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27 April 2020

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ABSTRACT

This paper modifies single assumption in the base of classical option pricing model and derives further extensions for the Black-Scholes-Merton equation. We regard the price as the ratio of the cost and the volume of market transaction and apply classical assumptions on stochastic Brownian motion not to the price but to the cost and the volume. This simple replacement leads to 2-dimensional BSM-like equation with two constant volatilities. We argue that decisions on the cost and the volume of market transactions are made under agents expectations. Random perturbations of expectations impact the market transactions and through them induce stochastic behavior of the underlying price. We derive BSM-like equation driven by Brownian motion of agents expectations. Agents expectations can be based on option trading data. We show how such expectations can lead to nonlinear BSM-like equations. Further we show that the Heston stochastic volatility option pricing model can be applied to our approximations and as example derive 3-dimensional BSM-like equation that describes option pricing with stochastic cost volatility and constant volume volatility. Diversity of BSM-like equations with 2 – 5 or more dimensions emphasizes complexity of option pricing problem. Such variety states the problem of reasonable balance between the accuracy of asset and option price description and the complexity of the equations under consideration. We hope that some of BSM-like equations derived in this paper may be useful for further development of assets and option market modeling.

Keywords: Option Pricing, Black-Scholes-Merton Equations, Stochastic Volatility, Market Transactions, Expectations, nonlinear equations

JEL: C10, G110, G120, G130[1]

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1. Introduction.

We argue classical option pricing (Black and Scholes, 1973; Merton, 1973) (BSM) and stochastic volatility (Heston, 1993) models and proposes some extensions of model equations. Classical papers by Black, Scholes (1973) and Merton (1973) were published almost 50 years ago and nevertheless their results impact current development and further research of assets and option pricing models. Hundreds researches contribute to this important financial problems. We refer only few papers by Bates (1996), Merton (1997), Scholes (1997) and some who contribute to general treatment of options pricing and financial methods - Figlewski, (1998), Shiryaev (1999), Hull (2009). Many researchers provide extension of the BSM model: Black, Derman and Toy (1990), Hull and White (2001). Extensions of BSM from constant to stochastic volatility were developed by Hull and White (1987), Heston (1993), Ball and Roma (1994), Saikat (1996), Britten-Jones and Neuberger (2000), Engle and Figlewski (2014), Cohen and Tegner (2018), multiple assets option pricing models by Broadie and Detemple (1997), Rapuch and Roncalli (2004), Carmona and Durrleman (2006), Li, Deng and Zhou, (2010), application of Non-Gaussian processes by Borland (2004), extension of diffusion by Kleinert and Korbel (2016). Some collections of different approaches to BSM model are presented in Choi (2018). We have no intend to give any reasonable review of current state of derivatives and option pricing theory. We only indicate some directions for classical BSM model extensions – extension from constant to stochastic volatility models, multiple assets options pricing, non-Brownian random processes and etc. It seems that during these 50 years after Black and Scholes (1973), and Merton (1973) studies almost all possible methods and directions for options pricing modeling are described.

Nevertheless we regard the classical BSM model as an endless source for further development and extensions. Time by time it is useful to state a simple questions to the classical models. It may help to find out the way for further progress. In this paper we consider the classical BSM model and state a simple question – why assumption on Brownian random behavior of the underlying asset's price was the initial for BSM model? What economic factors define price evolution?

Indeed, the asset's price is not the March Hare from Lewis Carroll's "Alice's Adventures in Wonderland" that can jump randomly. The price don't behave arbitrary like "The Cat that walked by himself" by R. Kipling. The price is not a stand-alone financial notion. The price dynamics is determined by numerous economic and financial factors. Stochastic behavior of

these factors impact stochastic behavior of the price. Assumptions in the ground of option pricing should correspond to general relations between economic and financial variables and market transactions that describe random market evolution and define stochastic price dynamics. Otherwise some studies on option pricing become very similar to intellectual math attempt to guess the correct form for the stochastic process that govern the price or its volatility in particular without efforts to understand financial reasons of such dynamics. Numerous such attempts improve classical BSM model and enhance the option price studies but give a little for the understanding – what are the hidden financial relations that govern the asset's random price evolution?

