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Cao, Dan and Evans, Martin and Lua, Wenlan

Georgetown University, Georgetown University, Tsinghua University

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Dan Cao†
Georgetown University

Martin D.D. Evans‡
Georgetown University

Wenlan Luo§
Tsinghua University

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Abstract

We examine how medium-term movements in real exchange rates and GDP vary with international financial conditions. For this purpose, we study the international transmission of productivity shocks across a variety of IRBC models that incorporate different assumptions about the persistence of productivity shocks, the degree of international risk sharing and access to international asset markets. Using a new global solution method, we demonstrate that the transmission of productivity shocks depends critically on the proximity of a national economy to its international borrowing limit. We then show that this implication of the IRBC model is consistent with the behavior of the US-UK real exchange rate and GDP over the past 200 years. The model also produces a negative correlation between relative consumption growth and real depreciation rate consistent with more recent data, and hence offers a resolution of the Backus-Smith puzzle.

JEL: C60, F30, F31, F41, F44, G11

Keywords: Real Exchange Rates, International Real Business Cycles, Global Solution Methods

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†Department of Economics, Georgetown University. Email:dc448@georgetown.edu

‡Corresponding Author: Department of Economics, Georgetown University. Email:evansmdd@gmail.com

§School of Economics & Management, Tsinghua University. Email:luowenlan@gmail.com

Introduction

Research on the behavior of real exchange rates spans a broad frequency spectrum. At one end, macroeconomic models emphasize how changing interest rates (driven by monetary policy and other variables) induce monthly and quarterly variations in real exchange-rates via their impact on nominal exchange-rates in the presence of price-stickiness (see, e.g., Engel et al., 2008).[1] At the other end of the spectrum, following Harrod (1933), Balassa (1964) and Samuelson (1964), a large empirical literature examines how very low frequency or secular movements in real exchange rates are linked to sectoral shifts in productivity and other factors. This paper focuses on the behavior of real exchange rates in the less studied center of the frequency spectrum; that is the middle ground between business-cycles and the very long run. In particular, we examine how medium-term co-movements in real exchange rates and GDP vary with international financial conditions.

We begin by documenting the empirical importance of medium-term co-movements in the real exchange rate and GDP (per capita) using more than 200 years of US and UK data. We show that shocks driving the cyclical components of GDP have persistent exchange-rate effects that last for approximately ten years, and that these shocks account for a significant fraction of the real US-UK depreciation rate over horizons ranging from five to fifteen years. Our empirical estimates also show that shocks producing an increase in US (UK) GDP *generally* induce an initial real appreciation (depreciation) of the US dollar which is amplified during the next few years.

Next, we compare these empirical findings with the theoretical predictions of International Real Business Cycle (IRBC) models. In particular, we examine whether the medium-term co-movements in GDP and the real exchange-rate are consistent with the international transmission and propagation of productivity shocks.

Our theoretical analysis has two noteworthy features. First, we use a newly developed solution method that accurately captures the nonlinear equilibrium dynamics of IRBC models globally; that is to say, not just in the neighborhood of the steady state. Our analysis of these equilibrium dynamics reveals that the co-movements in GDP and the real exchange rate induced by productivity shocks vary according to prevailing financial conditions; specifically, the proximity of a national economy to its international borrowing limit.

Second, we consider several different "varieties" of the IRBC model with a common core: There are two countries, which we call the US and UK, each populated by a large number of households. Each country hosts a representative firm that produces a tradable good using domestic labor and

[1] More recent work such as Gabaix and Maggiori (2015) and Itskhoki and Mukhin (2017) highlights the importance of financial intermediaries and financial frictions in the short-run variations of nominal exchange rates.

capital. The firm hires labor and undertakes investment to maximize its total value to its shareholders. Households consume a basket of US and UK tradable goods, provide labor to domestic firms, and hold a portfolio of financial assets. Beyond these core features, we examine models that make different assumptions about the persistence of productivity shocks, the degree of international risk sharing and access to international asset markets. This enables us to identify features of the international transmission mechanism that apply across a range of models in the IRBC class.

Our analysis reveals the following characterization of the transmission mechanism. When international risk-sharing is incomplete, a positive productivity shock to US firms will generally raise the wealth of US households (who hold the firms' equity) more than UK households. This, in turn, increases the demand for the baskets of US and UK traded goods by all households. On the production side, the productivity shock raises the marginal product of US capital and thereby creates an incentive for US firms to undertake more physical investment. So the transmission mechanism must reconcile higher investment by US firms with greater consumption by US and UK households. This reconciliation takes place via movements in the relative prices of US and UK goods faced by US and UK households. In particular, the relative price of US goods rises for UK households so that they substitute UK for US goods in their consumption baskets. This reduces world demand for US goods, thereby facilitating US investment, and increases world demand for the UK good. Consequently, US net exports fall. The change in relative prices also reduces the marginal revenue product of UK capital, so the increase in demand for UK goods is accommodated at the existing level of production by a fall in UK investment. Thus, taken together, the productivity shock produces an immediate change in relative prices, which is reflected in a real appreciation of the US dollar, that facilitates a rise in US investment and a fall in UK investment accompanied by a deterioration in the US net foreign asset position. Relative prices also adjust through time as the effects of the productivity shock diminish. These adjustments facilitate the return of the US net foreign asset position and the US and UK capital stocks to the levels consistent with the long-run levels of productivity.

The operation of the transmission mechanism relies on several structural features: the persistence of productivity shocks, the elasticity of substitution between domestic and foreign goods in households' consumption baskets, and the impediments to risk-sharing. It also only operates when US households' existing asset holdings enable them to freely borrow internationally. When households are close to or at their international borrowing limit, US positive productivity shocks induce a real depreciation of the US dollar rather than an appreciation. As US households cannot borrow to finance more investment and consumption, the increase in demand for US goods is outweighed by the rise in the supply of US goods, which depresses the relative price of US goods inducing the real depreciation.

To check whether there is any empirical support for this transmission mechanism, we return to the US-UK data. Using a chronology from Reinhart and Rogoff (2008), we show that annual co-movements in GDP and the real exchange rate depend on whether either the US or UK was in a banking crisis. In the years without crises, the co-movements in GDP and real exchange rates are consistent with the IRBC transmission of productivity shocks away from the international borrowing limit. In years with either a US or UK banking crisis, which hinders international borrowing and lending, the co-movements in GDP and the real exchange rate are consistent with the operation of the IRBC transmission mechanism near the borrowing constraint.

This paper contributes to very large literatures that study the behavior of real exchange rates and international macroeconomics using IRBC models.[2] We therefore focus on papers from these literatures that are most directly relate to our study. Our empirical analysis of the US-UK real exchange rate builds on Lothian and Taylor (2008) insofar as we use US and UK per capita GDP to identify the long-run real exchange rate.[3] Consistent with Ricci et al. (2013), our empirical model takes the form of an error-correction model, but we also allow for state-dependency to capture the effects of banking crises. Our theoretical analysis is most closely related to work on the exchange-rate implications of IRBC models. Backus and Smith (1993) and Kollmann (1995) pointed out that real exchange rates should be perfectly correlated with relative cross-country consumption under complete risk-sharing, but the empirical correlation is negative or close to zero for many country pairs. This finding is commonly referred to as the Backus-Smith puzzle. Subsequent research has explored how incomplete risk-sharing affects the correlation (see, e.g., Corsetti et al., 2008, Kollmann, 2012, Benigno and Thoenissen, 2008 and Benigno and Kucuk, 2012).[4] We also analyze the implications of the IRBC model with incomplete risk-sharing, but our attention is mainly on the joint behavior of the real exchange rate and GDP. Our focus on the medium-run is also shared by Rabanal and Rubio-Ramírez (2015), who note that most of the variation in real exchange rates occurs at lower than business-cycle frequencies. Finally, our analysis extends existing research on solving international macro models with incomplete risk-sharing and portfolio choice. Existing

[2]For surveys of the real exchange-rate literature; see, e.g., Froot and Rogoff (1995) and Edwards and Savastano (2000). Important early contributions in international macroeconomics using IRBC models include; Backus et al. (1992), Baxter and Crucini (1995) and Kehoe and Perri (2002).

[3]Other contributions to the literature on the long-term determinants of real exchange rates include Edwards and Ostry (1990), De Gregorio et al. (1994), Ostry (1994), De Gregorio et al. (1994), Froot and Rogoff (1995), Goldfajn and Valdes (1999), Chen and Rogoff (2003), Cashin et al. (2004), and Galstyan and Lane (2009), among others.

[4]Recent research on the Backus-Smith puzzle that departs from the canonical IRBC setup includes: Colacito and Croce (2013), Kollmann (2016) and Tretvoll (2018) who change preferences; Bodenstein (2008) who includes limited enforcement of financial contracts, and Rouillard (2018) who adds financial frictions on firms.

methods developed by Devereux and Sutherland (2010), Tille and van Wincoop (2010) and Evans and Hnatkovska (2012) approximate the equilibrium around the steady state, whereas we adapt the global solution method in Cao (2010, 2018) to accommodate portfolio choice.[5,6] The use of a global solution method allows us to examine how the exchange-rate implications of productivity shocks critically depend on the economy's proximity to the borrowing constraint.

The paper is structured as follows. We begin, in Section 1, with our preliminary analysis of the US-UK data. Section 2 presents our benchmark version of the IRBC model, that includes the core features of all model variants we study. We then provide a description of the models' equilibrium and our solution method in Section 3. In Section 4 we examine the transmission mechanism in the benchmark version of the IRBC model and compare the implications for the behavior of real exchange rates and GDP with the US-UK data. Section 5 examines the transmission mechanism across different variants of the IRBC model and investigates the models' implications for resolving the Backus-Smith puzzle. Section 6 concludes.

1 Empirical Analysis

We focus on the medium-term behavior of real exchange rates; that is in the middle ground between business-cycles and the very long run. For this purpose, our analysis uses more than 200 years of annual data (1880 - 2016) on exchange rates, prices and GDP from the US and UK. These data extend the time series of exchange rates and prices constructed by Lothian and Taylor (1996). The data for per capita GDP is obtained from The Maddison Project (2018). In this section, we present a time-series model that characterizes the medium-term co-movements in the data. We will examine how these co-movements depend on financial conditions in Subsection 4.3 below.

For the purpose of this study, we define the real exchange rate as the relative price of the basket of goods consumed in the UK in terms of the price of a basket of goods consumed in the US. Mathematically, the real exchange rate in period t is calculated as

$$\mathcal{E}_t = \frac{S_t \hat{P}_t}{P_t}, \tag{1}$$

[5]These solution methods have been used to study external adjustment (see, e.g., Tille and van Wincoop, 2010 and Evans, 2014) and origins of home bias in international portfolio holdings (see, e.g., Coeurdacier and Gourinchas, 2008, Hnatkovska, 2009, Engel and Matsumoto, 2009, Coeurdacier et al., 2010, Devereux and Sutherland, 2010, and Coeurdacier and Rey, 2012).

[6]We extend the solution method in Cao (2010, 2018) to allow for endogenous capital accumulation, imperfect substitution between traded goods, and up to five continuous state variables.

where P_t and $\hat{P}_t$ are US and UK period-t price indices and S_t denotes the period-t nominal exchange rate measured as the price of a UK pound in US dollars. (Hereafter we use hats to denote UK variables.) According to this definition, a depreciation (appreciation) in the real value of the US dollar corresponds to a rise (fall) in $\mathcal{E}_t$.

The first step in constructing our empirical model accounts for the fact that both the US and UK economies have undergone secular changes during the past 200 years which have produced long-run exchange-rate effects. Consistent with the empirical literature, these long-run effects appear to be well-represented by cointegration between the log real exchange rate $\varepsilon_t = \log \mathcal{E}_t$ and log per capita real GDP in the US and UK, $y_t = \log Y_t$ and $\hat{y}_t = \log \hat{Y}_t$. In particular, we estimate the cointegrating relation to be

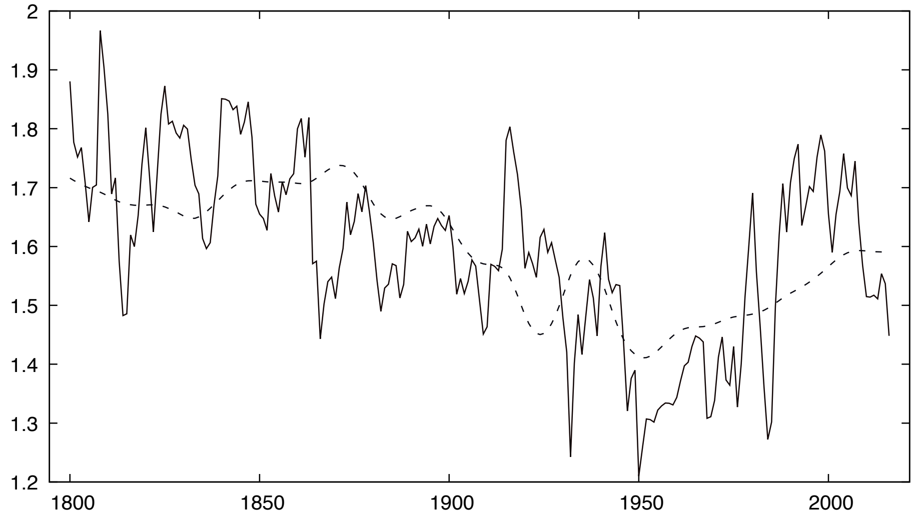
$$\varepsilon_t = \underset{(0.367)}{0.624} + \underset{(0.118)}{0.558} \hat{y}_t^{Trend} - \underset{(0.079)}{0.439} y_t^{Trend}, \tag{2}$$

where y_t^{Trend} and $\hat{y}_t^{Trend}$ are the trends in y_t and $\hat{y}_t$ computed from the HP filter. Equation (2) shows the cointegrating coefficient estimates and their standard errors computed by DOLS following Stock and Watson (1993).[7] Estimating the cointegration regression between the log real exchange rate and the logs of per capital GDP rather than the HP trends produces very similar results. We prefer to use the HP trends because the long-run level for ε_t implied by the cointegration estimates has less year-by-year variation, which seems more economically plausible. Figure 1 plots the log real exchange rate ε_t and these long-run estimates ε_t^{LR} over our sample period. We are agnostic about the underlying economic reason for cointegration in the data. It could represent the long-term effects of sectoral shifts in productivity, changes in the terms of trade, government consumption, trade policy or a structural shift. Our focus is instead on the medium-term variations in ε_t that are not included in ε_t^{LR}.[8]

[7] Statistical tests reveal that we cannot reject null of a unit root in ε_t , y_t^{Trend} and $\hat{y}_t^{Trend}$ individually, and we can reject the null of no cointegration between the three variables. In particular, the ADF tests for the null hypothesis that ε_t contains a unit root produce p-values of 0.167 and 0.4512, with and without a constant, respectively, while the KPSS test for the null hypothesis that ε_t is stationary produces a p-value of less than 0.01. In the case of y_t^{Trend} and $\hat{y}_t^{Trend}$, the ADT tests produce p-values of over 0.99, and the KPSS tests produce p-values of less than 0.01. The ADF test for the null hypothesis of no cointegration produces a p-value of less than 0.01. We obtain very similar results using per capita GDP, y_t and $\hat{y}_t$, rather than the HP trends.

[8] Much of the research on the long-term determinants of real exchange rates originates with the Harrod-Balassa-Samuelson (HBS) model (Harrod, 1933 Balassa, 1964 and Samuelson, 1964). For example, Lothian and Taylor (2008) use the HBS model as the rationale for defining the long-run US-UK rate in terms of the difference between US and UK log per capita GDP. Other contributions by De Gregorio et al. (1994), Chen and Rogoff (2003), Cashin et al. (2004), and Ricci et al. (2013), among others, focus on the terms of trade. The role of government consumption is considered in Froot and Rogoff (1995), Ostry (1994), De Gregorio et al. (1994), and Galstyan and Lane (2009); while Edwards and Ostry (1990) and Goldfajn and Valdes (1999) study the effects of trade policy. Ricci et al. (2013) examine all these factors in a panel cointegration model covering 24 years and 48 countries.

Figure 1: US/UK Real Exchange Rate



Notes: Log Real Exchange Rate ε_t (solid) and log long-run level ε_t^{LR} implied by cointegration (dashed): annual data 1880-2016.

Our empirical model comprises three equations: one for the annual real depreciation rate $\Delta\varepsilon_t = \varepsilon_t - \varepsilon_{t-1}$, and one each for the cyclical component of US and UK per capita GDP computed from the HP filter; denoted by y_t^{Cycle} and $\hat{y}_t^{Cycle}$, respectively

$$\Delta\varepsilon_t = \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} y_t^{Cycle} \\ \hat{y}_t^{Cycle} \end{bmatrix} + \gamma gap_{t-1} + e_t, \tag{3}$$

$$\begin{bmatrix} y_t^{Cycle} \\ \hat{y}_t^{Cycle} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} y_{t-1}^{Cycle} \\ \hat{y}_{t-1}^{Cycle} \end{bmatrix} + \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \Delta\varepsilon_{t-1} + \begin{bmatrix} v_t \\ \hat{v}_t \end{bmatrix}, \tag{4}$$

where $gap_t = \varepsilon_t - 0.624 - 0.558\hat{y}_t^{Trend} + 0.439 y_t^{Trend}$ is the error-correction term from the cointegrating regression. Equation (3) relates the current real depreciation rate to the cyclical components of GDP in the US and UK and the lagged error-correction term. Equation (4) characterizes the joint dynamics of the cyclical components of GDP in terms of past cyclical GDP and the real depreciation rate. These equations allow us to empirically characterize the joint dynamics of the real exchange rate and GDP in a parsimonious manner. Estimating alternative specifications that include, for example, lagged depreciation rates in (3) and error-correction terms in (4), produce statistically insignificant coefficients on the additional variables.

Panel I of Table 1 reports the OLS estimates of (3) and (4). Both regression equations appear well-specified; the majority of the coefficient estimates are highly statistically significant and the is no evidence of residual correlation. Column (i) reports the estimates of the depreciation rate equation in (3). The estimates imply that shocks that raise cyclical GDP in the US tend to also produce a real appreciation of the US dollar, whereas shocks that increase cyclical GDP in the UK produce a real depreciation. Columns (ii) and (iii) report the estimates of the GDP equations (4). Here we see that there is significant cross-country dependency via lagged GDP in the case of the US equation and via the lagged real depreciation rate in the UK equation.

