Spatial Pillage Game

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Abstract

A pillage game is a coalitional game as a model of *Hobbesian anarchy.* The *spatial pillage game* introduces a *spatial feature* into the pillage game. Players are located in regions and can travel from one region to another. The players can form a coalition and combine their power only within their destination regions, which limits the exertion of the power of each coalition. Under this spatial restriction, a coalition can pillage less powerful coalitions without any cost. The feasibility of pillages between coalitions determines the *dominance relation* that defines stable states in which powers among the players are endogenously balanced. With the spatial restriction, the set of stable states changes. However, if the players have forecasting ability, then the set of stable states does not change with the spatial restriction. *Core, stable set, and farsighted core* are adopted as alternative solution concepts.

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1 Introduction

Hobbesian anarchy is a societal state prior to the formation of a government that ensures property rights. Without a governmental organization, no individuals are safe to secure their wealth. Individuals could be tempted to pillage others whenever possible and beneficial. Although a coalition could be formed to secure their wealth, some members of the coalition may still be tempted to betray others and to take their wealth. Consequently, in Hobbesian anarchy, the possibility of the stable distribution of wealth is questionable.

A substantial amount of literature on *allocation by force* has been devoted to this possibility. Skaperdas (1992) showed that a cooperative outcome is possible in equilibrium if the probability of winning in conflict is sufficiently robust against each individual's action. Hirshleifer (1995) found the conditions under which Hobbesian anarchy is stable. Also, Hirshleifer (1991), Konrad and Skaperdas (1998), and Muthoo (1991) studied the situations in which property right is partially secured. These studies analyzed noncooperative models in which the formation of coalitions is limited or not allowed.

In contrast to the previous models, Piccione and Rubinstein (2007) and Jordan (2006) developed models of Hobbesian anarchy that allow the formation of coalitions. Piccione and Rubinstein introduced *the jungle* in which coercion governs economic transactions and they compared the equilibrium allocation of the jungle with the equilibrium allocation of an exchange economy. Jordan introduced *pillage games* and examined stable sets of allocations in which the power of pillaging balances endogenously.

The *spatial pillage game* is an extended version of a pillage game. In most literature on "allocation by force" including the studies above, there is no restrictions on the use of power. Thus, any individual or coalitions can pillage another individual or other coalitions if one is more powerful than others. However, the acts of pillaging and defending are inevitably conditioned under spatial restriction. Members of a coalition, if they move together, cannot simultaneously pillage two less powerful coalitions that are far apart from each other. Likewise, two coalitions cannot combine their power to defend themselves together against another powerful coalition unless they are close enough to each other. The spatial pillage game introduces a *space feature*, which conditions power usage based on location, into a Hobbesian anarchy model that allows the formation of coalitions, in the hope of understanding how spatial restriction affects stable distributions of wealth.

The spatial pillage game internalizes the space feature through the following assumptions. There are regions and each player can stay in only one of the regions. Players can change their regions to pillage others. The regions are connected with one another, and thus players can travel from a region to another in one move. Players can form a coalition and combine their power only after getting together in a common region. If coalitions are in different regions, they cannot combine their power. The influence of the power of each coalition is limited within its region. Therefore, a coalition cannot pillage two other coalitions in different regions simultaneously.

The other assumptions in this spatial pillage game are the same as in the original pillage games. A fixed amount of wealth is allocated among a finite number of players. Some players can form a coalition under the spatial restriction. A coalition can pillage less powerful coalitions within its region without any cost. An increase in the wealth of a coalition causes an increase in its power. Since the power of each coalition is endogenously determined, the spatial pillage game cannot have a characteristic function, which exogenously determines the power of each coalition.

The pillage games are characterized by *power functions* that determine the feasibility of pillages between coalitions. Jordan (2006) presented three power functions classified by the degree of their dependence on the sizes of coalitions. *Wealth is power* is one of the power functions and specifies the power of each coalition as its total wealth. Therefore, "wealth is power" is characterized as independent of the sizes of coalitions. Only the pillage game with this power function has a stable set in every possible case. Therefore, the spatial pillage game adopts "wealth is power" as a power function so that if there exist solutions in this spatial pillage game, then we can compare it with the solutions in the original pillage game and can find out how the spatial restriction affects a stable distribution of wealth.

