A New Nonlinear Wavelet-Based Unit Root Test with Structural Breaks

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Abstract

In the literature, there are no nonlinear wavelet-based unit root tests with structural breaks. To fill this gap in the literature, this study proposes new wavelet-based unit root tests that take into account nonlinearity and structural breaks. According to Monte Carlo simulations results, the proposed tests show better size and power properties as the sample size increases. Moreover, the results indicate that the Fourier Wavelet-based KSS (FWKSS) unit root test is more powerful than the WKSS test in the presences of structural breaks.

JEL Classification: C12, C22

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1. Introduction

Analysis of time series with complex structures is usually carried out using the time dimension. But time-dimensional analyses often do not provide enough information about a time series. As a result, the information contained in some series cannot be read completely, and it is impossible to obtain reliable results under these conditions (Masset P., 2008). In such situations, it is necessary to use frequency-dimensional analysis. Repetitive movements in a time series can be revealed with the help of mathematical functions used in the frequency dimension. The results from frequency-dimensional analysis are more detailed than those from time-dimensional analysis. For this reason, it is necessary to use frequency-dimensional analyses with high-frequency financial series.

Traditional unit root tests have very weak power properties, especially in the case of negative moving averages (MA). As Granger (1966) states, the power characteristics of a time series peak at low-frequency components and decrease exponentially as frequency increases (Fan and Gençay, 2010). Accordingly, it can be said that the low-frequency components of a time series contain a significant amount of information about that series. This idea expresses the basic logic in the development of wavelet-based unit root tests. Although there have been recent studies on wavelet-based unit root tests, there are few unit root tests using wavelets in the literature. The first of the wavelet-based unit root tests were developed by Fan and Gençay (2010). This test is based on the variance ratio approach, and the test statistics are obtained simply by proportioning the variance of the low-frequency scaling coefficients to the whole variance of the series. For Fan and Gençay's (2010) test, there are three different situations applied to raw, demanded and detrended series. Furthermore, Eroğlu and Soybilgen (2018) developed Augmented Dickey–Fuller (ADF, 1981) and Phillips–Perron (PP, 1988) versions of the wavelet-based unit root test.

The wavelet-based unit root test generally uses the discrete wavelet transform for wavelet decomposition (Fan and Gençay, 2010). The wavelet coefficients in the discrete wavelet transform are as follows:

$$w_{1,t} = \sum_{l=0}^{L-1} h_l x_{2t+1-l \bmod N} \qquad t = 0,1,....,N/2-1 \tag{1}$$

The scale coefficients for the discrete wavelet transform are as follows:

$$v_{1,t} = \sum_{l=0}^{L-1} g_l x_{2t+1-l \bmod N} \qquad t = 0,1,....,N/2-1 \tag{2}$$

An important different between the equations are the filters used. While an h_l filter is used for the wavelet coefficients, a g_l filter is used for the scale coefficients.

Wavelet-based unit root tests in the literature are carried out using a linear data generation process. However, in the literature, there is no wavelet-based unit root test based on a nonlinear data generation process. To fill this gap, this study provides a wavelet-based nonlinear unit root test based on the unit root methodology in Kapetanios, Shin and Snell (KSS, 2003). Moreover, this proposed test has been extended to take into account structural breaks.

2. Nonlinear Unit Root Test with Wavelets

Kapetanios et al. (2003) used the following exponential smooth transition autoregressive (ESTAR) process for the nonlinear unit root test:

$$\Delta y_t = \phi y_{t-1} + \gamma y_{t-1}\left[1 - e^{(-\theta y_{t-d}^2)}\right] + \varepsilon_t \tag{3}$$

where $\left[1 - e^{(-\theta y_{t-d}^2)}\right]$ is the exponential transition function and it is bounded between 0 and 1. Kapetanios et al. (2003) proposed the parameter constraints $\phi = 0$ and $d = 1$ for Eq. 3.

$$\Delta y_t = \gamma y_{t-1}\left[1 - e^{(-\theta y_{t-1}^2)}\right] + \varepsilon_t \tag{4}$$

They tested the null hypothesis $H_0 : \theta = 0$ against the alternative hypothesis $H_0 : \theta > 0$. But the presence of the unknown parameter γ makes it difficult to test the null hypothesis. Therefore, Kapetanios et al. (2003) computed the first-order Taylor series approximation to the ESTAR model, and they got the following auxiliary regression.

$$\Delta y_t = \delta y_{t-1}^3 + \varepsilon_t \tag{5}$$

The test statistic for the nonlinear unit root test is as follows:

$$KSS = \hat{\delta} / s.h.(\hat{\delta}) \tag{6}$$

It is possible to use information in the time and frequency dimensions together with wavelet transforms. Therefore, wavelet transform should be applied to the series. There are two types of wavelet transforms in the literature: the continuous and discrete wavelet transforms.