To provide more perspective on the implications of the model estimates, Figure 2 plots the dynamic response of the log real exchange rate to a shock that induces a one percent increase in either y_t^{Cycle} or $\hat{y}_t^{Cycle}$.[9] The plots show that positive shocks to the cyclical components of US and UK GDP have opposite initial effects on the real exchange rate; producing a real appreciation (depreciation) when the shock affects US (UK) GDP. These effects are amplified in the next one to two years as they produce feedback effects on cyclical GDP in both countries, before slowly disappearing over the next decade.

The persistence of the dynamic responses in Figure 2 is also reflected in the variance decompositions for the real depreciation rate reported in Panel II of Table 1. The upper row of statistics shows the contribution of cyclical and trend GDP to the variance of the real depreciation rate over horizons ranging from one to twenty years.[10] These statistics show that GDP contributes most to the variations in the real exchange rate at around the ten-year horizon (the exact maximum contribution is estimated to be 22 percent at 11 years). The cyclical components of GDP account for

[9]The goal here is simply to characterize the joint dynamics of the $\Delta\varepsilon_t$, $y_t^{Cycle}$and $\hat{y}_t^{Cycle}$ rather than to identify the effects of particular economic shocks. Mathematically, (3) and (4) can be rewritten as a Vector Error Correction Model (VECM) for $\Delta\varepsilon_t$, $y_t^{Cycle}$and $\hat{y}_t^{Cycle}$ with zero restrictions on several of the coefficients. The plots in Figure 2 correspond to the impulse responses functions (IRFs) from the constrained VECM to positive v_t and $\hat{v}_t$ shocks. We have also computed (IRFs) that account for the estimated correlation between v_t and $\hat{v}_t$ shocks, using a standard Cholesky identification scheme with different orderings. The IRFs from these identification schemes are very similar to those plotted in Figure 2.

[10]To compute these variance contributions, we use the estimates of (3) and (4) to decompose the time series for ε_t into three components: one driven by the v_t and $\hat{v}_t$ shocks to y_t^{Cycle} and $\hat{y}_t^{Cycle}$, denoted by ε_t^{Cycle}; one driven by the changes in y_t^{Trend} and $\hat{y}_t^{Trend}$, denoted by ε_t^{Trend}; and one driven by the e_t shocks, denoted by ε_t^e. By construction, $\varepsilon_t = \varepsilon_t^{Cycle} + \varepsilon_t^{Trend} + \varepsilon_t^e$, so the variance of the k-year depreciation rate can be written as $Var(\Delta^k\varepsilon_t) = Cov(\Delta^k\varepsilon_t, \Delta^k\varepsilon_t^{Cycle}) + Cov(\Delta^k\varepsilon_t, \Delta^k\varepsilon_t^{Trend}) + Cov(\Delta^k\varepsilon_t, \Delta^k\varepsilon_t^e)$, where Δ^kdenotes the k'th difference operator. The variance contribution of the cyclical and trend GDP components is $\left(Cov(\Delta^k\varepsilon_t, \Delta^k\varepsilon_t^{Cycle}) + Cov(\Delta^k\varepsilon_t, \Delta^k\varepsilon_t^{Trend})\right)/Var(\Delta^k\varepsilon_t)$, which we compute as the slope coefficient from a regression of $\Delta^k\varepsilon_t^{Cycle} + \Delta^k\varepsilon_t^{Trend}$ on $\Delta^k\varepsilon_t$. The variance contribution of the trend GDP components is computed analogously from the regression of $\Delta^k\varepsilon_t^{Trend}$ on $\Delta^k\varepsilon_t$. Standard errors for the variance contributions are computed from the regressions with the Newey-West estimator which allows for heteroskedasticity and overlapping data.

Table 1: Model Estimates

I: Estimates Equation	$\Delta\varepsilon_t$		y_t^{Cycle} (US)		$\hat{y}_t^{Cycle}$ (UK)
	(i)		(ii)		(iii)
Regressors					
y^{Cycle} (US)	-0.262*		0.496***		-0.003
	(0.147)		(0.082)		(0.052)
$\hat{y}^{Cycle}$ (UK)	0.526***		0.278***		0.536***
	(0.159)		(0.099)		(0.095)
gap	-0.175***				
	(0.034)				
$\Delta\varepsilon$			0.043		0.063***
			(0.033)		(0.024)
SEE	0.064		0.033		0.025
R^2	0.137		0.388		0.324
II: Variance Contribution			Horizon (years)		
	1.	5.	10.	15.	20.
GDP Contribution	0.062	0.167	0.217	0.192	0.144
(standard error)	(0.019)	(0.024)	(0.030)	(0.036)	(0.025)
Trend GDP Contribution	0.011	0.036	0.082	0.099	0.077
(standard error)	(0.007)	(0.012)	(0.019)	(0.027)	(0.023)

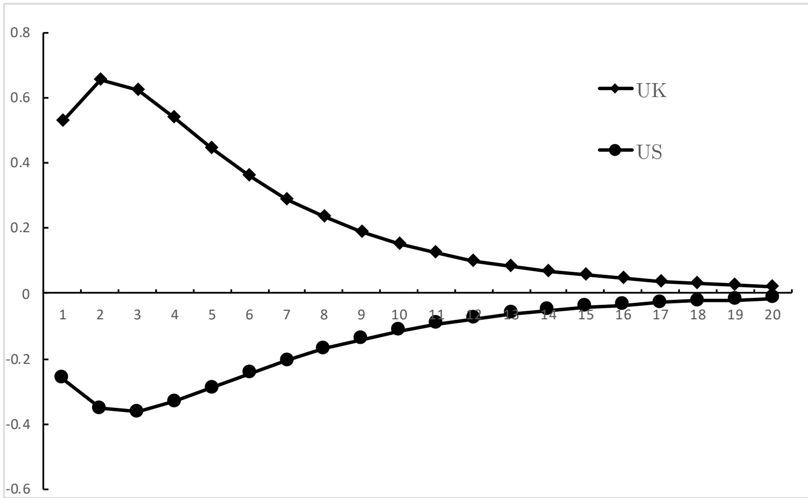
Notes: Panel I reports estimates of (3) and (4). Estimates are computed by OLS from annual data 1802-2016 (215 observations), and robust standard errors are shown in parenthesis below the coefficient estimates. Statistical significance at the 10, 5 and 1% levels is indicated by *, ** and ***, respectively. Panel II reports the estimated contribution of US and UK GDP (trend and cyclical components) to the variance of the real depreciation rate over horizons from one to 20 years, and the variance contributions of the trend components in US and UK GDP alone. See footnote 10 for details.

most of these contributions. When we recalculate the variance contributions without the cyclical components (i.e., imposing the restriction that $\beta_1 = \beta_2 = 0$), the variance contributions of GDP fall by at least 50 percent, as is shown by the lower row of statistics.[11]

To summarize, our empirical analysis reveals that variations in cyclical GDP become more economically important as proximate drivers of the real depreciation rate beyond business-cycle

[11] Our results are consistent with the finding in Rabanal and Rubio-Ramírez (2015) that most of the variation in real exchange rates occurs at lower than business-cycle frequencies. While they used spectral methods to decomposed the variance of the real exchange rates by frequency, we focus on the contribution of shocks that also drive cyclical GDP.

Figure 2: Real Exchange Rate Response



Notes: The figure plots the dynamic response of the log real exchange rate to a shock that induces a one percent increase in either y_t^{Cycle} (US) or $\hat{y}_t^{Cycle}$(UK).

frequencies. Shocks driving the cyclical components of the GDP appear to have a sizable and persistent impact on the real US-UK exchange rate, and account for more than ten percent of the variations in the real depreciation rate over horizons ranging from five to fifteen years.

2 The Model

We now examine whether the transmission and propagation of productivity shocks in a variety of IRBC models would produce co-movements in real exchange rates and GDP consistent with the US-UK data. Our analysis is centered on a benchmark model with two symmetric countries and stationary productivity shocks. Each country is populated by a large number of identical utility-maximizing households who have access to financial markets for equities and bonds. We depart from the complete markets/complete risk-sharing assumption made in Backus et al. (1992) by limiting international financial asset trades. Households can trade equity issued by domestic firms and internationally-traded bonds, but they cannot trade equity issued by foreign firms. We deliberately omit features found in other open economy models, such as non-traded goods, distribution sectors, and sticky prices, in order to focus on the key IRBC transmission mechanisms. Nevertheless, the benchmark model allows for a variety of complex interactions between the consumption, savings and portfolio decisions of households, and the production decisions of firms. Our solution method allows us to investigate how these interactions contribute to the transmission of productivity shocks across the entire state-space (i.e., globally), which in turn enables us to identify the origins of state-dependence in the dynamics of the real exchange rate.

This section describes the structure of the benchmark model, defines the equilibrium, and presents the calibration we use in our solution method. In subsequent sections, we show that the key economic elements of the transmission mechanism in our benchmark model apply across other IRBC variants. These variants include models with financial autarky, models which allow households greater access to international financial markets, and models with co-integrated non-stationary productivity shocks.

2.1 Households

We refer to the two countries in our models as the US and UK. The US is populated by a continuum of identical households distributed on the interval [0,1/2) with preferences defined over a consumption basket of traded goods. In particular, the expected utility of a representative US household in period t is given by

$$\mathbb{U}_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \left\{ \tfrac{1}{1-\gamma} C_{t+i}^{1-\gamma} \right\},$$

where $\gamma > 0$ is the coefficient of relative risk-aversion and $1 > \beta > 0$ is the subjective discount factor. $\mathbb{E}_t$ denotes expectations conditioned on period t information. C_t is the consumption index defined over two consumption goods; a US-produced good, C_t^{US}, and a UK-produced good, C_t^{UK}. We assume that the index takes the CES form so $C_t = \mathcal{C}(C_t^{\mathrm{US}}, C_t^{\mathrm{UK}})$ where $\mathcal{C}$ is defined as

$$\mathcal{C}(a,b) = \left(\alpha^{\frac{1}{\theta}} a^{\frac{\theta-1}{\theta}} + (1-\alpha)^{\frac{1}{\theta}} b^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}, \tag{5}$$

with elasticity parameter θ, and share parameter α for the US good. We also assume consumption home-bias: $\alpha > 1/2$.

P_t is the associated consumption price index for US households in US dollars

$$P_t = \left(\alpha (P_t^{\mathrm{US}})^{1-\theta} + (1-\alpha) (P_t^{\mathrm{UK}})^{1-\theta} \right)^{\frac{1}{1-\theta}}, \tag{6}$$

where P_t^{US} and P_t^{UK} are the dollar prices at which the US households purchase US and UK goods respectively. US households are also endowed with one unit of labor and derive nominal wage income of W_t from working for US firms.

In our benchmark model, US households can trade equity shares of the representative US firm and a bond. The bond payoff is denominated in the US price index so it is risk-free from the perspective of US households. The budget constraint of US households is therefore given by

$$P_t C_t + Q_t \chi_t + Q_t^b b_t \leq (D_t + Q_t)\chi_{t-1} + P_t b_{t-1} + W_t, \tag{7a}$$

where: χ_t is the number of shares of equity issued by US firms held by the US household; Q_t is the nominal price of equity issued by US firms; D_t is the nominal dividend paid by US firms; b_t is the number of bonds held by the US household; Q_t^b is nominal price of a bond (in US dollars). Dividing both sides of the budget constraint (7a) by P_t, we obtain the constraint in real terms

$$C_t + q_t \chi_t + q_t^b b_t \leq (d_t + q_t)\chi_{t-1} + b_{t-1} + w_t, \tag{7b}$$

where q_t, q_t^b, and w_t are the corresponding real values of Q_t, Q_t^b, and W_t. We also impose exogenous constraints on real bond holdings

$$b_t \geq \underline{b}. \tag{8}$$

These borrowing limits will be adjusted by the current level of productivity when productivity shocks are not stationary.

Similarly, the UK is populated by a continuum of households distributed on the interval [1/2,1]. Here the preferences of the representative UK household have an analogous form to the one for the representative US household except that the consumption index, $\hat{C}_t$, replaces C_t:

$$\hat{\mathbb{U}}_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \tfrac{1}{1-\gamma} \hat{C}_{t+i}^{1-\gamma}.$$

UK and US households have symmetric preferences with respect to individual traded goods, so $\hat{C}_t = \mathcal{C}(\hat{C}_t^{\mathrm{UK}}, \hat{C}_t^{\mathrm{US}})$ where $\mathcal{C}$ is given in (5) and $\hat{C}_t^{\mathrm{UK}}$ and $\hat{C}_t^{\mathrm{US}}$ denote the consumption of UK and US-produced goods by UK households. The price index for UK households in pounds is given by

$$\hat{P}_t = \left(\hat{\alpha} (\hat{P}_t^{\mathrm{US}})^{1-\theta} + (1-\hat{\alpha}) \, (\hat{P}_t^{\mathrm{UK}})^{1-\theta} \right)^{\frac{1}{1-\theta}}. \tag{9}$$

where $\hat{\alpha} = 1 - \alpha$. UK households can trade equity shares of the representative UK firm and the US bond. Their budget constraints in nominal and real terms are given by

$$\hat{P}_t\hat{C}_t + \hat{Q}_t\hat{\chi}_t + \frac{Q_t^b}{S_t}\hat{b}_t \leq (\hat{D}_t + \hat{Q}_t)\hat{\chi}_{t-1} + \frac{P_t}{S_t}\hat{b}_{t-1} + \hat{W}_t, \tag{10a}$$

and

$$\hat{C}_t + \hat{q}_t\chi_t + (q_{,t}^b/\mathcal{E}_t)b_t \leq (\hat{d}_t + \hat{q}_t)\hat{\chi}_{t-1} + (1/\mathcal{E}_t)\hat{b}_{t-1} + \hat{w}_t, \tag{10b}$$

where $\hat{\chi}_t$ and $\hat{b}_t$ are the number of UK equity shares and US bonds held by UK households, and recall that S_t and $\mathcal{E}_t$ are the nominal and real US-UK exchange rates. We also impose exogenous constraints on real bond holdings of UK households

$$\hat{b}_t \geq \underline{b}. \tag{11}$$

We assume that the law of one price holds for both traded goods, so $P_t^{\mathrm{UK}} = S_t\hat{P}_t^{\mathrm{UK}}$ and $P_t^{\mathrm{US}} = S_t\hat{P}_t^{\mathrm{US}}$. Let $p_t^{\mathrm{US}} = P_t^{\mathrm{US}}/P_t$ and $p_t^{\mathrm{UK}} = P_t^{\mathrm{UK}}/P_t$ denote the relative prices of US and UK goods faced by US households, and $\hat{p}_t^{\mathrm{US}} = \hat{P}_t^{\mathrm{US}}/\hat{P}_t$ and $\hat{p}_t^{\mathrm{UK}} = \hat{P}_t^{\mathrm{UK}}/\hat{P}_t$ denote the relative prices faced by UK households. Then, by the definition of real exchange rate in (1), $p_t^{\mathrm{US}} = \mathcal{E}_t\hat{p}_t^{\mathrm{US}}$ and $p_t^{\mathrm{UK}} = \mathcal{E}_t\hat{p}_t^{\mathrm{UK}}$. From the definitions of P_t and $\hat{P}_t$ in (6) and (9), we obtain $1 = \alpha(p_t^{\mathrm{US}})^{1-\theta} + (1-\alpha)(p_t^{\mathrm{UK}})^{1-\theta}$ and $1 = \hat{\alpha}(\hat{p}_t^{\mathrm{US}})^{1-\theta} + (1-\hat{\alpha})(\hat{p}_t^{\mathrm{UK}})^{1-\theta}$. Combining these equations with (1) produces the following expression for the real exchange rate

$$\mathcal{E}_t = \left(\frac{1-\alpha}{1-\hat{\alpha}} + \frac{(\alpha-\hat{\alpha})(\hat{p}_t^{\mathrm{US}})^{1-\theta}}{1-\hat{\alpha}}\right)^{\frac{1}{\theta-1}}. \tag{12}$$

Notice that this expression for $\mathcal{E}_t$ is decreasing in $\hat{p}_t^{\mathrm{US}}$ when there is consumption home bias ($\alpha > \hat{\alpha}$). In other words, a depreciation of the real exchange rate lowers the relative price of US goods facing UK households when there is consumption home bias. We will make use of this link between the real exchange rate and relative prices below.

2.2 Firms

There is a single industry in each country, and each industry is populated by a continuum of identical firms distributed on the interval [0,1]. A representative US firm owns all of its capital stock, K_t, and hires labor L_t to produce output of US goods, Y_t, according to $Y_t = F(A_t, K_t, L_t)$ where F is a constant-returns-to-scale production function, and A_t denotes the state of productivity. The output of UK goods by a representative UK firm, $\hat{Y}_t$, is given by an identical production function using its own capital, $\hat{K}_t$, and hiring labor $\hat{L}_t$, with productivity $\hat{A}_t$.

Firms in each country choose production and investment plans to maximize their total value to shareholders. In particular, a representative US firm solves

$$\max_{\{D_{t+i}, I_{t+i}, K_{t+i+1}\}} \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \frac{\lambda_{t+i}}{\lambda_t} \frac{D_{t+i}}{P_{t+i}}$$

s.t.

$$D_t = P_t^{\mathrm{US}} F(A_t, K_t, L_t) - W_t L_t - P_t^{\mathrm{US}} I_t \quad \text{and} \quad K_{t+1} = (1-\delta)K_t + I_t,$$

where: I_t is investment, $1 > \delta > 0$ is the depreciation rate, and λ_t is the stochastic discount factor of the firm's shareholders. Since equity can only be traded in domestic asset markets, the firm's shareholders are US households, so $\lambda_t = C_t^{-\gamma}$.

After substituting out investment, the problem of a representative US firm can be rewritten in real terms as

$$\max_{\{K_{t+i+1}, L_{t+i}\}} \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \frac{\lambda_{t+i}}{\lambda_t} \Big\{ p_t F(A_{t+i}, K_{t+i}, L_{t+i}) - w_{t+i} L_{t+i} - p_t [K_{t+i+1} - (1-\delta) K_{t+i}] \Big) \Big\}.$$

Similarly, a representative UK firm solves

$$\max_{\{\hat{K}_{t+i+1}, \hat{L}_{t+i}\}} \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \frac{\hat{\lambda}_{t+i}}{\hat{\lambda}_t} \Big\{ \hat{p}_{t+i} F(\hat{A}_{t+i}, \hat{K}_{t+i}, \hat{L}_{t+i}) - \hat{w}_{t+i} \hat{L}_{t+i} - \hat{p}_{t+i} [\hat{K}_{t+i+1} - (1-\delta) \hat{K}_{t+i}] \Big) \Big\},$$

where the stochastic discount factor is given by the marginal utility of UK households, $\hat{\lambda}_t = \hat{C}_t^{-\gamma}$.

