A Note on the Application of Schubert Calculus in Heterogeneous Economies With Pure Exchange

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Abstract

In this paper we find a geometrical characterization of some components of the theory of equilibrium Walrasian and the existence of these in a pure exchange economy with heterogeneous agents where there are r goods and m agents. It will be proven that this economics admit stratification, for this we will consider the Grassmannian varieties. These are of greater importance for construct varieties of flags that contain information for obtaining the core of the economy and the existence of equilibrium.

1 Introduction

In certain cases the problem of finding the equilibrium point it becomes difficult in economic theory since there are spaces that don't have good properties like compactness or connected. The discussion extends to problem versions of more general equilibrium exposed by Keiding [2017]. In this work, we use the Grassmannian varieties mentioned in Griffiths and Harris [2014] and Fulton and Harris [2013] to describe some components of economies with pure exchange with heterogeneous agents.
Suppose an economy with pure exchange with agents heterogeneous. Groups of heterogeneous individuals are presented which are homogeneous among themselves with goods to be traded. We want to study the pure exchange produced by these groups.
We could ask ourselves the following questions as the motivation for this paper. It is possible to describe the core in this type of economics with the tools known in the economics of homogenous type?. Under what circumstances is it possible to find a equilibrium between the representative agents of these groups?.
Equilibrium conditions are studied for an economy of pure exchange assuming convexity, monotonicity between other characteristics of the utility functions of each individual, Iusem and Sosa [2003].

This paper presents a characterization of equilibrium in economies with pure exchange when agents have aspects of heterogeneity. Agents may be different in their beliefs, their level of risk aversion, continuity in its function useful or intertemporal preferences and endowments.
The objective is to describe this type of economy in high dimensions using the Schubert calculus. The latter has a basis for the Schubert varieties described in Section 2.
Our contribution is the geometric and algebraic description of economies with pure exchange and behaviour heterogeneity between the agent's addition, the fees are not necessary marginal substitution and therefore the marginal relations of substitution. Which we attribute an important leap in the literature.
Theorems Rizvi [2006] implies that strong assumptions on preferences, such as homogeneity among agents or heterogeneity is not sufficient for stability or uniqueness of equilibrium[1].

We study the impact of heterogeneous agents on the equilibrium properties. We will use Grassmannian varieties and varieties of flags as fundamental tools to describe several aspects of a heterogeneous economy with exchange pure. In Main Theorem we show that the set of fixed points is not empty in a minimal flag, which means that indeed in a heterogeneous exchange economy pure equilibrium is not unique.

This paper is composed as follows: Section (2) a short motivation and some key concepts for the realization of this paper. Here we will show an example with explicit characteristics of individuals, followed by some properties that have an exchange space and some important facts about Grassmannian varieties and flag varieties. Section

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[1] Mantel [1974] shows that the theorems of Rizvi [2006] results for homotetic preferences, and Kirman and Koch [1986] shows that this results with identical preferences and non-linear endowments

(2) it is dedicated to the main results obtained in this paper as well as some important definitions such as Schubert cells and varieties. In particular, Theorem (3.8) shows in effect that the set of fixed points in a minimal flag is not empty using the Borel fixed point Theorem. Section (4) shows the main proof's of this paper.

2 Motivation and preliminary

Let E k-dimensional vectorial space on $\mathbf{R}$ characterized to agents contained of the economy. We assume that there is a partition of E in subsets $\{E_q\}_q$, $q = 1, \dots, k$ of individuals which are homogeneous among themselves with goods to exchange. Suppose in a principle that we know the utility functions (level of risk aversion, substitution elasticity, the weight of each utility function), relative prices and initial endowments.

There are several types of preferences among agents, for example, Bernoulli type preferences, quadratic preferences, quasi-linear preferences and additive separable preferences. With homotetic preferences, it is well known that if the preferences they are identical, Gorman [1953], collinear endowments ,Chipman [1974], or relatively constant risk aversion (CRRA) at most 1, Hens and Loeffler [1995], then the equilibrium is unique. The following example shows an explicit motivation of what you want to show in this work, multiple equilibrium are built between agents with different endowments and weights on assets with CRRA at least 1. You can see more of this type of examples in Toda and Walsh [2017].

