socially too little advertising is derived.

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Tirole (1988, pp. 291-294) presents a model of duopolists ($i = 1, 2$) located on the ends of a Hotelling line who chose advertising (Φ_i) and price (p_i) simultaneously, and derives the pure-strategy symmetric equilibrium. The purpose is pedagogical, partly to highlight the forces that could lead to excessive or insufficient advertising. Tirole (1988) notes that "these conclusions, of course, are only valid in the competitive range," *i.e.*, only to the extent that a competitive equilibrium exists and so assumptions are made to ensure the existence of the equilibrium.[1] Here, it is shown that such an outcome exists.

1. Tirole's Model

Consumers are distributed uniformly along a unit length with density 1, have unit demand with gross surplus $\overline{s}$ from consuming the good. They have linear transportation cost t. They do not know of the existence of either product unless they receive an ad from a firm; then they learn that firm's location and price. Advertising Φ_i is the fraction of consumers that firm i reaches with an advertisement. Consumers have equal chances of receiving a given ad (implicitly this is independent of each firm). The cost to firm i to reach the fraction Φ_i of the consumers is quadratic: $A(\Phi_i) = a\Phi_i^2/2$. Tirole (1988, Ch.7, Fn. 27) assumes that $a > t/2$ so that the firms choose in equilibrium $\Phi < 1$.

Assumption 1. $a > t/2$.

Production is on demand with constant marginal cost c. Implicitly it is assumed that all potential exchanges are efficient: $\overline{s} - c - t \geq 0$.

For the consumers firm 1 reaches, a fraction $1 - \Phi_2$ are not reached by its rival and so firm 1 is a monopolist in this case. For the remaining fraction of consumers that firm 1 reaches, they are reached by firm 2, which occurs with probability Φ_2. These latter consumers are fully informed and the demand for firm 1 in this case is presented as (Tirole, 1988, p. 293, top col. 1)

$$(p_2 - p_1 + t)/2t.$$ [2]

Thus, demand for firm 1 is (Tirole, 1988, p. 293, col. 1: D_1)

$$D_1 = \Phi_1 \left[(1 - \Phi_2) \times 1 + \Phi_2 \frac{(p_2 - p_1 + t)}{2t} \right]. \tag{1}$$

1.1. Competitive Equilibrium.

Profit for firm 1 is (Tirole, 1988, p. 293)

$$\Phi_1 \left[(1 - \Phi_2) \times 1 + \Phi_2 \frac{(p_2 - p_1 + t)}{2t} \right] (p_1 - c) - a\frac{\Phi_1^2}{2} \tag{2}$$

Differentiating with respect to p_i and Φ_i and imposing symmetry yields the competitive equilibrium price (with the equation numbering as in Tirole, 1988 to ease comparison)

$$p^c = c + (2at)^{1/2}, \tag{7.15}$$

and the competitive equilibrium advertising

$$\Phi^c = \frac{2}{1 + (2a/t)^{1/2}}. \tag{7.16}$$

Substituting these equilibrium values into the profit expression (2) yields the competitive equilibrium profit for firm 1

$$\Pi^c = \frac{2a}{(1 + (2a/t)^{1/2})^2}. \tag{7.17}$$

[1] For example, Tirole (1988, Fn. 27), notes that advertising costs cannot be "too high" in order to rule out a firm's incentive to charge a high price and focus "on one's own turf."

[2] More precisely, since the maximum demand is 1 and the minimum is 0, the demand function is $\min\left\{1, \max\left\{0, (p_2 - p_1 + t)/2t\right\}\right\}$, but this is implicit given earlier derivations in (Tirole, 1988, p. 98). For ease in following the derivations in Tirole (1988), expressions here follow those in Tirole (1988).

This pure strategy symmetric equilibrium is the only equilibrium presented in Tirole (1988), though of course others may exist.

Returning to the demand function (1), there is another standard assumption (implicitly) made. For the consumers that firm 1 reaches, who are not reached by firm 2, firm 1 is a monopolist. Given reaching these consumers, the demand in this case is assumed equal to 1: the 1 in $(1 - \Phi_2) \times 1$ on the RHS of (1). This means that conditional on the firm reaching the consumer and its rival not reaching the consumer, the firm has a sale with probability 1, that is, all consumers accept the offer (in contrast, for the second term the firm may only sell to a fraction of the consumer it reaches). This is a variation of the "covered market" assumption and implies that the $\bar{s}$ is large enough and t is low enough so that the furthest consumer purchases. This implies that the competitive equilibrium price (denoted p^c in (7.15)) is such that the furthest consumer buys.[3]

Assumption 2. *Covered Market Assumption:* $p \le \bar{s} - t$.