Production Functions and Productivity Processes We assume that production functions are Cobb-Douglas: $F(A, K, L) \equiv A K^{\eta} L^{1-\eta}$, with $1 > \eta \geq 0$. We also assume, following Backus et al. (1992), that log productivity in each country follows stationary AR(1) processes:

$$\log A_t = \rho \log A_{t-1} + \epsilon_t \tag{13a}$$

and

$$\log \hat{A}_t = \rho \log \hat{A}_{t-1} + \hat{\epsilon}_t, \tag{13b}$$

with $1 > \rho > 0$, where ϵ_t and $\hat{\epsilon}_t$ are I.I.D productivity shocks. In subsequent sections, we examine the implications of alternative productivity processes; including specifications that allow for correlations between the shocks as in Heathcote and Perri (2002), and unit-root specifications as in Rabanal et al. (2011).

2.3 Equilibrium

We study a standard sequential competitive equilibrium: in equilibrium, prices and allocations are such that allocations solve the households' and firms' optimization problems and all markets clear in each history of shocks. Numerically, we solve for sequential competitive equilibrium with a particular recursive structure: a recursive equilibrium is a sequential competitive equilibrium in which prices and allocations are (single-valued) functions of two exogenous state variables; the levels of US and UK productivity $(A_t, \hat{A}_t)$, and three endogenous state variables; the US and UK capital stocks, and US bond holdings $(K_t, \hat{K}_t, b_{t-1})$. In our benchmark model, the only internationally traded asset is the US bond. This bond is in zero net supply, so $0 = b_t + \hat{b}_t$ by market clearing. Thus b_{t-1} identifies the US net foreign asset position at the start of period t. In the other IRBC models we study, the recursive equilibrium requires a larger number of endogenous state variables, which we describe below.

To calculate the moments from the solution of the model, we use the concept of a stationary recursive equilibrium in Duffie et al. (1994) and Cao (2020). A stationary recursive equilibrium is a recursive equilibrium with a stationary ergodic distribution resulting from the transition functions for the state variables (i.e., $(A_t, \hat{A}_t, K_t, \hat{K}_t, b_{t-1})$ in our benchmark model). We study the equilibrium dynamics of variables with impulse response functions (IRFs) that are calculated by simulating draws from the ergodic distribution. Precise definitions for these equilibrium concepts and a description of the global solution method are given in Section 3.

2.4 Calibration

Since we focus on the equilibrium dynamics beyond the business cycles, we use annual frequencies for our model (instead of the quarterly frequencies used in the IRBC literature starting with Backus et al., 1992).[12] We use the preference and technology parameters from Corsetti et al. (2008) who provide annual estimates of these parameters. The parameters are given in Table 2. We use the elasticity of substitution $\theta = 10$ in the benchmark calibration. This elasticity is in the mid-range of the estimates found in the trade literature as reported in Obstfeld and Rogoff (2000, Section 2.3).

[12] In Appendix D, we use the quarterly productivity processes estimated in Heathcote and Perri (2002) and Rabanal et al. (2011).

For the borrowing limit, we use $\underline{b} = -1$. The calibration of this lower bound implies that households can borrow up to approximately 60% of their income (including capital and labor income).

Table 2: **Benchmark Calibration**

Parameters	**Descriptions**	**Values**
β	discount factor	0.96
γ	risk aversion	2
$\alpha = 1 - \hat{\alpha}$	home consumption share	0.72
θ	consumption elasticity	10
δ	depreciation rate	0.1
η	capital share	0.36
$\underline{b}$	borrowing limit	-1
$corr(\ln A_t, \ln \hat{A}_t)$	productivity correlation	0
ρ	productivity persistence	0.82
σ	productivity shock std.	0.022

2.5 Moments and Distributions

It is useful to briefly describe some of the characteristics of the equilibrium in our benchmark model. To this end, Table 3 compares the variability of consumption, investment and productivity in post-war US data (available from the St Louis Fed's database, FRED), and the equilibrium of the benchmark model. We follow the standard practice (Backus et al., 1992) of comparing the standard deviation of each variable relative to the standard deviation of output. The benchmark model's statistics are calculated using long time series (40,000 years) simulated from the ergodic distribution of the recursive equilibrium. For comparison purposes, we also report statistics from the equilibrium of an IRBC model with complete risk-sharing that has all the core features of our benchmark except the restrictions on asset trade.

As the table shows, the benchmark model comes close to replicating the relative variability of consumption in the data, but somewhat understates (overstates) the relative variability of investment (productivity). These discrepancies between the data and model are most likely due to the simplicity of our specification with respect to consumption and production.[13] They do not appear

[13] Recall that, to highlight the transmission mechanism in our model, we choose to omit many ingredients that would bring the business cycles moments closer to those in the data, such as non-traded goods or distribution services. Corsetti et al. (2008) show that these ingredients are important quantitatively. For example, we use the standard deviation for productivity shocks from the traded good sectors in Corsetti et al. (2008), which implies larger output volatility than in the data. In the data, output includes both traded and non-traded goods.

to be primarily attributable to the restrictions on asset trade because the relative variability of investment and productivity under complete risk-sharing are almost identical.[14] However, as we would expect, the variability of consumption under complete risk-sharing is considerably lower than in our benchmark model.

Table 3: **Business Cycle Moments**

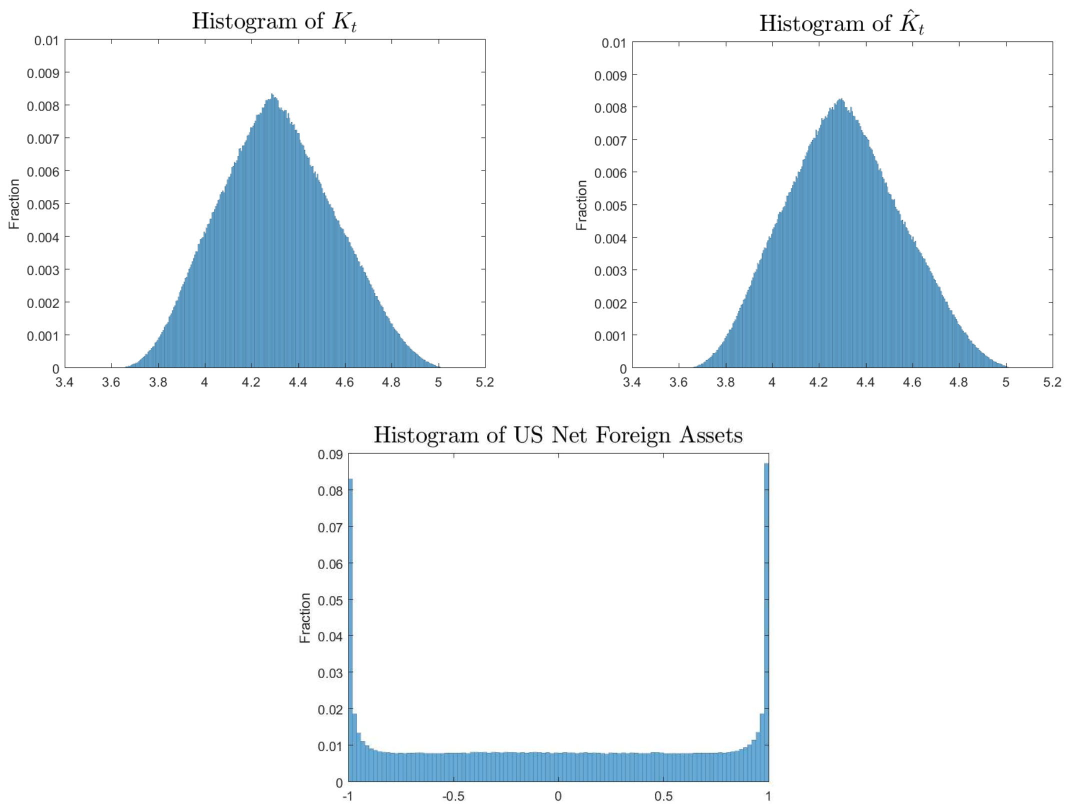
	Relative Variability: $Std(X)/Sdt(Y)$		
$X =$	Consumption	Investment	Productivity
Data	0.87	3.86	0.46
Benchmark	0.82	2.12	0.73
Complete Markets	0.54	2.14	0.74

We identify the recursive equilibrium in the benchmark model with three endogenous state variables: the US and UK capital stocks, and US bond holdings; $K_t, \hat{K}_t$ and b_{t-1}. Figure 3 shows the marginal distributions of these endogenous state variables computed by simulation from the stationary recursive equilibrium. The distributions for capital are unimodal and symmetric. In contrast, the distribution for b_{t-1}, which identifies the US net foreign asset (NFA) position, is bimodal with peaks at the US and UK borrowing limits. This feature of the equilibrium reflects the fact that shocks have persistent effects on the international distribution of bond holding away from the borrowing limits: it appears in all the IRBC models we study with incomplete risk-sharing.[15] As we shall see, the proximity of an economy to its international borrowing limit has important implications for the transmission and propagation of productivity shocks.

[14] The relative volatility of productivity is really measuring the volatility of equilibrium output because productivity is exogenous.

[15] The existing literature removes the feature by altering the model with stationary cardinal utility (Corsetti et al., 2008) or with quadratic bond-holding cost (Heathcote and Perri, 2002). These modifications make it possible to solve the model using local, linearization methods. But they also remove the state-dependent equilibrium properties, which we focus on in the present paper.

Figure 3: Histograms of State Variables



Note: This figure is generated by the solution of the baseline model with parameters given in Table 2.

3 Equilibrium Definitions and Solution Method

In this section, we provide a more detailed description of the equilibrium and the global solution method. Readers wishing to skip these technical aspects of our work may proceed to Section 4 without loss of continuity.

3.1 Equilibrium

We use a standard definition of sequential competitive equilibrium: Given initial state variables $K_0, b_{-1}, \hat{K}_0, \hat{b}_{-1} = -\hat{b}_1$, a sequential competitive equilibrium (SCE) consists of *stochastic* sequences of allocations

$$\left\{C_t^{\mathrm{US}}, C_t^{\mathrm{UK}}, \chi_t, b_t, K_{t+1}, L_t, \hat{C}_t^{\mathrm{US}}, \hat{C}_t^{\mathrm{UK}}, \hat{\chi}_t, \hat{b}_t, \hat{K}_{t+1}, \hat{L}_t\right\},$$

and prices

$$\left\{p_t^{\mathrm{US}}, p_t^{\mathrm{UK}}, q_t, q_t^b, \hat{p}_t^{\mathrm{US}}, \hat{p}_t^{\mathrm{UK}}, \hat{q}_t, \mathcal{E}_t\right\},$$

such that the allocations solve households' and firms' optimization problems and markets clear.

Market clearing in the two goods markets requires that

$$\begin{aligned} Y_t &= A_t K_t^{\eta} L_t^{1-\eta} = C_t^{\mathrm{US}} + \hat{C}_t^{\mathrm{US}} + K_{t+1} - (1-\delta)K_t, \quad \text{and} \\ \hat{Y}_t &= \hat{A}_t \hat{K}_t^{\eta} \hat{L}_t^{1-\eta} = C_t^{\mathrm{UK}} + \hat{C}_t^{\mathrm{UK}} + \hat{K}_{t+1} - (1-\delta)\hat{K}_t. \end{aligned}$$

Labor is supplied inelastically by domestic households. We normalize the labor supply to unity in both counties so labor market clearing requires,

$$1 = L_t \qquad \text{and} \qquad 1 = \hat{L}_t.$$

Finally, the total equity issued by US and UK firms is normalized to unity, and bonds are in zero net supply, so the market clearing conditions in the four financial markets are

$$1 = \chi_t, \qquad 1 = \hat{\chi}_t, \qquad \text{and} \qquad 0 = b_t + \hat{b}_t.$$

In order to solve for SCEs numerically, we look for recursive equilibria with a natural state space that includes the following variables:

$$x_t = \left(A_t, \hat{A}_t, K_t, \hat{K}_t, b_{t-1}\right),$$

when productivity processes are stationary. Notice that from the financial market clearing conditions, equity holdings are always equal to 1 and the bond holdings of UK households are implied by the bond holdings of US households. In a recursive equilibrium, the allocations and prices are functions of the state variables.

Definition 1. A recursive equilibrium (RE) is a SCE in which the allocations and prices are functions of the state variable x_t:

$$z_t = \mathcal{Z}(x_t)$$

for each equilibrium variable z

$$z = \left(C^{\mathrm{US}}, C^{\mathrm{UK}}, b', K', \hat{C}^{\mathrm{US}}, \hat{C}^{\mathrm{UK}}, \hat{b}', \hat{K}', p^{\mathrm{US}}, p^{\mathrm{UK}}, q, q^b, \hat{p}^{\mathrm{US}}, \hat{p}^{\mathrm{UK}}, \hat{q}, \mathcal{E}\right), \tag{14}$$

where b', K' and $\hat{b}', \hat{K}'$ stand for next period bond holdings and capital stocks.

For later reference, we use the notation $\mathcal{K}(x), \hat{\mathcal{K}}(x), \mathcal{B}(x)$ for the $K', \hat{K}', b'$ components of $\mathcal{Z}(x)$. In Subsection 3.2, we present the solution method used to compute these REs. A related equilibrium concept is the stationary recursive equilibrium which corresponds to a recursive equilibrium with a stationary distribution over the state variables (see Duffie et al., 1994 and Cao, 2020).

Definition 2. An stationary recursive equilibrium is a RE with a stationary distribution Δ over the state variable $x \in \mathbb{R}^5$ such that Δ is a fixed point of the transition function $\mathcal{F}$ implied by the RE policy functions:

$$\left(\mathcal{K}(x), \hat{\mathcal{K}}(x), \mathcal{B}(x)\right),$$

and exogenous stochastic law of motion for A and $\hat{A}$.

In an stationary recursive equilibrium, by the Ergodic Theorem, the long-run moments implied by the model can be computed as deterministic functionals of the policy functions $\mathcal{Z}$'s, transition function, $\mathcal{F}$, and stationary distribution Δ.

3.2 Global Solution Method

We solve for REs using policy-function iteration, a global method developed in Cao (2010, 2018) and Cao and Nie (2017). The method is well-suited for dynamic stochastic general equilibrium (DSGE) models with portfolio choices.[16] We extend the method to allow for endogenous capital accumulation, imperfect substitutions between traded goods, and up to five continuous state variables.[17] In the recent international finance literature, Rabitsch et al. (2015) and Stepanchuk and Tsyrennikov (2015) also solve a two-country model with portfolio choice using policy-function iteration, but they only consider endowment economies, which allow them to describe the solution with a single continuous state variable (the wealth share). Coeurdacier et al. (2019) allow for up to

[16] Earlier work using policy-function iterations for DSGE economies includes Coleman (1990) and Judd et al. (2000).

[17] The incomplete markets two-bond model with co-integrated random walk productivity processes in Appendix D.2 features five continuous state variables: relative productivity, two capital stocks, and two bond positions. Because of the presence of the two bonds, US and UK households need to solve portfolio choice problems. The bonds' returns are also close to being collinear because the variation of the real exchange rate is small when the elasticity of substitution, θ, is high.

three continuous state variables (two capital stocks and a bond position) but their model does not feature portfolio choice or imperfect substitution between traded goods. We describe in detail our global solution method below.

The optimization problems facing households and firms are concave maximization problems (concave objectives and convex constraints). Therefore, to guarantee that the allocations solve the agents' optimization problems, it is necessary and sufficient that the allocations satisfy the first-order conditions (F.O.C.s) at equilibrium prices. The Lagrange multiplier λ_t on the US households' budget constraint, (7b), is shown below to be equal to their marginal utility, and let μ_t denote the multiplier on the US households' borrowing constraint, (8). For US households, the F.O.C.s are:

$$\begin{aligned}
\{C_t\} :& C_t^{-\gamma} = \lambda_t, \\
\{b_t\} :& -\lambda_t q_t^b + \beta \mathbb{E}_t[\lambda_{t+1}] + \mu_t = 0, \\
\{b_t, \mu_t\} :& \mu_t(b_t - \underline{b}) = 0, \mu_t \geq 0 \qquad \text{and} \\
\{\chi_t\} :& -q_t \lambda_t + \beta \mathbb{E}_t[\lambda_{t+1}(q_{t+1} + d_{t+1})] = 0.
\end{aligned}$$

Similarly, we obtain the F.O.C.s for the UK households using the hat variables.

For US firms, the F.O.C in K_{t+1} (I_{t+1}) is:

$$\mathbb{E}_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \frac{p_{t+1}^{\text{US}}}{p_t^{\text{US}}} \left(\eta A_{t+1} K_{t+1}^{\eta-1} + 1 - \delta \right) - 1 \right] = 0,$$

and for UK firms, the F.O.C. in $\hat{K}_{t+1}$ ($\hat{I}_{t+1}$) is:

$$\mathbb{E}_t \left[\beta \frac{\hat{\lambda}_{t+1}}{\hat{\lambda}_t} \frac{\hat{p}_{t+1}^{\text{UK}}}{\hat{p}_t^{\text{UK}}} \left(\eta \hat{A}_{t+1} \hat{K}_{t+1}^{\eta-1} + 1 - \delta \right) - 1 \right] = 0.$$

To solve for the the recursive equilibrium, we look for policy functions, $\{\mathcal{P}^{(n)}\}_{n=1}^{N}$ that map from state variables

$$x = \left(A, \hat{A}, K, \hat{K}, b \right),$$

to allocations and prices z given in (14), as well as the Lagrangian multipliers $\lambda, \mu, \hat{\lambda}, \hat{\mu}$. The Lagrange multipliers appear in the households' F.O.C.s to ensure that the allocations are optimal solutions to the households' consumption-saving/investment problems.