Example 2.1. Consider $m = 4$ with $E = \mathbf{R}^4$. Let E_1 and E_2 be two subspaces of E such that both are convex, not empty and disjoint, without loss of generality suppose E_1 open. For the first geometrical form of Hanh-Banach there is a hyperplane H that separates E_1 and E_2, both of equal or no cardinality. Let $i_1 \in E_1$ and $i_2 \in E_2$ be the representative agents in each subspace and $I \subset \mathbf{R}^2$ some exchange space. Assume that each agent has equal risk aversion and utility fuctions of the Bernoulli type, that is,

$$U_{i_1}(x_1, x_2) = \frac{1}{1-\gamma}\left(\alpha^\gamma x_1^{1-\gamma} + (1-\alpha)^\gamma x_2^{1-\gamma}\right)$$

$$U_{i_2}(x_1, x_2) = \frac{1}{1-\gamma}\left((1-\alpha)^\gamma x_1^{1-\gamma} + \alpha^\gamma x_2^{1-\gamma}\right)$$

where $\gamma > 1$ is the level of risk aversion and $0 < \alpha < 1$ determines the weight of the utility function.

The initial endowments are $e_1 = (e, 1-e)$ and $e_2 = (1-e, e)$ where $0 < e < 1$. Let $p_1 = 1$, $p_2 = p$, the prices and $x_{ij}(p)$ demand of agent i for good j, and

$$z_j(p) = \textstyle\sum_{i=1}^{2}(x_{ij}(p) - e_{ij})$$

excess demand for good j. Let $\epsilon = \frac{1}{\gamma} < 1$ the elasticity of substitution.

Lemma 2.2. *Suppose that*

$$\epsilon < 1 - \frac{1}{2}\left(\frac{e}{\alpha} + \frac{1-e}{1-\alpha}\right) \tag{1}$$

Then the economy has at least three equilibria. In particular, for any $\gamma > 2$ we can construct an economy with three equilibria.

Proof. Let $w_1 = e + (1-e)p$ the wealth of the agent 1. Since the preferences are identical to the relative risk aversion, this demand for good 1 is

$$x_{11} = \frac{\alpha p_1^{-\epsilon} w_1}{\alpha p_1^{1-\epsilon} + (1-\alpha)p_2^{1-\epsilon}} = \frac{\alpha(e + (1-e)p)}{\alpha + (1-\alpha)p^{1-\epsilon}}$$

Since agents are symmetric, agent 2's demand for good 1 can be obtained by changing $\alpha \to 1-\alpha$ adn $e \to 1-e$, Therefore the aggresate excesso demand for good 1 is

$$\begin{aligned} z_1(p) =& x_{11} + x_{21} - e - (1-e) \\ =& \frac{\alpha(e + (1-e)p)}{\alpha + (1-\alpha)p^{1-\epsilon}} + \frac{(1-\alpha)(1-e+ep)}{1-\alpha+\alpha p^{1-\epsilon}} - 1 \end{aligned}$$

By symmetry, $p = 1$ is a equilibrium price. Since $0 < \epsilon < 1$, by direct substitution we have $z_1(1) = \alpha + (1-\alpha) - 1 = 0$, and $z(\cdot)$ is continue implies $\lim_{p\to\infty} z(p) = \infty$. Therefore if $z_1'(1) < 0$, we have $z_1(p) < 0$ while $p > 1$. As $\lim_{p\to\infty} z(p) = \infty$ by the intermediate value theorem there exist $p* > 1$ such that $z_1(p*) = 0$, so $p*$ is an equilibrium price. By symmetry, if p^* is a equilibrium price them $\frac{1}{p*}$ it is.

Therefore in order to show the existence of multiple equilibria, it suffices to show $z_1'(1) < 0$. Differentiating $x_{11}(p)$, we obtain

$$x_{11}'(p) = \alpha(1 - e - (1 - \alpha)(1 - \epsilon))$$

at $p = 1$. Changing $\alpha \to 1 - \alpha$ and $e \to 1 - e$, the economy has at least three equilibria if

$$z_1'(1) = x_{11}' + x_{12}' = \alpha(1 - e) + (1 - \alpha)e - 2\alpha(1 - \alpha)(1 - \epsilon) < 0$$

iff

$$\epsilon < 1 - \frac{1}{2}\left(\frac{e}{\alpha} + \frac{1 - e}{1 - \alpha}\right)$$

which is (1).