It is obvious that description of additional factors that impact asset's price would increase complexity of the model. To reduce the complexity we start only with two factors that directly define the asset's price. The price p is not an additive variable but determined as evident ratio of two additive variables – the cost C and the volume V of market transaction. Any theory should have ground on properties of additive variables. Aggregation of additive cost C_Δ and volume V_Δ of transactions performed during time term Δ define mean asset price as $p_\Delta = C_\Delta/V_\Delta$. Random properties of the market transactions, random dynamics of the cost C and the volume V of the market transactions should define stochastic behavior of the price. The cost C_Δ and the volume V_Δ of asset transactions during different time terms Δ may have different stochastic properties and that induces variations of price random properties. We regard the assumptions on random behavior of the market transactions, their cost and volume as initial factors that determine stochastic properties of the price. Below we show how these obvious and simple relations between the cost, the volume and the price of the market transactions allow contribute to classical BSM and stochastic volatility models.

2. Market transactions and option pricing

Let's take the classical BSM model. According to the BSM model (Black and Scholes, 1973; Merton, 1973; Hull, 2009) the underling asset price $p(t)$ follows the standard Brownian motion $dW(t)$ as:

$$dp(t) = p(t)\,\mu\,dt + p(t)\,\sigma dW(t) \tag{1.1}$$

$$< dW(t) > = 0; \ < dW(t)dW(t) > = dt \tag{1.2}$$

Here μ – linear trend, σ - dispersion and r – risk-free rate and $\mu,\ \sigma,\ r$ – are constant. We use notion <...> to define averaging of random process. Option price $S(t,p)$ follows the classical BSM equation (1.3):

$$\frac{\partial S}{\partial t} + rp\frac{\partial S}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2 S}{\partial p^2} = rS \qquad (1.3)$$

Let's maintain all assumptions of the classical BSM model except the main one: we don't take assumptions (1.1, 1.2) on the price $p(t)$.

We suppose that stochastic behavior of the price $p(t)$ should be determined by random properties of market transactions with underling assets and in particular by random dynamics of the cost $C(t)$ and the volume $V(t)$ of the transactions. Trivial relations define the cost $C(t)$ and the volume $V(t)$ of the market transaction $M(t)$ and the price $p(t)$ as:

$$M(t) = \big(C(t), V(t)\big); \quad p(t) = \frac{C(t)}{V(t)} \qquad (2)$$

Relations (2) are trivial but they replace initial assumptions (1.1, 1.2) on random properties of the price $p(t)$ by assumptions on random properties of the cost $C(t)$ and the volume $V(t)$ of the transaction $M(t)$. To keep simplicity of BSM model as a first approximation let's study the standard Brownian processes (1.2) and take all coefficients like trends, dispersions and rates as constant. Due to (2) let's replace assumptions (1.1, 1.2) on the price $p(t)$ by assumptions on the Brownian motion dW_c of the cost $C(t)$ and the Brownian motion dW_v of the volume $V(t)$ similar to (1.1):

$$dC(t) = C(t)\,\mu_c\,dt + C(t)\,\sigma_c\,dW_c(t) \qquad (3.1)$$

$$dV(t) = V(t)\,\mu_v\,dt + V(t)\,\sigma_v\,dW_v(t) \qquad (3.2)$$

$$<dW_c(t)> = 0; \;\; <dW_c(t)dW_c(t)> = dt \qquad (3.3)$$

$$<dW_v(t)> = 0; \;\; <dW_v(t)dW_v(t)> = dt \qquad (3.4)$$

Let's take that Brownian processes dW_c and dW_v are correlated as:

$$<dW_c(t)dW_v(t)> = \lambda\,dt \qquad (3.5)$$

These assumptions determine the random behavior of the price $p(t)$ (2) as:

$$dp(t) = \frac{dC(t)}{V(t)} - \frac{C(t)}{V(t)}\frac{dV(t)}{V(t)} \qquad (3.6)$$

Relations (3.1, 3.3, 3.6) define $dp(t)$:

$$dp(t) = p(t)[(\mu_c - \mu_v)\,dt + (\sigma_c\,dW_c(t) - \sigma_v dW_v(t))] \qquad (3.7)$$

