A Simple Model of Competitive Testing

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Abstract

A number of candidates are competing for a prize. Each candidate is privately informed about his type. The decision-maker who allocates the prize wants to give it to the candidate with the highest type. Each candidate can take a test that reveals his type at a cost. I show that an increase in competition increases information revelation when the cost is high, and reduces it when the cost is low. Nevertheless, the decision-maker always benefits from greater competition. Candidates can be better off if the cost is higher. Mandatory disclosure is Pareto-dominated by voluntary disclosure unless competition is low. Finally, when the test is noisier, candidates are more likely to take it.

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1 Introduction

In many situations, competing privately informed agents have access to a costly exogenous test that credibly reveals their private information to a

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principal. For example, students competing for a scholarship allocated on the basis of ability can individually choose to reveal their ability by taking a GRE test at some cost. Firms competing for a fixed-price procurement contract can reveal the quality of their products by paying an independent agency to certify it. Political candidates facing an electorate that wants to select the most competent candidate can invest in a costly media campaign to communicate their competence. Job applicants competing for an opening can take an optional test to demonstrate their skill.

To analyse these and similar settings, the paper models a group of candidates who are competing for a prize of fixed value. Each candidate is privately informed about his type, which is drawn from the unit interval. The prize is allocated by a decision-maker, who would like to give it to the candidate with the highest type. Each candidate can reveal his type by taking a test. The test is costly, and candidates simultaneously decide whether to take it. Other than through the test, candidates cannot send information to the decision-maker.

Since the test is costly, not all candidates take it. Instead, there is a symmetric equilibrium, in which a candidate takes the test if and only if his type is above some threshold. In that case, he wins the prize if no other candidate has a higher type. On the other hand, if a candidate does not take the test, the decision-maker learns that his type is below the threshold. Then the candidate can only get the prize if nobody else takes the test, in which case the decision-maker allocates the prize at random.

The first result of the paper shows that competition affects information revelation in a non-monotone way. When the cost of the test is high, increasing the number of candidates makes the decision-maker more likely to learn the type of the best candidate. But when the cost is low, an increase in the number of candidates makes the decision-maker less informed. At the same time, even when the number of candidates goes to infinity, the probability that some information is revealed remains distinct from zero and from one.

To see the intuition, consider a candidate i whose type is at the threshold. An increase in the number of candidates has two effects. First, it reduces i's payoff if he does not take the test, because even if no other candidate takes it, the decision-maker will randomise over a larger number of candidates – hence, i becomes less likely to win the prize. Second, it reduces i's payoff from taking the test, since it makes it more likely that some other candidate has a type above the threshold, takes the test, and wins over i. But if the cost of the test is low, the threshold is low as well – so the probability that some candidate has a type above the threshold increases at a high rate. As a result, the second effect dominates the first, and i becomes less willing to take the test.

Second, I show that even though competition can result in less information revelation, the decision-maker always benefits from an increase in competition. On the other hand, an increase in the cost of the test hurts the decision-maker but can make candidates better off by reducing inefficient testing.

Third, the paper examines the effect of the decision-maker committing not to give the prize to any candidate who does not take the test. There is a substantial literature focusing on mandatory disclosure as a way of making decision-makers better off[1]. But does the decision-maker benefit from making disclosure mandatory when information is disclosed through a costly exogenous test? On the one hand, such a rule reduces the payoff of a candidate who does not take the test to zero. Hence, candidates become more willing to take it, and the decision-maker receives more information. On the other hand, if no candidate takes the test, such a commitment leaves the decision-maker unable to allocate the prize, reducing her utility. Which of the two effects dominates? I show that when the number of candidates is above some cutoff, the second effect is stronger, and hence the decision-maker is worse off – intuitively, the reason is that when competition is strong, the payoff of a candidate who does not take the test is sufficiently low even without such a commitment. Since candidates are always hurt by mandatory disclosure, this implies that mandatory disclosure is strictly Pareto-dominated by voluntary disclosure unless the number of candidates is small. For example, when candidates' types are uniformly distributed, I show that making the test voluntary is better whenever the number of candidates is at least 3.

Fourth, I consider what happens when the test sends an imperfect signal about a candidate's type. The paper shows that candidates are more likely to take the test when it is noisier. Intuitively, without noise, if a candidate whose type is at the threshold takes the test, he can only win if no other candidate has a higher type. With noise, he can also win if some candidate has a higher type but does worse on the test. Hence, his incentive to take the test increases.

To see the implications of these results, consider several firms competing for a procurement contract with a fixed price. Each firm can credibly reveal

[1]See an overview in Dranove and Jin (2010).

the quality of its product by asking an independent body to certify it at some cost. If more firms enter, does the buyer become more informed? The first result suggests that an increase in competition will reduce information revelation when certification is cheap relative to profit margins in the market, but not when it is costly.

Alternatively, consider an election contested by several candidates. Each candidate can choose to organise a press conference, or take part in a public debate. These actions send a signal about the candidate's competence, which are reported to voters by media. The signals reach the electorate with some noise, which is larger when the quality of journalism is lower, or when media penetration is low (so voters tend to learn media stories through their friends, rather than directly). The fourth result suggests that in such situations, candidates will be more likely to send signals about their quality.

Furthermore, consider university applicants that can reveal their ability through a standardised test such as SAT or GRE. Should universities make submission of test scores optional rather than mandatory? While negative effects of highly competitive university admission tests on applicants have been noted before[2], the third result of the paper suggests that in some situations, universities too can be better off if submission of test scores is optional.

Finally, consider the problem of a firm running a standardised test. The firm wants to maximise its profit, and can choose the noise level of the test. The fourth result implies that increasing noise increases the expected number of test takers, and hence the firm's expected revenue. While it is possible that a more precise test is more costly to run, this suggests that the firm can intentionally make the test imprecise even in the absence of this factor.

The rest of this section discusses the related literature. Section 2 describes the baseline model. Section 3 examines the effect of competition on disclosure. Section 4 discusses how players' utilities are affected by competition and cost of the test. Section 5 analyses the effect of making the test mandatory for receiving the prize. Section 6 analyses the effect of test precision. Section 7 discusses asymmetric equilibria, as well as the case when candidates have heterogeneous costs of taking the test. Section 8 concludes. The appendix contains all proofs.

Related literature. The paper is related to several strands of literature. A number of papers, starting with Spence (1973), look at senders who sig-

[2] See a discussion in Olszewski and Siegel (2016).

nal their types by taking costly actions. In these models, multiple equilibria typically exist, including a pooling equilibrium. Signalling is different from testing modelled in this paper, because a test directly reveals the type to the decision-maker. Hence, in my paper a candidate with a low type cannot mimic a candidate with a high type – thus, taking the test imposes separation, and only candidates who do not take the test can pool. This ensures the existence of a unique and tractable symmetric equilibrium. Within the signalling literature, Feltovich et al. (2002), Alós-Ferrer and Prat (2012), and Daley and Green (2014) examine settings in which a Spence-type signal is complemented by an additional exogenous signal that, like test score in my paper, is correlated with the sender's type. In these papers, however, the additional signal is costless to the sender and is transmitted regardless of the sender's action – only the Spence-type signal is chosen by the sender. On the other hand, in my setup candidates choose whether to send a test score, at a cost. This enables the analysis of the effect of competition and noise on their choice, as well as of the welfare effects of making disclosure mandatory.

Another literature has looked at principals that design mechanisms to induce agents to take a test. In Rosar (2017) a principal faces a single agent and designs an experiment that sends a signal about the agent's type; the agent can choose whether to take part in the experiment. In that setting, the experiment is costless. In Alonso (2017), there are two principals, who compete over a pool of workers that are imperfectly informed about the value of her match with each principal. The principals design recruitment interviews that send imperfect signals about match values. The value of being recruited is endogenous, and is determined by bargaining. Ginzburg (2020) analyses a firm that sets the price of a test that it administers; unlike the decision-maker in this paper, it is interested in maximising agents' expenditure on testing, rather than in allocating a prize optimally. Ben-Porath et al. (2014) study a principal who needs to allocate an object to one of a number of privately informed agents. An agent's type can be verified at a cost. Unlike this paper, the cost is paid by the principal (the paper also discusses an extension in which, in addition, verification also imposes a cost on the agent). The optimal mechanism involves selecting a favoured agent, asking agents about their types, and verifying the types of agents who report higher types than the favoured agent does. In contrast, my paper considers a constrained setting in which there is no communication between candidates and the decision-maker other than signals from the test, and the decision-maker cannot favour some candidates over others.

The paper is also related to models of auctions with endogenous costly entry[3]. While these models do not deal with information revelation, taking the test in my paper can be framed as entering a common-value auction. Two key differences between my model and that literature underlie the different results that my model produces. First, in an auction with entry, an agent who does not enter the auction cannot win the object, and receives zero payoff. Since the expected payoff from entry is decreasing with competition, this implies that greater competition decreases entry. In my model, on the other hand, a candidate can win the prize without taking the test – moreover, Section 5 shows that this setup Pareto-dominates an "auction-like" setup in which taking the test is required for winning the prize. This implies that payoffs from taking the test and from not taking the test both change with the level of competition, although at varying rates. As a result, the probability of taking the test is nonmonotone in the number of candidates. Second, in an auction, the winning bidder and the seller's payoff are determined by agents' endogenously chosen bids. These bids depend on the number of agents – as a result, in an auction the seller gains from restricting entry (Levin and Smith, 1994). In my paper, on the other hand, the winner and the payoff of the decision-maker are determined by candidates' exogenous types. This implies that the decision-maker always gains both from increased competition, and from more candidates taking the test.

Another related literature studies all-pay auctions or contests (see Corchón and Serena, 2018, for an overview), in which agents compete by choosing bids or effort levels. My model can be seen as a contest in which candidates choose between high effort (taking the test) and low effort (not taking the test). However, this "contest" has very a particular structure: the winner is determined not by candidates' endogenously chosen effort, but by their exogenous types[4] (for those who take the test), or randomly (when nobody takes it). This produces very different results. First, there is a unique pure-

[3]See, for example, McAfee and McMillan (1987), Levin and Smith (1994), Moreno and Wooders (2011), Sogo et al. (2016), Cao et al. (2018), and others.