Finally, set us show that we can construct an economy with at least three equilibria when $\gamma > 2$. Set $e = \alpha^2$. Then (1) become

$$\epsilon < 1 - \frac{1}{2}\Big(\alpha + (1 + \alpha)\Big) = \frac{1}{2} - \alpha$$

Hence for any $\gamma > 2$ ($\epsilon < \frac{1}{2}$), by choosing α such that $0 < \alpha < \frac{1}{2} - \epsilon$ and setting $e = \alpha^2$, we get an economy with at least three equilibria. □

Suppose now that we don't know any of the data previous and that we have chosen representative individuals of each E_q and we want to produce an equilibrium between these agents in an exchange space I such that $\dim(I) < k$.

Some relevant aspects about the exchange space I are the following.

Lemma 2.3. *The exchange space I is a topological vectorial space not empty, separable and finite-dimensional.*

Proof. We can consider I as a metric space. Very metric subspace of a separable metric space is separable. □

We will assume that I is m-dimensional. The characteristic function of I, χ_I is defined for each agent i

$$\chi_I(i) = \begin{cases} 1, & i \in I; \\ 0, & i \notin I. \end{cases}$$

Lemma 2.4. *The volume*

$$v(I) = \int_I \chi_I(i)di$$

of the exchange space I is non-zero.

Lemma 2.5. *The exchange space is a measurement space* [2] *(I, Σ, μ) Σ-finite where I is the exchange space, Σ is a Σ-algebra of subsets of I whose elements are exchanges and μ is a measure on Σ.*

Remark 2.6. Since space I has non-zero volume, it implies that I has measure non-zero.

Let $r, m \in \mathbf{N}$, $r < m$. The Grassmannian varieties [3] $\mathrm{Gr}_r(I)$ is defined as the set of all vector subspaces of dimension r on $\mathbf{R}$, this is:

$$\mathrm{Gr}_r(I) := \{P \subset I : \dim_{\mathbf{R}} P = r\}$$

Fixed r. Let θ *Plücker* application

$$\theta : \mathrm{Gr}_r(I) \to \mathbf{P}(\bigwedge^r I) : P = \langle u_1, \ldots, u_r\rangle \mapsto [u_1 \wedge \ldots \wedge u_r]$$

where $P \in \mathrm{Gr}_r(I)$ and $\{u_1, \ldots, u_r\}$ is a base for P. Then θ is an injection. And therefore the Grassmannian varieties they are projective varieties.

Definition 2.7. A subset of vectors $\omega \in \bigwedge^r I$ is is totally decomposable if $\omega = v_1 \wedge v_2 \wedge \ldots \wedge v_r$ for certain $v_1, v_2, \ldots, v_r \in I$.

[2] Note that if μ is the counting measure defined as

$$\mu(A) = \begin{cases} |A|, & \text{if A is finite;} \\ +\infty, & \text{if A is infinite.} \end{cases}$$

$\forall A \subset \Sigma$, where $|A|$ denotes the cardinality of A.

[3] The $Gl(r, \mathbf{R})$ group of invertible matrices act transitively on the array of dimensions of $m \times r$ by multiplication to the right and this action does not change the column generators, that is, given $P = \langle u_1, \cdots, u_r\rangle$, a r-plane $\mathrm{Gr}_r(I) = Gl(r, \mathbf{R})/H$

And therefore, the dimension of $\mathrm{Gr}_r(I)$ as a variety is equal to the dimension of $\mathrm{Gl}(r, \mathbf{R})$ minus the size of the one-point stabilizer.

The following result shows the relationship between the bivectors and the polynomial equation that define exchange value. This result generalizes the result obtained in Danilov and Sotskov [1990] on exchange values in a pure exchange economy [4].

Theorem 2.8. *The image of* $\mathrm{Gr}_r(I)$ *under the application* θ *is defined by a homogeneous polynomial equation on* $\mathbf{P}(\bigwedge^r I)$.