Two Brownian processes (3.1; 3.2) cause that option price S should depend on time t and two variable and we take them as the price p and the volume V. Many papers present option pricing under action of multiple Brownian processes (Broadie and Detemple, 1997; Rapuch and Roncalli, 2004; Carmona and Durrleman, 2006; Li, Deng and Zhou, 2010; Hull and White, 1987; Heston, 1993; Ball and Roma, 1994; Britten-Jones and Neuberger, 2000; Cohen and Tegner, 2018). Our aim is not the derivation of option pricing equation under action of two Brownian processes. For the simplest assumptions of classical BSM model we

demonstrate that randomness of the cost and the volume of market transactions induces 2-dimensional option pricing equation. It is easy to show (Hull, 2009; Poon, 2005) that (3.2-3.5; 3.7) cause:

$$\frac{\partial S}{\partial t} + rp\frac{\partial S}{\partial p} + rV\frac{\partial S}{\partial V} + \frac{1}{2}p^2\sigma^2\frac{\partial^2 S}{\partial p^2} + \frac{1}{2}V^2\sigma_v^2\frac{\partial^2 S}{\partial V^2} + pV\varrho\frac{\partial^2 S}{\partial V\partial p} = rS \tag{4.1}$$

$$\sigma^2 = \sigma_c^2 - 2\lambda\sigma_c\sigma_v + \sigma_v^2 \ ; \ \varrho = \lambda\sigma_c\sigma_v - \sigma_v^2 \tag{4.2}$$

We use risk-free portfolio Π and risk-free rate r.

$$\Pi(t) = S - \alpha p - \beta V \ ; \quad d\Pi = r\Pi dt \tag{4.3}$$

The derivation of (4.1; 4.2) is standard (Hull, 2009; Poon, 2005) and we omit it here. Equation (4.1) describes option price dynamics on 2-dimensional space (p,V). If Brownian motion dW_v (3.2) is identical to (3.1) dW_c with (3.5) $\lambda=1$ then only one Brownian process govern random price and (3.7) can be presented as

$$dp(t) = p(t)[(\mu_c - \mu_v)\,dt + (\sigma_c - \sigma_v)\,dW_c(t))] \tag{4.4}$$

and option equation (4.1) is reduced to the classical BSM equation (1.3) with σ^2

$$\sigma^2 = (\sigma_c - \sigma_v)^2$$

The same reduction from (4.1) to (1.3) follows if volume $V(t)$ is a regular function or constant. Thus classical BSM model describes option pricing in the assumption that cost C and volume V of the market transactions follow the identical Brownian motion or cost C or volume V are regular functions or constant. We omit here the change of variables that leads to the equation as it is simple and gives no new meaning.

If the cost $C(t)$ (3.1) and the volume $V(t)$ (3.2) are described by different Brownian processes and the price $p(t)$ follows (3.7) then the option price S should obey the 2-dimensional BSM-like equation (4.1).

3. Expectations and option pricing

As we show above the simple relations (2) that define the price of transactions increase "space" dimension of the BSM equation. However the market transactions are performed by economic agents, and these agents take decisions on the cost, the volume and the price of the transactions under personal expectations. Thus expectations those approve market transactions impact evolution of transactions cost, volume and price. Different agents may take their decisions on base of different expectations and random perturbations of numerous expectations may cause random disturbances of the cost, the volume and the price of transactions. Studies of expectations and their impact on economic and financial markets have a long history and we mention only some starting with Keynes (1936), Muth (1961) and

Lucas (1972) and further research by (Sargent and Wallace 1976; Hansen and Sargent 1979; Kydland and Prescott 1980; Blume and Easley 1984; Greenwood and Shleifer 2014; Manski 2017). Usually expectations are treated as agent's forecasts of trends and values of economic and financial variables, inflation and bank rates, income and prices, technology and weather forecasts and etc. As expectation we regard agents assumptions on future state and dynamics of any economic variables or factors that can impact economic development. Variability and diversity of factors and variables that establish agent's expectations make them the major source of the randomness that impact decisions on market transactions and through them the major randomness impact on market and price dynamics. Expectations are most ambiguous economic issues. To simplify the problem as much as possible let's take the following assumptions. Let's propose that each market transaction is performed under expectations those approve decisions on cost and volume of transactions taken by two agents involved into transaction. Let's define as x_j, $j=1,..4$ expectations of agents involved into market transaction. Let's take that x_1, x_2 – describe expectations on cost and volume of the first agent – the seller and x_3, x_4 describe expectations on cost and volume of the second agent – the buyer. Let's assume that expectations impact each other and the cost and the volume of the transaction depend on all expectations of both agents. We assume that random changes of expectations dx_j cause random change of transaction's volume V and cost C as

$$dC = C \sum_{j=1,4} A_j \, dx_j \quad ; \quad dV = V \sum_{j=1,4} B_j \, dx_j \tag{5.1}$$

$$dp = p \sum_{j=1,4} D_j \, dx_j \; ; \quad D_j = A_j - B_j \; ; j = 1,..4 \tag{5.2}$$

$$A_j = \frac{\partial lnC}{\partial x_j} \quad ; \quad B_j = \frac{\partial lnV}{\partial x_j} \tag{5.3}$$