As criteria for stable distributions of wealth and players, three solution concepts are explored; core, stable set, and farsighted core. First, the *core* is the collection of states at which pillage is not possible, thus it is one of the most persuasive solution concepts. However, due to its strong requirement, the core is too small to represent stable states as shown in Theorem 1. Second, the *stable set* is much bigger than the core if it exists, as shown in Proposition 1 and Example 1. A stable set is a collection of states that is both internally stable and externally stable. Internal stability requires that pillage not be possible between states in the collection and external stability requires that pillage at a state outside the collection result in another state inside the collection. In some cases, however, no stable set exists and even when they do exist, they contain implausible states. Third, the *farsighted core*, which was introduced by Jordan (2006), solves these problems with stable sets, as shown in Theorem 2. A farsighted core admits the assumption that a player has forecasting ability and is defined as a collection of states at which *pillage in expectation* is not possible in the sense that some members of the pillage would end up being worse off, and consequently they would not join the pillage.

In section 2, we search for the core and stable sets. The core is not affected by the spatial restriction since allocations in the core does not change under the spatial restriction. A stable set, on the other hand, is affected by the spatial restriction. Thus, the stable set, if it exists, is much bigger than a stable set in Jordan's model. In section 3, we study the farsighted core. There exists the unique farsighted core and it is similar to the farsighted core in Jordan's model. Since Jordan's model, without the spatial concept, induces similar result to the one in this spatial pillage game, we conclude that if the players have the forecasting ability, the assumption that the farsighted core bases on, then the stable distributions of wealth do not change with or without the spatial restriction.

2 Core and stable set

The environment of the spatial pillage game is defined in Definitions 1 and 2. We normalize the total wealth to unity.

Definition 1 [1] *The finite set I is the set of* ***players****. A* ***coalition*** *is a subset of I. The set $A = \{w \in \mathbb{R}^{\#I} : w_i \geq 0$ for all $i \in I$ and $\sum_{z \in I} w_z = 1\}$ is the set of* ***allocations****.*

The definitions below concern the spatial environment.

Definition 2 *The finite set R is the set of* ***regions*** *and the Cartesian product $R^{\#I}$ is the set of* ***distributions****. Given a distribution $p \in R^{\#I}$, the coalition $p^r = \{i \in I : p_i = r\}$ is the* ***population*** *at region r.*

A distribution is short for a population distribution and denotes how players are distributed over the regions. For example, the distribution $p = (1, 1, 2)$ expresses that players 1 and 2 are at region 1 and player 3 is at region 2. Also, it means $p^1 = \{1, 2\}$ and $p^2 = \{3\}$.

A **state** denotes both the allocation and distribution of the status quo.

Definition 3 *The Cartesian product $X = A \times R^{\#I}$ is the set of* ***states****.*

For instance, the ordered pair $(w, p) = ((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (1, 1, 2))$ is a state in a three-player and two-region model. The state (w, p) expresses that player 1 has $\frac{1}{2}$ and player 2 has $\frac{1}{4}$ while staying at region 1 and player 3 has $\frac{1}{4}$ while staying at region 2.

The **dominance relation** between states is defined as follows;

Definition 4 *Given states (w, p) and (w', p'), define $W = \{i : w'_i > w_i\}$ and $L = \{i : w'_i < w_i\}$. Suppose that for some $r \in R$, i) $\{i : w'_i \neq w_i\} \subset p^r$; ii) for all $q \neq r$, $p'^q = p^q \setminus W$; and iii) $\sum_{i \in W} w_i > \sum_{i \in L} w_i$. Then, the state (w', p')* ***dominates*** *the state (w, p).*