[4]Some contest models incorporate exogenous heterogeneity among contestants by allowing for asymmetric costs of effort (Moldovanu and Sela, 2001; Liu et al., 2018) or commonly known head starts (Siegel, 2009; 2014). Unlike costs of effort, candidates' types in my model do not affect the cost of taking the test, but directly determine candidates' chances of winning in case they do. At the same time, unlike head starts, candidates' types are their private information; furthermore, they only affect the chance of winning for candidates who take the test.

strategy equilibrium, whereas in a standard all-pay auction a pure strategy equilibrium typically does not exist. Second, a contest designer aims to maximise contestants' effort[5]. This makes it optimal for her to commit to give zero reward to a contestant who exerts the minimum level of effort. Here, on the other hand, the decision-maker is not directly interested in candidates taking the test – instead, she aims to correctly select the candidate with the highest type. As a result, as Section 5 shows, the decision-maker prefers not to commit to make test taking mandatory for receiving the prize. Third, as discussed above, the fact that the payoff of a candidate who does not take the test is positive implies that competition has a non-monotone effect on test participation, in contrast to a standard outcome in contest models.

Finally, a number of papers (Janssen and Roy, 2015; Levin et al., 2009; Board, 2009; Forand, 2013) have looked at competing firms who choose whether to reveal their product quality at a cost. In particular, Cheong and Kim (2004) and Guo and Zhao (2009) examine the effect of an increase in the number of firms. In these papers, firms compete by endogenously setting prices; a firm that reveals its product quality is able to charge a higher price, while a firm that does not receives zero profit. Since increased number of firms lowers equilibrium prices, it reduces incentives to reveal information. In contrast, in my paper candidates can only choose whether to take the test – the value of the prize is exogenously fixed. At the same time, a candidate that does not take the test receives a positive payoff that depends on the number of candidates. As a result, competition has a non-monotone effect on information revelation.

2 Model

There are $n > 1$ candidates (male) that are competing for a prize allocated by a decision-maker (female). The value of the prize to each candidate is 1. Each candidate i has a type $x_i \in [0, 1]$, which is his private information. Types are drawn independently from a distribution F with an associated density f. Each candidate can decide to take a test at a cost $c \in (0, 1)$. The test, if taken, perfectly reveals his type to the decision-maker[6].

[5] Typically aggregate effort, or, as in Denter and Sisak (2016), the effort of the winner.

[6] Section 6 considers the case when the test is noisy, while Section 7.2 analyses the case when the cost is a function of a candidate's type.

The decision-maker receives a payoff x_i if she allocates the prize to candidate i. Hence, the decision-maker would like to allocate the prize to a candidate with the highest type. The decision-maker is not allowed to favour some candidates over others – thus, if her posterior belief is such that several candidates have the highest expected type, she randomises between them uniformly.

The timing is as follows. First, nature draws x_i for every candidate i. Each candidate learns his type. Candidates then simultaneously decide whether to take the test. The decision-maker learns the types of candidates who took it. She then chooses a candidate that receives the prize. Most of the paper focuses on symmetric equilibria – thus, the strategy of every candidate i is a function $h : [0,1] \to [0,1]$ which maps the candidate's type to the probability of taking the test. I discuss asymmetric equilibria in Section 7.1.

3 Effect of Competition

3.1 Equilibrium

The decision-maker will allocate the prize to a candidate whose ex post expected type is the highest. Intuitively, this implies that after taking the test, a candidate is more likely to win if his type is higher. Then candidates with higher types should be more willing to take the test. Hence, there should exist a cutoff such that a candidate takes the test if and only if his type is above it. This intuition underlies the following lemma:

Lemma 1. *At every symmetric equilibrium, there exists a threshold $b \in (0,1]$ such that $h(x) = 1$ for all $x > b$, and $\Pr[h(x) > 0 \mid x \leq b] = 0$.*

In words, any symmetric equilibrium is characterised by a threshold b such that candidates whose types are above b always take the test, while candidates whose types are below b never take the test – except for, possibly, some set of types whose mass is zero[7]. This last possibility is irrelevant, because the paper examines what happens in expectation. I will thus focus

[7]The reason for the latter possibility is that, for types between $\mathrm{E}(x \mid x < b)$ and b, the probability of winning the prize after taking the test is constant – but only as long as the mass of candidates with these types who take the test is zero.

on the pure-strategy equilibrium in which each candidate takes the test if and only if his type is above b.

The decision-maker's expected payoff equals the expected type of the candidate whom she gives the prize. At the equilibrium, if a candidate has a type above b, he takes the test and the decision-maker learns his type. This happens with probability $1 - F(b)$. Hence, if at least one candidate takes the test, the decision-maker is able to allocate the prize to the best candidate with certainty. In these situations, I will say that the decision-maker *makes an informed decision*. The probability of this event is $1 - F(b)^n$.

If candidate i does not take the test, the decision-maker's expectation of i's type equals $\mathrm{E}[x \mid x < b]$. Since $b > \mathrm{E}[x \mid x < b]$, a candidate who does not take the test has a lower ex-post expected type than any candidate who does. He can thus only win the prize if nobody else takes the test, which happens with probability $F(b)^{n-1}$. In that case, the decision-maker gives him the prize with probability $\frac{1}{n}$. Thus, if a candidate does not take the test, his expected payoff is $F(b)^{n-1} \frac{1}{n}$. On the other hand, a candidate with type $x_i > b$ takes the test and wins the prize with certainty if every other candidate has a lower type, which happens with probability $F(x_i)^{n-1}$.

Suppose that $c \leq \frac{n-1}{n}$. At $x_i = b$, candidate i must be indifferent between taking and not taking the test, which yields the equation

$$F(b)^{n-1} - c = F(b)^{n-1} \frac{1}{n} \tag{1}$$

On the other hand, if $c > \frac{n-1}{n}$, then the left-hand side of (1) is smaller than the right-hand side for all $b > 0$. Then at the equilibrium no candidate wants to take the test, so $b = 1$. Hence, the equilibrium threshold b is characterised as follows:

Lemma 2. *The unique symmetric equilibrium is given by* $F(b) = \min\left\{\left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}}, 1\right\}$.

3.2 Effect of Competition on Disclosure

How does an increase in n affect information revelation? In particular, we can look at the probability $1 - F(b)^n$ that the decision-maker makes an informed decision. To determine the effect of n on that probability, the following result will be useful:

Lemma 3. *The probability of an informed decision strictly decreases with n if $c < \frac{n-1}{ne}$, and strictly increases with n if $\frac{n-1}{ne} < c < \frac{n-1}{n}$. If $c > \frac{n-1}{n}$, then the probability of an informed decision equals* 0.

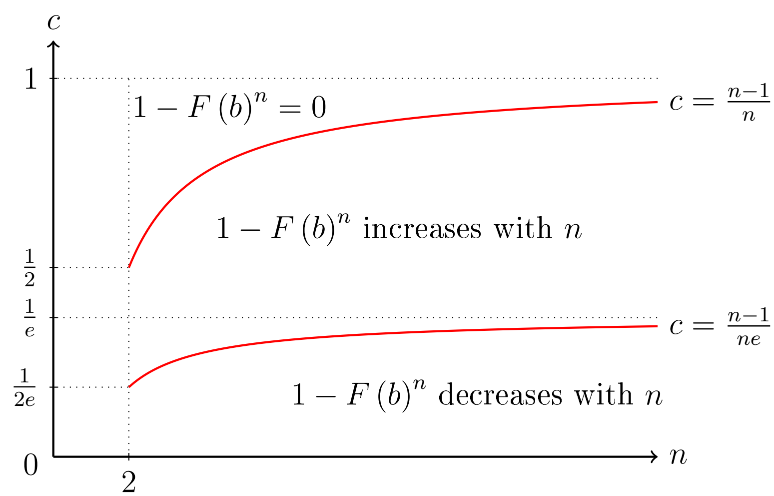

Figure 1: Effect of competition on $F(b)^n$.

Lemma 3 shows that the effect of n on the probability that no candidate takes the test depends on how large c is relative to boundaries that depend on n. Figure 1 illustrates this result.

As we can see from Figure 1, if c is high, the probability that no candidate takes the test is either constant at 1, or decreases with n. If c is low, then that probability increases with n. Overall, the following result characterises the effect of competition on the probability of an informed decision for different levels of c:

Proposition 1. *If $c > \frac{1}{e}$, the probability of an informed decision weakly increases with n. If $c < \frac{1}{2e}$, the probability of an informed decision strictly decreases with n. If $c \in \left(\frac{1}{2e}, \frac{1}{e}\right)$, the probability of an informed decision strictly increases with n if $n < \frac{1}{1-ce}$, and strictly decreases with n if $n > \frac{1}{1-ce}$.*

To see the intuition behind this result, consider the marginal candidate, whose type equals b. For him, increasing n has two effects. First, the expected payoff from taking the test falls, because $F(b)^{n-1}$, the probability that no other candidate has a higher type, decreases. Second, the expected payoff from not taking the test falls as well, because $\frac{1}{n}$, the probability of being randomly selected to receive the prize when nobody takes the test, becomes smaller. But if c is low, then b is low as well. In that case, the impact of increasing the number of candidates on $F(b)^{n-1}$ is relatively large. Thus, the first effect dominates the second, and the marginal candidate becomes less willing to take the test. On the other hand, if c is high (but not so high that nobody takes the test), then $F(b)$ is close to 1. Then increasing n does not change $F(b)^{n-1}$ much, and so the second effect dominates the first. Finally, if c is very large, then no candidate takes the test, and a further increase in n has no impact on information disclosure.