We initialize the policy function sequence with $\mathcal{P}^{(1)}$ which corresponds to the equilibrium in the 1-period version of the model. Assuming that we have solved for $\mathcal{P}^{(n)}$, we look for $\mathcal{P}^{(n+1)}$

by solving the system of equilibrium equations at each collocation point of the exogenous and endogenous state variables. The system of equations involves the 20 unknowns listed above and 20 equations (F.O.C.s, market-clearing conditions, relative prices and real exchange-rate restrictions). $\mathcal{P}^{(n)}$ corresponds to the equilibrium in the n-period version of the model and the limit of $\{\mathcal{P}^{(n)}\}$ corresponds to the equilibrium of the infinite-horizon model.[18]

To accommodate the high-dimension of the problem, we resort to the adaptive sparse grid method developed in Ma and Zabaras (2009) and recently applied in economic applications by Brumm and Scheidegger (2017). A unique aspect of the problem is that we need to solve for equilibria with multiple assets and portfolio choice in an environment with incomplete markets where there is trade in both US and UK denominated bonds (Subsection 5.2). This problem is challenging since returns to assets in certain regions of the state space are close to collinear, requiring the Jacobian matrix of the equilibrium system to be evaluated very accurately. To tackle this problem, we adopt an automatic differentiation method (Baydin et al., 2017) which allows us to evaluate the Jacobian matrix up to machine precision with speed comparable to analytical gradients.[19] Nevertheless, to solve the models accurately for many different calibrations requires intensive computation, which we undertake with numerical implementations in C++ and parallel programing on a 48 CPU core machine.

Ergodic Distribution and Impulse Response Functions In order to compute the ergodic distribution over the (exogenous and endogenous) state variables

$$x = \left(A, \hat{A}, K, \hat{K}, b\right),$$

we simulate 100 samples for $50,000$ periods using the nonlinear policy functions

$$\left(\mathcal{K}(x), \hat{\mathcal{K}}(x), \mathcal{B}(x)\right)$$

from the RE solution.[20] We drop the first $10,000$ burn-in periods from each sample. Therefore, in total we have 4 million observations for the state variables which we use to construct the histogram of the ergodic distribution.

[18]See Duffie et al. (1994) and Cao (2018, 2020) for examples of existence proofs using the limit of equilibria in finite-horizon economies.

[19]We use an efficient (Jacobian) gradient-based equation solver that respects boxed bounds with an interior-point method, following Powell (1970), Coleman and Li (1996), and Bellavia et al. (2012).

[20]By the Ergodic Theorem, one long sample is enough to approximate the ergodic distribution but using 100 samples allows us to parallelize the simulations.

We use impulse response functions to understand the transmission mechanism in our model. To compute the unconditional impulse response functions (UIRFs), we pick $N = 200,000$ draws from the ergodic distribution (the last $2,000$ draws from each of the 100 samples) and produce two replica sets of the draws. Starting from each draw

$$\left(A^{(n)}, \hat{A}^{(n)}, K^{(n)}, \hat{K}^{(n)}, b^{(n)}\right)_{n=1}^{N}$$

in the first set, we simulate forward 100 periods (years, or quarters depending on the model). We discretize the productivity processes in (13) using 3-point discrete Markov chains. From each draw in the second set, we change $A^{(n)}$ to the highest productivity level, and simulate forward 100 periods. The impulse response of a variable of interest, among the policy variables z, is the difference of the averages of the variable values from the two sets.

To compute the conditional impulse response functions (CIRFs), we pick a subset of N draws that satisfies the conditionality (e.g., b lies in the bottom 5% of its marginal ergodic distribution). Then we simulate forward the two sets of draws and calculate the difference of the averages of the variable of interest as we do for the UIRFs.

4 The Transmission Mechanism in the Benchmark Model

In this section, we examine the transmission and propagation of productivity shocks in the benchmark model. We first show that *under most circumstances* the transmission mechanism produces initial co-movements in the real exchange rate and output that are consistent with the empirical evidence from the US-UK data, presented in Section 1. Next, we demonstrate that a very different transmission mechanism operates in the model when either the US or UK is close to or at the international borrowing constraint. We then return to the US-UK data to empirically investigate whether this alternative transmission mechanism was ever operable. We extend our analysis to other variants of the IRBC model in Section 5.

4.1 General Transmission

According to our empirical model, shocks raising the cyclical component of US GDP produced a real appreciation in the US dollar, whereas shocks increasing the cyclical component of UK GDP produced a real depreciation (see Figure 2). We now compare these implications with the dynamic responses of output and the real exchange rate to productivity shocks in the benchmark model.

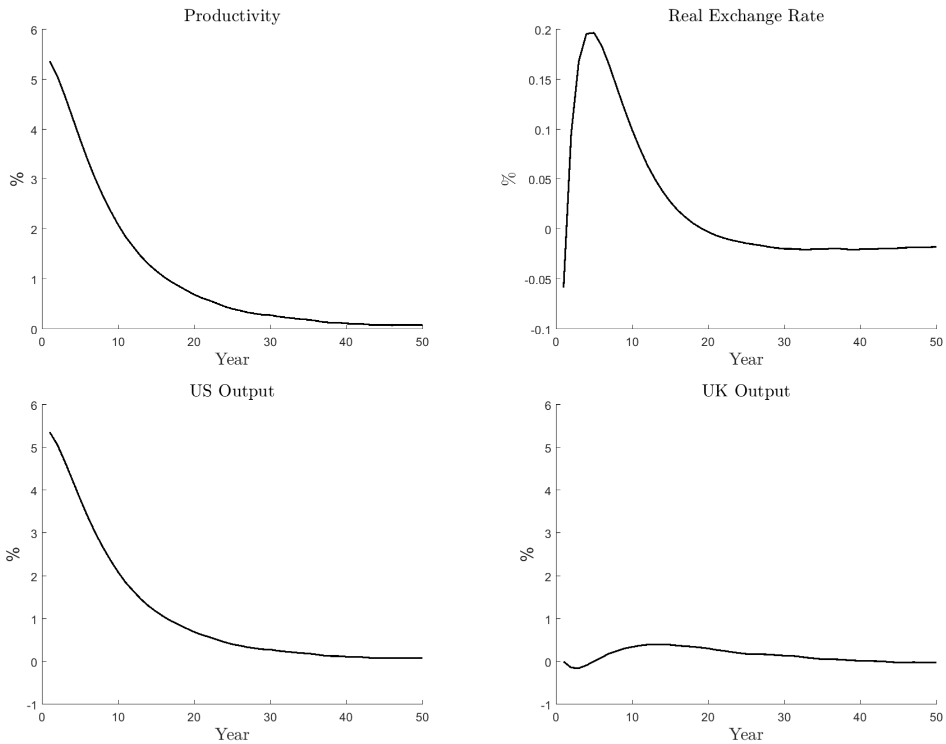
Our global method produces a set of non-linear equations that characterize the equilibrium dynamics in the model, so the dynamic effects of productivity shocks depend on the prevailing values of the endogenous state variables. To account for this form of state-dependency, we simulate the equilibrium effects of the productivity shocks over 100 periods starting from 2000 values for the endogenous state variables that are randomly chosen from their joint ergodic distribution. We then compute the average dynamic effect of the productivity shock for each of the 100 periods across the simulations. We refer to these averages as the Unconditional Impulse Response Functions (UIRFs) because the dynamic effects of the productivity shocks are not conditioned on particular values of the prevailing endogenous state variables.

Figure 4 plots the UIRFs for log US productivity, the log real exchange rate, and the logs of US and UK output, induced by a positive US productivity shock.[21] The plots show that while the productivity shock produces an immediate and persistent rise in US output, it has minimal effects on UK output (there is a small fall and then rise in the years following the shock). The shock also produces an immediate real appreciation of the US dollar, followed by a persistent depreciation that lasts for approximately 15 years. We note that the immediate effects of the productivity shock on US output and the real exchange rate are consistent with the empirical responses in Figure 2, but thereafter the UIRFS and empirical responses diverge. We discuss the reasons for this divergence in Section 6.

At first glance, the immediate exchange-rate effects of productivity shocks shown in Figure 4 may seem counterintuitive. Since positive US productivity shocks increase the production of US goods, it seems that the relative price of these goods should fall, which would be reflected in a depreciation of the real exchange rate (see equation (12)). To uncover the flaw in this intuition, we need to understand how productivity shocks affect the demand for US (and UK) goods. Recall that households are prohibited from holding foreign equity, which inhibits risk-sharing. As a consequence, productivity shocks to US firms have larger wealth effects on US households than UK households (even though all households choose their portfolios optimally). This means that a positive US productivity shock increases the world consumption demand for US goods relative to UK goods *in the absence of any change in relative prices* (because household preferences exhibit home bias in the consumption). Productivity shocks also affect the investment demand for goods by firms through their impact on the marginal product of capital. In particular, because a positive shock produces a persistent increase in US productivity (see Figure 2), it also creates a strong incentive for US firms to increase investment when the shock occurs. In sum, therefore, the initial effect of

[21] The UIFR for productivity is a standard IRF because the process in (13) is linear.

Figure 4: UIRFs for US productivity, real exchange rate, and US, UK output

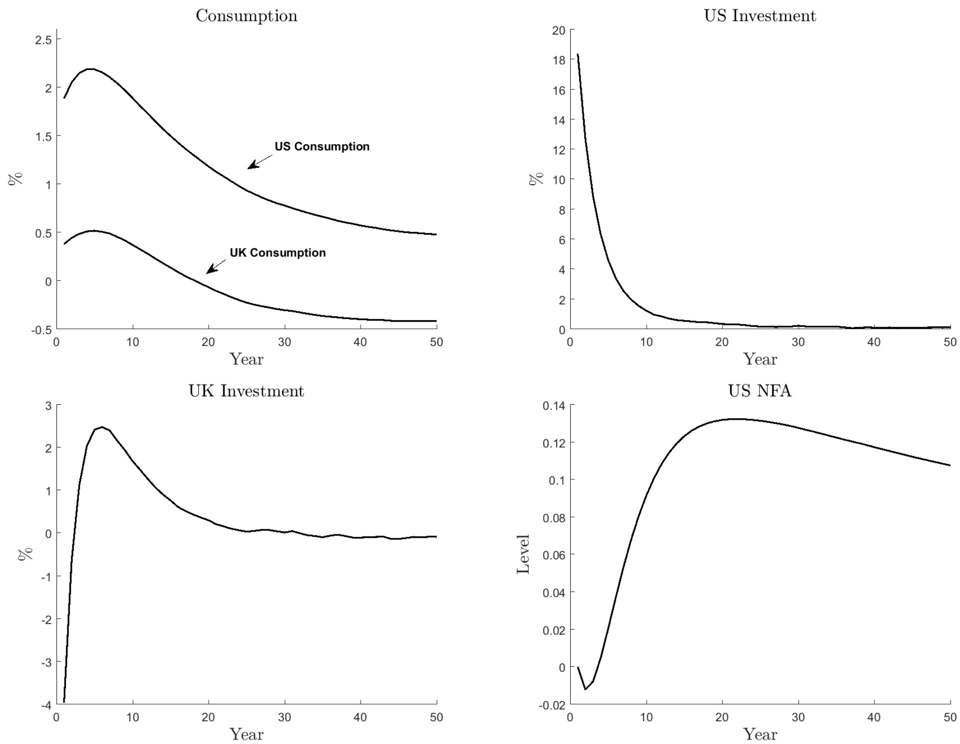


Note: This figure is generated by the solution of the baseline model with parameters given in Table 2

a positive US productivity shock is to increase the demand for US goods for both consumption and investment purposes. Of course, the productivity shock also makes existing capital (and labor) more productive, which increases the supply of US goods. While it is *possible* that the higher supply of US goods matches the increase in demand, so markets will clear without any change in relative prices, this is not the case in our calibration of the benchmark model. Instead, the relative price of US goods rises to reduce the consumption demand for US goods so that total demand matches the production capacity of US firms while they undertake more investment. The real appreciation of the dollar induced by the positive US productivity shock reflects this change in relative prices.

Figure 5 provides more information on the transmission of productivity shocks. The upper panel plots the UIRFs for the US and UK consumption indices and US real investment. These plots show that the real appreciation of the dollar reconciles the differing wealth effects of the productivity

Figure 5: IRFs for US productivity, real exchange rate, and US, UK output



Note: This figure is generated by the solution of the baseline model with parameters given in Table 2

shock on US and UK households produced by incomplete risk sharing with the desire of US firms to increase real investment. The increase in the relative price of US goods induces both US and UK households to substitute UK goods for US goods in their consumption baskets, which facilitates market clearing in the US goods market, but it also increases the world consumption demand for UK goods. Since there is no immediate change in the productive capacity of UK firms, this increase in demand must be accommodated by a fall in UK investment, as is shown by the UIRF in the lower left-hand panel of Figure 5. This takes place because the expected increase in UK consumption following the shock (shown in the upper left-hand plot) raises the required real return on UK capital. Thus, in effect, the appreciation of the real exchange rate facilitates a shift in the pattern of international real investment towards US production and away from UK production.[22]

[22]It is clear from the discussion that if US firms can substitute UK goods for US goods when undertaking investment in their capital stock, as assumed in Backus et al. (1993) and the subsequent literature, then our transmission

The US productivity shock also has implications for trade flows and the US NFA position (b_{t-1} in the US budget constraint (7b)). In particular, the initial higher consumption of UK goods by US households produces a US trade deficit and a deterioration in the US NFA position. This is shown by the UIRF for the US NFA position plotted in the lower panel of Figure 5. Then, as US investment falls relative to output, the real exchange rate depreciates and consumption shifts back towards US goods, producing a US trade surplus. As the plot shows, these trade surpluses generate an improvement in the US NFA position that lasts approximately twenty years.

The response of the US NFA position in Figure 5 also makes clear that in order to garner the full benefit of a US productivity shock, US households must be able to borrow internationally, or equivalently run a trade deficit. Because the UIRFs in Figures 4 and 5 are average responses computed from the entire ergodic distribution of the endogenous state variables, they do not reflect the transmission of productivity shocks when the US NFA position is close to the lower bound of the ergodic distribution shown in Figure 3. We consider this in the next Subsection.

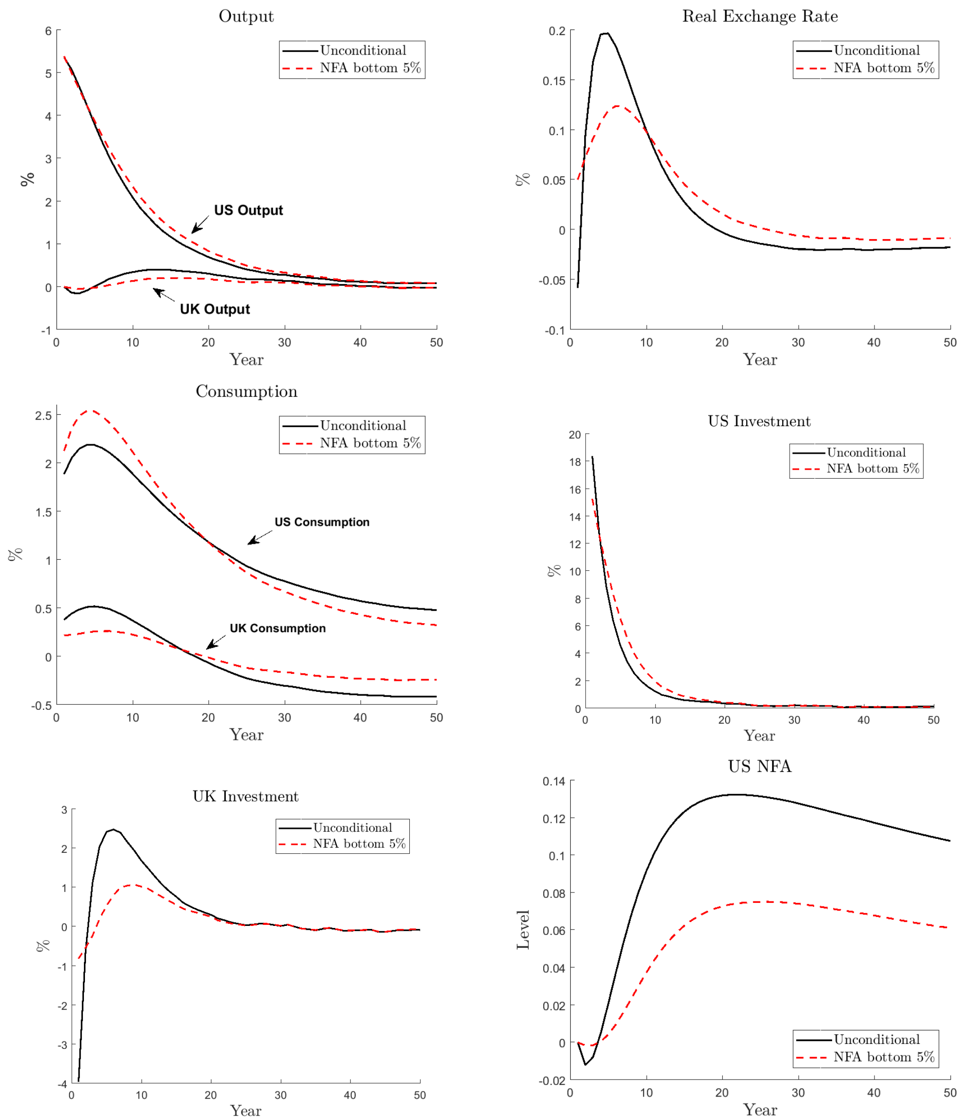
4.2 Transmission Near the Borrowing Constraint

In order to study the effects of US productivity shocks when US households are near their international borrowing limit, we need to compute a second set of impulse response functions. For this purpose, we again simulate the equilibrium effects of the productivity shocks over 100 periods starting from 2000 values for the endogenous state variables but now they are randomly chosen from a portion of their joint ergodic distribution where the US NFA position lies in the fifth percentile of its marginal distribution. We then compute the average dynamic effect of the productivity shock for each of the 100 periods across the simulations. We refer to these averages as the Conditional Impulse Response Functions (CIRFs) because the dynamic effects of the productivity shocks are conditioned on US households being in the proximity of their international borrowing constraint.

Figure 6 plots the CIRFs and the UIRFs for log output, the log real exchange rate, the US NFA position and log consumption induced by a positive US productivity shock. The plots in the upper left-hand panel show that proximity to the borrowing constraint does not significantly affect how US productivity shocks affect either US or UK output. In particular, there are no visible differences between the CIRF and UIRF for US output up to five years following the shock. In contrast, the upper right-hand panel shows that proximity to the borrowing limit changes the impact of the

mechanism is weakened. Backus et al. (1993) assume that investment goods have exactly the same home-bias and substitution patterns as consumption goods. However, Burstein et al. (2004) find that investment goods have very different domestic-imported contents from consumption goods. Our model features this dissimilarity between investment and consumption.