Example 2.9. Let $I = \mathbf{R}^m$, suppose 2 goods. An bivector $\omega \in \bigwedge^2 \mathbf{R}^m$ is decomposable iff $\omega \wedge \omega = 0$

Proof. For a, b goods. We will say $a \prec b$ iff b is preferred to a.
Let

$$\omega = \sum_{a \prec b} p_{ab} e_a \wedge e_b$$

a bivector. By Theorem 2.8, ω is decomposable iff

$$\begin{aligned} \omega \wedge \omega &= (\sum_{a \prec b} p_{ab} e_a \wedge e_b) \wedge (\sum_{c \prec d} p_{cd} e_c \wedge e_d) \\ &= \sum_{a \prec b \prec c \prec d} [p_{ab}p_{cd} - p_{ac}p_{bd} + p_{ad}p_{bd}] e_a \wedge e_b \wedge e_c \wedge e_d = 0 \end{aligned}$$

this means that $p_{ab}p_{cd} - p_{ad}p_{bd} + p_{ac}p_{bd} = 0$, which is the system wanted. □

Remark 2.10. When $r = m - 1$, we obtain the case of complete markets, and $P \in \mathrm{Gr}_r(I)$ is the hyperplane which defines the price vector, with free disposal and the whole of established budget restrictions are found under this hyperplane. A consequence of the law of Walras is when $r < m - 1$ budget constraints correspond to the case of incomplete markets ($m - r - 1$ is the number of "missing markets"), or more generally in which agents face a set of linear restrictions on the level of prices an the specification of asset returns; thus change in P may correspond to changes both in the level of prices and in asset returns, this produces the region of mutual negotiation in the cashier from Edgewordth.

Let now $r = m - n$ the total number of different goods of each individual, where to $P \subset I$, $\dim(P) = r$ and $\mathrm{codim}(P) = n$. A complete flag is a sequence of vector spaces

$$F_\bullet = \Big\{\{0\} \subset F_1 \subset F_2 \subset \ldots \subset F_m = I\Big\} \tag{2}$$

with $\dim(F_i) = i$ for each agent i. The set of all flags is called a variety of flags and is denoted $\mathcal{F}$. There is a canonical inclusion

$$\mathcal{F} \hookrightarrow \mathrm{Gr}_1(I) \times \mathrm{Gr}_2(I) \times \ldots \times \mathrm{Gr}_{r-1}(I)$$

sending $F_\bullet \mapsto (F_1, \ldots, F_{r-1})$. It's

$$\mathcal{F} = \{(F_1, \ldots, F_{r-1}) \in \prod_r \mathrm{Gr}_r(I) : F_1 \subset F_2 \subset \ldots \subset F_{m-1}\}$$

This relation is algebraic, implies that the flags are in closed sub-varieties effect of $\coprod_r \mathrm{Gr}_r(I)$ and therefore projective.
Using (2) we say that a flag is minimal if it is composed by a point (q), a line (l) that contains to q and a hyperplane (P) determined by q and l. Here, pictorially the not countable lines that pass through q are the possible contract curves of each individual and the hyperplane P determine the core of this type of economy. Under this point of view an equilibrium problem in this type economics is the realization of a 'good' flag. Therefore, the existence of a equilibrium is determined if the flag is minimal.

3 Main results

Lemma 3.1. *Given a vector space* I *of dimension* m *on* $\mathbf{R}$, *then there are minimal flags.*

Each $P \in \mathrm{Gr}_r(I)$ produces a new sequence of vector spaces of the form

$$\{0\} = F_0 \cap P \subset F_1 \cap P \subset \ldots \subset F_m \cap P = P$$

which defines a sequence increasing of integers $d_i = \dim(P \cap F_i)$ $\forall i$ that satisfy

[4] Fixed a set K goods, bounded. An exchange value is given by a collection of positive integers $\{p_{jk}\}$, $j, k \in K$. Here p_{jk} denotes how many units of j are given by a unit of k. These numbers satisfy the following natural relationship

$$p_{jk} \cdot p_{kl} \cdot p_{ij} = 1, \quad j, k, l \in K.$$

1. $d_0 = 0$,
2. $d_m = r$,
3. $d_i \leq d_{i+1} \leq d_i + 1$.