In addition, we can also look at the effect of competition on the decision of a given candidate to take the test. The probability that a candidate takes the test is $1 - F(b)$. The effect of n on this probability is described by the following result:

Lemma 4. *The probability that a candidate takes the test strictly decreases with n if $c < \frac{n-1}{n}e^{-\frac{1}{n}}$, strictly increases with n if $\frac{n-1}{n}e^{-\frac{1}{n}} < c < \frac{n-1}{n}$, and equals 0 if $c > \frac{n-1}{n}$.*

This result is similar in spirit to the previous one: when n increases, candidates become less likely to take the test when its cost is low, and more likely to take the test when its cost is high.

3.3 Asymptotic Results

The results above suggest that when n is very large, the probability of an informed decision is strictly increasing in n if c is above a cutoff that approaches $\frac{1}{e}$, and strictly decreasing in n if c is below that cutoff. Where does this probability converge as $n \to \infty$?

Since $\lim_{n\to\infty} \left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}} = \lim_{n\to\infty} \left(\frac{c}{1-\frac{1}{n}}\right)^{\frac{1}{n-1}} = 1$, Lemma 2 implies that $\lim_{n\to\infty} F(b) = 1$, i.e. the probability that a given candidate takes the test goes to zero. Intuitively, when $n \to \infty$, for every type $x < 1$ there is almost surely a candidate with a higher type, and thus for every $x < 1$ it is optimal not to take the test.

Nevertheless, the probability that the decision-maker makes an informed decision remains distinct from zero and from one even when n is very large, as the following result shows:

Proposition 2. *When n approaches infinity, the probability of an informed decision approaches $1 - c$.*

Intuitively, suppose that $F(b)^n$ approached 1 as $n \to \infty$. Then in the limit the expected type of a candidate who does not take the test would equal

$\mathrm{E}[x]$. But then any candidate with a type above $\mathrm{E}[x]$ would win the prize with probability 1 if he took the test. Hence, he would deviate, contradicting the initial assumption.

4 Utilities and Welfare

4.1 Effect of Competition

Does the decision-maker gain from an increase in competition? Increasing n has two effects on her payoff. On the one hand, since the type of each candidate is an independent draw from the distribution F, increasing the number of draws increases the expected type of the best candidate. In a perfect information setting, this would make the decision-maker better off. However, when c is small, greater competition can also increase the probability that even the best candidate does not take the test. If that happens, the decision-maker will have to allocate the prize at random, which means that the prize will not necessarily go to the best candidate. This creates a negative effect of competition on the decision-maker's utility. Nevertheless, the following will show that the first effect always dominates the second, and hence greater competition is always better for the decision-maker.

Let v be the decision-maker's expected utility. With probability $1-F(b)^n$, the decision-maker makes an informed decision, and allocates the prize to the candidate with the highest type. With probability $F(b)^n$, no candidate takes the test, and the decision-maker allocates the prize to a random candidate. Hence, the decision-maker's expected payoff equals

$$v = [1 - F(b)^n]\, \mathrm{E}[\max\{x\} \mid \max\{x\} > b] + F(b)^n\, \mathrm{E}[x \mid x < b] \qquad (2)$$

If $c \geq \frac{n}{n-1}$ (or, equivalently, if $n \leq \frac{1}{1-c}$), then $b = 1$, and thus $v = \mathrm{E}[x]$, which does not depend on n. Intuitively, when no candidate takes the test, the decision-maker has to allocate the prize at random regardless of n. Otherwise, we have the following result:

Proposition 3. *If $n > \frac{1}{1-c}$, an increase in n strictly increases v. Otherwise, an increase in n has no effect on v.*

Hence, an increase in competition makes the decision-maker strictly better off, unless no candidates take the test (in which case the decision-maker's payoff is not affected by n). Intuitively, while greater competition can reduce the probability that the decision-maker makes an informed decision, this can only occur when the cost of the test is low, as Proposition 1 states. But if c is low, then so is $F(b)$. Hence, each new candidate is likely to take the test, so the positive effect of an increase in the number of draws from F is large, outweighing the negative effect.

4.2 Effect of Cost

The cost c of the test can affect the decision-maker's utility by affecting b, and hence the amount of information that is revealed. If $c > \frac{n-1}{n}$, then $b = 1$, and a further increase in c does not change it. If $c < \frac{n-1}{n}$, then an increase in c increases b, making candidates less likely to reveal their types, and hence reducing the decision-maker's expected payoff.

What about candidates? Since candidates are symmetric, a randomly selected candidate wins the prize with an ex ante probability of $\frac{1}{n}$. With probability $1 - F(b)$ he also takes the test and pays the cost c. Thus, his overall expected utility equals

$$u = \frac{1}{n} - c\left[1 - F(b)\right] \tag{3}$$

This yields the following result:

Proposition 4. *An increase in c decreases u if $c < \left(\frac{n-1}{n}\right)^n$, increases u if $\left(\frac{n-1}{n}\right)^n < c < \frac{n-1}{n}$, and does not affect u if $c > \frac{n-1}{n}$.*

Hence, making the test more costly can increase candidates' expected utility. This happens when the of the test cost is moderately high.

Intuitively, for candidates the test is a deadweight loss – it only serves to reallocate the prize between candidates at a cost to those who take the test. If $c < \frac{n-1}{n}$, an increase in c has two opposite effects. On the one hand, by raising the threshold b, it reduces the expected number of candidates who take the test, thus increasing candidates' utility. On the other hand, those candidates who do take the test have to pay a higher cost. If c is sufficiently small, then the effect of increasing c on b is small as well, so the second effect dominates the first. The opposite is true when c is moderately large. Finally, if $c > \frac{n-1}{n}$, then no candidate takes the test, and increasing c has no effect.

Since the decision-maker always prefers a lower cost unless $c > \frac{n-1}{n}$, this implies that lowering the cost increases welfare if $c < \left(\frac{n-1}{n}\right)^n$, and has no effect on welfare if $c > \frac{n-1}{n}$. When $c \in \left(\left(\frac{n-1}{n}\right)^n, \frac{n-1}{n}\right)$, lowering the cost makes candidates worse off and the decision-maker better off.

5 Optimality of Voluntary Disclosure

Can the decision-maker change the rules of the game to increase her welfare? As before, suppose that the decision-maker cannot favour some candidates over others, and candidates cannot communicate to the decision-maker other than through the test[8]. One option that is available to the decision-maker is to commit to never give the prize to a candidate who does not take the test. For example, universities can require every applicant to take the test for his or her application to be considered. By making such a commitment, the decision-maker reduces the payoff of a candidate who does not take to zero. Is such a commitment optimal?

By an argument similar to the one in Lemma 1, when testing is mandatory for receiving the prize, the equilibrium has a similar threshold form to the one described earlier:

Lemma 5. *Under mandatory disclosure, at every symmetric equilibrium, there exists a threshold $\hat{b} \in [0,1]$ such that $h(x) = 1$ for all $x > \hat{b}$, and* $\Pr\left[h(x) > 0 \mid x < \hat{b}\right] = 0$.

At the threshold, a candidate receives a payoff of $F\left(\hat{b}\right)^{n-1} - c$ if he takes the test. Since he is indifferent between taking and not taking the test, the equilibrium threshold is given as

$$F\left(\hat{b}\right) = c^{\frac{1}{n-1}}$$

It is easy to see that $F\left(\hat{b}\right)$ and $F\left(\hat{b}\right)^n = c^{\frac{n}{n-1}}$ are strictly increasing in n for any $c \in (0,1)$. Thus, in contrast to the case without commitment, under mandatory disclosure an increase in competition has a monotone effect on the probability that a candidate takes the test, and on the probability of an informed decision.

[8] If if costless communication between the decision-maker and candidates is possible, then one optimal mechanism available to the decision-maker would be to ask all candidates to report their types, and then ask the candidate with the highest reported type to take the test, promising to give him the prize if and only if the test confirms the type.

Since the payoff of a candidate who does not take the test is zero under mandatory testing, and positive without it, mandatory testing increases the incentive of candidates to take the test. Hence, $F\left(\hat{b}\right) < F(b)$, so a given candidate is more likely to take the test when disclosure is mandatory.

In the baseline model – that is, under voluntary testing – the expected payoff of a randomly selected candidate is given by (3). Under mandatory testing, it equals $\frac{1}{n} - c\left[1 - F\left(\hat{b}\right)\right]$. Since $F\left(\hat{b}\right) < F(b)$, mandatory testing reduces the expected payoff of a random candidate. The intuition is the same as in Section 4.2: the cost of the test is a deadweight loss, and making candidates more likely to take it reduces their utility.

What about the decision-maker's payoff? Without commitment, it is given by (2). On the other hand, when the test is mandatory, the decision-maker's payoff equals the type of the best candidate if the best candidate takes the test, which happens with probability $1 - F\left(\hat{b}\right)^n$. If the best candidate does not take the test, the decision-maker's payoff is zero. Let $\hat{v}$ be the decision-maker's payoff under mandatory testing. It equals

$$\hat{v} = \left[1 - F\left(\hat{b}\right)^n\right] \mathrm{E}\left[\max\{x\} \mid \max\{x\} > \hat{b}\right] \tag{4}$$

Is this larger than v – that is, is the decision-maker better off under mandatory testing than under voluntary testing? In general, this depends on the shape of F. However, we can show that for any F, there exists a cutoff such that whenever n is larger than that cutoff, voluntary testing is better for the decision-maker, and hence (since candidates always prefer voluntary testing) is better for all players. Formally, we have the following result:

Proposition 5. *For all $c > 0$, and any F, there exists $\bar{n}$ such that voluntary testing strictly Pareto-dominates mandatory testing for all $n \geq \bar{n}$.*

Intuitively, a commitment to only select the winner from candidates who took the test has two effects. On the one hand, since candidates become more likely to take the test, mandatory testing increases the probability that the best candidate reveals his type. Hence, the decision-maker is more likely to make an informed decision, which increases her expected payoff. On the other hand, in the event that nobody takes the test, this commitment leaves the decision-maker unable to allocate the prize, reducing her payoff.