Relations (5.3) model dependence of transaction's cost C and volume V on expectations x_j. As we argue above agent's expectations may be based on any economic or financial variables or factors those impact economic development and may determine decisions on making the market transactions with underlying assets. Let's regard cost C and volume V of market transactions and coefficients A_j and B_j as functions of expectations x_j. That imply possible dependence of coefficients (5.3) on agents expectations x_j, $j=1,..4$. As we show below this simple assumption permit argue non-linear models for option pricing.

For simplicity let's take that random expectations dx_j follow the standard Brownian motion

$$dx_j = \mu_j dt + \sigma_j dW_j \; ; \; j = 1,..4 \tag{5.4}$$

$$< dW_i dW_j > = \lambda_{jk} dt \; ; \lambda_{jj} = 1 ; |\lambda_{jk}| \leq 1 ; \; j, k = 1,..4 \tag{5.5}$$

Four Brownian processes imply that option price should depend on four variables and we take price p and three expectations x_j, $j=1,2,3$ (5.4; 5.5) as independent variables. BSM-like equation on option price $S=S(t,p,x_1,x_2,x_3)$ takes form

$$\frac{\partial S}{\partial t} + rp\frac{\partial S}{\partial p} + rx_j\frac{\partial S}{\partial x_j} + \frac{1}{2}p^2\sigma_p^2\frac{\partial^2 S}{\partial p^2} + \frac{1}{2}\sum_{j,k=1,2,3}\lambda_{jk}\,\sigma_j\sigma_k\frac{\partial^2 S}{\partial x_j\partial x_k} + p\varrho_j\frac{\partial^2 S}{\partial x_j\partial p} = rS \quad (6.1)$$

$$\sigma_p{}^2 = \sum_{j,k=1,.4}\lambda_{jk}\,D_jD_k\,\sigma_j\sigma_k \quad (6.2)$$

$$\varrho_j = \sum_{k=1,.4}\lambda_{jk}D_k\,\sigma_k\sigma_j \quad (6.3)$$

It is clear that seller and buyer may perform market transactions with underlying asset on base of numerous expectations. Agent's expectations are most uncertain and most influential economic issues that deliver major randomness to financial markets and economics as a whole. Ensemble of these expectations and their disturbances deliver additional uncertainty to asset price $p(t)$ and through it to option pricing. In (Olkhov, 2019) we present a simple model that describes direct impact of small fluctuations of expectations on price and return fluctuations and price-volume relations. Similar model can be used to model impact of ensemble of expectations on option pricing.

4. Non-linear option pricing models

Expectations that govern the underlying asset transactions may concern options pricing dynamics. In other words – market traders may perform transactions with underlying assets on base of information and assessments of corresponding option trading data. Actually we believe that experienced investors and professional traders use all market information available to them to establish their expectations and to take the preferable market transaction. Thus the cost C and the volume V of market transactions with underlying depend on agent's expectations and coefficients (5.3) may depend on agent's expectations formed by current option price S or its derivatives by time t or by price p and etc. For example, relations (5.3) that describe direct dependence on option price S may model nonlinear dependence of option pricing models. Non-linear option pricing models are studied for more than 25 years (Bensaid, et.al., 1992; Sircar and Papanicolaou, 1998; Frey, 2008; Frey and Polte, 2011; Loeper, 2018).

Expectations of investors and traders on option pricing data impact their market transactions on underlying assets and cause non-linear option pricing equations. As a toy model let's regard dependence of transactions cost $C(t)$ on expectation x that is determined by option price S

$$C = C(t,x) \; ; \quad A(t,S) = \frac{\partial lnC}{\partial x} \quad (7.1)$$

Let's take for simplicity that transactions cost $C(t)$ depend on single expectation x and volume $dV=0$. Let's assume that investors and traders forecast change dx of their expectation due to standard Brownian motion and

$$dx = \mu dt + \sigma dW \; ; \; <dWdW> = dt \qquad (7.2)$$

Then, due to (3.6; 3.7) change of price dp can be presented as:

$$dp\,(t) = pA(S)dx = pA(S)[\mu dt + \sigma dW] \; ; \; dV = 0 \qquad (7.3)$$

and the equation (4.1) is reduced to classical BSM equation (1.3) with non-linear term and takes the form:

$$\frac{\partial S}{\partial t} + rp\frac{\partial S}{\partial p} + \frac{1}{2}A^2(S)\sigma^2 p^2 \frac{\partial^2 S}{\partial p^2} = rS \qquad (7.4)$$