Gençay et al. (2002) stated that it is appropriate to use the discrete wavelet transform with economic and financial time series since it is not possible to analyze the coefficients of the continuous wavelet transform for all frequencies. For this reason, the discrete wavelet transform is used in this study for wavelet-based nonlinear unit root test. Furthermore, we follow Fan and Gençay (2010) and use the Haar filter for the wavelet transform. The wavelet and scaling coefficients that were separated using a Haar filter are as follows:

$$W_{1,t} = \frac{1}{\sqrt{2}}\left(y_{2t} - y_{2t-1}\right), \qquad t = 1,2,...,T/2 \tag{7}$$

$$V_{1,t} = \frac{1}{\sqrt{2}}\left(y_{2t} + y_{2t-1}\right), \qquad t = 1,2,...,T/2 \tag{8}$$

where the wavelet coefficients $W_{1,t}$ contain high-frequency $\left[1/2,1\right]$ properties of the series, while the scaling coefficients $V_{1,t}$ include low-frequency $\left[0,1/2\right]$ properties. Low-frequency coefficients are known to contain more information about the series; in other words, scaling coefficients are used in place of the raw series in the unit root process. Accordingly, the model used for wavelet-based nonlinear unit root test is written as follows (Aydin, 2019):

$$\Delta V_{1,t} = \sum_{j=1}^{p} \rho_j \Delta V_{1,t-j} + \delta V_{1,t-1}^{3} + \varepsilon_t \tag{9}$$

where $V_{1,t}$ are the scaling coefficients. The value of 1 used as a sub-index shows the level of decomposition. Equation 9 shows the model for the KSS unit root test with wavelet transform. The equation states that the series analyzed by the null hypothesis includes a unit root. The test statistic for the nonlinear wavelet-based KSS unit root test is as follows:

$$WKSS = \hat{\delta} / s.h.(\hat{\delta}) \tag{10}$$

Since wavelet transforms do not affect the asymptotic distributions of the tests, all assumptions made for the test statistics used remain valid (Fan and Gençay, 2010).

Theorem 1: Under the null hypothesis, the WKSS test shows compliance with the following asymptotic distribution (Kapetanios et al. 2003):

$$t_{NL} \Rightarrow \frac{\left\{\frac{1}{4}W(1)^4 - \frac{3}{2}\int_0^1 W(r)^2 dr\right\}}{\sqrt{\int W(r)^6 dr}} \tag{11}$$

where $W(r)$ is the standard Brownian motion feature. In the presence of correlated errors, the asymptotic distribution of the WKSS test is like that of the non-correlated errors. Therefore, there is no need to determine a separate asymptotic distribution.

2.1.Nonlinear Wavelet Unit Root Test with Fourier Function

Yazgan and Ozkan (2015) used Fourier functions to detect structural breaks in wavelet transformations. They demonstrated that the data generation process to be used in the detection of structural breaks is as follows:

$$y_t = \mu(t) + \varepsilon_t \tag{12}$$

where the deterministic term is expressed as follows (Yazgan and Ozkan, 2015):

$$\mu(t) \cong \alpha \sum_{i=1}^{n} \left\{ (2i-1)^{-1} \sin\left[\frac{2\pi(2i-1)kt}{T} \right] \right\} \tag{13}$$

where n is the frequency of the deterministic term and k represents the frequency value of the Fourier function. According to Yazgan and Ozkan (2015), as the frequency value of the deterministic term increases, the form of the breaks shifts from smooth to sharp. We take the value n = 1 and recommend the following model for the Fourier WKSS (FWKSS) test:

$$\Delta V_{1,t} = \sum_{j=1}^{p} \rho_j \Delta V_{1,t-j} + \delta V_{1,t-1}^3 + \beta \sin(2\pi kt / T) + \varepsilon_t \tag{14}$$

where $V_{1,t}$ represents the scaling coefficients. Following Enders and Lee (2012), we recommend the following steps of the FWKSS test:

Step 1**:** Eq. 14 is estimated in the $1 \leq k \leq 5$ range. The model with the smallest sum of residual squares is chosen as the appropriate model.