However, if n is large, a candidate who does not take take the test is unlikely to win the prize even without commitment. Thus, his expected payoff $F(b)^{n-1}\frac{1}{n} = \frac{c}{n-1}$ is close to zero, that is, close to the payoff of a candidate who does not take the test under mandatory testing. Hence, the first effect of commitment is small when n is sufficiently large. On the other hand, the second, negative effect of commitment remains large: even as $n \to \infty$, the probability that no candidate takes the test remains strictly positive, as Proposition 2 has established. Thus, the overall gain from the commitment is negative when n is sufficiently large.

The size of the cutoff above which voluntary testing Pareto-dominates mandatory testing depends on the shape of F. In the benchmark case when F is uniform, the cutoff is fairly low, as the next result shows:

Corollary 1. *If F is uniform, then voluntary testing strictly Pareto-dominates mandatory testing if and only if $n \geq 3$.*

Recall that by assumption, $n \geq 2$. Hence, when the distribution of types is uniform, keeping the test voluntary is better whenever the number of candidates is larger than the minimum.

6 Noisy Test

So far we have assumed that the test perfectly reveals the candidate's type. Suppose, however, that the test is imperfect. For example, a student applying for a scholarship can take a standardised test, but the test can be a noisy signal of his ability. Similarly, an election candidate may invest in campaigning to inform voters about his competence, but media reporting can add noise to her message.

Specifically, suppose that rather than revealing candidate i's type x_i, the test reveals a test score $s_i = x_i + z_i$, where z_i is noise. When a candidate decides to take the test, he knows his type x_i, but not the realisation of the noise. After candidate i takes the test, nature draws z_i from some distribution G with smooth logconcave density g that has full support on $\mathbb{R}$. Different candidates' noise realisations are drawn independently.

We can show that the decision-maker prefers a candidate with a higher test score. Specifically, the following lemma proves that when g is logconcave, the distribution of $s \mid x$ satisfies the monotone likelihood ratio condition (Milgrom, 1981), and hence a higher score is a more favourable signal about the candidate's type:

Lemma 6. *If candidates i and j take the test, and $s_i > s_j$, then the decision-maker has a higher expected utility from giving the prize to candidate i then to candidate j.*

Then we can show that, as in the baseline model, the equilibrium symmetric strategy is of a threshold form:

Lemma 7. *At every symmetric equilibrium, for any G, there exists a threshold $b \in [0, 1]$ such that $h(x) = 1$ for all $x > b$, and $h(x) = 0$ for all $x < b$.*

If $b = 1$, nobody takes the test. Consider instead the case when $b < 1$. If a candidate does not take the test, the decision-maker knows that his type is below b; while if he takes the test, the decision-maker knows that his type is above b. Thus, a candidate who does not take the test can never win over a candidate who takes it. If two or more candidates take the test, Lemma 6 ensures that the decision-maker will give the prize to the candidate with the highest test score.

By varying G, we can vary how noisy the test is. In particular, consider a family of distributions of the form $G_\lambda(z) \equiv G(\lambda z)$ for different values of $\lambda \in (0, +\infty)$. Increasing (decreasing) λ makes the noise more (less) concentrated around zero, and hence makes the test less (more) noisy[9]. How does a change in λ affect the equilibrium?

It turns out that candidates are more likely to take the test when the test is noisier. Formally, we have the following result:

Proposition 6. *For any F and G, and any values of n and c, decreasing λ decreases b.*

To see the intuition behind this result, take the case when there is no noise, and consider candidate i whose type equals b. If i takes the test, he wins the prize if and only if all other candidates have types below b, and don't take the test. Now make the test noisy, and suppose that b were held constant. If i takes the test, he still wins over anyone who does not take it, since not taking the test reveals that one's type is below b. But now i can also win over a competitor who has a higher type, if that competitor takes the test and receives a lower score than i. Hence, i's chance of winning the prize after taking the test increases. Thus, i becomes more willing to take the test, and the threshold b decreases.

[9] For example, if G is a normal distribution, then λ is proportional to the inverse of its variance.

Thus, candidates are more likely to take the test when the test is less informative. One could think that as $\lambda \to 0$ (i.e. as the test becomes "infinitely noisy"), the test ceases to carry any information, and candidates should be unwilling to take it. However, this only happens in the limit – for any positive λ, a higher test score still indicates that the candidate has a higher type. Hence, a $\lambda \to 0$, along the sequence candidates with sufficiently high types prefer to take it.

Since each candidate is more likely to take the test when it is noisier, greater noise increases aggregate expenditure on the test, as the following result states:

Corollary 2. *Expected total spending on the test increases if λ decreases.*

Thus, if the test is run by a monopolist (as is, for example, the GRE) and its price is fixed, the monopolist has an incentive to make the test less precise.

7 Extensions

7.1 Asymmetric Equilibria

The previous analysis has focused on equilibria that are symmetric in the sense that all candidates have the same cutoff b that determines whether a candidate takes the test. This section will discuss equilibria that can emerge if candidates are allowed to have asymmetric thresholds.

Consider an strategy profile given by T different thresholds b_t, with $b_1 < b_2 ... < b_T$. For each $t \in \{1, 2, ...T\}$, let k_t be the number of candidates that act according to threshold b_t (that is, take the test if and only if their type is above b_t), with $\sum_{t=1}^{T} k_t = n$. In general, a large set of such equilibria exists. However, we can show that all of them must satisfy the following condition:

Proposition 7. *Any equilibrium with asymmetric thresholds $\{b_t\}_{t=1}^{T}$ has $T \leq 3$. Furthermore, if $T = 3$, then $k_1 = 1$.*

Hence, while multiple asymmetric equilibria are generally possible, Proposition 7 shows that in an equilibrium with asymmetric thresholds, candidates follow either two or three different strategies. Specifically, any such equilibrium has the following shape: some candidates have a high threshold b_3, some other candidates have medium threshold b_2, and at most one candidate has a low threshold b_1.

7.2 Heterogeneous Costs

The basic model assumed that the cost of taking the test is the same for all candidates. This section will show that the basic results of the paper also hold when the cost is allowed to depend on candidate's type. For a specific application, consider again the example of students competing for a scholarship, and suppose that the effort required to take the test is higher when the applicant's ability is lower.

Formally, suppose that for a candidate with type x, the cost of taking the test is $c(x) \in [\underline{c}, \overline{c}]$, where $0 < \underline{c} < \overline{c} < 1$. Let $c(x)$ be continuously differentiable and strictly decreasing in type. Thus, the cost is lower for candidates with higher types.

As before, we can show that the symmetric equilibrium is of threshold form:

Lemma 8. *At every symmetric equilibrium, there exists a threshold* $b \in (0, 1]$ *such that* $h(x) = 1$ *for all* $x > b$*, and* $h(x) = 0$ *for all* $x < b$*.*

If the test is not mandatory, a candidate with type b receives an expected payoff of $F(b)^{n-1} - c(b)$ if he takes the test, and $F(b)^{n-1} \frac{1}{n}$ if he does not. If $b < 1$, the equilibrium is given by the indifference condition

$$F(b) = \left[\frac{c(b)\, n}{n-1}\right]^{\frac{1}{n-1}} \tag{5}$$

This is an equilibrium whenever $c(1) = \underline{c} \leq \frac{n-1}{n}$. If $\underline{c} > \frac{n-1}{n}$, then $F(b)^{n-1} - c(b) < F(b)^{n-1} \frac{1}{n}$ for all b, and hence the equilibrium is given by $b = 1$. Note that the left-hand side of (5) is increasing in b, and the right-hand side is decreasing in b – hence, the equilibrium is unique.

The effect of increasing the number of candidates on information revelation is then captured by the following result:

Proposition 8. *Suppose that* $\underline{c} < \frac{n-1}{n}$*. At the equilibrium,* b *is increasing in* n *if and only if* $c(b) < \frac{n-1}{n} e^{-\frac{1}{n}}$*.*

To interpret this result, consider the case when $\overline{c}$ is low, and hence the test is cheap (relative to the value of the prize) even for candidates with low type. In particular, suppose that $\overline{c} < \frac{n-1}{n} e^{-\frac{1}{n}}$. Since $c(b) \leq \overline{c}$, the condition in the proposition is satisfied. Then $\frac{db}{dn} > 0$, and hence an increase in competition reduces the probability that a given candidate takes the test. On the other hand, suppose that $\underline{c}$ is high, and hence the test is costly for all candidates. If $\underline{c} > \frac{n-1}{n}e^{-\frac{1}{n}}$ then, since $c(b) \geq \underline{c}$, we have $\frac{db}{dn} \leq 0$, and thus an increase in competition increases the probability that a given candidate takes the test.

Hence, the basic logic of the results from Section 3 – that competition increases (reduces) information disclosure when the cost of the test is relatively high (low) holds when the cost of the test is heterogeneous.

What if the decision-maker commits not to give the prize to anyone who does not take the test? As before, the expected payoff of a candidate who does not take it equals zero, and the equilibrium threshold $\hat{b}$ is given by

$$F\left(\hat{b}\right) = c\left(\hat{b}\right)^{\frac{1}{n-1}}$$

We can verify that, as before, $b > \hat{b}$. To see that, define $z(x) \equiv \frac{F(x)^{n-1}}{c(x)}$. Then $z(b) = \frac{n}{n-1} > 1 = z\left(\hat{b}\right)$. Since $z(\cdot)$ is an increasing function, this implies that $b > \hat{b}$.

Given this, we can show that the decision-maker's expected gain from making the test mandatory is negative when competition is sufficiently strong.

Proposition 9. *For all $c(\cdot)$, and any F, there exists $\bar{n}$ such that $v > \hat{v}$ for all $n \geq \bar{n}$.*

Hence, the result from Section 5 holds in a more general setting in which the cost of the test depends on the candidate's type.