Definite dependence of A^2 on option price S on its derivatives should be studied for each particular case separately. We don't study here any particular non-linear BSM-like equation but outline only a simple and direct way to take into the consideration impact of expectations on option pricing and method for derivation of corresponding the non-linear BSM-like equations. The similar assumptions can induce more sophisticated non-linear BSM-like equations in the dimension two, three or four that take into account impact of two, three or four expectations starting with (6.1). Such non-linear BSM-like equations can describe dependence of coefficients (5.3) on option price S or its derivatives by time t or by price p.

Description of impact of expectations on transactions and their cost C and volume V is rather complex problem. There are numerous agents involved into market transactions with underlying or with options. Different agents perform their transactions under numerous expectations. As we mentioned above agents may establish their expectations on base of any economic and financial variables, market and tax trends, technology and climate forecasts, and on base of any social or psychology factors that may impact agent's mood. Thus description of option pricing as well as description of asset pricing should take into account definite "mean" action of various expectations that impact decisions of different agents. Such "mean" expectations as well as fluctuations from "mean" expectations impact underlying and option pricing. The methods for description of distribution of expectations of different agents and modeling "mean" expectations are presented in Olkhov (2019). These methods introduce distributions of agents, transactions and expectations that help describe price-volume and return-volume disturbances for asset pricing. The approach to economic modeling developed by (Olkhov, 2016a; 2016b) gives opportunity to argue some hidden problems of option pricing (Olkhov, 2016c). We refer for these studies for further details.

5. Stochastic volatility

It is well known that assumption on constant volatility of the classical BSM model doesn't match the market reality. Numerous extensions of BSM equations were proposed to describe impact of stochastic volatility on option pricing. Stochastic volatility models were developed starting at least with Cox and Ross (1976) and then followed by Hull and White (1987), Heston (1993), Ball and Roma (1994), Saikat (1996), Poon (2005), Engle and Figlewski (2014), Cohen and Tegner (2018). Further studies of stochastic volatility models concern usage of various assumptions on properties of stochastic processes that may describe real properties of market volatility variations. Stochastic volatility models (Heston, 1993, Poon, 2005) describe transition from 1-dimensional BSM equation to 2-dimensional heat-type equation. As we show impact of the cost and the volume of transactions with underlying induce 2-dimensional BSM-like equation (4.1; 4.2). If one takes into account impact of expectations those approve decisions on market transactions then option pricing may obey two, three or four-dimensional BSM-like equations with constant volatilities. Extensions of (4.1; 4.2) equations to stochastic volatility model introduce two additional random variables: random cost volatility σ^2_c random volume and σ^2_v that follow Brownian motion $dW_{\sigma c}$ and $dW_{\sigma v}$. Let's define

$$x \ = \ \sigma_c{}^2 \quad ; \quad y \ = \ \sigma_v^2 \tag{8.1}$$

Relations (3.1-3.5) those define equation (4.1) stochastic volatility model are complemented by additional relations (Heston, 1993; Poon, 2005)

$$dx = \alpha_x(\theta_x - x)dt + \sigma_x\sqrt{x}\, dW_x \tag{8.2}$$

$$dy = \alpha_y(\theta_y - y)dt + \sigma_y\sqrt{y}\, dW_y \tag{8.3}$$

Relations (3.1-3.5) and (8.1; 8.2) define four independent Brownian motions and induce corresponding 4-dimension BSM-like equation. To avoid excess complexity let's present 3-dimensional Heston-like equation $S=S(t,p,V,x)$ that model the stochastic cost volatility $x=\sigma^2_c$ and keep the volume volatility $y= \sigma_v^2$ - constant.