Using (2) and (3), $d_j \geq j - n$. The sub-indices of the elements of the sequence d_j are in bijection with certain partitions. Next we will describe which. For each good j and agent i, be

$$j_1 \quad \text{the first index such that} \quad d_{j_1} = 1$$

$$\vdots$$

$$j_i \quad \text{the first 'index such that} \quad d_{j_i} = i,$$

it's $j_1 \leq n+1$, $j_2 \leq n+2$,.., $j_i \leq n+i$, implies $j_1 = n+1-\lambda_1$, $j_2 = n+2-\lambda_2$, ..., $j_i = n+i-\lambda_i$, for certain $\lambda_i \geq 0\ \forall i$. This means $j_1 < j_2 < \ldots < j_i$ so $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. This produces a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$. Which contains all the information on the goods j's exchanged by the agents i's. To this partition we associate a Young's diagram with r rows and n columns, this consists of an arrangement of boxes placed by rows so that each row has a smaller or equal number of boxes that the previous row. Thus, listing the number of boxes in each row provides a decreasing succession of positive integers. The members of the partition contain the information of goods exchanged by the agents in this economy in said period of time. Finally, fixed the integers of the dimensions and a partition λ, the Schubert cells[5] are defined X_λ° and the Schubert varieties[6] X_λ, both subset of $\mathrm{Gr}_r(I)$, as

$$X_\lambda^\circ = \{P \in \mathrm{Gr}_r(I) : \dim(P \cap F_i) = j \text{ for } n+j-\lambda_j \leq i \leq n+j-\lambda_{j+1}, 0 \leq j \leq r\}$$

y

$$X_\lambda = \{P \in \mathrm{Gr}_r(I) : \dim(P \cap F_{n+j-\lambda_j}) \geq j, 1 \leq j < r\}$$

Remark 3.2. The co-dimension$(X_\lambda) = |\lambda|$, where $|\lambda|$ corresponds to the number of boxes of the partition λ.

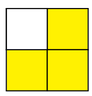
Lemma 3.3. *If $\{e_1, \ldots, e_m\}$ is a standard base for I, exist a unique base $\{u_i\}_{i=1}^r$ for $P \in X_\lambda^\circ$ such that*

1. $u_i = e_{n+k-\lambda_i} + \sum_{j<n+i-\lambda_i} x_{ij} e_j$
2. $x_{n+l-\lambda_l} = 0\ \forall l < i$.

Example 3.4. Let $P \in \mathrm{Gr}_2(I)$ y $I = \mathbf{R}^4$, P is a 2-plane in $\mathbf{R}^4$ we have the following sequence

$$\{0\} = F_0 \cap P \subset F_1 \cap P \subset F_2 \cap P \subset F_3 \cap P \subset F_4 \cap P = P.$$

Now $F_0 = \{0\}$, $F_1 = \langle e_1 \rangle$, $F_2 = \langle e_1, e_2 \rangle$, $F_3 = \langle e_1, e_2, e_3 \rangle$ y $F_4 = \mathbf{R}^4$. Suppose that $P = \langle e_1 + e_2, e_3 + e_4 \rangle$, then $P \cap F_0 = \{0\}$, $P \cap F_1 = \{0\}$, $P \cap F_2 = \langle e_1 + e_2 \rangle$, $P \cap F_3 = \langle e_1 + e_2 \rangle$, $P \cap F_4 = P$, therefore $d_0 = 0$, $d_1 = 0$, $d_2 = 1$, $d_3 = 1$, $d_4 = 2$, the the associated partition is $\lambda = (1)$



This means that for $i_1 \in E_1$ and $i_2 \in E_2$ (Assuming Hanh-Banach Theorem of Separation) with $i_1, i_2 \in I$ the exchange between agents consists of a single period in which a good is exchanged. Analogously to the other agents.