Step 2**:** The presence of the nonlinearity is investigated using the standard t-test. However, in the case of the unit root null hypothesis, new critical values for the $\tau(\hat{k})$ test (in Table 5) were calculated since the classical critical values could not be used. Critical values were obtained by following Becker et al. (2004). If the Fourier function is not significant, it is recommended to use the WKSS unit root test.

3. Monte Carlo Simulation Results

In this section, we examine the small sample properties of the nonlinear wavelet-based unit root tests. Critical values were calculated for the proposed unit root tests for the raw (case 1), demeaned (case 2), and detrended (case 3) series. The computed critical values of the WKSS and FWKSS tests are presented in Table 1 and Table 2-4, respectively. All critical values were obtained with 50,000 replications for $T = \{50, 100, 250\}$ and $k = \{1, 2, 3, 4, 5\}$.

3.1. Finite-sample size

The size properties of the proposed tests were obtained using the following data generation process:

$$y_t = \alpha \sin\left(\frac{2\pi kt}{T}\right) + v_t \tag{15}$$

$$v_t = v_{t-1} + \varepsilon_t \tag{16}$$

The nominal size used for size and power analysis was set at 0.05. All power and size properties were obtained with 50,000 replications. The sample sizes were determined to be $T = \{50, 100, 250\}$. The results of size properties of proposed unit root tests are presented in Table 6-9. Although the results show size defects in small samples in both tests, it appears that the size defects are reduced as the sample size increases. Moreover, the size of FWKSS test is close to 0.05 in all cases with different values of k.

3.2. Finite-sample power

The following ESTAR data generation process was used to determine power properties:

$$y_t = \alpha \sin\left(\frac{2\pi kt}{T}\right) + v_t \tag{17}$$

$$\Delta v_t = \gamma v_{t-1}\left[1 - e^{(-\theta v_{t-1}^2)}\right] + \varepsilon_t \tag{18}$$

where ε_t is defined as $\varepsilon_t \sim N(0,1)$. The γ term is known as the stationary parameter, and when it takes a value in the $-2 < \gamma < 0$ range, it ensures that the ESTAR process is stationary (Kapetanios et al., 2003). However, θ is used as a smoothness parameter and the model becomes approximately linear when this parameter grows large. In this study, parameter limits were determined as $\alpha = 1$, $k = 1$, $\gamma = \{-1.5, -1, -0.5, -0.1\}$, and $\theta = \{0.01, 0.05, 0.1, 1\}$. The results of power analyses of proposed unit root tests are presented in Table 10-12. The results show that FWKSS test is more powerful than the WKSS test in all cases.

4. Conclusion

This study proposes new wavelet-based unit root tests that take into account nonlinearity and structural breaks. The small sample properties of the proposed unit root tests were investigated using Monte Carlo simulations. The results show that the FWKSS test is more powerful than the WKSS when structural breaks are included in the nonlinear data generation process. The unit root tests proposed in this study are the first wavelet-based nonlinear unit root tests that take into account breaks. In this sense, the study fills this gap in the literature.

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Appendix:

Table 1: Critical Values for WKSS Test

	%	*T=50*	*T=100*	*T=250*
Case 1	1	-2.434	-2.433	-2.403
	5	-1.846	-1.871	-1.896
	10	-1.575	-1.611	-1.636
Case 2	1	-3.340	-3.296	-3.266
	5	-2.679	-2.681	-2.685
	10	-2.368	-2.395	-2.406
Case 3	1	-3.454	-3.248	-3.179
	5	-2.795	-2.714	-2.720
	10	-2.497	-2.458	-2.480