[1] *We follow notations in Jordan (2006).*

The dominance relation shows the states to which the status quo can move. It must satisfy both *physical* and *spatial conditions.* The physical condition requires that the winning coalition W have enough power to pillage the losing coalition L. Definition 4 presents this condition at *iii*). Jordan (2006) introduced a variety of physical conditions. The condition *iii*) above accords with the physical condition of the *wealth is power* in Jordan (2006). The spatial condition requires that the acts of pillaging satisfy spatial restriction. This condition is expressed at *i*) and *ii*) in the definition above. The condition *i*) means that transfers of wealth happen only in the destination region r where the pillage happens. The condition *ii*) denotes that only the winners travel. That is, the spatial restriction in dominance relation is that W can gather into a common region and can combine their power in order to pillage L. Note that if there is only one region, then this definition of dominance relation coincides with the definition of the *wealth is power* in Jordan (2006). So, this definition can be considered as a spatial version of the *wealth is power.*

In this section, we adopt the solution concepts of **core** and **stable set**. The definition stated below follows Lucas (1992) and Jordan (2006).

Definition 5 *The set of undominated states is the* ***core*** C*. For any set* E *of states, let the set* $U(E)$ *be the set of states that are not dominated by any state in* E*. A set* S *of states is a* ***stable set*** *if it satisfies both* $S \subset U(S)$*, which means internal stability, and* $S \supset U(S)$*, which means external stability.*

Theorem 1 embodies the core.

Theorem 1 *The set* $C = \{(w, p) \in X : \text{for each } i \in I,\ w_i = 1, \frac{1}{2}, \text{ or } 0\}$ *is the core.*

Proof. Suppose $(w, p) \in C$. For any $i \in I$, if $w_i > 0$ then $w_i \geq \frac{1}{2}$ and $\sum_{j \neq i} w_j \leq \frac{1}{2}$. Therefore, any player can prevent others from pillaging itself. This means that (w, p) is not dominated. Since (w, p) is arbitrary, every state in C is not dominated. Suppose $(w, p) \notin C$. Then, there exists i such that $w_i \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. If $w_i \in (0, \frac{1}{2})$, then the coalition $W = \{j : j \neq i\}$ can pillage player i since $\sum_{j \neq i} w_j > \frac{1}{2} > w_i$. If $w_i \in (\frac{1}{2}, 1)$, then $w_i > \frac{1}{2} > \sum_{j \neq i} w_j$ and thus player i can pillage others. This means that (w, p) is dominated by some state in X. Since (w, p) is arbitrary, every state in $X \setminus C$ is dominated. ■

Theorem 1 shows the core does not change under the spatial restriction since allocations in C does not change under the spatial restriction. This result is natural in that the core is the set of states that are not dominated by another state. Without the spatial restriction, if a state is not dominated by another state, then it must be undominated under the spatial restriction. Also, if a state is dominated by some state without the spatial restriction, then it can be dominated under the spatial restriction by pillaging only one player in the losing coalition. So, it cannot be in the core under the spatial restriction. Therefore, core allocations do not change under the spatial restriction.

Stable set is more involved than the core. First, we start with the trivial case in which there is only one region. Definition 6 introduces a **dyadic state** and the set of dyadic states $\boldsymbol{D}$. Proposition 1 establishes that the set D is the unique stable set in an one-region model. Note that Definition 6 and Proposition 1 are adapted from Jordan (2006) for the spatial pillage game.

Definition 6 *An allocation $w \in A$ is dyadic if for each i, $w_i = 0$ or $(\frac{1}{2})^{k_i}$ for some non-*

negative integer k_i. A state (w,p) is ***dyadic*** *if w is dyadic. The set* $\mathbf{D}$ *denotes the set of dyadic states.*

Proposition 1 (Theorem 3.3 in Jordan, 2006) *In an one-region model, the unique stable set is D.*

Next, under the spatial restriction, an example is shown to illustrate some features about the stable set.

Example 1 (Existence of a stable set in three-player models) *Define the set of states $S' \equiv \{(w,p) \in X :$ for distinct players $i,j,k \in I$, i) $w_i \geq \frac{1}{2}$, ii) $w_j = w_k = \frac{1-w_i}{2}$, and iii) $p_j \neq p_k\}$. Then, the set of states $D \cup S'$ is a stable set in three-player models*[1]*.*

Proposition 2 states nonexistence of the stable set in the four-player and two-region model.