1.2. Welfare Optimum.

The planner chooses Φ (that is, the planner has both firms set the same level) to maximize (Tirole, 1988, Fn. 29, p. 294)[4]

$$\Phi^2(\bar{s} - c - t/4) + 2\Phi(1 - \Phi)(\bar{s} - c - t/2) - 2\left(a\Phi^2/2\right).$$

The first term reflects when a consumer receives ads from both firms. Their average transportation cost is $t/4$. When they receive only one ad, their average transportation cost is $1/2$. The maximization yields (Tirole, 1988, Fn. 29)

$$\Phi^* = \frac{2(\bar{s} - c) - t}{2(\bar{s} - c) - 3t/2 + 2a}. \tag{3}$$

Intuitively, Φ^* is increasing in $\bar{s}$ and straightforward calculus confirms this.

1.3. Two Implications.

Lemma 1. *Given Assumptions 1 and 2, the equilibrium price (7.15) implies that* $c + 2t \le \bar{s}$.

Proof. Combining the equilibrium price (7.15) and Assumptions 2 we have

$$c + (2at)^{1/2} \le \bar{s} - t.$$

Solving for a obtains

$$a \le \frac{(\bar{s} - t - c)^2}{2t}. \tag{4}$$

From Assumption 1, (4) becomes

$$\frac{t}{2} < a \le \frac{(\bar{s} - t - c)^2}{2t}$$
$$\frac{t}{2} < \frac{(\bar{s} - t - c)^2}{2t}.$$

Solving for $\bar{s}$ yields

$$c + 2t \le \bar{s}.$$

□

Note that Lemma 1 is a necessary condition, but not necessarily a sufficient one. Lemma 1, in turn, has an implication regarding the monopoly price.

Lemma 2. *The monopoly price is the corner solution:* $p^m = \bar{s} - t$.

[3]The assumption could also be inferred from the statement Tirole (1988, Bottom p. 292, col. 2) "we look at equilibria with overlapping market areas for firms among the fully informed consumers."

[4]Implicitly it is assumed that the price the planner sets is such that all consumers are willing to buy since all potential exchanges are assumed efficient.

Proof. From Lemma 1

$$\begin{aligned}\bar{s} &\geq c + 2t \\ \bar{s} - 2t &\geq c \\ 2\bar{s} - 2t &\geq \bar{s} + c \\ \bar{s} - t &\geq \frac{\bar{s} + c}{2},\end{aligned}$$

with the RHS of the last inequality being the solution to the monopoly profit-maximization problem assuming an interior solution to the concave problem (that is, at that price not all consumers buy). (Though straightforward, for completeness the derivation of this price is in Appendix A). Given that quantity demanded at $p = \bar{s} - t$ equals 1, so too is quantity demanded at the lower price $(\bar{s} + c)/2$ (that is, at the latter price quantity demanded is bounded by the unit length of the city). And so, profits are greater at $p = \bar{s} - t$. □

2. Existence of socially insufficient advertising in equilibrium

2.1. The condition for socially insufficient advertising.

Intuitively, there is insufficient advertising when $\bar{s}$ is sufficiently large: as $\bar{s}$ increases, the planner values advertising more, but increases in $\bar{s}$ do not affect the competitive equilibrium level of advertising (7.16).

Lemma 3. *For there to be socially insufficient advertising, $\bar{s}$ must be greater than*

$$\bar{s}^* \equiv \frac{(c + t/2)((2a/t)^{1/2} - 1) + 2a - t/2}{(2a/t)^{1/2} - 1}. \tag{5}$$

Proof. Differencing the socially optimal level of advertising (3) from (7.16) yields

$$\Phi^* - \Phi^c = 2\frac{(2a/t)^{1/2}(2s - 2c - t) - 2(s - c - t) - 4a}{(4s - 4c - 3t + 4a)[1 + (2a/t)^{1/2}]},$$

which, since Φ^* is increasing in $\bar{s}$ while Φ^c is constant in $\bar{s}$, is increasing in $\bar{s}$. Solving for $\bar{s}$ such that the above is zero obtains

$$\frac{(c + t/2)((2a/t)^{1/2} - 1) + 2a - t/2}{(2a/t)^{1/2} - 1} \equiv \bar{s}^*.$$

Note that since by Assumption 1 $a > t/2$, then $(2a/t)^{1/2} - 1 > 0$. This ensures that $\bar{s}^* > 0$. □