Table 2: Critical Values for FWKSS Test (Case 1)

	k	1%	5%	10%
T=50	1	-3.05052	-2.29388	-1.92532
	2	-2.68806	-2.00453	-1.67529
	3	-2.53757	-1.88082	-1.58411
	4	-2.46119	-1.85630	-1.56419
	5	-2.40138	-1.81439	-1.53254
T=100	1	-2.98066	-2.29193	-1.94726
	2	-2.64552	-2.03791	-1.73307
	3	-2.52943	-1.97073	-1.67499
	4	-2.45754	-1.90868	-1.62888
	5	-2.44972	-1.89588	-1.61092
T=250	1	-2.92151	-2.29368	-1.97860
	2	-2.63397	-2.05940	-1.77253
	3	-2.55352	-1.98344	-1.70310
	4	-2.51819	-1.96189	-1.68112
	5	-2.48238	-1.92438	-1.64632

Table 3: Critical Values for FWKSS Test (Case 2)

	k	1%	5%	10%
T=50	1	-3.98595	-3.18276	-2.80172
	2	-3.64432	-2.89720	-2.54612
	3	-3.49847	-2.78775	-2.45756
	4	-3.44304	-2.73772	-2.39819
	5	-3.36784	-2.69313	-2.36630
T=100	1	-3.71111	-3.08782	-2.74968
	2	-3.45895	-2.85586	-2.54006
	3	-3.37481	-2.77631	-2.48483
	4	-3.36668	-2.73584	-2.44395
	5	-3.32173	-2.71804	-2.42175
T=250	1	-3.65867	-3.04487	-2.75268
	2	-3.44837	-2.86755	-2.57248
	3	-3.37627	-2.79518	-2.50230
	4	-3.33531	-2.75460	-2.47181
	5	-3.30800	-2.73365	-2.45916

Table 4: Critical Values for FWKSS Test (Case 3)

	k	1%	5%	10%
T=50	1	-4.00652	-3.24548	-2.89865
	2	-4.10191	-3.34092	-2.98095
	3	-4.13302	-3.30730	-2.93209
	4	-4.04479	-3.24543	-2.88283
	5	-4.00463	-3.20481	-2.84387
T=100	1	-3.69878	-3.12823	-2.85104
	2	-3.81167	-3.22051	-2.92112
	3	-3.81277	-3.18987	-2.89184
	4	-3.74323	-3.14523	-2.84324
	5	-3.73721	-3.12078	-2.82709
T=250	1	-3.67911	-3.13346	-2.86565
	2	-3.76104	-3.21348	-2.93216
	3	-3.70844	-3.16260	-2.88876
	4	-3.66961	-3.13496	-2.85599
	5	-3.66422	-3.11150	-2.83869

Table 5: Critical Values for $\tau(\hat{k})$ Test

	1%	5%	10%
Case 1			
T=50	-2.58042	-1.76263	-1.34616
T=100	-2.72846	-1.88017	-1.46066
T=250	-2.69923	-1.93011	-1.50902
Case 2			
T=50	-2.63157	-1.77420	-1.35733
T=100	-2.73127	-1.91745	-1.48128
T=250	-2.76360	-1.95903	-1.54099
Case 3			
T=50	-2.71988	-1.82605	-1.40301
T=100	-2.81003	-1.95665	-1.51516
T=250	-2.76299	-1.99781	-1.55978

Table 6: Size Properties for WKSS Test

	T=50	***T=100***	***T=250***
Case 1	0.047	0.048	0.05
Case 2	0.058	0.052	0.049
Case 3	0.069	0.057	0.049

Table 7: Size Properties of FWKSS Test for Case 1

	k=1	***k=2***	***k=3***	***k=4***	***k=5***
T=50	0.053	0.055	0.059	0.060	0.067
T=100	0.048	0.049	0.052	0.055	0.057
T=250	0.049	0.052	0.053	0.050	0.051

Table 8: Size Properties of FWKSS Test for Case 2

	k=1	***k=2***	***k=3***	***k=4***	***k=5***
T=50	0.064	0.067	0.069	0.071	0.073
T=100	0.051	0.053	0.055	0.057	0.060
T=250	0.048	0.049	0.050	0.050	0.052

Table 9: Size Properties of FWKSS Test for Case 3

	k=1	***k=2***	***k=3***	***k=4***	***k=5***
T=50	0.065	0.085	0.078	0.040	0.034
T=100	0.059	0.061	0.062	0.063	0.065
T=250	0.049	0.049	0.051	0.053	0.055