[5]For each r-plane P en $\mathrm{Gr}_r(I)$. P is precisely the T-fixed point in $\mathrm{Gr}_r(I)$, where

$$T = \left\{ \begin{pmatrix} \alpha_{1,1} & 0 & \cdots & 0 \\ 0 & \alpha_{2,2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \alpha_{r,r} \end{pmatrix} \right\} \subset Gl(r, \mathbf{R})$$

it is the subgroup of diagonal matrices (this is a maximal torus of $Gl(r, \mathbf{R})$). If we consider now

$$B = \left\{ \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \ldots & \alpha_{1,4} \\ 0 & \alpha_{2,2} & \ldots & \alpha_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha_{r,r} \end{pmatrix} \right\} \subset Gl(r, \mathbf{R})$$

the upper triangular matrix subgroup (this is the Borel subgroup of $Gl(r, \mathbf{R})$), so Schubert cells are BP.

[6]The Schubert varieties X_λ are the clausure of the Schubert cells X_λ°, i.e. $X_\lambda = \overline{X_\lambda^\circ}$ with the Zarisky topology.

One of the main advantages of this type of coding of the information on the exchanges made by the agents of the economy, is the obtaining of the number of goods exchanged in each period of time waiting for greater utility in each of these. This fact is relevant in this paper assuming that it can be extended to versions with applications to Game Theory under uncertainty. The following result shows that there is indeed a stratification on of this type of economy mainly given by Schubert varieties and this stratification can be represented by Young diagrams.

Lemma 3.5. *Given λ and μ partitions contained in a rectangle of dimensions $r \times n$. Then $X_\lambda \cap X_\mu \neq \emptyset$ iff $\mu \subset \lambda$.*

Example 3.6. Let $I = \mathbf{R}^4$, $r = 2$. We have then 4 heterogeneous agents and 2 exchange goods. Let's show some characteristics of Schubert cells and varieties for $\mathrm{Gr}_2(\mathbf{R}^4)$. λ is a partition we can make with the set $\{0, 1, 2\}$ and $j \in \{1, 2\}$ for interesting cases, then the cells of Schubert are: X°_{00}, X°_{10}, X°_{20}, X°_{21}, X°_{11} and X°_{22}. The following table shows the relation between the dimensions and co-dimensions of Schubert cells, together with the sequence of integers determined by the flag and partition obtained.

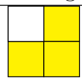
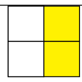
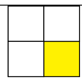
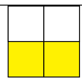
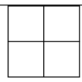
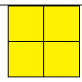
j	$n+j-\lambda_j$	$n+j-\lambda_{j+1}$	λ	$d_\bullet$	Young	
1	2	3	10	00112		dim 3, codim 1
2	4	4				
1	2	2	11	00122		dim 2, codim 2
2	3	4				
1	1	2	21	01122		dim 1, codim 3
2	3	4				
1	1	3	20	01112		dim 2, codim 2
2	4	4				
1	1	1	22	01222		dim 0, codim 4
2	2	4				
1	3	3	00	00012		dim 4, codim 0
2	4	4				

Table 1: Representation of Schubert cells.

Note also that for $\mathrm{Gr}_2(\mathbf{R}^4)$ the parametrization of Schubert cells are as follows:

$$X^\circ_{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad X^\circ_{11} = \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix}$$

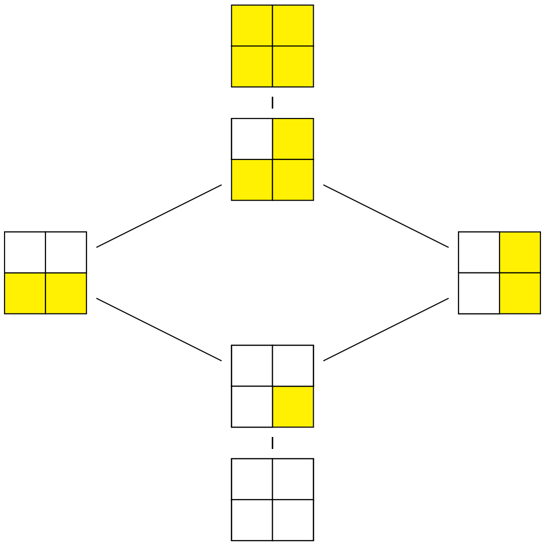
$$X^\circ_{10} = \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad X^\circ_{21} = \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}$$

$$X^\circ_{20} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix} \qquad X^\circ_{22} = \begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}$$