$$\frac{\partial S}{\partial t} + rp\frac{\partial S}{\partial p} + rV\frac{\partial S}{\partial V} + [\alpha_x(\theta_x - x) - \vartheta x]\frac{\partial S}{\partial x} + \frac{1}{2}p^2(x - 2\lambda\sigma_v\sqrt{x} + \sigma_v^2)\frac{\partial^2 S}{\partial p^2} + \frac{1}{2}V^2\sigma_v^2\frac{\partial^2 S}{\partial V^2} +$$

$$\frac{1}{2}\sigma_x^2 x\frac{\partial^2 S}{\partial x^2} + p\sigma_x\sqrt{x}\left[\sqrt{x}\lambda_{cx} - \sigma_v\lambda_{vx}\right]\frac{\partial^2 S}{\partial x\partial p} + V\sigma_v\sigma_x\sqrt{x}\,\lambda_{vx}\frac{\partial^2 S}{\partial V\partial x} + pV\varrho\frac{\partial^2 S}{\partial V\partial p} = rS \tag{8.4}$$

$$< dW_x(t)dW_c(t) > = \lambda_{xc}\, dt \quad ; \quad < dW_x(t)dW_v(t) > = \lambda_{xv}\, dt \tag{8.5}$$

If dW_v is identical to dW_c or for is V-const then the equation (4.1) is reduced to the classical (1.3) and (8.4) is reduced to the Heston stochastic volatility equation (8.6) (Heston, 1993)

$$\frac{\partial S}{\partial t} + rp\frac{\partial S}{\partial p} + [\alpha_x(\theta_x - x) - \vartheta x]\frac{\partial S}{\partial x} + \frac{1}{2}p^2 x\frac{\partial^2 S}{\partial p^2} + \frac{1}{2}\sigma_x^2 x\frac{\partial^2 S}{\partial x^2} + \lambda_{xc}\sigma_x px\frac{\partial^2 S}{\partial x\partial p} = rS \tag{8.6}$$

During more then 25 years transitions of the option pricing from constant to stochastic volatility models in the Heston approximation were described in numerous papers (Heston, 1993; Poon, 2005; Cohen and Tegner, 2018) and we are not going to reproduce them once more. We just show that description of stochastic volatility of the cost and the volume or stochastic volatility of expectations increase dimension of the equations (4.1) and (6.1) and add extra complexity for option pricing modeling. Nonlinear equations (7.4) with constant volatilities σ^2 also can be starting points for extension to stochastic volatility approximation. Description of stochastic volatility increase dimension of the equation (7.4) and add extra complexity for solving high dimensional nonlinear equations.

6. Discussion

Let's argue some internal problems of option pricing modeling. Studies of these problems may clarify relations between forecasting of option pricing on base of the BSM-like equations and real market data.

High frequency trading can deliver thousands records of market transactions per second. This information is very useful for short-term market assessment during one hour or one day. But even intraday volatility assessments may require initial aggregation of high frequency market data for seconds or minutes. Option pricing assessments for weeks and months may need time data aggregation during hours or days. Aggregation of market data by time term t_1 means that the option price evolution model has internal time scale t_1. This time scale t_1 may impact random properties of the underlying price and on option price model dynamics. The second time scale $t_2 > t_1$ is responsible for the averaging procedure <..> to assess the mean value and volatility of Brownian processes (1.2). The scale $1/t_2$ define maximum frequency scale for the problem under consideration. Usage of different time scales t_1, t_2 to solve the same options pricing problem with scale - time to maturity T may cause different properties, as distinctive frequency scales will be different. On the other hand usage of time scales t_1 and t_2 causes aggregation of economic and financial variables of the problem. In particular it means aggregation of cost and volume of the transactions with underlying during time term t_2 to obtain correct values for price $p(t)$ due to relations (3.6; 4.1). Roughly speaking, one should measure the sum of the cost $C(t)$ of sum of the volume $V(t)$ of all transactions during time term t_2 and their ratio (3.6) should define the price $p(t)$. Hence the solution $S(T,p,V)$ of the option price equation (4.1) should depend on the internal time scale t_2.