Table 10: Power Properties for Case 1

γ	θ	***T=50***		***T=100***		***T=250***	
		WKSS	FWKSS	WKSS	FWKSS	WKSS	FWKSS
-0.1	0.01	0.057	0.075	0.111	0.117	0.171	0.246
	0.05	0.110	0.124	0.204	0.224	0.319	0.516
	0.1	0.144	0.144	0.238	0.275	0.446	0.610
	1	0.172	0.184	0.312	0.341	0.536	0.672
-0.5	0.01	0.142	0.147	0.249	0.318	0.394	0.684
	0.05	0.290	0.314	0.524	0.695	0.696	0.950
	0.1	0.405	0.416	0.606	0.823	0.756	0.982
	1	0.634	0.681	0.781	0.962	0.857	0.999
-1	0.01	0.145	0.215	0.463	0.500	0.559	0.859
	0.05	0.353	0.470	0.731	0.870	0.890	0.990
	0.1	0.472	0.619	0.871	0.948	0.940	0.998
	1	0.695	0.909	0.959	0.998	0.990	1.000
-1.5	0.01	0.127	0.271	0.567	0.617	0.724	0.924
	0.05	0.446	0.583	0.664	0.938	0.897	0.997
	0.1	0.559	0.742	0.889	0.984	0.957	1.000
	1	0.740	0.967	0.985	1.000	0.998	1.000

Table 11: Power Properties for Case 2

γ	θ	*T=50*		*T=100*		*T=250*	
		WKSS	FWKSS	WKSS	FWKSS	WKSS	FWKSS
-0.1	0.01	0.054	0.075	0.062	0.076	0.089	0.109
	0.05	0.066	0.087	0.080	0.086	0.102	0.251
	0.1	0.073	0.083	0.089	0.091	0.239	0.341
	1	0.080	0.082	0.103	0.108	0.329	0.429
-0.5	0.01	0.078	0.091	0.109	0.110	0.345	0.430
	0.05	0.116	0.124	0.156	0.260	0.298	0.925
	0.1	0.137	0.156	0.375	0.400	0.475	0.981
	1	0.247	0.317	0.536	0.748	0.842	0.999
-1	0.01	0.103	0.104	0.127	0.157	0.427	0.732
	0.05	0.144	0.177	0.273	0.478	0.541	0.993
	0.1	0.236	0.264	0.430	0.681	0.698	0.999
	1	0.774	0.626	0.798	0.966	0.910	1.000
-1.5	0.01	0.101	0.115	0.204	0.214	0.780	0.868
	0.05	0.137	0.248	0.408	0.646	0.612	0.999
	0.1	0.270	0.386	0.685	0.837	0.836	1.000
	1	0.658	0.779	0.851	0.993	0.958	1.000

Table 12: Power Properties for Case 3

γ	θ	*T=50*		*T=100*		*T=250*	
		WKSS	FWKSS	WKSS	FWKSS	WKSS	FWKSS
-0.1	0.01	0.030	0.034	0.033	0.035	0.047	0.062
	0.05	0.038	0.043	0.049	0.053	0.106	0.171
	0.1	0.044	0.046	0.058	0.067	0.198	0.237
	1	0.050	0.056	0.080	0.091	0.257	0.325
-0.5	0.01	0.039	0.048	0.069	0.074	0.186	0.297
	0.05	0.064	0.089	0.187	0.211	0.662	0.806
	0.1	0.105	0.125	0.288	0.327	0.889	0.923
	1	0.144	0.280	0.453	0.658	0.905	0.995
-1	0.01	0.055	0.064	0.094	0.120	0.445	0.561
	0.05	0.109	0.149	0.258	0.398	0.891	0.957
	0.1	0.153	0.235	0.457	0.579	0.903	0.991
	1	0.456	0.543	0.666	0.917	0.980	1.000
-1.5	0.01	0.094	0.110	0.123	0.170	0.697	0.720
	0.05	0.156	0.244	0.396	0.543	0.909	0.988
	0.1	0.258	0.382	0.510	0.744	0.990	0.999
	1	0.557	0.643	0.877	0.954	1.000	1.000