On the other hand for Schubert varieties we have the following information:

$$\begin{aligned} \text{codim} \quad 1 &: X_{10} = \{P : \dim(P \cap F_2) \geq 1\} \\ \text{codim} \quad 2 &: X_{11} = \{P : (P \cap F_3) \geq 2\} \\ X_{20} &= \{P : F_1 \subset P\} \\ \text{codim} \quad 3 &: X_{21} = \{P : F_1 \subset P \subset F_3\}. \end{aligned}$$

Here, Schubert varieties of co-dimension 1 and therefore partition (1) encodes that a single good exchanged between the agents and in a single period of time. The varieties of Schubert of co-dimension 2, produce 2 partitions (1,1) and (2,0), therefore the agents have two periods of time to exchange in which they can either exchange 1 either in each period or the entire of your assets in a first period. Schubert's variety of co-dimension 3, produce a partition (2,1) which encodes a exchange in all of your assets in a first period of time and 1 well exchanged in a second period. Therefore, the stratification of this economy is produced by the following representation:



Now as $\mathrm{Gr}_2(\mathbf{R}^4)$ is the variety of projective lines in $\mathbf{P}^3$, then a flag in $\mathbf{P}^3$ corresponds a $q \in l \subset P \subset \mathbf{P}^3$, where q is a point, l is a line and P a hyperplane in $\mathbf{P}^3$, then

(1) The Schubert variety of co-dimension 1, X_{10}: Parametrize the lines that intersect l.

(2) The Schubert variety of co-dimension 2, X_{20}: Parametrizes the lines containing q.

(3) The Schubert variety of co-dimension 2, X_{11}: Parametrizes the lines that are contained in P.

(4) The Schubert variety of co-dimension 3, X_{21}: Parametrizes the lines that are contained in P and containing q.

Following the same notation, let $\{e_1, e_2, \ldots, e_m\}$ a basis for I and set a flag $\mathcal{F} : \{0\} \subset F_1 \subset F_2 \subset \ldots \subset F_m = I$, where $F_i = \langle e_1, \ldots, e_i \rangle$. The opposite flag $\widetilde{\mathcal{F}}$ is defined $\widetilde{F}_i = \langle e_m, \ldots, e_{m-i+1} \rangle$. Let's denote Schubert cells and varieties of the opposite flag by $\widetilde{X}^\circ_\mu$ and $\widetilde{X}_\mu$ respectively.
There is a characterization about the exchange of two agents heterogeneous, corresponding to the varieties of Schubert.

Lemma 3.7. *Consider 2 heterogeneous agents in I. λ is the partition associated to a agent and μ to other agent. $X_\lambda \cap \widetilde{X}_\mu \neq \emptyset$ iff $\lambda_i + \mu_{r+1-i} \leq n$.*

Theorem 3.8. *(Main Theorem.) The set of fixed points in a minimal flag is not empty.*

The proof is consequently of the following discussion about the varieties of flags.

Definition 3.9. An algebraic variety X is complete if $p_2 : X \times Y \to Y$ for any algebraic variety Y is a closed map (the universally closed property), i.e. the image of a closed set is closed in Y.

Remark 3.10. Projective varieties are complete.

The "completeness" property in the category of algebraic varieties is analogous to the "compactness" property in the category of Hausdorff topological spaces.

Lemma 3.11. *The dimension of a complete, affine variety X is zero.*

Theorem 3.12. *(Borel Fixed Point Theorem). If a connected solvable group H acts on a non-empty complete variety, then the fixed point*

$$X^H := \{P \in X : \forall B \in H, BP = P\}$$

is non-empty[7].