Proposition 2 *No stable set exists in the four-player and two-region model.*

Proof. A proof is omitted but is available from the author. ■

These results on the stable sets show that the stable sets are affected by the spatial restriction. Without the spatial restriction, the stable set is the set of dyadic states D. However, under the spatial restriction, the stable sets change so that it includes more states than D because of the limited feasibility of the dominance relation. Moreover, in four-player and two-region model, a stable set does not exist.

[1] *Complete characterization of stable sets in three-player models is available from the author.*

As shown in Example 1 and Proposition 2, the stable set with respect to the dominance relation is not regarded as a plausible solution to the spatial pillage game. In the four-player and two-region model, no stable set exists. In the three-player models, there exist stable sets. However, they contain implausible states, such as some states in the set of states $X_{\#I} = \{(w,p) : \text{for some player } i,\ 1 > w_i > \frac{1}{2}\}$. According to the interpretation about a stable set in Harsanyi (1974), no state in $X_{\#I}$ can be a plausible prediction because one of the players has enough power to pillage the others, so eventually the player will pillage the rest of the wealth. That is, any state in $X_{\#I}$ is directly or indirectly dominated by the core and thus cannot be a stable state.

These problems with the stable set with respect to the dominance relation are caused by the limited feasibility of dominance relation under the spatial restriction. This limited feasibility of dominance relation, in turn, makes the conditions of the stable set, both internal stability and external stability, improper to be requirements for a proper solution to the spatial pillage game. The external stability requires that any state outside a stable set be directly dominated by some state in the stable set. With respect to this limited dominance relation, some states in $X_{\#I}$ are directly dominated only by other states in $X_{\#I}$, thus a stable set must contain some states in $X_{\#I}$ to satisfy external stability. Also, with respect to this limited dominance relation, the core cannot directly dominate every state in $X_{\#I}$, and thus an internally stable set can include both the core and some states in $X_{\#I}$. This explains why stable sets in three-player models contain some states in $X_{\#I}$.

In the four-player and two-region model, if an internally stable set S'' includes a set, of states, that dominates every state in $X_{\#I} \setminus S''$, then due to the limited feasibility of the dominance relation, S'' contains improperly many states so that there exists some state $(w, p) \notin S''$ such that S'' inevitably dominates every state that dominates (w, p). Thus, by the internal stability of S'', S'' cannot dominate (w, p), which is not in S''. That is, there is no set of states that satisfies both internal stability and external stability. This explains why no stable set exists in these models.

Jordan (2006) introduced a new solution concept, **farsighted core**. This farsighted core is defined based on an advanced concept of dominance relation, **Dominance in Expectation**. In this *dominance in expectation*, players make an **expectation** about how each state proceeds, and they pillage or defend according to their expectation. Naturally, this advanced concept of dominance relation allows broader feasibility of dominance relation while satisfying the spatial restriction. As a result, this solution concept based on the *dominance in expectation* solves the problems with the stable set and provides the unique solution which represents "an endogenous balance of power," as Jordan (2006) mentioned.

3 Core in expectation

The farsighted core solution concept is defined as follows. An *expectation* is a belief that all players have in common and indicates how each state proceeds.

Definition 7 *An* ***expectation*** *is a function $f : X \longrightarrow X$ satisfying, for some integer $k \geq 2$, $f^k = f^{k-1}$ where $f^k = f \circ f^{k-1}$. Let $f_w(w, p)$ and $f_p(w, p)$ denote the allocation and* *the distribution at $f(w, p)$, respectively.*

Jordan (2006) considered only one-step *expectation* where every state reaches its stable state within one step, i.e. $f = f^2$. Here, the expectation is extended as a finite-step *expectation* where some states take finite steps, possibly more than one step, to reach their stable states. Based on this extended expectation, this study shows the same result, Corollary 1, as the result in Jordan (2006).

Dominance in Expectation between states indicates possible states into which the present state can change, provided that players follow the expectation after the changes. Just like in the previous dominance relation, both *physical* and *spatial conditions* should be satisfied in order for a *winning coalition in expectation*, who end up being better off, to change its present state through defeating a *losing coalition in expectation*, who end up being worse off. Physical condition is reflected on the conditions *iii*) and *iv*) in Definition 8 and spatial condition is reflected on the conditions *i*) and *ii*).