8 Conclusions

In many situations, agents cannot communicate directly to a principal, but can reveal information through a costly exogenous procedure. This paper developed a model of competing agents who choose whether to take a test that sends a signal about their types to a decision-maker.

Several results were derived. First, greater competition reduces information disclosure if the cost of disclosure is low, but increases it if the cost is high. Second, the decision-maker always gains from greater competition; at the same time, an increase in the cost of the test can make candidates better off. Third, mandatory disclosure is strictly Pareto-dominated by voluntary disclosure unless competition is low. Third, greater test noise makes candidates more likely to take the test.

An important feature of the model was the fact that the candidates are competing for a single prize. In some settings – such as competition for political office, a company seeking to fill a single vacancy, or a university that needs to allocate a single scholarship – this assumption is naturally satisfied. In other settings, the number of prizes can be greater than one. Future work can extend the analysis to account for this possibility.

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Appendix

Proof of Lemma 1. Suppose $h(x) = 1, \forall x \in [0,1]$. Then the payoff of a candidate with type x who takes the test equals $F(x)^{n-1} - c$. This is negative for sufficiently small x, so these candidates prefer to deviate. Thus, at equilibrium, $h(x) < 1$ for some $x \in [0,1]$.

Let m be the expected type of a candidate who does not take the test. Denote by $\pi(x)$ the probability that a candidate with type x wins the prize. Then a candidate with type x weakly prefers to take the test whenever $\pi(x) - c \geq \pi(m)$, and weakly prefers not taking it whenever $\pi(x) - c \leq \pi(m)$.

Note that $\pi(\cdot)$ is nondecreasing. Then whenever $x \leq m$, we have $\pi(x) - c < \pi(m)$, so any candidate whose type is below m strictly prefers not to take the test. Thus, $h(x) = 0$ for all $x \leq m$. For a candidate with type $x > m$, $\pi(x)$ is the probability that every other candidate either does not take the test, or takes it and has a type below x. Hence, for all $x > m$,

$$\pi(x) = \left(\int_0^1 [1 - h(u)] \, dF(u) + \int_0^x h(u) \, dF(u) \right)^{n-1} \tag{6}$$

Let $b \equiv \sup \{x \mid h(x) < 1\}$. Then we must have $\pi(b) - c \leq \pi(m)$. Now take some $\hat{x} < b$, and suppose that $h(\hat{x}) > 0$. Then we must have $\pi(\hat{x}) - c \geq \pi(m)$. This implies that $\pi(b) \leq \pi(\hat{x})$. But $\pi(\cdot)$ is nondecreasing, so $\pi(b) = \pi(\hat{x})$. Substituting the expressions for $\pi(b)$ and $\pi(\hat{x})$ from (6) and simplifying, we get $\int_0^b h(u) \, dF(u) = \int_0^{\hat{x}} h(u) \, dF(u)$. Therefore, $\int_{\hat{x}}^b h(u) \, dF(u) = 0$. This should hold for all $\hat{x} < b$ such that $h(\hat{x}) > 0$.

Hence, there exists $b \in [0,1]$ such that $h(x) = 1$ for all $x > b$, and $h(x) = 0$ for almost all $x < b$. By the reasoning given in the beginning of the proof, $b > 0$. □

Proof of Lemma 2. If $c \leq \frac{n-1}{n}$, then (1) implies that $F(b) = \left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}} \leq 1$. If $c > \frac{n-1}{n}$, then $b = 1$, so $F(b) = 1$. Uniqueness follows from the fact that the right-hand side of the expression in the lemma is constant in b. □

Proof of Lemma 3. If $c > \frac{n-1}{n}$, then $F(b)^n = 1$ by Lemma 2. If $c < \frac{n-1}{n}$, then $F(b)^n = \left(\frac{cn}{n-1}\right)^{\frac{n}{n-1}} = e^{\frac{n}{n-1}\ln\frac{cn}{n-1}}$. Approximating n by a continuous variable and differentiating yields

$$\begin{aligned}\frac{d}{dn}F(b)^n &= F(b)^n\left[-\frac{1}{(n-1)^2}\ln\frac{cn}{n-1} + \frac{n}{n-1}\frac{n-1}{cn}\frac{-c}{(n-1)^2}\right]\\ &= -\frac{F(b)^n}{(n-1)^2}\left[\ln\left(\frac{cn}{n-1}\right)+1\right]\end{aligned}$$

which is positive whenever $\ln\left(\frac{cn}{n-1}\right) < -1$, i.e. whenever $c < \frac{n-1}{ne}$. □

Proof of Proposition 1. If $c > \frac{1}{e}$, then $c > \frac{n-1}{ne}, \forall n$, hence by Lemma 3, $F(b)^n$ is weakly decreasing in n. If $c < \frac{1}{2e}$, then $c < \frac{n-1}{ne}, \forall n \geq 2$, hence by Lemma 3, $F(b)^n$ is strictly increasing in n. If $c \in \left(\frac{1}{2e}, \frac{1}{e}\right)$, then by 3, $F(b)^n$ is decreasing in n if and only if $c > \frac{n-1}{ne}$, that is, if and only if $n < \frac{1}{1-ce}$. □

Proof of Lemma 4. If $c > \frac{n-1}{n}$, then $F(b) = 1$ by Lemma 2. If $c < \frac{n-1}{n}$, then $F(b) = \left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}} = e^{\frac{1}{n-1}\ln\frac{cn}{n-1}}$. Approximating n by a continuous variable and differentiating yields

$$\begin{aligned}\frac{d}{dn}F(b) &= F(b)\left[-\frac{1}{(n-1)^2}\ln\frac{cn}{n-1} + \frac{1}{n-1}\frac{n-1}{cn}\frac{-c}{(n-1)^2}\right]\\ &= -\frac{F(b)}{(n-1)^2}\left[\ln\left(\frac{cn}{n-1}\right)+\frac{1}{n}\right]\end{aligned}$$

which is positive whenever $\ln\left(\frac{cn}{n-1}\right) < -\frac{1}{n}$, i.e. whenever $c < \frac{n-1}{n}e^{-\frac{1}{n}}$. □

Proof of Proposition 2. $\lim_{n\to\infty}\left(\frac{cn}{n-1}\right)^{\frac{n}{n-1}} = \lim_{n\to\infty}\left(\frac{c}{1-\frac{1}{n}}\right)^{\frac{1}{1-\frac{1}{n}}} = c$. Since $F(b)^n = \min\left\{\left(\frac{cn}{n-1}\right)^{\frac{n}{n-1}}, 1\right\}$, $\lim_{n\to\infty} F(b)^n = c$. □

Proof of Proposition 3. We have

$$\mathrm{E}\left[\max\{x\} \mid \max\{x\} > b\right] = \frac{\int_b^1 xd\left[F(x)^n\right]}{1-F(b)^n}$$

which uses the fact that the cdf of $\max\{x\}$ is $F(x)^n$. At the same time, we have

$$\mathrm{E}\left[x \mid x < b\right] = \frac{\int_0^b xd\left[F(x)\right]}{F(b)}$$

Substituting these into (2), simplifying, and integrating by parts, we obtain

$$\begin{aligned}
v =& \int_b^1 xd\left[F(x)^n\right] + F(b)^{n-1}\int_0^b xd\left[F(x)\right] \\
=& \, xF(x)^n\big|_b^1 - \int_b^1 F(x)^n\,dx + F(b)^{n-1}\left[xF(x)\big|_0^b - \int_0^b F(x)\,dx\right] \\
=& 1 - bF(b)^n - \int_b^1 F(x)^n\,dx + bF(b)^n - F(b)^{n-1}\int_0^b F(x)\,dx \\
=& 1 - \int_b^1 F(x)^n\,dx - F(b)^{n-1}\int_0^b F(x)\,dx
\end{aligned}$$

Substituting $F(b)^{n-1} = \frac{cn}{n-1}$ and differentiating with respect to n, approximating n by a continuous variable, yields

$$\begin{aligned}
\frac{dv}{dn} =& F(b)^n\frac{db}{dn} - \int_b^1 F(x)^n \ln\left[F(x)\right]dx + \frac{c}{(n-1)^2}\int_0^b F(x)\,dx - F(b)^n\frac{db}{dn} \\
=& -\int_b^1 F(x)^n \ln\left[F(x)\right]dx + \frac{c}{(n-1)^2}\int_0^b F(x)\,dx > 0 \\
>& 0
\end{aligned}$$

□

Proof of Proposition 4. If $c \geq \frac{n-1}{n}$, then $F(b) = 1$, and a further increase in c does not affect u. Suppose that $c < \frac{n-1}{n}$. Substituting the expression for $F(b)$ from Lemma 2 into (3) yields

$$u = \frac{1}{n} - c\left[1 - \left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}}\right] = \frac{1}{n} - c + c^{\frac{n}{n-1}}\left(\frac{n}{n-1}\right)^{\frac{1}{n-1}}$$

Differentiating with respect to c, we obtain

$$\frac{du}{dc} = -1 + \frac{n}{n-1}c^{\frac{1}{n-1}}\left(\frac{n}{n-1}\right)^{\frac{1}{n-1}} = -1 + c^{\frac{1}{n-1}}\left(\frac{n}{n-1}\right)^{\frac{n}{n-1}}$$

which is positive if and only if $c > \left(\frac{n-1}{n}\right)^n$. □

Proof of Lemma 5. Identical to the proof of Lemma 1 with m and $\pi(m)$ replaced by zero. □

Proof of Proposition 5. We have $\hat{v} = \int_{\hat{b}}^1 xd\left[F(x^n)\right]$. At the same time, from the proof of Proposition 3, we have $v = \int_b^1 xd\left[F(x)^n\right] + F(b)^{n-1}\int_0^b xd\left[F(x)\right]$. Subtracting and integrating by parts yields:

$$\begin{aligned}
\hat{v} - v = &\ \int_{\hat{b}}^b xd\left[F(x)^n\right] - F(b)^{n-1}\int_0^b xd\left[F(x)\right] \\
= &\ bF(b)^n - \hat{b}F\left(\hat{b}\right)^n - \int_{\hat{b}}^b F(x)^n\,dx - F(b)^{n-1}\left[bF(b) - \int_0^b F(x)\,dx\right] \\
= &\ -\hat{b}F\left(\hat{b}\right)^n - \int_{\hat{b}}^b F(x)^n\,dx + F(b)^{n-1}\int_0^b F(x)\,dx
\end{aligned}$$