Figure 6: IRFs for US productivity, real exchange rate, and US, UK output



Note: This figure is generated by the solution of the baseline model with parameters given in Table 2. The solid lines are UIRFs and the dashed lines are CIRFs, conditional on US NFA being in the fifth percentile of its marginal distribution.

shock on the real exchange rate; producing an initial real depreciation of the US dollar rather than an appreciation. The plot also shows that proximity to the borrowing constraint dampens the real exchange-rate response over the next ten years.

To understand why the exchange rate effects of the productivity shock are so different in the proximity of borrowing constraint, recall that at prevailing relative prices, the initial effect of the shock is to increases consumption and investment demand for US goods beyond the productivity capacity of US firms. When US households are able to borrow internationally, markets clear because the relative price of US goods rises so that US households substitute UK for US goods in their consumption basket. Since this adjustment mechanism produces a US trade deficit, it is inoperable when US households are at the borrowing constraint. Instead, US households are restricted from consuming more UK goods that cannot be purchased from the proceeds of higher US exports to the UK. The relative price of UK goods must rise to reconcile this balanced trade restriction with the wealth effect of the productivity shock on US households. The middle panel of Figure 6 shows that the shock produces an increase in US households' composite consumption via the wealth effect, so US households need to shift the composition of their consumption basket from UK to US goods to avoid running a trade deficit. Of course, this shift means that less of the prevailing productive capacity of US firms is available for real investment, so the higher demand for US investment produced by the productivity shock must be tempered by an increase in the required return on capital. These effects can be seen by comparing the CIFR and UIFR for US investment and consumption in the middle panel of Figure 6. The productivity shock induces a smaller increase in US investment and higher *growth* in US consumption. The lower panel of Figure 6 shows two further implications. First, the initial fall in UK investment is reduced because the borrowing constraint limits export demand.[23] Second, the international borrowing constraint stops the US NFA position from falling in the immediate aftermath of the shock and dampens its rise in subsequent years. Indeed, a comparison of the CIRFs and UIRFs for US consumption and the NFA position makes clear that proximity to the international borrowing constraint limits the ability of US households to inter-temporally smooth consumption.

In summary, our analysis shows that the transmission mechanism for productivity shocks depends critically on the ability of a country benefiting from the shock to borrow internationally. When there are no restrictions on further borrowing, the exchange rate moves to facilitate the international reallocation of investment to its most productive use, but when further borrowing is restricted, the real exchange rate moves to mitigate the effects of the restriction on welfare.

[23]The appreciation of the dollar also lowers the value of UK bond holdings, so that the wealth effect of the productivity shock on UK households is smaller; see the CIRF and UIRF for UK composite consumption in the middle panel of Figure 6.

4.3 Empirical Evidence

We now return to the US-UK data to investigate whether the co-movements of GDP and the real exchange rate vary according to the proximity of either country to its international borrowing limit. For this purpose, we examine whether the coefficients in the empirical model of equations (3) and (4) differ in years with banking crises. More precisely, we construct a dummy variable s_t that equals one in years when there is either a US or UK banking crisis, based on the chronology in Reinhart and Rogoff (2008, Table A.3). We then re-estimate equations (3) and (4) allowing for the coefficients to vary with s_t. In the case of the cyclical GDP equations (4), there was no statistically significant evidence of state-dependency in any of the coefficients. However, in the case of the real depreciation rate equation (3), there are statistically significant differences in all the coefficients between crisis and non-crisis years.

Panel I of Table 4 reports the estimates of the depreciation equation (3) allowing for the effects of crises. The coefficient estimates in the left-hand column, computed in years without a crisis, are similar to those in Table 1. In particular, these estimates imply that in the absence of a banking crisis, shocks that raise US GDP tend to also produce a real appreciation of the US dollar, whereas shocks that increase UK GDP produce a real depreciation. These contemporaneous co-movements are consistent with the initial UIRFs implied by our theoretical model (see Figure 4). The right-hand column reports the estimated difference between the coefficients in crisis and non-crisis years. As the table shows, all of these differences are statistically significant at conventional levels. The immediate exchange-rate effects of a shock in a crisis are determined by the sum of the coefficients in each row. These estimated coefficients imply that shocks producing an increase in USD GDP induce a real depreciation of the dollar, and shocks increasing UK GDP are accompanied by a real appreciation. Again, these co-movements are consistent with the initial CIRFs implied by the model when a country is close to its international borrowing limit (see Figure 6).

The estimates in Table 4 also have implications for the persistence of co-movements in cyclical GDP and the real exchange rate because the coefficients on the error-correction term differ significantly between crisis and no-crisis years. To appreciate these effects, Figure 7 plots the dynamic response of the log real exchange rate to shocks that increase cyclical US and UK GDP in crisis and non-crisis years. The responses in non-crisis years are similar to those in Figure 2. The initial impact of a shock is amplified for a couple of years, and then dissipates slowly. The response patterns are quite different in crisis years. Not only are the initial exchange-rate effects of the shock reversed (i.e., depreciations replace appreciations, and vise versa), but there is no amplification and the effects die more quickly.

Table 4: State-Dependent Model Estimates

I: Estimates		Real Depreciation Rate Equation: $\Delta\varepsilon$	
		No Crisis	Difference in Crisis
Regressors			
y^{Cycle} (US)		-0.294*	0.806**
		(0.155)	(0.386)
$\hat{y}^{Cycle}$ (UK)		0.609***	-1.702***
		(0.163)	(0.589)
gap		-0.148***	-0.313***
		(0.034)	(0.097)
SEE	0.063		
R^2	0.183		

II: Variance Contribution	Horizon (years)				
	1.	5.	10.	15.	20.
GDP Contribution	0.068	0.185	0.234	0.201	0.149
(standard error)	(0.021)	(0.026)	(0.032)	(0.038)	(0.026)
Trend GDP Contribution	0.009	0.032	0.075	0.091	0.072
(standard error)	(0.006)	(0.012)	(0.018)	(0.027)	(0.023)

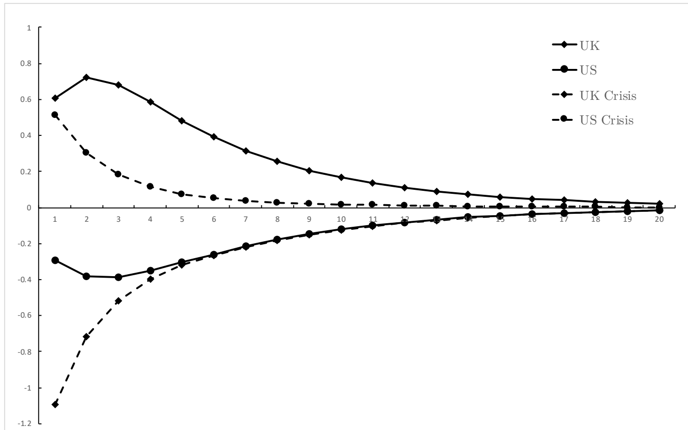
Notes: Panel I reports estimates of the real depreciation rate equation

$$\Delta\varepsilon_t = \left(\begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} + \begin{bmatrix} \beta_1^{\text{diff}} & \beta_2^{\text{diff}} \end{bmatrix} s_t\right) \begin{bmatrix} y_t^{Cycle} \\ \hat{y}_t^{Cycle} \end{bmatrix} + [\gamma + \gamma^{\text{diff}} s_t] gap_{t-1} + e_t,$$

where $\beta_1^{\text{diff}}, \beta_2^{\text{diff}}$ and γ^{diff} identify the difference in the corresponding coefficient between crisis and non-crisis years. The left-hand column reports OLS estimates of β_1, β_2 and γ, and the right-hand column reports estimates of $\beta_1^{\text{diff}}, \beta_2^{\text{diff}}$ and γ^{diff} . Robust standard errors are shown in parenthesis. Estimates are computed from annual data 1802-2016 (215 observations). Statistical significance at the 10, 5 and 1% levels is indicated by *, ** and ***, respectively. Panel II reports the estimated contribution of US and UK GDP (trend and cyclical components) to the variance of the real depreciation rate over horizons from one to 20 years, and the variance contributions of the trend components in US and UK GDP alone. See footnote 10 for details.

Crises do not appear to materially affect how the cyclical and trend components of GDP contributed to the variance of the depreciation rates across different horizons. Panel II of Table 4 reports the variance contributions of GDP to depreciation rates allowing for the effects of crises. These estimated contributions are similar to those in Table 1. Even though the exchange-rate implications of shocks differ between crisis and non-crisis years, crisis occur relatively infrequently (crisis years cover only seven percent of the sample period), so the variance decompositions largely

Figure 7: Conditional Real Exchange Rate Responses



Notes: The figure plots the dynamic response of the log real exchange rate to a shock that induces a one percent increase in either y_t^{Cycle} (US) or $\hat{y}_t^{Cycle}$(UK). Solid plots for responses in no-crisis years, dashed plots for crisis years.

reflect the co-movements in GDP and depreciation rates in non-crisis years.

In summary, the estimates in Table 1 and plots in Figure 4 clearly show that co-movements in the US-UK real exchange rate and US and UK GDP over the past two centuries have differed according to whether one or both countries were experiencing a banking crisis. We interpret these findings to be broadly consistent with the results of our theoretical analysis under the reasonable assumption that banking crises in either the US or UK impaired international borrowing and lending.

5 Is the Transmission Mechanism Robust?

To this point, we have examined the transmission mechanism for productivity shocks in a particular specification of the IRBC model. In this section, we study the transmission mechanism in several different specifications in order to identify the key features of IRBC models that are needed for the mechanism to operate. We also investigate how these features contribute to resolutions of the widely-studied Backus-Smith puzzle.

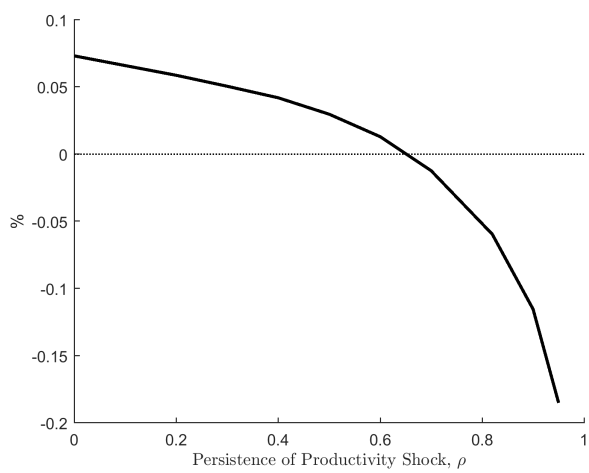
5.1 Alternative Calibrations of the Benchmark Model

We begin our investigation by considering alternative calibrations of our benchmark specification. In particular, we examine the robustness of the transmission mechanism to different degrees of persistence in productivity, and different elasticities of substitution between US and UK goods in households' preferences.

Productivity Persistence We solved the benchmark model for a range of values for the persistence parameter ρ (with other parameters unchanged), and computed the UIRFs to a positive US productivity shock from each solution. The results of this exercise are shown in Figure 8, where we plot the initial UIRF for the real exchange rate against the persistence parameter ρ. Here we see that positive US productivity shocks only produce an initial real appreciation of the dollar in calibrations where the persistence parameter ρ is above 0.6; when shocks have less persistence, they produce a real depreciation of the dollar.

The persistence of productivity shocks plays a key role in the transmission mechanism because it has a significant impact on US firms' investment decisions: Ceteris paribus, firms will want to invest less when positive productivity shocks have low persistence. Under these circumstances,

Figure 8: Varying Productivity Persistence



Notes: The figure plots the initial UIRF for the real exchange rate against the persistent of productivity computed from the solutions to alternative calibrations of the benchmark IRBC model with differing persistence parameters.

when the shock hits the prevailing productive capacity of US firms can exceed the investment and consumption demand for US goods so the relative price of US goods falls to clear markets, which is reflected in a real depreciation of the dollar. Appendix A.1 provides a more detailed discussion of the transmission mechanism when productivity is less persistent, including a comparison of the UIRFs with our benchmark specification.

Elasticity of Substitution We undertook an analogous exercise to investigate the role played by the elasticity of substitution between US and UK goods in households' preferences, θ. In this case we computed the UIRFs from solutions of the benchmark model for a range of values for θ. Figure 9 plots the initial UIRF for the real exchange rate against the elasticity parameter θ. Here we see that positive US productivity shocks produce a real appreciation of the dollar in the calibrations of the model with θ above 6; and a real depreciation when the elasticity of substitution is lower.

The elasticity of substitution plays a key role in the transmission mechanism because it affects the willingness of US households to alter the composition of their consumption baskets. In our benchmark specification (where $\theta = 10$), UK goods are good substitutes of US goods, so a small fall in the relative price of UK goods is sufficient to induce households to consume proportionately more UK goods in response to the productivity shock, which facilitates market-clearing when US real investment is high. This mechanism breaks down when the elasticity of substitution is below a certain level because the required change in relative prices is so large that it deters US real investment via its effect on the marginal product of capital.[24] Instead, the transmission mechanism appears to operate as though US households are close to the international borrowing constraint. When UK goods are poor substitutes for US goods, US households react to the productivity shock *as if* they were facing a borrowing constraint that limited the increase in their consumption of UK goods. As we show in Appendix A.2, the UIRFs in an equilibrium with low elasticity ($\theta = 2$) are very similar to the CIRFs in Figure 6 .

5.2 Alternatives to the Benchmark Model

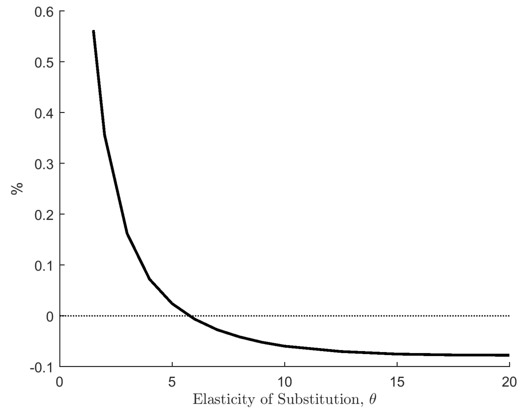
Next, we next investigate how the transmission mechanism operates in IRBC models that have more substantial differences with our benchmark specification. In particular, we consider models

[24] The first-order condition for US investment is given by

$$\mathbb{E}_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{p_{t+1}^{\mathrm{US}}}{p_t^{\mathrm{US}}}\left(\eta A_{t+1}K_{t+1}^{\eta-1}+1-\delta\right)-1\right]=0.$$

If a productivity shock induces a very large rise in the relative price of US goods, $p_{t+1}^{\mathrm{US}}/p_t^{\mathrm{US}}$ falls, which can offset the effects of the shock on productivity, A_{t+1}.

Figure 9: Varying Elasticity of Substitution



Notes: The figure plots the initial UIRF for the real exchange rate against the elasticity of substitution θ computed from the solutions to alternative calibrations of the benchmark IRBC model with differing elasticity parameters.

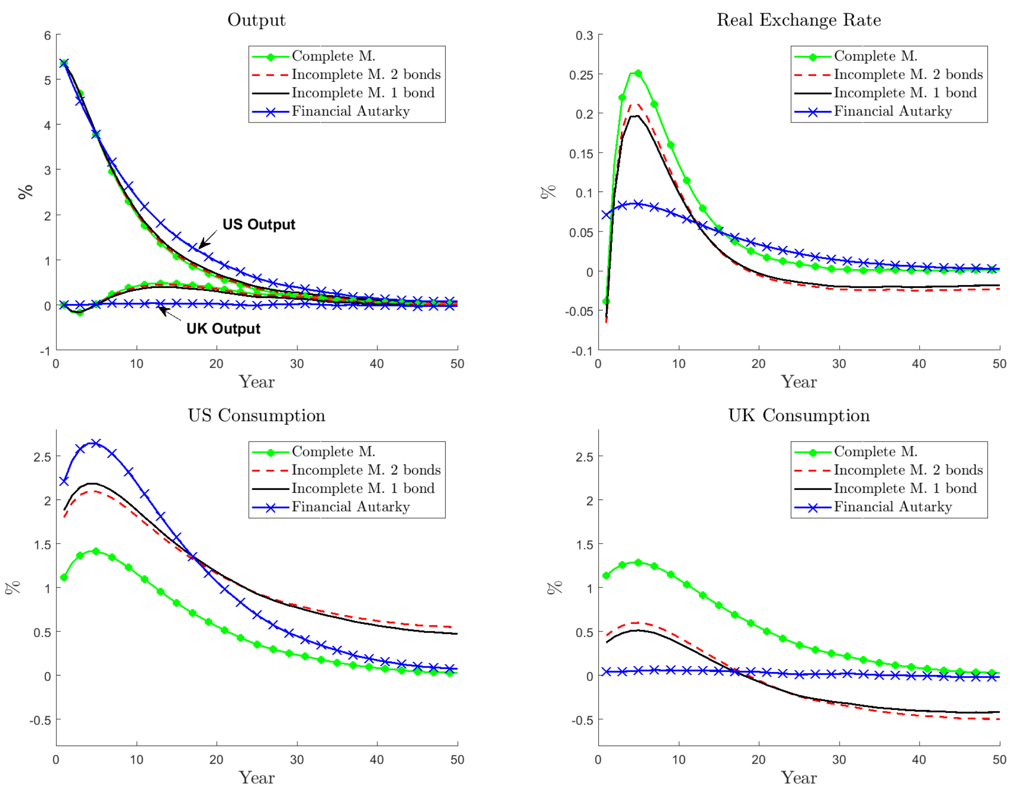
with different financial market structures, production technologies and productivity processes.

Financial Market Structure Our benchmark model only allows for international asset trade in a single bond. As a result, markets are incomplete and risk-sharing between US and UK households is impaired. To this point, we have not emphasized how incomplete risk-sharing contributes to the transmission mechanism other than to point out that it produces different wealth effects across US and UK households. To investigate how the *degree* of risk-sharing affects the transmission mechanism, we considered solutions to three variants on our benchmark specification: one with complete markets, one with financial autarky, and one where households can trade two bonds internationally. In the two bond model, households face a more substantive portfolio choice problem of allocating their wealth between domestic equity, US bonds and UK bonds. A complete description of these models can be found in Appendix B.