Definition 8 *Let an expectation f satisfy $f^k = f^{k+1}$. Given states $(\bar{w}, \bar{p})$ and $(w(n), p(n))$, for each $n \in \mathbb{N}$, define $W_f^{(n)} = \{i : f_w^k(\bar{w}, \bar{p})_i > w(n)_i\}$ and $L_f^{(n)} = \{i : f_w^k(\bar{w}, \bar{p})_i < w(n)_i\}$. Then, a state $(\bar{w}, \bar{p})$* ***dominates*** *(w, p)* ***in expectation*** *if there exists a sequence of states $\{(w(n), p(n))\}_{n=1}^N$ that has $(w(1), p(1)) = (w, p)$ and $(w(N), p(N)) = (\bar{w}, \bar{p})$ such that for each $1 \leq n \leq N-1$ and for some $r \in R$, i) $\{i : w(n+1)_i \neq w(n)_i\} \subset p(n+1)^r$; ii) for all $q \neq r$, $p(n+1)^q = p(n)^q \setminus (W_f^{(n)} \cap p(n+1)^r)$; iii) $\sum_{i \in W_f^{(n)} \cap p(n+1)^r} w(n)_i > \sum_{i \in L_f^{(n)} \cap p(n+1)^r} w(n)_i$; and iv) $\sum_{i \in W_f^{(n)}} w(n)_i > \sum_{i \in L_f^{(n)}} w(n)_i$.*

This dominance relation concept considers players' ability to forecast how each state proceeds. When the players have forecasting ability, they maximize their allocations in a final state. Thus, if some players expect that they belong to a losing coalition in expectation, $L_f^{(n)}$, who will be pillaged and so will be worse off in the final state, then they might have an incentive to get together in a common region and combine their powers in order to defend themselves against a winning coalition in expectation, $W_f^{(n)}$, who will be better off in the final state. However, under the condition iv), $L_f^{(n)}$ basically has no power to deter $W_f^{(n)}$ from pillaging $L_f^{(n)}$ even when all members of $L_f^{(n)}$ gather and combine their powers. This is because $W_f^{(n)}$ can also gather and combine their powers to pillage $L_f^{(n)}$. As a result, under the condition iv), $L_f^{(n)}$ has no incentive to take any defensive action and therefore, this condition is necessary that $W_f^{(n)}$ successfully pillages $L_f^{(n)}$ when the players have the forecasting ability.

The condition iv), however, is not sufficient that $W_f^{(n)}$ practically executes its plan to pillage $L_f^{(n)}$. This is because $W_f^{(n)}$ can exert its power only under the spatial restriction. Together with the condition iv), the conditions i), ii), and iii) represent sufficient conditions that $W_f^{(n)}$ executes its plan to pillage $L_f^{(n)}$ under the spatial restriction. These conditions are similar to the conditions in Definition 4. So, the condition i) means that in each step of the pillaging process, transfers of wealth happen only in one region r where pillage actually happens. The condition ii) states that only members of $W_f^{(n)}$ travel. Finally, the condition iii) denotes that members of $W_f^{(n)}$ in the region r have enough power to pillage members of $L_f^{(n)}$ in r. So, this pillage by the members of $W_f^{(n)}$ is feasible within r.

This definition differs from the definition of *dominance in expectation* in Jordan (2006) in that this definition generalizes the number of steps that the dominance relation can take. Jordan (2006) introduced one-step *dominance in expectation* in which every plan to change a state can be completed within one step, i.e. $(w(1), p(1)) = (w, p)$ and $(w(2), p(2)) = (\bar{w}, \bar{p})$. This Jordan's definition is suitable for the one-step expectation since it can be organized according to a binary relation derived from the one-step *dominance in expectation*. However, the finite-step expectations, except one-step expectations, cannot be organized according to the binary relation. Since the present study extends the expectation as the finite-step expectation, the *dominance in expectation* must also be generalized as the finite-step *dominance in expectation* in which plans to change a state can take more than one steps before it ends, i.e. $(w(N), p(N)) = (\bar{w}, \bar{p})$ for some $N \geq 2$. If an expectation is organized in accord with a relation derived from the *dominance in expectation*, it is called a *consistent expectation*. Jordan (2006) interpreted consistency as "a rational expectation property." He said that "an expectation is consistent if only rational acts of pillage are expected, and an allocation is expected to persist only if no rational pillage is possible."

