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ON THE IMPOSSIBILITY OF EVENTS OF ZERO PROBABILITY

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ABSTRACT. A logically consistent way of maintaining the events of probability zero are actually impossible is presented.

Consider a random variable X uniformly distributed on the unit interval [0,1]. For any $x \varepsilon [0,1]$, the probability that $X=x$ is zero, and yet one of these events must occur. This appears to preclude interpretation of an event of probability zero as an impossible event. Several authors of elementary probability texts comment on this puzzling situation. Levi (1980) argues from this that events of probability zero may nonetheless be 'seriously possible'. The object of this note is to present a logically consistent way of maintaining that events of probability zero are impossible.

We first present an analogy with the Axiom of Choice. Let E_i, i = l,2,..., be a countable collection of sets. Let $S_i$be the statement "E_i is not empty". The axiom of choice states that if for each i, S_i is true, then the infinite conjunction of them is also true. It is important to note that while the meaning of the infinite conjunction (i.e. the simultaneous truth of all of the sentences) is easily understood intuitively, it is not a sentence according to the usual rules of logic, which permits only finite combinations of sentences. It is possible to introduce a new primitive sentence equivalent in meaning to the infinite conjunction. However, as logicians have discovered, this sentence is not logically entailed by the truth of each of sentences S_i. More precisely, a man who believes each S_i to be true while maintaining that the infinite conjunction of them is false will not arrive at a logical contradiction (unless his counterpart who believes the infinite conjunction to be true can also arrive at a contraduction). A lucid presentation of this and related matters is available in Cohen (1966). This phenomenon (S_i is true for each i while the infinite conjunction is false) is called ω-inconsistency (where ω stands for the first infinite ordinal). Let $Q(x)$ be the sentence "X will take the value x". For any subset S of [0,1], define the disjunction $Q^*(S) = \vee_{x \varepsilon S} Q(x)$. In equating the probability of an event with the possibility of the same event, we are asserting that Q*(S) is false for any set of (outer) probability zero, while it is true for any set of (inner) probability one. As explained earlier, this is not a logically inconsistent position, despite appearances. It is, rather, Ω-inconsistent, where

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Ω is the first ordinal with the cardinality of the set [0,1]. Since subsets of cardinality less than Ω must have probability zero, it is clear that inconsistency will not arise for such collections of sentences. The air of paradox arises because of our implicit supposition that Q*([0,l]) logically entails the truth of one of the statements *Q(x).* This is not true unless we assume Ω-consistency, which is a rather strong axiom.

On reflection, it appears quite plausible to display Ω-inconsistency in this situation. We are prepared to assert, for any fixed $x \varepsilon$ [0,1], that we will not observe the event X=*x* regardless of how many (up to a countable number of) repetitions of X we observe. Thus it is reasonable to believe *Q(x)* to false for any *x*, while not denying the truth of *Q*[0,l]*.

The perceptive reader will observe that by rejecting Ω-consistency we have avoided logical inconsistency at the stage prior to observing *X.* After observing $X=x_0$ we appear to face the difficulty of having to change the truth value of $Q(x_o)$ from false to true. This problem can be avoided by proper interpretation of the event $X=x_o$. We continue to maintain that $Q(x_0)$ is false, but now regard the disjunction $Q^*([x_0 - \varepsilon, x_o + \varepsilon])$ as being true for any value of $\varepsilon > 0$. This gives us a logically consistent identification of probability 0 with impossibility, and also demonstrates once again the pitfalls of intuitive reasoning about infinities.

In conclusion, we remark that the above interpretation permits a clarification of the logic of the method of maximum likelihood. We seek to find a parameter value which maximizes the likelihood of the observed event. When the observed event $X=x_o$ is regarded as $Q(x_0)$, this does not make sense as stated (with continuous variables), and must be justified heuristically. However, it does make sense to ask for a parameter value maximizing the probability of event $Q^*([x_o - \varepsilon, x_0 + \varepsilon])$ for ε small. This is the logic of the method of maximum probability estimators, due to Weiss and Wolfowitz (1974).

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