The upper panel of Figure 10 plots the UIRFs of output and the real exchange rate from a positive US productivity shock for the benchmark model and the three variants. These plots show that the degree of risk-sharing has small effects on the behavior of output. The UIRFs are essentially indistinguishable between the models with international trade in one or two bonds or complete markets. Under financial autarky, US productivity shocks have more persistent effects on US output and minimal effects on UK output. The UIRFs also show that the impact of the shock on the real exchange rate is similar in the models with complete markets and international asset trade (with one or two bonds); there is an initial real appreciation of the dollar followed by a period of depreciation that lasts approximately 15 years. In contrast, there is an initial real depreciation of the dollar in the model with financial autarky, which persists for a very long time. These differences show that the ability to borrow and lend internationally is more important than the degree of risk-sharing per se. Indeed, the initial response of the exchange rate under financial autarky is very similar to the response we examined in the benchmark model when US households were close to the international borrowing limit.

Risk-sharing has larger effects on the behavior of consumption. As the lower panel of Figure 10 shows, there are significant differences between the UIRFs for US and UK consumption across the

Figure 10: Different Financial Market Structures



Note: This figure is generated by the solution of the baseline model with parameters given in Table 2 under different financial market structures: complete markets, incomplete markets with one or two bonds, and financial autarky.

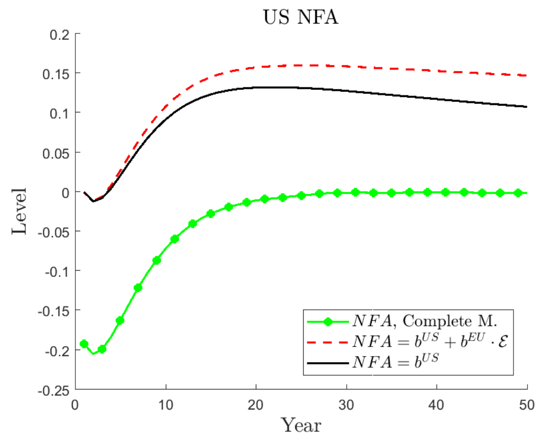
models with financial autarky, incomplete markets (with one or two bonds) and complete markets. Under financial autarky, the effects of the US productivity shock are almost entirely confined to US consumption because the real depreciation of the dollar keeps the US demand for UK goods essentially unchanged. Productivity shocks have similar effects on US and UK consumption in the models with international asset trade in one or two bonds. In both models, the initial effect of the shock is to increase US consumption more than UK consumption because the wealth effects are concentrated in US equity. In the complete markets model, the wealth effects of the productivity shock are more evenly distributed between US and UK households, so the shock has a smaller (larger) initial impact on US (UK) consumption than when markets are incomplete. Although the UIRFs for US and UK consumption appear very similar, in reality, UK consumption initially rises by more than US consumption, for reasons we discuss below.

The plots in Figure 10 make clear that the co-movements in output and the real exchange rate produced by productivity shocks are generally robust to the degree of international risk-sharing. Away from the international borrowing constraint (and financial autarky), positive US productivity shocks produce a real appreciation in the dollar and an increase in US output under complete or incomplete risk-sharing. In contrast, our analysis also reveals that co-movements in the real exchange rate and consumption depend on the degree of risk-sharing. When risk-sharing is impaired (i.e., in either the one or two bond models), positive US productivity shocks induce a real appreciation and a rise in US consumption relative to UK consumption. More generally, in our benchmark specification productivity shocks induce a negative correlation between the real depreciation rate $\Delta\varepsilon_{t+1}$ and relative consumption growth $\Delta \ln C_{t+1} - \Delta \ln \hat{C}_{t+1}$ equal to -0.208. This implied correlation is similar to the estimated correlation of -0.39 documented in Corsetti et al. (2008) using recent US and UK annual data. Of course, as Backus and Smith (1993) originally stressed, the correlation is one under complete markets.

Figure 11 provides a perspective on why the consumption responses to productive shocks differ in the models with complete and incomplete markets. The figure plots the UIRFs for the US NFA position in the model with complete markets and in the models with one and two internationally traded bonds. Here we see that a positive US productivity shock has little impact on the US NFA position in the models with incomplete markets.[25] So the wealth effects of the shock originate from the capital gains of households' domestic equity holdings. In the complete market model, the

[25] In the benchmark model, NFA positions are defined in terms of US bond holdings, so an appreciation of the real exchange rate implies a capital gain on UK NFA, but has no effect on US NFA. In the two bond model, an appreciation of the real exchange rate represents a capital loss on US holdings of UK bonds, and a capital gain on UK holdings of US bonds. The UIRF for US NFA shows that these valuation effects are typically very small.

Figure 11: NFA UIRFs under complete and incomplete markets



Note: This figure is generated by the solution of the baseline model with parameters given in Table 2 under different financial market structures: complete markets, incomplete markets with one or two bonds. Under financial autarky, US NFA is always zero so we omit it from the figure.

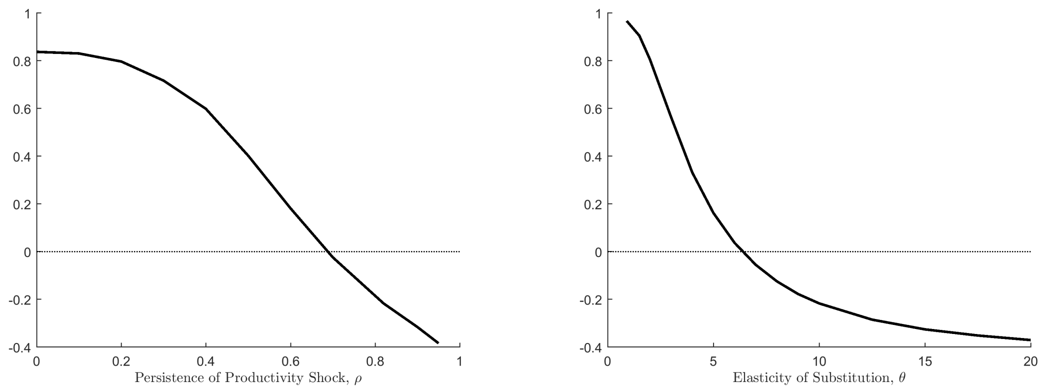
productivity shock lowers the US NFA position, which dampens the wealth effect on US households, and amplifies the effect on UK households.[26] Intuitively, households hold a portfolio of state-contingent securities that provides a hedge against adverse domestic productivity shocks (because there is home bias in consumption), so US households enjoy a smaller capital gain in response to positive US productivity shocks than UK households. As a consequence, the shock produces a larger increase in UK consumption than US consumption, as we noted in our discussion of Figure 10.

Our results demonstrate that the joint dynamics of consumption and real exchange rate depend on whether international trade in assets allows for complete risk-sharing or not. This point is also emphasized in Corsetti et al. (2008). Benigno and Kucuk (2012) argue (in the context of the Corsetti et al.'s model) that the sign of the Backus-Smith correlation between the real depreciation rate and relative consumption growth depends on whether one or two bonds are traded internationally. In contrast, we find that differences in the degree of risk-sharing produced by expanding the number of internationally traded bonds have little effect on the Backus-Smith correlation. Rather, the key factors affecting the correlation are the persistence of productivity shocks ρ and the elasticity of

[26]Under complete markets, households trade state-contingent Arrow securities which pay off depending on the realization of productivity shocks. The US NFA position is defined as the ex-post payoff of US households' Arrow security portfolio. The exact notation and definition are given in Appendix B.

substitution θ. This can be clearly seen in Figure 12 which plots the correlation against ρ and θ computed from alternative calibrations of the benchmark specification.

Figure 12: Backus-Smith Correlation



Notes: The figure plots the correlation between the relative consumption growth and real deprecation rate, computed from the solutions to alternative calibrations of the benchmark IRBC model, against the persistence of productivity shock ρ (left panel) and elasticity of substitution θ (right panel).

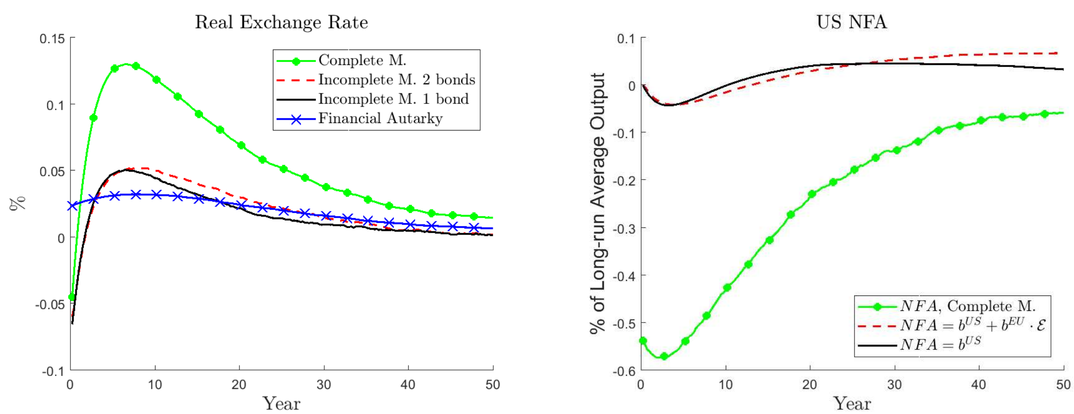
Production and Investment Firms' investment decisions play a central role in the transmission mechanism we have described. To further emphasize this point, we considered a variant of the benchmark model that eliminates investment decisions and fixes each firm's capital stock at its steady-state value. Appendix C reports the UIRFs from US productivity shocks under complete markets, international asset trade in one and two bonds, and financial autarky. Under all these different market structures, positive US productivity shocks raise US output and produce a real *depreciation* of the dollar. These results confirm that (away from the international borrowing constraint) the real exchange rate plays a central role in facilitating the international allocation of real investment.

Productivity When markets are incomplete, the effects of productivity shocks may well depend on the correlation of shocks across countries and whether they have persistent (unit root) effects on the level of productivity. We considered two variants of the benchmark model to investigate these possibilities. In the first variant we replace the productivity process in (13) with the stationary bi-variate process for $\log A_t$ and $\log \hat{A}_t$ estimated in Heathcote and Perri (2002) that allows for cross-country correlation in productivity shocks. In the second variant we assumed that $\log A_t$ and

$\log \hat{A}_t$ follow co-integrated unit-root process as in Rabanal et al. (2011). Appendix D provides a detail description and analysis of these models.

We find that our main results concerning the correlations between real depreciation rates and output (and consumption) are robust to the presence of cross-country correlations or unit-roots across the different market structures considered above. We also find that productivity shocks have more persistent effects on the real exchange rate and the US NFA position in models with a unit-root process. This can be seen in Figure 13 where we plot the UIRFS for these variables under different market structures.[27] Under incomplete markets, positive US productivity shocks produce an initial real appreciation of the dollar that lasts for 2 years, and a deterioration in the US NFA position that lasts for 10 years. This is closer to the degree of persistence we find in the empirical model (see Figure 7).

Figure 13: UIRFs with Co-integrated Unit-Root Productivity Processes



6 Conclusions

This paper has examined how medium-term movements in real exchange rates and GDP vary with international financial conditions. Empirically, we have shown that shocks producing cyclical variations in GDP account for a significant fraction of the real US-UK depreciation rate over horizons ranging from five to fifteen years. We then showed that productivity shocks can induce co-movements in the real exchange rates, GDP and consumption that are consistent with the

[27] The model is in quarterly frequency, to be consistent with the estimation in Rabanal et al. (2011), but we display the UIRFs in annual frequency to make it visually comparable to the UIRFs from the baseline (annual) model.

data in a variety of IRBC models. One important finding to emerge from this analysis is that the co-movements depend on prevailing financial conditions; more specifically the proximity of a country to its international borrowing constraint. We find evidence consistent with this form of state-dependency in the US-UK data.

Unfortunately, the IRBC models we study do not account for all the features of the data. In particular, a common feature of every model is that the initial effects of productivity shocks on both real exchange rates and NFA positions are subsequently reversed. There is no evidence of these reversals in the US-UK data. In a sense this short-coming of IRBC models is not a surprise; there is no empirical evidence that shocks affecting the US NFA positions are subsequently reversed either (see, e.g., Evans, 2017). We conjecture that whatever economic mechanisms weaken mean-reversion in the US NFA position also inhibits the reversals in the exchange-rate effects of productivity shocks.[28]

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[28]This conjecture is confirmed in the alternative version of the benchmark model with co-integrated random walk productivity processes discussed in Subsection 5.2.

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APPENDIX

A Alternative Calibrations

In this appendix, we study how the model predictions depend on productivity persistence and the elasticity of substitution.

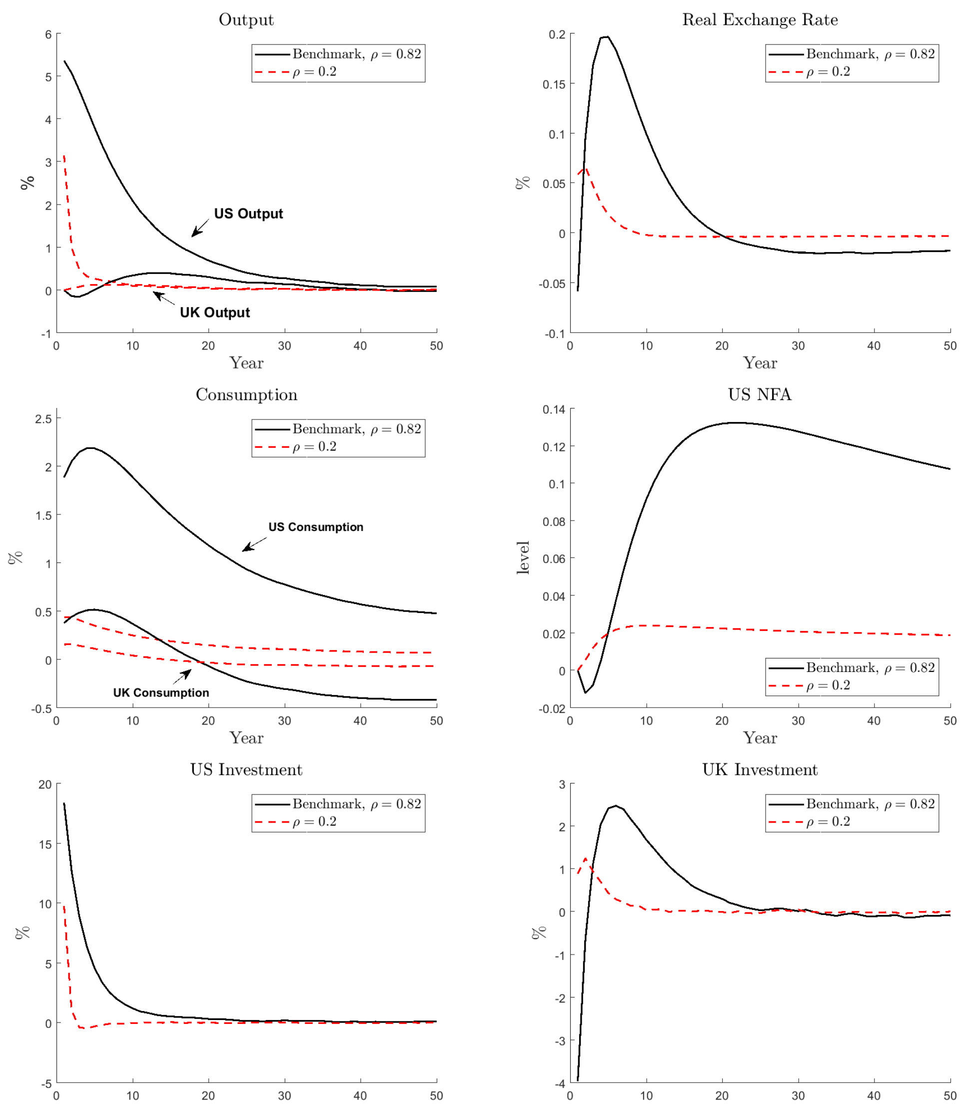
A.1 Less Persistent Productivity Shocks

Figure 14 shows the UIRFs for equilibrium variables upon a positive productivity shock to the US. The shock is less persistent, ($\rho = 0.2$, dashed red lines) than the benchmark calibration ($\rho = 0.82$, solid black lines). In this case, the real exchange rate depreciates, rather than appreciates as in the benchmark model. One reason is that investment in the US increases by much less than it does under the benchmark calibration. The increase in investment demand is not enough to offset the increase in the supply of US goods.