Definition 9 *An expectation f is* ***consistent*** *if $f(w, p)$ dominates (w, p) in expectation whenever $f(w, p) \neq (w, p)$ and if (w, p) is undominated in expectation whenever $f(w, p) = (w, p)$.*

Farsighted core and *farsighted supercore*[2] are defined as follows.

[2] *Farsighted supercore* is named after Roth's (1976) *supercore*.

Definition 10 *Given a consistent expectation f, the* ***farsighted core*** *under the expectation f is the set of states $K_f = \{(w,p) \in X :$ under the expectation f, no state in X dominates (w,p) in expectation$\}$. The* ***farsighted supercore*** *C_S is the intersection of all farsighted cores.*

A farsighted core is a set of stable states under some consistent expectation. The farsighted supercore is the set of stable states for all consistent expectations. Theorem 2 states that for any consistent expectation, the set of dyadic states D is the unique farsighted core and therefore is the farsighted supercore.

Theorem 2 *A consistent expectation f exists and the farsighted core K_f under f is the set of dyadic states, D. Therefore, the farsighted supercore C_S is also D.*

The proof of Theorem 2 uses the following lemmas.

Lemma 1 *For any state (w,p), if an allocation w' satisfies $\sum_{i\in\{i:w'_i>w_i\}} w_i > \sum_{i\in\{i:w'_i<w_i\}} w_i$ and $f(w',p^*) = (w',p^*)$ for every distribution p^*, then there exists a distribution p' such that (w',p') dominates (w,p) in expectation.*

Proof. Suppose that a state (w,p) and an allocation w' satisfy the premise of this lemma. To prove this lemma, it suffices to construct a sequence of states $\{(w(n),p(n))\}_{n=1}^{N}$ that can make (w',p') dominate (w,p) in expectation for some p'.

Let $W'_f = \{i : w'_i > w_i\}$ and $L'_f = \{i : w'_i < w_i\}$. Select $(w(2),p(2))$ such that $(w(2),p(2))$ results from W'_f's pillaging all members of L'_f in the region $\min\{p_i : i \in L'_f\}$ and also from W'_f's proportioning their wealth to w'. Similarly, select states $(w(n),p(n))$ for $n \in \mathbb{N}$ until $w(N) = w'$ for some N. Then, the sequence of the states $\{(w(n),p(n))\}_{n=1}^{N}$ makes (w',p') dominate (w,p) in expectation for some p'. ∎

Lemma 2 (Lemma 3.10 in Jordan, 2006) *For some positive integer k, let w be a dyadic allocation such that for each i, if $w_i > 0$ then $w_i \geq 2^{-(k+1)}$. If an allocation w' satisfies that $\sum_{z\in\{i:w'_i>w_i\}} w_z > \sum_{z\in\{i:w'_i<w_i\}} w_z$, then there exists a dyadic allocation w'' such that $\sum_{z\in\{i:w''_i>w'_i\}} w'_z > \sum_{z\in\{i:w''_i<w'_i\}} w'_z$ and for each i, if $w''_i > 0$ then $w''_i \geq 2^k$.*

Proof of Theorem 2. First, we are going to construct a consistent expectation that has a farsighted core D and then will prove the uniqueness of the farsighted core.

Construct an expectation f as follows. If $(w,p) \in D$, then $f(w,p) = (w,p)$. If $(w,p) \notin D$, then select a dyadic allocation w' such that $\sum_{i\in\{i:w'_i>w_i\}} w_i > \sum_{i\in\{i:w'_i<w_i\}} w_i$. Proposition 1 assures the existence of w'. Let $W'_f = \{i : w'_i > w_i\}$ and $L'_f = \{i : w'_i < w_i\}$. Construct $f(w,p)$ such that $f(w,p)$ results from W'_f's pillaging all members of L'_f in the region $\min\{p_i : i \in L'_f\}$ and also from W'_f's proportioning their wealth to w'. Similarly, construct $f^n(w,p)$ for $n \in \mathbb{N}$ until $f^N_w(w,p) = w'$ for some N.