Existence of the internal time scales $t_1 < t_2$ for the option pricing and the requirement to use relations (3.6) to define price $p(t)$ at moment t arises the problem of impact of expectations on

asset and option pricing. Indeed, relations (3.6) for cost $C(t)$ and volume $V(t)$ aggregated by all transaction during time term t_2 define so-called weighted average price $p(t)$ weighted by volumes of transactions. However expectations of agents those approve agents decisions on value of cost and the volume of the performed transactions can be based on any factors that might impact agent's will to make a deal. For example, agent's expectations can be based on assessment of price $\pi(t)$ as simple average during same time term t_2. The difference between two assessments of price may be great. For example let's take a price equals 10S per share at first term and during the second term price equals 30$ per share. Simple average during two time terms define mean price $\pi(t)=(10\$+30\$)/2=20\$$ per share. But the volume V_1 of transactions during the first time term was $V_1 =100$ shares and the volume V_2 of transactions during the second time term was $V_2 =1$ share. Thus total cost equals C=1030$ and total volume equals V=101 shares. Hence weighted average price $p(t)$ equals 10,2$. The difference between weighted average price $p(t)$ equals 10,2$ and "simple" average price $\pi(t)=20\$$ is sufficient enough to add significant perturbations in trading of the asset and the price trend. We don't argue what price definition should be treated as "correct" for pricing modeling. We only underline that impact of agents expectations on asset and option pricing can be based on such factors as different treatments of the same financial issues like price or other variables.

Above issues and variety forms and dimensions of possible linear or nonlinear equations (4.1; 6.1; 7.4; 8.4) that can model options pricing in different assumptions on constant or stochastic volatility or on impact of expectations arise the problem of sufficiency requirements of option modeling. The classical BSM equations (1.3) deliver simple 1-dimensional model that describe the option price dynamics with reasonable accuracy. Classical BSM model has only two parameters. One should only wonder how such a simple equation (1.3) gives so good model forecast for option pricing. Any further extensions of the classical BSM model add more accuracy but that cost extra complexity.

We consider the classical BSM model assumptions and replace the only one. We take into account the dependence of underlying price on cost and volume of market transactions. This trivial point gives sufficient reasons to extend the 1-dimensional BSM equations (1.3) to 2-dimensional equations (4.1). But such extension even for the case with constant parameters costs extra volume volatility and correlation coefficients. Transition from constant to stochastic volatility approximations turn the model much more complex 3- or 4-dimensional equations (8.4).

Common understanding that market transactions are governed by agents expectations and infinite diversity of these expectations transfers the relatively simple 2-dimensional equation

(4.1) to 4-dimensional equation (6.1) with constant coefficients. Any attempt to model stochastic volatility of underlying asset price of stochastic volatility of expectations would increase the dimension and the complexity of the equations. Moreover any sophistication of option pricing equations requires additional sophistication of market econometrics. Actually market data on transactions cost and volume are among the usual and we hope that we don't add excess complexity to option price description on base of (4.1). As well the attempt to take into considerations the impact of agents random expectations on market transactions and use corresponding econometric data may be not too simple.

We regard the balance between the accuracy of the description and complexity of econometrics as main problem for asset and option pricing modeling. We don't see any place for unique "correct and perfect" option pricing model equation but propose that different approximations should serve for different option markets. Any step towards accuracy will be compensated by two steps back to complexity due to changes in agents expectations and their impact on asset and option pricing. Nonlinear relations even in simplest form modeled by (7.4) reflect the tip of the complex mutual dependence between underlying and options.

7. Conclusion

After almost 50 years since publication the classical BSM options pricing model remains the source for further investigations. Change of only one initial assumption of the BSM model that concerns proposals on price random behavior allows extends BSM equation (1.3) from one to two dimensional BSM-like equation (4.1). This extension is a result of simple presentation of price $p(t)$ as ratio of cost $C(t)$ and volume $V(t)$ of market transaction with the underlying. We present simple conditions on properties of random behavior of the cost and the volume those reduce equation (4.1) to classical BSM equation (1.3). Simple trick with the cost and the volume of market transactions opens the way for further extension of BSM-like equation (4.1) to 4-dimensinal equation (6.1) that describe action of expectations on decisions of performed transactions with particular cost and volume and through it describe impact of expectations on underlying and option price. Equation (6.1) serves as starting point for modeling (7.4) non-linear relations between underlying and option market and opens the way for equation (8.4) that describe the Heston stochastic volatility model impact on option pricing. The set of the BSM-like equations (4.1; 6.1; 7.4; 8.4) have only common origin – simple proposal to define the price as the ratio of the cost and the volume of the market transactions.

These equations don't simplify the description of the option pricing but may adopt impact of the real economic and financial factors on the underlying and the option pricing. Complex market relations require complex equation to describe real market processes and we hope that usage of equations (4.1; 6.1; 7.4; 8.4) and obvious directions for their extensions may improve current asset and option pricing modeling.

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