A.2 Lower Elasticity of Substitution

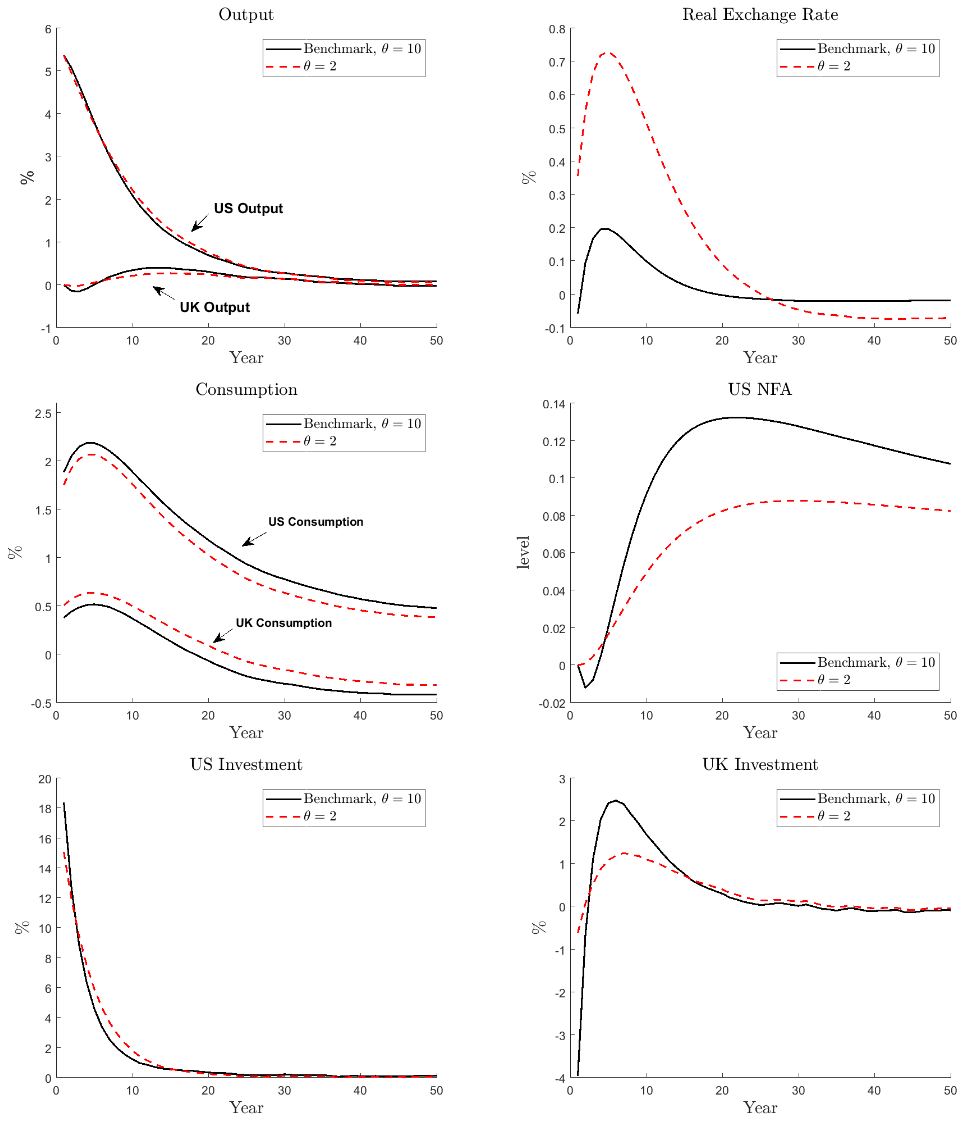
Figure 15 shows the UIRFs for equilibrium variables upon a positive productivity shock to the US. The elasticity of substitution is lower, $\theta = 2$, (dashed red lines) than the benchmark calibration ($\theta = 10$, solid black lines). Again, in this case, the real exchange rate depreciates. One reason is that investment in the US increases by less than it does under the benchmark calibration. In addition, US consumption increases by less. UK consumption increases by more but because of consumption home bias, this increase does not lead to a large increase in the demand of UK households for US goods. As a result, the increase in the demand for US goods is outweighed by the increase in the supply of US goods and, hence, the real exchange rate depreciates.

Figure 14: UIRFs under Different Values of Productivity Persistence



Note: This figure is generated by the solution of the baseline model with parameters given in Table 2 (solid lines) and the solution of the baseline model but with less persistent productivity shocks (dashed lines).

Figure 15: UIRFs under Different Values of Elasticity of Substitution



Note: This figure is generated by the solution of the baseline model with parameters given in Table 2 (solid lines) and the solution of the baseline model but with a lower elasticity of substitution (dashed lines).

B Alternative Market Structures

In this appendix, we examine the equilibrium dynamics under different financial market structures: complete markets, incomplete markets with one or two bonds, and financial autarky. Under financial autarky, US and UK households can trade in the good markets but not in financial markets. Therefore, the current account must be zero at all times.

B.1 Complete Markets

Under complete markets, households have access to the complete set of state-contingent Arrow securities, besides domestic equity. At time t and history z^t, let $\mathcal{Q}_t^A(z_{t+1}|z^t)$ denote the price of the Arrow security that pays off one US dollar at $t+1$ if and only if state z_{t+1} is realized. Similarly let $\hat{\mathcal{Q}}_t^A(z_{t+1}|z^t)$ denote the price of the Arrow security that pays off one UK pound at $t+1$ if and only if state z_{t+1} is realized. To simplify notation, we omit the history dependence. It follows from no-arbitrage that

$$\hat{\mathcal{Q}}_t^A(z_{t+1}) = \mathcal{Q}_t^A(z_{t+1})S_{t+1}(z_{t+1})/S_t.$$

Given the Arrow securities and prices, US households' budget constraint, (7a), becomes

$$P_tC_t + Q_t\chi_t + \sum_{z_{t+1}} \mathcal{Q}_t^A(z_{t+1})\mathcal{A}_t(z_{t+1}) \leq (D_t + Q_t)\chi_{t-1} + \mathcal{A}_{t-1}(z_t) + W_tL_t,$$

where $\mathcal{A}_t(z_{t+1})$'s are the holdings of state-contingent Arrow securities which payoff in US dollar. Let $a_{t-1} = \frac{\mathcal{A}_{t-1}(z_t)}{P_t}$ denote the real holding and $q_{t+1}^A(z_{t+1}) = \mathcal{Q}_t^A(z_{t+1})\frac{P_{t+1}(z_{t+1})}{P_t}$ denote the real price of Arrow securities. The US households' budget constraint can be written in real terms:

$$C_t + q_t\chi_t + \sum_{z_{t+1}} q_{t+1}^A(z_{t+1})a_t(z_{t+1}) \leq (d_t + q_t)\chi_{t-1} + a_{t-1}(z_t) + w_tL_t.$$

The US NFA at time t is defined as $a_{t-1}(z_t)$.

Similar, UK households' budget constraint, (10a), becomes

$$\hat{P}_t\hat{C}_t + \hat{Q}_t\hat{\chi}_t + \sum_{z_{t+1}} \hat{\mathcal{Q}}_t^A(z_{t+1})\hat{\mathcal{A}}_t(z_{t+1}) \leq \left(\hat{D}_t + \hat{Q}_t\right)\hat{\chi}_{t-1} + \hat{\mathcal{A}}_{t-1}(z_t) + \hat{W}_t\hat{L}_t,$$

where $\hat{\mathcal{A}}_t(z_{t+1})$'s are the holdings of state-contingent Arrow securities which payoff in UK pounds and (10b) becomes

$$\hat{C}_t + \hat{q}_t\hat{\chi}_t + \sum_{z_{t+1}} \hat{q}_t^A(z_{t+1})\hat{a}_t(z_{t+1}) \leq \left(\hat{d}_t + \hat{q}_t\right)\hat{\chi}_{t-1} + \hat{a}_{t-1}(z_t) + \hat{w}_t\hat{L}_t.$$

B.2 Incomplete Markets with Two Bonds

In the benchmark model, the two countries are not symmetric because we assume that the international bond is denominated in a particular currency (USD). In this subsection, we consider a completely symmetric model in which there are two bonds: a US bond and a UK bond. The US bond payoff is denominated in the US price index so that it is risk-free from the perspectives of US households. The UK bond payoffs is denominated in the UK price index $\hat{P}_t$, defined similarly to P_t. So the budget constraint of the US households, (7a), is modified to:

$$P_tC_t + Q_t\chi_t + Q^b_{\mathrm{US},t}b_t^{\mathrm{US}} + S_t\hat{Q}^b_{\mathrm{UK},t}b_t^{\mathrm{UK}} \leq (D_t + Q_t)\chi_{t-1} + P_tb_{t-1}^{\mathrm{US}} + S_t\hat{P}_tb_{t-1}^{\mathrm{UK}} + W_t$$

where b_t^{US} is the number of US bonds held by the US household; $Q^b_{\mathrm{US}\ ,t}$ is nominal price of a US bond (in US dollars); b_t^{UK} is the number of UK bonds held by the US household; $\hat{Q}^b_{\mathrm{UK},t}$ is the nominal price of an UK bond (in pounds). Dividing both sides of the nominal budget constraint by P_t, we obtain the constraint in real terms

$$C_t + q_t\chi_t + q^b_{\mathrm{US},t}b_t^{\mathrm{US}} + \mathcal{E}_t\hat{q}^b_{\mathrm{UK},t}b_t^{\mathrm{UK}} \leq (d_t + q_t)\chi_{t-1} + b_{t-1}^{\mathrm{US}} + \mathcal{E}_tb_{t-1}^{\mathrm{UK}} + w_t$$

where $q_t, q^b_{\mathrm{US},t}$, and w_t are the corresponding real values of $Q_t, Q^b_{\mathrm{US},t}$, and W_t; $\hat{q}^b_{\mathrm{UK},t} = \hat{Q}^b_{\mathrm{UK},t}/\hat{P}_t$ is the real price of UK bonds in the UK. We also impose exogenous constraints on real bond holdings

$$b_t^{\mathrm{US}}, b_t^{\mathrm{UK}} \geq \underline{b}.$$

The US NFA at time t is defined as $b_{t-1}^{\mathrm{US}} + \mathcal{E}_tb_{t-1}^{\mathrm{UK}}$.

Similarly, the budget constraints of the UK households in nominal and real terms are modified to

$$\hat{P}_t\hat{C}_t + \hat{Q}_t\hat{\chi}_t + \frac{Q^b_{\mathrm{US},t}}{S_t}\hat{b}_t^{\mathrm{US}} + \hat{Q}^b_{\mathrm{UK}\ ,t}\hat{b}_t^{\mathrm{UK}} \leq (\hat{D}_t + \hat{Q}_t)\hat{\chi}_{t-1} + \frac{P_t}{S_t}\hat{b}_{t-1}^{\mathrm{US}} + \hat{P}_t\hat{b}_{t-1}^{\mathrm{UK}} + \hat{W}_t$$

and

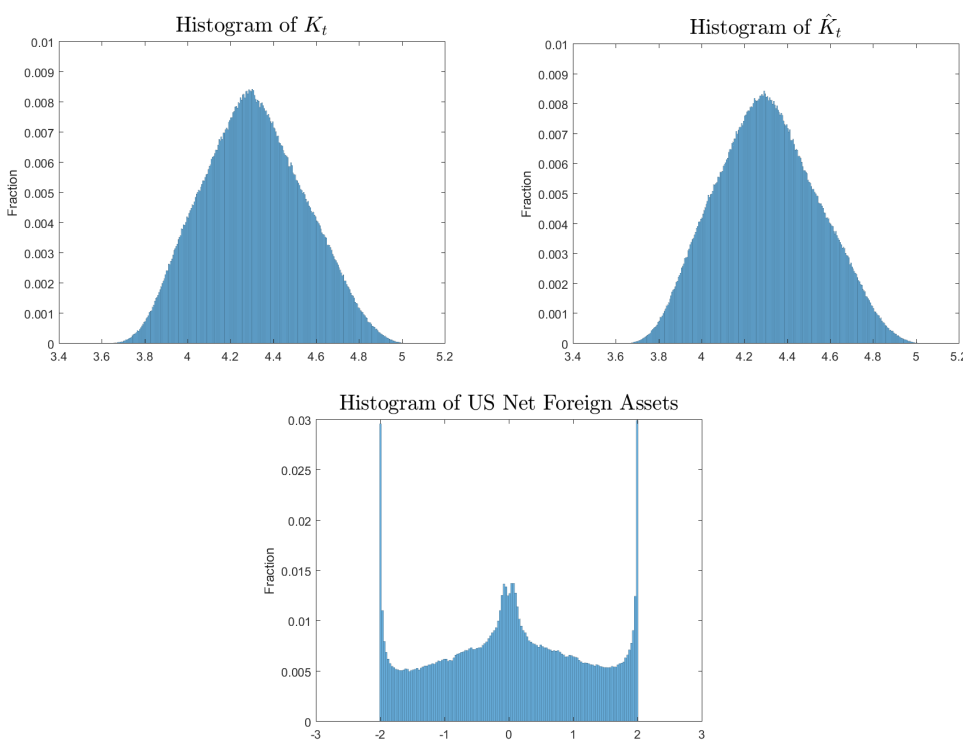
$$\hat{C}_t + \hat{q}_t\hat{\chi}_t + (q^b_{\mathrm{US},t}/\mathcal{E}_t)\hat{b}_t^{\mathrm{US}} + \hat{q}^b_{\mathrm{UK},t}\hat{b}_t^{\mathrm{EU}} \leq (\hat{d}_t + \hat{q}_t)\hat{\chi}_{t-1} + (1/\mathcal{E}_t)\hat{b}_{t-1}^{\mathrm{US}} + \hat{b}_{t-1}^{\mathrm{UK}} + \hat{w}_t$$

Notice that we do not assume that there are transaction costs associated with holding foreign bonds (see, e.g., Ghironi et al., 2009), or similarly that the domestic return on foreign bonds exogenously depends on the level of international debt.

B.3 Results under Different Market Structures

Figure 16 shows the histograms of capital and the US NFA $b_{t-1}^{\mathrm{US}} + \mathcal{E}_t b_{t-1}^{\mathrm{UK}}$ in the two-bond incomplete markets model. Compared to the histograms for the benchmark one-bond model in Figure 3, the distribution of NFA in the two-bond model has an additional mode at 0. This reflects the fact that having two assets help countries insure against shocks better than having just one asset. However, the insurance is not perfect so NFAs still move around significantly and households in each country run up against both of their borrowing constraints (NFA being around 2 or -2), for US households and for UK households, quite frequently.

Figure 16: Histograms of State Variables in the Two-Bond Model



Note: This figure is generated by the solution of the two-bond model with parameters given in Table 2.

Table 5 shows that the correlation between output growth and the real deprecation rate under complete markets and under incomplete markets with one or two bonds is negative. But the correlation is positive under financial autarky. In addition, the correlation between relative consumption growth and the real depreciation rate is negative under incomplete markets, but is one under complete markets and is almost one under financial autarky.

Table 5: International Co-movements

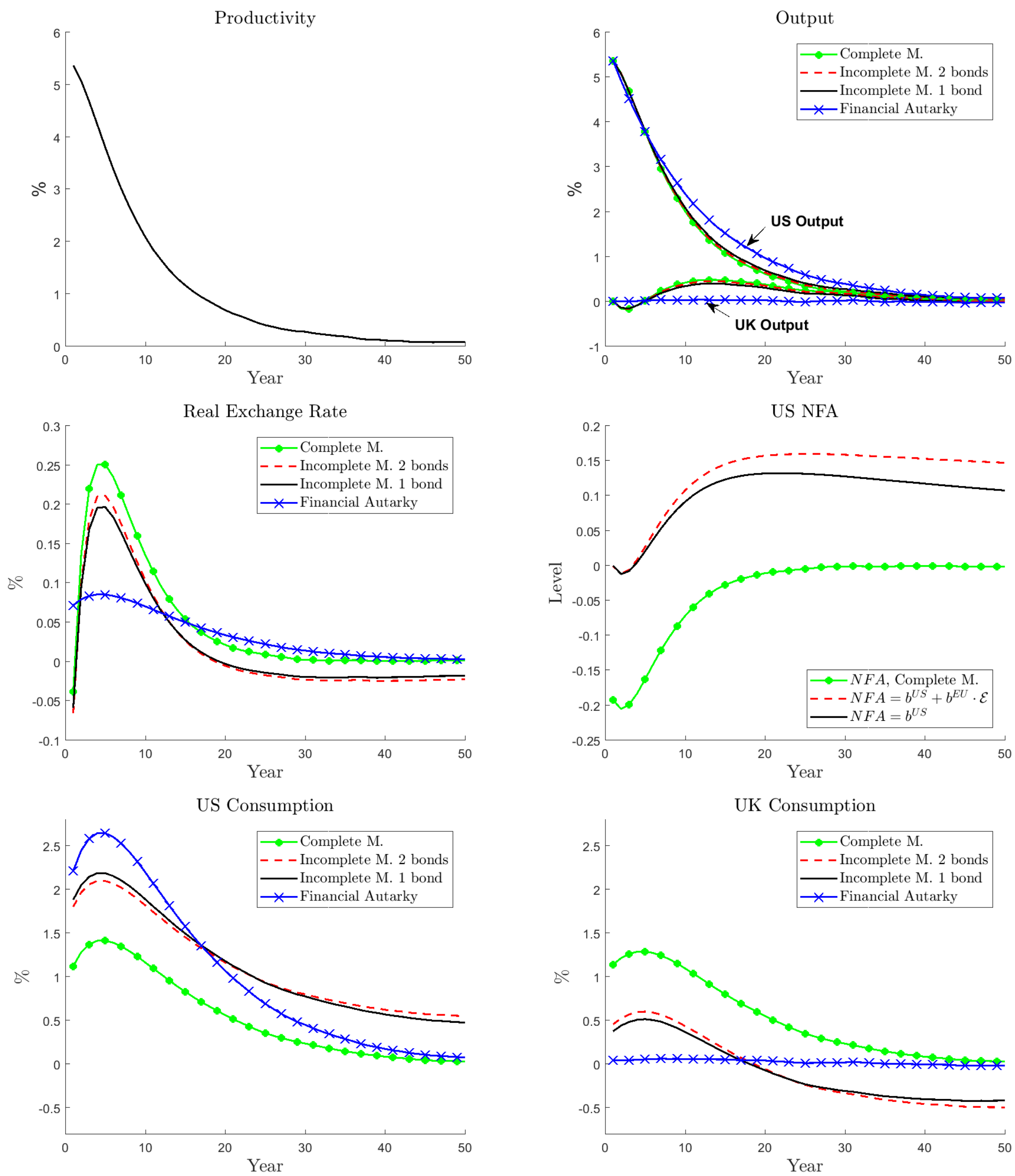
Economy	RER-Output Corr $corr\left(\Delta \log Y_{t+1}, \Delta \log \mathcal{E}_{t+1}\right)$	Backus-Smith Corr $corr\left(\Delta \log C_{t+1} - \Delta \log \hat{C}_{t+1}, \Delta \log \mathcal{E}_{t+1}\right)$
Data		
US vs. UK		Negative
Specifications		
Complete Markets	-0.133	1.000
Incomplete Markets (2 bonds)	-0.223	-0.234
Incomplete Markets (1 bond)	-0.208	-0.211
Financial Autarky	0.681	1.000

To shed light on these results, Figure 17 shows the IRFs for the main equilibrium variables under different market structures. The equilibrium dynamics are very similar under incomplete markets with one bond, our benchmark model, or two bonds. Under complete markets, the real exchange rate appreciates when productivity increases in the US. However, US consumption increases by less than UK consumption, leading to a positive correlation between the relative consumption growth and the real depreciation rate. Under financial autarky, the real exchange rate depreciates, similar to the responses under incomplete markets with one bond and when the US households are close to their borrowing limit.