Now, we need to show the expectation f constructed above is consistent. If $(w,p) \notin D$, then for each $n \leq N-1$, we have that $\sum_{i\in W^{(n)}_f} f^n_w(w,p)_i > \sum_{i\in L^{(n)}_f} f^n_w(w,p)_i$ where $W^{(n)}_f = \{i : w'_i > f^n_w(w,p)_i\}$ and $L^{(n)}_f = \{i : w'_i < f^n_w(w,p)_i\}$. That is, the fourth condition in Definition 8 is satisfied. Also, it is easily seen that the expectation f is designed to satisfy the other three conditions in Definition 8. Consequently, for each $n \leq N$, a state $f^n(w,p)$ dominates $f^{n-1}(w,p)$ in expectation where $f^0(w,p) = (w,p)$. In addition, no state (w,p) in D is dominated in expectation by another state (w',p') in D because $\sum_{i\in\{i:w'_i>w_i\}} w_i \leq \sum_{i\in\{i:w'_i<w_i\}} w_i$ by Proposition 1. That is, if $(w,p) \in D$ and thus $f(w,p) = (w,p)$, then (w,p) is not dominated in expectation. Therefore, the expectation f is consistent.

To prove the uniqueness of a farsighted core, let f be a consistent expectation with the farsighted core K_f. Also, for each non-negative integer n, define $D_n = \{(w,p) \in D : w_i = 0$ or $\geq (\frac{1}{2})^n\}$. Then, we have $D_0 \subset K_f$. Suppose, by way of induction, that for some n, we have $D_n \subset K_f$. Note that if $(w,p) \in D_n$, then $(w,p') \in D_n$ for any distribution p'. Thus, by

Lemmas 1 and 2, any state (w', p') that dominates some state $(w, p) \in D_{n+1}$ in expectation is dominated in expectation by some state (w'', p'') in D_n because the allocation w' satisfies that $\sum_{i \in \{i: w'_i > w_i\}} w_i > \sum_{i \in \{i: w'_i < w_i\}} w_i$. Since f is consistent, we have that $D_{n+1} \subset K_f$. By induction, we have $D \subset K_f$. In addition, D dominates all states outside D by Proposition 1 and by Lemma 1. Again, since f is consistent, if $(w, p) \notin D$, then $(w, p) \notin K_f$, that is, $K_f \subset D$. Therefore, we have $K_f = D$, and since f is an arbitrary consistent expectation, we have $C_S = D$. ■

This result is similar to the result in Jordan (2006), which stated that D is the unique far-sighted core in one region models, which, in turn, do not have the spatial restriction. Clearly, dominance relation with respect to *dominance in expectation* changes if we introduce the spatial restriction. For example, let's consider the dominance relation between the following two states; $(\bar{w}, \bar{p}) = ((1,0),(1,1))$ and $(w, p) = ((\frac{3}{4}, \frac{1}{4}),(2,2))$. Then, $(\bar{w}, \bar{p})$ and (w, p) satisfy the physical condition for the dominance relation because $\sum_{i \in \{i: \bar{w}_i > w_i\}} w_i > \sum_{i \in \{i: \bar{w}_i < w_i\}} w_i$. So, if there is no spatial restriction, then $(\bar{w}, \bar{p})$ dominates (w, p) in expectation. But, if we introduce the spatial restriction, then $(\bar{w}, \bar{p})$ does not dominate (w, p) in expectation because any possible pillaging movement from (w, p) will results in the distribution $(2, 2)$. That is, in this example, dominance relation with respect to *dominance in expectation* has changed under the spatial restriction.