Note that $\lim_{n\to\infty} F(b) = \lim_{n\to\infty}\left(\frac{c}{1-\frac{1}{n}}\right)^{\frac{1}{n-1}} = 1$, and $\lim_{n\to\infty} F\left(\hat{b}\right) = \lim_{n\to\infty} c^{\frac{1}{n-1}} = 1$. Hence, $\lim_{n\to\infty}\hat{b} = \lim_{n\to\infty} b = 1$, and thus $\lim_{n\to\infty}\int_{\hat{b}}^b F(x)^n\,dx = 0$. At the same time, $\lim_{n\to\infty} F\left(\hat{b}\right)^n = \lim_{n\to\infty} c^{\frac{n}{n-1}} = c$, and $\lim_{n\to\infty} F(b)^{n-1} = \lim_{n\to\infty}\frac{cn}{n-1} = c$. Thus,

$$\lim_{n\to\infty}\left[\hat{v} - v\right] = \lim_{n\to\infty}\left[-c + c\int_0^b F(x)\,dx\right] = -c\left[1 - \int_0^1 F(x)\,dx\right] < 0$$

which implies that voluntary testing is strictly better for the decision-maker when n is large enough. Together with the fact that candidates strictly prefer voluntary testing, this implies strict Pareto-dominance. □

Proof of Corollary 1. If $F(x) = x$, then, using the transformation of $\hat{v} - v$ done in the previous proof, we have

$$\begin{aligned}
\hat{v} - v =& -\hat{b}^{n+1} - \int_{\hat{b}}^{b} x^n dx + b^{n-1} \int_0^b x dx \\
=& -\hat{b}^{n+1} - \frac{1}{n+1} b^{n+1} + \frac{1}{n+1} \hat{b}^{n+1} + \frac{1}{2} b^{n+1} \\
=& \frac{n-1}{2(n+1)} b^{n+1} - \frac{n}{n+1} \hat{b}^{n+1}
\end{aligned}$$

Substituting $b = F(b) = \left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}}$ and $\hat{b} = F\left(\hat{b}\right) = c^{\frac{1}{n-1}}$ into the above expression yields

$$\hat{v} - v = \frac{n-1}{2(n+1)} \left(\frac{cn}{n-1}\right)^{\frac{n+1}{n-1}} - \frac{n}{n+1} c^{\frac{n+1}{n-1}} = c^{\frac{n+1}{n-1}} \frac{1}{n+1} \left[\frac{n-1}{2} \left(\frac{n}{n-1}\right)^{\frac{n+1}{n-1}} - n\right]$$

which has the same sign as $\frac{n-1}{2} \left(\frac{n}{n-1}\right)^{\frac{n+1}{n-1}} - n$, or as $\frac{1}{2} \left(\frac{n}{n-1}\right)^{\frac{2}{n-1}} - 1$. This is negative for all $n \geq 3$, and positive for $n = 2$. $\square$

Proof of Lemma 6. To prove the lemma, it is sufficient to show that $s_i > s_j$ implies that the conditional distribution of $x \mid s_i$ first-order stochastically dominates the conditional distribution of $x \mid s_j$. Milgrom (1981) shows that this holds if and only if the likelihood ratio $\frac{k(s|x)}{k(s|\bar{x})}$ is increasing in s for any x, $\bar{x}$ such that $x > \bar{x}$, where $k(s \mid x)$ is a conditional distribution of s given x. Given the additive structure of noise, $k(s \mid x) = g(s - x)$, and the above monotone likelihood ratio property is equivalent to the statement that $\frac{d}{ds} \left(\frac{g(s-x)}{g(s-\bar{x})}\right) > 0$, which is equivalent to

$$\frac{d}{ds} \left(\ln \left[\frac{g(s-x)}{g(s-\bar{x})}\right]\right) = \frac{d}{ds} \left[\ln g(s-x) - \ln g(s-\bar{x})\right] > 0$$

This holds if and only if $\frac{d}{ds} [\ln g(s-x)] > \frac{d}{ds} [\ln g(s-\bar{x})]$, $\forall x > \bar{x}$, i.e. if and only if $\frac{d}{ds} \ln g(\cdot)$ is decreasing, and hence if and only if $g(\cdot)$ is logconcave. $\square$

Proof of Lemma 7. Let $\tilde{\pi}$ be the probability that a candidate wins the prize after deciding not to take the test. Let $\pi(x)$ be the ex ante probability that a candidate whose type is x wins the prize. Then a candidate with type

x takes the test if $\pi(x) - c > \tilde{\pi}$, and does not take the test if $\pi(x) - c < \tilde{\pi}$. Note that $\tilde{\pi}$ does not depend on x. Thus, to show that h is of a threshold form as stated in the lemma, it is sufficient to demonstrate that $\pi(x)$ is strictly increasing in x.

At the equilibrium, let $\tilde{s}$ be the score such that the expected type conditional on having score $\tilde{s}$ equals the expected type of a candidate who did not take the test. Lemma 6 implies that $\tilde{s}$ is unique.

Suppose that candidate i with type x takes the test and receives a score $s = x + z$. Now take a competitor with type $\hat{x}$. If $s < \tilde{s}$ (and hence $z < \tilde{s} - x$), i has a higher expected type than this competitor if and only if the latter takes the test and receives a score $\hat{x} + \hat{z}$ that is less than $x + z$. Given the competitor's type $\hat{x}$, this happens with probability $h(\hat{x})\, G(x + z - \hat{x})$. Thus, the probability that i wins over a given competitor equals

$$\int_0^1 f(\hat{x})\, h(\hat{x})\, G(x + z - \hat{x})\, d\hat{x} \equiv L(x, z)$$

Hence, i's probability of winning the prize given z such that $z < \tilde{s} - x$ equals $L(x, z)^{n-1}$.

On the other hand, if $s > \tilde{s}$ (and hence $z > \tilde{s} - x$), i has a higher expected type than his competitor if an only if the latter either (i) takes the test and receives a score $\hat{x} + \hat{z}$ that is less than $x + z$, or (ii) does not take the test. Given the competitor's type $\hat{x}$, the former event happens with probability $h(\hat{x})\, G(x + z - \hat{x})$, while the latter event happens with probability $1 - h(\hat{x})$. Thus, the probability that i wins over a given competitor equals

$$\int_0^1 f(\hat{x}) \left[1 - h(\hat{x}) + h(\hat{x})\, G(x + z - \hat{x})\right] d\hat{x} = K + L(x, z)$$

where $K \equiv \int_0^1 f(\hat{x}) \left[1 - h(\hat{x})\right] d\hat{x} \geq 0$. Hence, i's probability of winning the prize given z such that $z > \tilde{s} - x$ equals $\left[K + L(x, z)\right]^{n-1}$

Then i's overall probability of winning the prize is

$$\pi(x) = \int_{-\infty}^{\tilde{s} - x} g(z) \left[L(x, z)\right]^{n-1} dz + \int_{\tilde{s} - x}^{+\infty} g(z) \left[K + L(x, z)\right]^{n-1} dz$$

Differentiating it with respect to x yields

$$\begin{aligned}\frac{d\pi(x)}{dx} =& -g(\tilde{s}-x)\left[L(x,\tilde{s}-x)\right]^{n-1} + \int_{-\infty}^{\tilde{s}-x} g(z)\frac{d}{dx}\left(\left[L(x,z)\right]^{n-1}\right)dz \\ &+g(\tilde{s}-x)\left[K+L(x,\tilde{s}-x)\right]^{n-1} + \int_{\tilde{s}-x}^{+\infty} g(z)\frac{d}{dx}\left(\left[K+L(x,z)\right]^{n-1}\right)dz \\ &\geq g(\tilde{s}-x)\left(\left[K+L(x,\tilde{s}-x)\right]^{n-1} - \left[L(x,\tilde{s}-x)\right]^{n-1}\right)\end{aligned}$$

where the inequality is due to the fact that $L(x,z)$ is increasing in x while K does not change with x, which implies that $\frac{d}{dx}\left(\left[L(x,z)\right]^{n-1}\right) \geq 0$ and $\frac{d}{dx}\left(\left[K+L(x,z)\right]^{n-1}\right) \geq 0$. At the same time, unless $h(\cdot)$ is zero everywhere or almost everywhere, $L(x,z)$ is strictly increasing in x, which implies that the inequality is strict, and hence

$$\frac{d\pi(x)}{dx} > g(\tilde{s}-x)\left(\left[K+L(x,\tilde{s}-x)\right]^{n-1} - \left[L(x,\tilde{s}-x)\right]^{n-1}\right) \geq 0$$

where the second inequality comes from the fact that $K \geq 0$. On the other hand, if $h(\cdot)$ is zero everywhere or almost everywhere, then $K = \int_0^1 f(\hat{x})\,d\hat{x} > 0$, and we have

$$\frac{d\pi(x)}{dx} \geq g(\tilde{s}-x)\left(\left[K+L(x,\tilde{s}-x)\right]^{n-1} - \left[L(x,\tilde{s}-x)\right]^{n-1}\right) > 0$$

Hence, in every symmetric equilibrium we must have $\frac{d\pi(x)}{dx} > 0, \forall x$. Therefore, any symmetric equilibrium must be of a threshold form. □