[7] Bich and Cornet [2004] prove a fixed point theorem for multivalued applications defined on a set of finite products of Grassmannian varieties and convex sets

4 Mains proof's

Proof. (Proof Lemma (3.1))

This is deduced by the fact that I is a Noetherian ring on $\mathbf{R}$ and the axiom of choice on the elements F_i $\square$

Proof. (Proof Lemma (3.3))

Let $P \in X_\lambda^\circ$. Let's fix a base $\{e_1, \dots, e_m\}$ for I with $F_i = \langle e_1, \dots, e_i \rangle$. Let $\langle u_1, \dots, u_r \rangle$ a base for P such that $P \cap F_{n+i-\lambda_i} = \langle u_1, \dots, u_i \rangle$ $\forall i$. Let u_1 be a generator for the line $P \cap F_{n+1-\lambda_1}$ normalized so that $\langle u_1, e_{n+1-\lambda_1} \rangle = 1$; i.e.,

$$u_1 = (*, *, \dots, *, 1, 0, \dots, 0).$$

Now take u_2 so that u_1 and u_2 together span $P \cap F_{n+2-\lambda_2}$, normalized so that

$$\langle u_2, e_{n+1-\lambda_1} \rangle = 0, \qquad \langle u_2, e_{n+2-\lambda_2} \rangle = 1.$$

Continue in this way, choosing u_i so that $u_1, \dots, u_i$ span $P \cap F_{n+i-\lambda_i}$ and such that

$$\langle u_i, e_{n+j-\lambda_j} \rangle = \begin{cases} 0, & j < i; \\ 1, & j = 1. \end{cases} \tag{3}$$

Clearly, the choice of u_i at each stage is completely specified by these conditions; thus the k-plane P has a unique matrix representative of the form

$$\begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_r \end{pmatrix} = \begin{pmatrix} * & * & * & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ * & * & * & 0 & * & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ * & * & * & 0 & * & 0 & * & * & * & 1 & 0 & 0 & & & 0 \\ \cdot & & & \cdot & & \cdot & & & & & 0 & & & & 0 \\ \cdot & & & \cdot & & \cdot & & & & & & & & & \\ \cdot & & & \cdot & & \cdot & & & & & & & & & \\ * & * & * & 0 & * & 0 & * & * & * & 0 & * & * & 1 & & 0 \\ & & & 0 & & 0 & & & & 0 & & & 0 & & \\ & & & \vdots & & \vdots & & & & \vdots & & & \vdots & & \vdots \\ * & * & * & 0 & * & 0 & * & * & * & 0 & * & \dots & * & * & * \end{pmatrix}$$

$\square$

Proof. (Proof Lemma (3.5))

An element in X_λ is generated by the row vectors of a reduced matrix of the form

$$\begin{pmatrix} * & \dots & * & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ * & \dots & * & 0 & * & 1 & \dots & \dots & \dots & \dots & \dots & 0 \\ * & \dots & * & 0 & * & 0 & * & \dots & * & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & \dots & * & 0 & * & 0 & * & \dots & 0 & \dots & 1 & 0 \end{pmatrix}$$

where the 1's are in the position $n + i - \lambda_i$ of row i-th, and an element in X_{mu} is generated by the rows vectors of a reduced matrix of the equal form where the 1's are in the position $n + i - \mu_i$ of row i-th. Now, $X_\lambda \subset X_\mu$ if and only if the position in which is the 1 most to the right of the i-th of the first matrix is less than or equal to the position in which the 1st most to the right of the i-th of the second matrix, that is, if and only if $n + i - \lambda_i \leq n + i - \mu_i$, i.e., $\mu_i \leq \lambda_i$ $\forall i$ therefore $\mu \subset \lambda$. $\square$

Proof. (Proof Lemma (3.7)) When we are considering the intersection of X_λ and $\widetilde{X}_\mu$, we will make frequent use of the following subspace:

$$A_i = F_{n+i-\lambda_i}, \quad B_i = \widetilde{F}_{n+i-\mu_i}.$$

Suppose P is a subspace that is in both X_λ and $\widetilde{X}_\mu$. Then for any i between 1 and r,

$$\dim(P \cap A_i) \geq i \quad \text{and} \quad \dim(P \cap B_{r+1-i}) \geq r + 1 - i.$$

Since these two intersections take place in the r-dimensional vector space I, and $i + (r + 1 - i) - r = 1$, their intersection must have dimension at least 1. In particular, the intersection of A_i and B_{r+1-i} must have dimension at least.

$\square$

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