Nevertheless, Theorem 2 shows that if the players have the forecasting ability, then only states in D are expected to persist even when there is the spatial restriction. This is because the forecasting ability enhances players' ability to pillage. As a result, the forecasting ability complements the limited feasibility of pillages under the spatial restriction so that this limited feasibility does not affect the set of stable states that represents an endogenous balance of power. Therefore, **under the farsighted-player assumption, the set of stable states does not change under the spatial restriction.**

Theorem 2 also shows that the *dominance in expectation* selectively reflects the concept of "indirect dominance" which was introduced by Harsanyi (1974) and formalized by Chwe (1994). The indirect dominance concept means that if (w, p) dominates (w', p') with respect to, for example, the dominance relation in Definition 4 and (w', p') dominates (w'', p''), then if (w, p) is a stable, (w'', p'') cannot be a stable state since (w, p) indirectly dominates (w'', p''). To see how the *dominance in expectation* selectively reflects this indirect dominance concept, let (w, p) only indirectly dominate (w'', p''), that is, (w, p) cannot dominate (w'', p'') at once, and there exists a state that is dominated by (w, p) and dominates (w'', p'') simultaneously. If there exists a route that connects from (w'', p'') to (w, p) and through which a winning coalition who prefers (w, p) to (w'', p'') can achieve (w, p) by pillaging a losing coalition who prefers (w'', p'') to (w, p), then (w, p) dominate (w'', p'') in expectation. Otherwise, (w, p) does not dominate (w'', p'') in expectation. Here, the route is a sequence of states in Definition 8 that satisfies four conditions above, and the four conditions are sufficient conditions to change a state when the players have the forecasting ability. Therefore, the *dominance in expectation* reflects the indirect dominance concept only if a dominance relation can be actualized by the players who have the forecasting ability. As a result, this *dominance in expectation* designates the set of dyadic states D as the unique set of stable states.

In addition, the set D can be considered as a self-enforcing "standard of behavior," as interpreted by von Neumann and Morgenstern (1947). This is because no state inside D is dominated in expectation by another state in D whereas every state outside D is dominated in expectation by some state in D. Therefore, we conclude that the concept of *dominance in expectation* in this spatial pillage game captures the combined concept from Harsanyi's indirect dominance and von Neumann and Morgenstern's self-enforcing standard of behavior according to the spatial restriction.

We conclude that the concept of dominance in expectation in this spatial pillage game captures the combined concept from Harsanyi's indirect dominance and von Neumann and Morgenstern's self-enforcing standard of behavior according to the spatial restriction.

Xue (1998) and Konishi and Ray (2003) also introduced solution concepts for a coalitional game. Their solution concepts, similar to the farsighted core, are defined based on a progress of states that shows how the status quo proceeds to a stable state under the farsighted-player assumption. However, in contrast to the farsighted core, their solution concepts focus mainly on the forecasting ability of a winning coalition, and thus their solution concepts might not capture the fact that a losing coalition also has the forecasting ability and so they can defend themselves according to their expectation. As a result, their solution concepts might not designate some states that would be regarded as stable states if all players' forecasting abilities are considered. For example, in their solution concepts, the progress of states $((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (1, 1, 1)) \longrightarrow ((\frac{3}{4}, 0, \frac{1}{4}), (1, 1, 1)) \longrightarrow ((1, 0, 0), (1, 1, 1))$ is possible. Thus, the state $((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (1, 1, 1))$ might not be a stable state according to their solution concepts. However, since a losing coalition has the forecasting ability, at the state $((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (1, 1, 1))$, player 3 will help player 2 in order to deter player 1 from pillaging player 2 in expectation that the second state $((\frac{3}{4}, 0, \frac{1}{4}), (1, 1, 1))$ will proceed to the third state $((1, 0, 0), (1, 1, 1))$. Accordingly, the state $((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (1, 1, 1))$ shows balanced power among the players and therefore must be regarded as a stable state under the farsighted-player assumption as it is under the farsighted core solution concept.

Finally, Corollary 1 states that definitions about the farsighted core in Jordan (2006) can be extended to the definitions in this study.

Corollary 1 *In one region models, a consistent expectation exists, and it has $K_f = D$. Therefore, C_S is also D.*

Jordan (2006) used the *one-step expectation* and the *one-step dominance in expectation* and showed the same result as Corollary 1. Therefore, the definitions in Jordan (2006) can be extended as the *finite-step expectation* and the *finite-step dominance in expectation*.

4 References

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