Proof of Proposition 6. When $b = 1$, no candidate takes the test, and a marginal change in the level of noise does not change b. Now consider the case when $b < 1$. First, we will derive expression that defines b. Take a candidate i, and suppose his type equals b. Suppose i takes the test, which produces noise z, so the decision-maker observes $b+z$. Then i wins the prize if each of his competitors either (i) does not take the test, or (ii) takes the test and receives a score below $b+z$. For a given competitor, the probability of the former event is $\Pr(x<b)$. The latter event happens if the competitor has a type $x > b$ and, after taking the test, receives a score $x+\hat{z} < b+z$. The probability of this is $\Pr(x > b \wedge \hat{z} < b+z-x)$. Since there are $n-1$ competitors, the probability that i wins the prize equals

$$\left[\Pr(x<b) + \Pr(x>b \wedge \hat{z} < b+z-x)\right]^{n-1} = \left[F(b) + \int_b^1 f(x)\,G(b+z-x)\,dx\right]^{n-1}$$

Then ex ante, before z is realised, i's probability of winning the prize after taking the test equals

$$\int_{-\infty}^{+\infty} g(z)\left[F(b)+\int_b^1 f(x)\,G(b+z-x)\,dx\right]^{n-1} dz$$

On the other hand, if i does not take the test, he wins the prize with probability $F(b)^{n-1}\frac{1}{n}$. Thus, when $b<1$, the equilibrium is determined by the indifference condition

$$\int_{-\infty}^{+\infty} g(z)\left[F(b)+\int_b^1 f(x)\,G(b+z-x)\,dx\right]^{n-1} dz - c = F(b)^{n-1}\frac{1}{n}$$

To determine the effect of λ, take a distribution $G_\lambda(z)=G(\lambda z)$, and note that its pdf equals $\lambda g(\lambda z)$. The equilibrium is then given by the condition

$$\int_{-\infty}^{+\infty} \lambda g(\lambda z)\left[F(b)+\int_b^1 f(x)\,G(\lambda[b+z-x])\,dx\right]^{n-1} dz - c = F(b)^{n-1}\frac{1}{n}$$

This can be written as:

$$\int_{-\infty}^{+\infty} \lambda g(\lambda z)\,[M(z)]^{n-1}\,dz - c = F(b)^{n-1}\frac{1}{n} \tag{7}$$

where

$$M(z) \equiv F(b)+\int_b^1 f(x)\,G(\lambda[b+z-x])\,dx$$

Since (7) should hold for any λ, we can differentiate it with respect to λ to obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty} g(\lambda z)\,[M(z)]^{n-1}\,dz + \int_{-\infty}^{\infty} \lambda z g'(\lambda z)\,[M(z)]^{n-1}\,dz \\
+&\int_{-\infty}^{+\infty} \lambda g(\lambda z)\,(n-1)\,[M(z)]^{n-2}\,\frac{dM(z)}{d\lambda}dz \\
=&\frac{n-1}{n}F(b)^{n-2}f(b)\,\frac{db}{d\lambda}
\end{aligned} \tag{8}$$

Note that we can write

$$\int_{-\infty}^{+\infty} \lambda z g'(\lambda z) [M(z)]^{n-1} dz$$

$$= zg(\lambda z) [M(z)]^{n-1}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} g(\lambda z) \left[\frac{\partial}{\partial z}\left(z [M(z)]^{n-1}\right)\right] dz$$

$$= -\int_{-\infty}^{+\infty} g(\lambda z) \left[\frac{\partial}{\partial z}\left(z [M(z)]^{n-1}\right)\right] dz$$

$$= -\int_{-\infty}^{+\infty} g(\lambda z) [M(z)]^{n-1} dz - \int_{-\infty}^{+\infty} g(\lambda z) z(n-1) [M(z)]^{n-2} \frac{\partial M(z)}{\partial z} dz$$

where the first equality is a result of differentiating by parts; the second comes from the fact that for a logconcave g, $\lim\limits_{z\to-\infty} zg(\lambda z) = \lim\limits_{z\to+\infty} zg(\lambda z) = 0$,[10] while $M(z)$ is bounded between zero and one; and the third comes from straightforward differentiation. We can substitute this into (8), which then becomes

$$-\int_{-\infty}^{+\infty} g(\lambda z) z(n-1) [M(z)]^{n-2} \frac{\partial M(z)}{\partial z} dz + \int_{-\infty}^{+\infty} \lambda g(\lambda z)(n-1) [M(z)]^{n-2} \frac{dM(z)}{d\lambda} dz$$

$$= \frac{n-1}{n} F(b)^{n-2} f(b) \frac{db}{d\lambda}$$

This simplifies to

$$\int_{-\infty}^{+\infty} g(\lambda z) [M(z)]^{n-2} \left[\lambda \frac{dM(z)}{d\lambda} - z \frac{\partial M(z)}{\partial z}\right] dz = \frac{1}{n} F(b)^{n-2} f(b) \frac{db}{d\lambda} \quad (9)$$

Now note that

$$\frac{\partial M(z)}{\partial z} = \lambda \int_b^1 f(x) g(\lambda [b+z-x]) dx > 0$$

[10] To see why this is the case, note that $\int_{-\infty}^{+\infty} \lambda g(\lambda x) dz = 1$. Together with the fact that $g(\lambda z)$ is logconcave, and hence decreasing for sufficiently high z, this means that $\forall \varepsilon > 0$ there exists δ such that (i) $\int_{\delta}^{+\infty} g(\lambda z) dz < \frac{\varepsilon}{2\lambda}$; and (ii) $g(\lambda z)$ is decreasing for $z > 2\delta$. Then for any $z > 2\delta$ we have $zg(\lambda z) = 2\left(z - \frac{z}{2}\right) g(\lambda z) = 2\int_{\frac{z}{2}}^{z} g(\lambda z) dt < 2\int_{\frac{z}{2}}^{z} g(\lambda t) dt < 2\int_{\frac{z}{2}}^{+\infty} g(\lambda t) dt < 2\int_{\delta}^{+\infty} g(\lambda t) dt < \frac{\varepsilon}{2\lambda}$, where the first inequality comes from the fact that $g(\lambda z)$ is decreasing. Hence, $\lim\limits_{z\to+\infty} zg(\lambda z) = 0$. The statement that $\lim\limits_{z\to-\infty} zg(\lambda z) = 0$ can be proven analogously.

and

$$\begin{aligned}\frac{dM(z)}{d\lambda} =& f(b)\frac{db}{d\lambda} - f(b)G(\lambda z)\frac{db}{d\lambda} + \int_b^1 f(x)g(\lambda[b+z-x])\left(b+z-x+\lambda\frac{db}{d\lambda}\right)dx \\ =& f(b)[1-G(\lambda z)]\frac{db}{d\lambda} + A(z) + \frac{z}{\lambda}\frac{\partial M(z)}{\partial z} + \frac{db}{d\lambda}\frac{\partial M(z)}{\partial z} \\ =& \left(f(b)[1-G(\lambda z)] + \frac{\partial M(z)}{\partial z}\right)\frac{db}{d\lambda} + A(z) + \frac{z}{\lambda}\frac{\partial M(z)}{\partial z}\end{aligned}$$

where $A(z) \equiv \int_b^1 f(x)g(\lambda[b+z-x])(b-x)\,dx < 0$. Then we have:

$$\lambda\frac{dM(z)}{d\lambda} - z\frac{\partial M(z)}{\partial z} = \lambda\left(f(b)[1-G(\lambda z)] + \frac{\partial M(z)}{\partial z}\right)\frac{db}{d\lambda} + \lambda A(z) \quad (10)$$

We can now substitute (10) into (9) to obtain:

$$\begin{aligned}&\frac{db}{d\lambda}\int_{-\infty}^{+\infty}\lambda g(\lambda z)[M(z)]^{n-2}\left(f(b)[1-G(\lambda z)] + \frac{\partial M(z)}{\partial z}\right)dz \\ &+ \int_{-\infty}^{+\infty}\lambda g(\lambda z)[M(z)]^{n-2}A(z)\,dz \\ =& \frac{1}{n}F(b)^{n-2}f(b)\frac{db}{d\lambda}\end{aligned}$$

Then we can express $\frac{db}{d\lambda}$ as

$$\frac{db}{d\lambda} = \frac{\int_{-\infty}^{+\infty}\lambda g(\lambda z)[M(z)]^{n-2}A(z)\,dz}{\frac{1}{n}F(b)^{n-2}f(b) - \int_{-\infty}^{+\infty}\lambda g(\lambda z)[M(z)]^{n-2}\left(f(b)[1-G(\lambda z)] + \frac{\partial M(z)}{\partial z}\right)dz}$$

Since $A(z) < 0$ and $M(z) > 0$, the numerator is negative. We can show that the denominator is strictly negative too. This is because:

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \lambda g(\lambda z)[M(z)]^{n-2}\left(f(b)[1-G(\lambda z)]+\frac{\partial M(z)}{\partial z}\right) dz \\
>&\int_{-\infty}^{+\infty} \lambda g(\lambda z)[M(z)]^{n-2} f(b)[1-G(\lambda z)]\, dz \\
>&\int_{-\infty}^{+\infty} \lambda g(\lambda z) F(b)^{n-2} f(b)[1-G(\lambda z)]\, dz \\
=&F(b)^{n-2} f(b) \int_{-\infty}^{+\infty} \lambda g(\lambda z)[1-G(\lambda z)]\, dz \\
=&F(b)^{n-2} f(b)\left[1-\int_{-\infty}^{+\infty} \lambda g(\lambda z) G(\lambda z)\, dz\right] \\
=&\frac{1}{2} F(b)^{n-2} f(b) \\
\geq&\frac{1}{n} F(b)^{n-2} f(b)
\end{aligned}
$$

In the above, the first inequality is due to the fact that $\frac{\partial M(z)}{\partial z}>0$. The second inequality holds because $M(z)=F(b)+\int_b^1 f(x) G(\lambda[b+z-x])\, dx > F(b)$. The first equality uses a simple rearrangement of terms, while the second uses the fact that $\int_{-\infty}^{+\infty} \lambda g(\lambda z)\, dz=\int_{-\infty}^{+\infty} dG_\lambda(z)=1$. The third equality is due to the fact that

$$
\int_{-\infty}^{+\infty} \lambda g(\lambda z) G(\lambda z)\, dz=\left.[G(\lambda z)]^2\right|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty} \lambda g(\lambda z) G(\lambda z)\, dz
$$

and hence $\int_{-\infty}^{+\infty} \lambda g(\lambda z) G(\lambda z)\, dz=\frac{1}{2}\left.[G(\lambda z)]^2\right|_{-\infty}^{+\infty}=\frac{1}{2}$. Finally, the last weak inequality uses the fact that $n \geq 2$.