2.2. A necessary condition for the existence of a competitive equilibrium.

When considering a candidate equilibrium of price and advertising levels, one possible deviation for a firm is to set a higher price. As the partial derivative $\frac{\partial^2 \pi_i(p_1,p_2,\Phi_1,\Phi_2)}{\partial p_i^2} = -\frac{\Phi_1\Phi_2}{t}$ is negative, no prices that are accepted by some contested buyers can dominate the candidate equilibrium price p^c in (7.15), in the candidate equilibrium. The focus can be restricted for prices that are only accepted by (some of) the captive buyers of the seller charging them, assuming that the competitor's does set price p^c. From Lemma 2 the monopoly price is $p^m = \bar{s} - t$. As $\bar{s}$ increases, the monopoly price becomes more profitable, while the competitive equilibrium price does not change (7.15) and so the competitive equilibrium profit (7.17) does not change. Thus, there exists a sufficiently large $\bar{s}$ at which the firm would deviate so long as $\Phi^c < 1$, which is ensured by Assumption 1. Specifically, if firm i deviates to $\bar{s} - t$ its profit, given Assumption 2, is

$$\Pi^m \equiv \Phi^c(1 - \Phi^c)(\bar{s} - t - c) - a\frac{\Phi^c}{2}. \tag{6}$$

Let $\overline{s}^m$ denote the $\overline{s}$ such that deviating to the monopoly price is more profitable (and so the competitive equilibrium does not exist). That is, for $\overline{s} \geq \overline{s}^m$, $\Pi^m \geq \Pi^c$. That is, a necessary, but not necessarily sufficient condition for the competitive equilibrium is that

Lemma 4. *For a firm not to deviate from the competitive price, $\overline{s}$ must be less than*

$$\hat{s} \equiv \frac{2a + (c+t)((2a/t)^{1/2} - 1)}{(2a/t)^{1/2} - 1}. \tag{7}$$

Proof. The competitive equilibrium price is dominated by the monopoly price whenever (6) is greater than (7.17), or subtracting the latter from the former, when the following is positive

$$\underline{\Pi}^m - \Pi^c = 2\frac{[(2a/t)^{1/2} - 1](\overline{s} - c - t) - a}{[1 + (2a/t)^{1/2}]^2},$$

which is increasing in $\overline{s}$. Solving for the $\overline{s}$ such this is zero obtains

$$\hat{s} \equiv \frac{2a + (c+t)((2a/t)^{1/2} - 1)}{(2a/t)^{1/2} - 1}.$$

For $\overline{s} > \hat{s}$, $\Pi^m \geq \underline{\Pi}^m > \Pi^c$, and the firm would deviate from the competitive price. □

Since as s increases deviating to the monopoly price becomes more attractive and the social planner would have more advertising, it is possible that the s sufficient for the social planner to choose more advertising than the competitive equilibrium would imply that the competitive equilibrium does not exist as firms would deviate to the monopoly price. However, this is not true: at s^* the competitive equilibrium still exists.

Proposition 1. *For Tirole's competitive equilibrium in a Hotelling model of advertising, advertising is socially insufficient when $s \in [\hat{s}, s^*)$ and the set is not empty.*

Proof. Subtracting (7) from (5) yields

$$s^* - \hat{s} = -\frac{(2at)^{1/2}}{2[(2a/t)^{1/2}) - 1]} < 0.$$

The inequality follows as the denominator is positive since by Assumption 1, $a > t/2$. Thus, $s^* < \hat{s} \geq \overline{s}^m$. □

That is, the $\overline{s}$ needed for socially insufficient advertising in the competitive equilibrium is greater than the maximum $\overline{s}$ possible for a firm not to deviate from the competitive equilibrium price.

The intuition for the result is straightforward. The competitive equilibrium price and advertising levels are independent of $\overline{s}$. However, larger $\overline{s}$ increases the social return from advertising; as $\overline{s}$ increases, the planner would increase the level of advertising. Thus, there is a threshold $\overline{s}$, s^*, such that if $\overline{s}$ is greater than this, then there would be insufficient advertising in the candidate equilibrium if it exists. However, as $\overline{s}$ increases, the incentives to deviate to the monopoly price increase. For this model and its assumptions, the $\overline{s}$ at which the firm would deviate from the equilibrium price is greater than s^*, and so a range exists in which there is socially insufficient advertising.

Appendix A. Monopoly Price

In the Hotelling model, given a monopolist at 0 that sets a price p, a consumer located at x is willing to buy if $\overline{s} - p - tx \geq 0$. If the $\tilde{x}$ such that $\overline{s} - p - t\tilde{x} = 0$ is less than one ($\tilde{x} < 1$), then the demand the firm faces is $D_m = (\overline{s} - p)/t$ and its profit is $(p - c)(\overline{s} - p)/t$. Maximizing this with respect to p yields $\mathring{p} = (\overline{s} + c)/2$, which is the profit-maximizing price so long as the quantity demanded associated with $\mathring{p}$ is less than 1; else the profit-maximizing price is $\overline{s} - t$ (since the firm could then raise its price to $\overline{s} - t$ without any change in the demand in its product).

References

Tirole, J, 1988, The Theory of Industrial Organization, MIT Press: Cambridge, MA.