C Exchange Economy

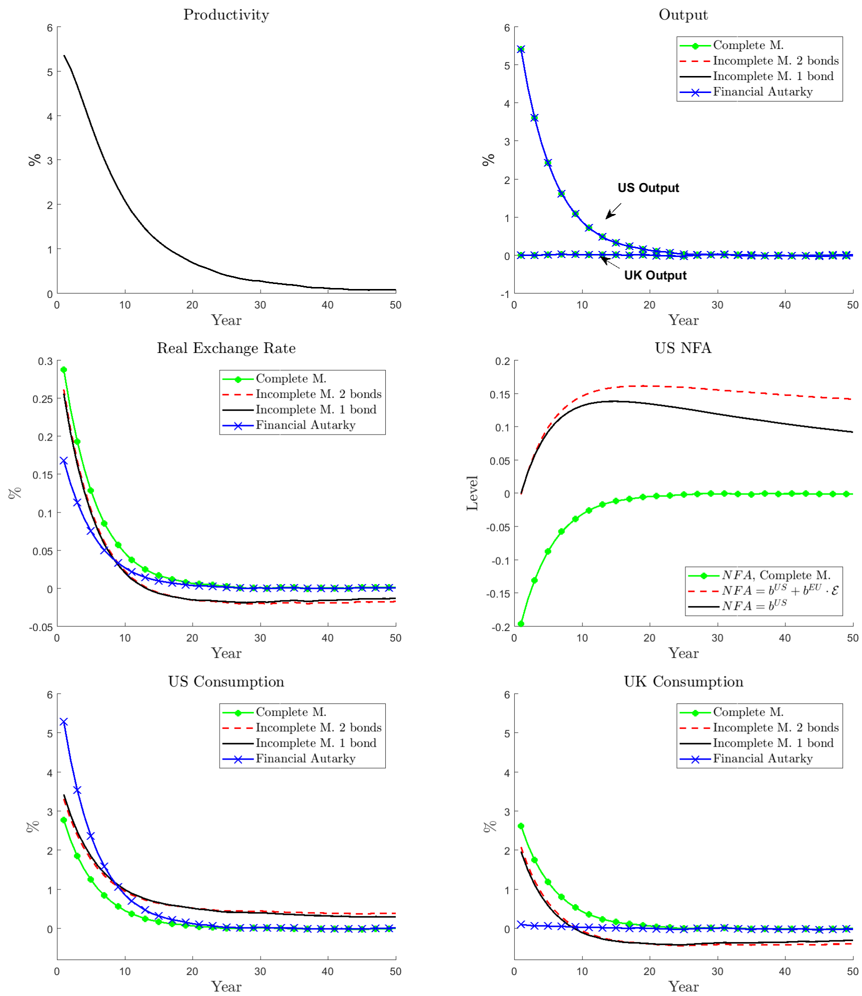
To understand the importance of endogenous capital accumulation and investment in our transmission mechanism, we consider a variant of our baseline model in which capital stocks are fixed at their steady-state values. This is effectively an endowment economy. Figure 18 displays the IRFs for the main equilibrium variables under different market structures. Despite the high elasticity of substitution and persistent endowment shocks, the real exchange rate depreciates upon an increase in endowment in the US. A greater supply of US traded goods depresses its price relative to UK traded goods. The results are robust across all financial market structures.

Figure 17: UIRFs under Different Market Structures



Note: This figure is generated by the solution of the baseline model with parameters given in Table 2 and under different financial market structures.

Figure 18: UIRFs in an Exchange Economy and under Different Market Structures



Note: This figure is generated by the solution of the exchange economy under different financial market structures with parameters given in Table 2 and capital stock fixed at the steady-state value of the baseline model.

D Alternative Productivity Processes

To highlight the endogenous transmission of productivity shocks, in the benchmark model, we assume that the productivity processes are stationary and productivity shocks are uncorrelated between the US and the UK. Recent estimates suggest that productivity shocks should be correlated and productivity processes can be non-stationary. In this section, we show that our mechanisms still apply under these alternative specifications for productivity processes. Recent estimates, including Heathcote and Perri (2002) and Rabanal et al. (2011) use quarterly data. Therefore, we use quarterly models and use several model parameters value from Backus et al. (1992). The model parameters are given in Table 6.

Table 6: Alternative Calibrations

Parameters	Descriptions	Values
β	discount factor	0.9875
γ	risk aversion	2
$\alpha = 1 - \hat{\alpha}$	home consumption share	0.85
θ	consumption elasticity	10
δ	depreciation rate	0.025
η	capital share	0.36
$\underline{b}$	borrowing limit	-1
Stationary Productivity w Correlated Shocks		
ρ_1	productivity persistence	0.97
ρ_2	productivity dependence	0.025
$corr_\epsilon$	correlation of innovation	0.29
σ_ϵ	std of innovation	0.0073
Non-Stationary Productivity		
κ	error-correction parameter	0.007
σ_ϵ	std of innovation	0.01

D.1 Higher Persistence and Correlated Shocks

We use the stationary productivity processes with the estimated persistence and standard deviation of innovation from Heathcote and Perri (2002). The dynamics of productivity, (13), are replaced by

$$\log A_t = \rho_1 \log A_{t-1} + \rho_2 \log \hat{A}_{t-1} + \epsilon_t$$

and

$$\log \hat{A}_t = \rho_1 \log \hat{A}_{t-1} + \rho_2 \log A_t + \hat{\epsilon}_t,$$

with the parameters given in Table 6. In addition to the explicit cross country dependence, $\rho_2 > 0$, the innovation ϵ_t and $\hat{\epsilon}_t$ are also correlated. Table 7 shows that the output-real exchange rate correlation and the Backus-Smith correlation are still significantly negative under the two incomplete markets specifications. Because the shocks are correlated, the meaning of UIRFs are different from the ones in the benchmark model. So instead of reporting the impact response of the real exchange rate, we report the correlation between the changes in log output, $\Delta \log Y_{t+1}$, and log depreciation rate, $\Delta \log \mathcal{E}_{t+1}$.

Table 7: Co-Movements under Alternative Productivity Processes

Economy	RER-Output Corr $corr\left(\Delta \log Y_{t+1}, \Delta \log \mathcal{E}_{t+1}\right)$	Backus-Smith Corr $corr\left(\Delta \log C_{t+1} - \Delta \log \hat{C}_{t+1}, \Delta \log \mathcal{E}_{t+1}\right)$
Stationary Pty (correlated shocks)		
Models:		
Complete Markets	−0.42	1.000
Incomplete Markets (2 bonds)	−0.53	−0.67
Incomplete Markets (1 bond)	−0.47	−0.63
Financial Autarky	0.70	1.000
Non-Stationary Pty		
Complete Markets	−0.44	1.000
Incomplete Markets (2 bonds)	−0.50	−0.728
Incomplete Markets (1 bond)	−0.40	−0.460
Financial Autarky	0.70	1.000

D.2 Non-Stationary Productivity Processes

As noted by Baxter and Crucini (1995), the impact of market incompleteness might depend on whether the productivity processes have unit roots. To investigate this possibility, we assume that productivity shocks are almost perfectly persistent and highly correlated: they follow co-integrated random walk processes.

In this case, the production function takes the form

$$F(A, K, L) = K^{\eta} (AL)^{1-\eta}.$$

We use the specification and estimates from Rabanal et al. (2011):

$$\log A_t = \log A_{t-1} + \kappa \log \left(\frac{\hat{A}_{t-1}}{A_{t-1}} \right) + \epsilon_t,$$

and

$$\log \hat{A}_t = \log \hat{A}_{t-1} + \kappa \log \left(\frac{A_{t-1}}{\hat{A}_{t-1}} \right) + \hat{\epsilon}_t,$$

where ϵ_t and $\hat{\epsilon}_t$ are I.I.D. and κ and $std(\epsilon_t) = std(\hat{\epsilon}_t) = \sigma_\epsilon$ are given in Table 6.

Because productivity processes are non-stationary, we need to adjust the borrowing limits to make them relevant as productivities evolve over time. For example, in the two-bond incomplete markets model, we assume that US households are subject to the following borrowing constraints:

$$b_t^{\mathrm{US}} \geq \underline{b} A_t, b_t^{\mathrm{UK}} \geq \underline{b} \hat{A}_t.$$

Similarly UK households are subject to the following borrowing constraints:

$$\hat{b}_t^{\mathrm{US}} \geq \underline{b} A_t, \hat{b}_t^{\mathrm{UK}} \geq \underline{b} \hat{A}_t.$$

We solve the model by normalizing the non-stationary variables for a country by its productivity. We need to keep track of an additional continuous state-variable which is the relatively productivity, $\frac{A_t}{\hat{A}_t}$. The relatively productivity follows a stationary process:

$$\log\left(\frac{A_t}{\hat{A}_t}\right) = (1-2\kappa)\log\left(\frac{A_{t-1}}{\hat{A}_{t-1}}\right) + \epsilon_t - \hat{\epsilon}_t.$$

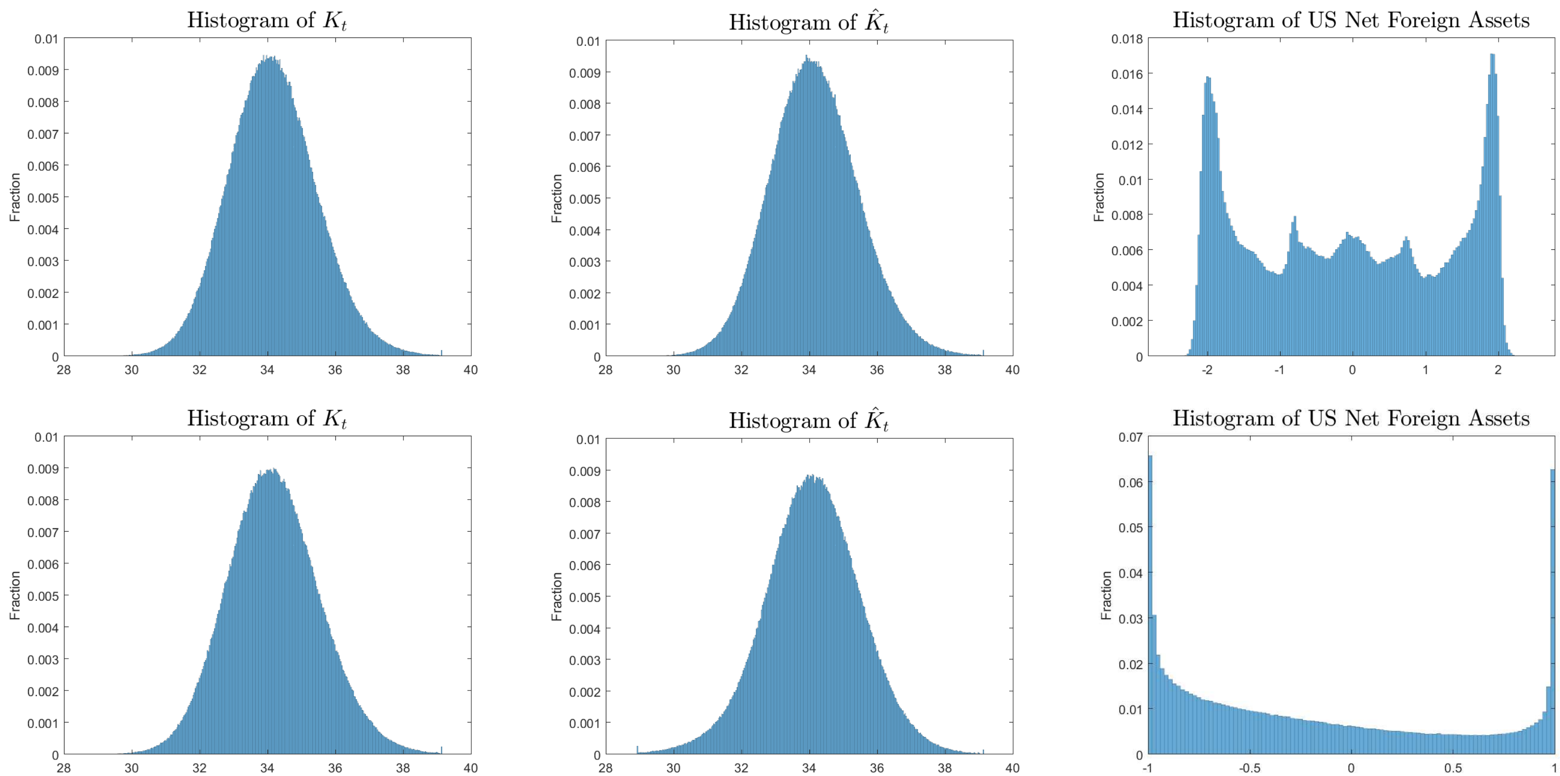
The two-bond model involves the largest number of continuous state variables: the relative productivity, two capital stocks, $\frac{K_t}{A_t}$, $\frac{\hat{K}_t}{\hat{A}_t}$, and two bond positions, $\frac{b_t^{\mathrm{US}}}{A_t}$, $\frac{b_t^{\mathrm{UK}}}{\hat{A}_t}$. This model is also challenging to solve because it involves the US and UK households' portfolio choices between USD denominated bonds and UK pound denominated bonds. The bonds' returns are also close to being collinear because the variation of the real exchange rate is small when the elasticity of substitution, θ, is high.

For non-stationary productivity processes, Figure 19 shows the marginal distributions of similar endogenous state variables under incomplete markets, normalized by the productivity levels to ensure stationarity. Having solved the model, we can examine its implications for exchange-rate, output, and consumption dynamics.

Table 7 shows that, under complete markets and incomplete markets specifications, the correlation between output growth and real exchange rate is negative. In addition, the incomplete markets specifications (with either one or two bonds) generate a negative correlation between relative consumption growth and the real depreciate rate, as in Backus-Smith's empirical findings. These results are close to the ones for stationary productivity processes shown discussed in Section 5.

To understand the results, we use the IRFs as in Section 5. The dynamics of equilibrium variables in Figure 20 upon a positive US productivity shock under different financial markets structures are similar to the stationary counterparts. However, one notable difference is that productivity, output, consumption, and investment in the US and the UK do not converge back to pre-shock levels in

Figure 19: Histograms of State Variables for Incomplete Markets and Growth Shocks

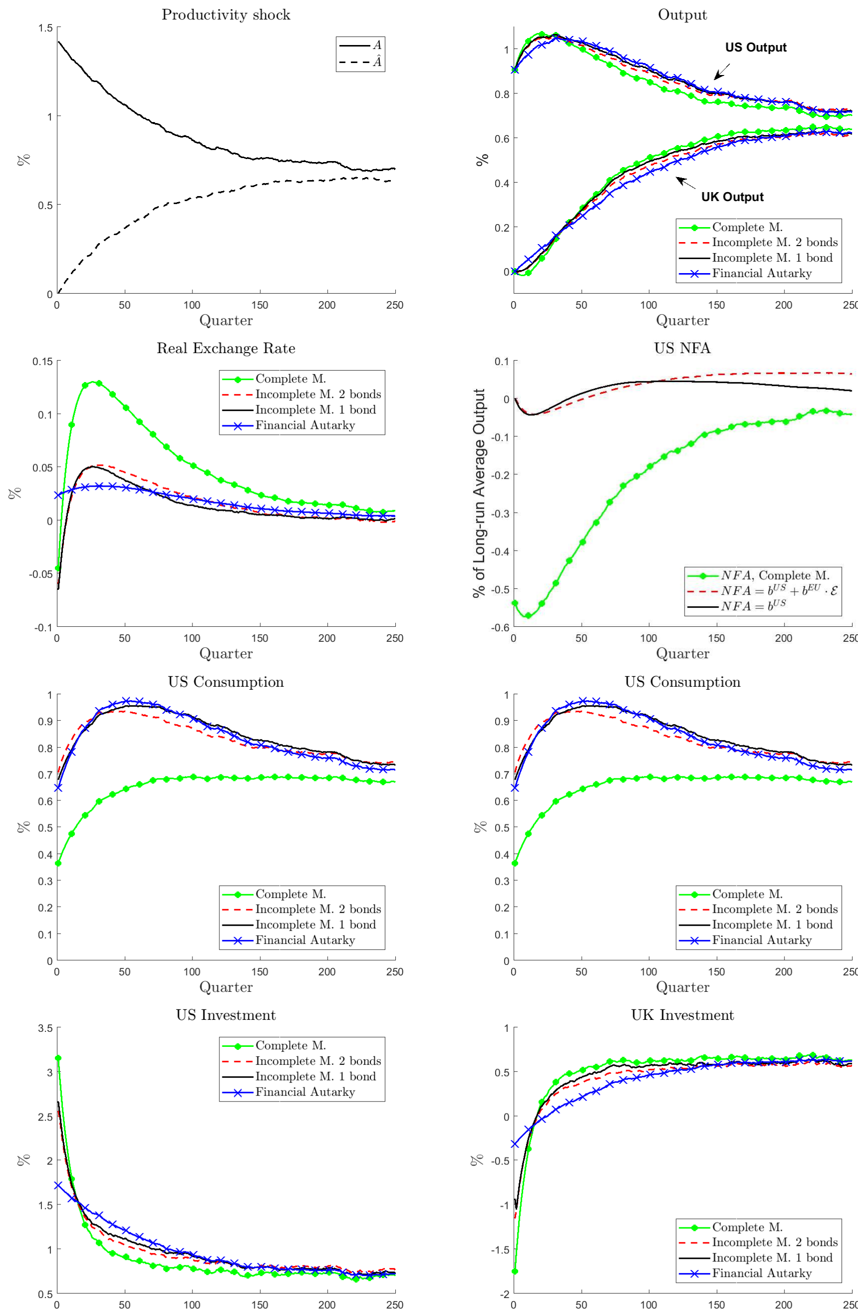


Note: This figure is generated by the solution of the model with co-integrated growth shocks. The upper panels are for the 2-bond incomplete markets model and the lower panels are for the 1-bond incomplete markets model.

the long run. This is because the productivity shocks are permanent according to the specification of productivity process described above.

Except for financial autarky, output in the US increases and real exchange rate appreciates when the US productivity increases, which leads to a negative correlation between output growth and the real depreciation rate. Under incomplete markets, US consumption increases at impact by more than UK consumption does and the real exchange rate appreciates, which leads to a negative Backus-Smith correlation. The transmission mechanism through the responses of different consumption components and investment shown in the figure is also the same as in the benchmark model with stationary shocks.

Figure 20: UIRFs under Non-Stationary Productivity Processes



Note: This figure is generated by the solution of the model with co-integrated random walks specification for productivity, under different financial market structures with parameters given in Table 6.