Since both the numerator and the denominator are negative, we conclude that $\frac{db}{d\lambda}>0$. □

Proof of Corollary 2. A candidate takes the test with probability $1-F(b)$. Hence, the expected spending on the test is $cn[1-F(b)]$, which increases as λ falls. □

Proof of Proposition 7. First, note that for any $t<T$ we have $\mathrm{E}[x \mid x<b_t] < \mathrm{E}[x \mid x<b_T]<b_T$. Hence, if a candidate that acts according to threshold

$t < T$ does not take the test, the decision-maker will have a lower expectation about his type than about the type of a candidate with threshold T, even if the latter does not take the test. Therefore, if a candidate with threshold $t < T$ does not take the test, he can never win the prize.

Furthermore, suppose a candidate has type $x < \mathrm{E}\left[x \mid x < b_T\right]$. If he takes the test, he cannot win the prize, since any candidate that acts according to threshold T will always win over him. Hence, for any candidate with type below $\mathrm{E}\left[x \mid x < b_T\right]$ it is never optimal to take the test. Therefore, at the equilibrium, we must have $b_1 \geq \mathrm{E}\left[x \mid x < b_T\right]$.

To prove the first statement, suppose there exists an equilibrium with more than three thresholds. Then we must have $b_T > b_{T-1} > b_{T-2} > b_{T-3} \geq \mathrm{E}\left[x \mid x < b_T\right]$.

If a candidate with threshold b_{T-1} has type b_{T-1} and takes the test, he wins if and only if (i) no candidate with threshold b_T takes the test; and (ii) all other candidates have types below b_{T-1}. If he does not take the test, then, by previous reasoning, his expected payoff is zero. Hence, b_{T-1} is defined by the indifference condition

$$F\left(b_T\right)^{k_T} F\left(b_{T-1}\right)^{n-k_T-1} - c = 0 \tag{11}$$

At the same time, if a candidate with threshold b_{T-2} takes the test, he wins if and only if (i) no candidate with threshold b_T or b_{T-1} takes the test; and (ii) all other candidates have types below b_{T-2}. Then b_{T-2} is defined by the indifference condition

$$F\left(b_T\right)^{k_T} F\left(b_{T-1}\right)^{k_{T-1}} F\left(b_{T-2}\right)^{n-k_T-k_{T-1}-1} - c = 0 \tag{12}$$

Putting (11) and (12) together yields

$$\begin{aligned} & F\left(b_T\right)^{k_T} F\left(b_{T-1}\right)^{n-k_T-1} = F\left(b_T\right)^{k_T} F\left(b_{T-1}\right)^{k_{T-1}} F\left(b_{T-2}\right)^{n-k_T-k_{T-1}-1} \\ \Longleftrightarrow\ & F\left(b_{T-1}\right)^{n-k_T-k_{T-1}-1} = F\left(b_{T-2}\right)^{n-k_T-k_{T-1}-1} \end{aligned}$$

Since $b_{T-1} > b_{T-2}$, this can only hold if $n - k_T - k_{T-1} = 1$. Hence, $k_{T-2} + k_{T-3} \leq 1$, which is impossible. Therefore, an equilibrium with more than three thresholds cannot exist.

To prove the second statement, take an equilibrium with three thresholds (b_1, b_2, b_3). Suppose $b_1 = \mathrm{E}\left[x \mid x < b_T\right]$. Then if a candidate with threshold b_1 takes the test, the decision-maker has the same posterior belief about his type as she has about the type of a candidate with threshold b_3 who does not take the test. Then the candidate with threshold b_1 wins the prize with probability $\frac{1}{k_3+1}$ if he takes the test and no other candidate does. Otherwise, he receives a payoff of zero. Hence, the indifference condition for candidate with threshold b_1 is

$$F(b_1)^{k_1-1} F(b_2)^{k_2} F(b_3)^{k_3} \frac{1}{k_3+1} - c = 0 \tag{13}$$

At the same time, a candidate with threshold b_2 wins the prize if and only if candidates with threshold b_3 do not take the test while all other candidates have types below b_2. Thus, his the indifference condition is

$$F(b_2)^{k_1+k_2-1} F(b_3)^{k_3} - c = 0 \tag{14}$$

Putting (13) and (14) together yields

$$\begin{aligned} & F(b_1)^{k_1-1} F(b_2)^{k_2} F(b_3)^{k_3} \frac{1}{k_3+1} = F(b_2)^{k_1+k_2-1} F(b_3)^{k_3} \\ \Longleftrightarrow\ & F(b_1)^{k_1-1} \frac{1}{k_3+1} = F(b_2)^{k_1-1} \end{aligned}$$

which is impossible, since $b_1 < b_2$, $k_1 \geq 1$, and $k_3 \geq 1$. Hence, at any equilibrium with three thresholds, we must have $b_1 > \mathrm{E}[x \mid x < b_T]$. Then a candidate with threshold b_1 wins the prize with certainty if and only if no other candidate takes the test. Therefore, his indifference condition is

$$F(b_1)^{k_1-1} F(b_2)^{k_2} F(b_3)^{k_3} - c = 0 \tag{15}$$

Putting (15) and (14) together implies

$$\begin{aligned} & F(b_1)^{k_1-1} F(b_2)^{k_2} F(b_3)^{k_3} = F(b_2)^{k_1+k_2-1} F(b_3)^{k_3} \\ \Longleftrightarrow\ & F(b_1)^{k_1-1} = F(b_2)^{k_1-1} \end{aligned}$$

Since $b_1 < b_2$, this can only hold if $k_1 = 1$. □

Proof of Lemma 8. As in the proof of Lemma 1, let $\pi(x)$ be the probability that a candidate with type x wins the prize after taking the test, and let $\pi(m)$ be the probability of winning the prize without taking the test. Then a candidate with type x is indifferent between taking and not taking the test when $\pi(x) - c(x) = \pi(m)$. Since $\pi(\cdot)$ is nondecreasing and $c(\cdot)$ is strictly decreasing, this equality holds for at most one type. If such a type exists, call it b. Otherwise, if $\pi(1) - c(1) < \pi(m)$, then $b = 1$. Note that $b = 0$ cannot be an equilibrium, because $\pi(0) - c(0) = -\bar{c} < 0 \leq \pi(m)$. □

Proof of Proposition 8. We can write (5) as $F(b)^{n-1} = \frac{n}{n-1}c(b)$. Approximating n by a continuous variable and differentiating with respect to it yields

$$F(b)^{n-1}\left[\ln F(b) + \frac{n-1}{F(b)}f(b)\frac{db}{dn}\right] = -\frac{1}{(n-1)^2}c(b) + \frac{n}{n-1}c'(b)\frac{db}{dn}$$

Hence,

$$\frac{db}{dn} = \frac{-\frac{1}{(n-1)^2}c(b) - F(b)^{n-1}\ln F(b)}{(n-1)F(b)^{n-2}f(b) - \frac{n}{n-1}c'(b)}$$

The denominator of the above is positive, since $c'(b) < 0$. Hence, $\frac{db}{dn} > 0$ whenever $F(b)^{n-1}\ln F(b) < -\frac{1}{(n-1)^2}c(b)$. Substituting $F(b)$ from (5), we find that $\frac{db}{dn} > 0$ if and only if $\frac{n}{n-1}c(b)\ln\left(\left[\frac{c(b)n}{n-1}\right]^{\frac{1}{n-1}}\right) < -\frac{1}{(n-1)^2}c(b)$, i.e. whenever $\ln\left[\frac{c(b)n}{n-1}\right] < -\frac{1}{n}$. This is true if and only if $c(b) < \frac{n-1}{n}e^{-\frac{1}{n}}$. $\square$

Proof of Proposition 9. Since the expressions for v and $\hat{v}$ are unchanged, using the same steps as in the proof of Proposition 5, we can write $\hat{v} - v = -\hat{b}F\left(\hat{b}\right)^n - \int_{\hat{b}}^{b}F(x)^n\,dx + F(b)^{n-1}\int_0^b F(x)\,dx$. We have $\lim_{n\to\infty}F(b) = \lim_{n\to\infty}\left[\frac{c(b)n}{n-1}\right]^{\frac{1}{n-1}} = \lim_{n\to\infty}[c(b)]^{\frac{1}{n-1}} = 1$, and $\lim_{n\to\infty}F\left(\hat{b}\right) = \lim_{n\to\infty}c\left(\hat{b}\right)^{\frac{1}{n-1}} = 1$, where the last equality in each case uses the fact that $c(b), c\left(\hat{b}\right) \in [\underline{c}, \overline{c}] \subseteq (0,1)$ at all values of n. Hence, $\lim_{n\to\infty}b = \lim_{n\to\infty}\hat{b} = 1$, and $\lim_{n\to\infty}\int_{\hat{b}}^{b}F(x)^n\,dx = 0$. At the same time, $\lim_{n\to\infty}F\left(\hat{b}\right)^n = \lim_{n\to\infty}c\left(\hat{b}\right)^{\frac{n}{n-1}} = c(1) = \underline{c}$, and $\lim_{n\to\infty}F(b)^{n-1} = \lim_{n\to\infty}\frac{c(b)n}{n-1} = c(1) = \underline{c}$. Thus, $\lim_{n\to+\infty}[\hat{v} - v] = -\underline{c} + \underline{c}\int_0^1 F(x)\,dx < 0$. $\square$