A folk theorem in infinitely repeated prisoner's dilemma with small observation cost

Hino, Yoshifumi

Vietnam-Japan University

10 December 2018

A folk theorem in infinitely repeated prisoner's dilemma with small observation cost*

Yoshifumi Hino †

Business Administration Program, Vietnam–Japan University

This version: April 30, 2020

Abstract

We consider an infinitely repeated prisoner's dilemma under costly observation. Players choose whether to observe the opponent or not after they choose their actions. If a player observes the opponent, he pays a small observation cost and he can observe the action chosen by his opponent in that period. Otherwise, he receives no signal or an inaccurate private signal. First, we prove an efficiency result that players can achieve a symmetric nearly Pareto efficient outcome. Then, we extend the idea with an interim public randomization device, which is realized just after players choose actions. Players can decide their observational decision after they see the interim public randomization device. We present a folk theorem for a sufficiently small observation cost when players are sufficiently patient.

Keywords Costly observation; Efficiency; Folk theorem; Prisoner's dilemma

JEL Classification: C72; C73; D82

1 Introduction

A standard insight in the theory of repeated games is that repetition enables players to obtain collusive and efficient outcomes. However, a common and important assumption behind such results is that the players in the repeated game can monitor each other's past behavior without any cost. We analyze an infinitely repeated prisoner's dilemma game where each player can only observe his opponent's previous action at a (small) cost after they choose actions. We establish an approximate efficient result. Then, we introduce an interim public randomization device, which is realized just after they choose actions, and show an approximate folk theorem.

*This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

†Business Administration Program, Vietnam–Japan University, Luu Huu Phuoc Street, My Dinh 1 Ward, Nam Tu Liem District, Hanoi, 12015, Vietnam
Tel: +84 24 7306 6001
Email: hino@vnu.edu.vn

In our model, we consider costly observation as a monitoring structure. Each player chooses his action, and then he makes an observational decision. If a player chooses to observe his opponent, then he can observe the action chosen by the opponent. The observational decision itself is unobservable. The player receives extremely inaccurate private signal.

Furthermore, no player can statistically identify the observational decision of his opponent. That is, our monitoring structure is neither almost-public private monitoring (Hörner and Olszewski (2009); Mailath and Morris (2002, 2006); Mailath and Olszewski (2011)), nor almost perfect private monitoring (Bhaskar and Obara (2002); Chen (2010); Ely and Välimäki (2002); Ely et al. (2005); Hörner and Olszewski (2006); Sekiguchi (1997); Piccione (2002); Yamamoto (2007, 2009))

We present two results. First, we show that the symmetric Pareto efficient payoff vector can be approximated by a sequential equilibrium under some assumptions regarding the payoff matrix when players are patient and the observation cost is small (efficiency). This first result shows that collusive outcomes can be approximated if it is symmetric. The second result is an approximate folk theorem. We introduce an interim public randomization device just after players choose actions. Players can see the public randomization before they choose their observational decisions. We present an approximate folk theorem under some assumptions regarding the payoff matrix when players are patient and the observation cost is small. We also show that a (standard) public randomization device which is realized at the end of stage game does not work instead of the interim public randomization device. This second result shows that any collusive outcomes can be approximated if an interim public randomization device is available.

The nature of our strategy is similar to the *keep-them-guessing strategies* in Chen (2010). In our strategy, each player i chooses C_i with certainty at the cooperation state, but randomizes the observational decision. Depending on the observation result, players change their actions from the next period. If the player plays C_i and observes C_j, he remains in a cooperation state. However, in other cases (for example, the player does not observe his opponent), player i moves out of the cooperation state and chooses D_i. From the perspective of player j, player i plays the game as if he randomizes C_i and D_i, even though player i chooses pure actions in each state. Such randomized observations create uncertainty about the opponents' state in each period and give an incentive to observe.

Our main contribution is the efficiency result and an approximate folk theorem in an infinitely repeated prisoner's dilemma. Some previous studies show that the efficiency result holds if communication or private signals are available. For example, Miyagawa et al. (2008) assume that some noisy information is available even if players do not observe their opponent. We discuss previous studies in Section 2. Our efficiency result holds in the least stringent setting compared with other studies.

Another contribution of the paper is a new approach to the construction of a sequential equilibrium. We consider randomization of observation, whereas previous studies confine their attention to randomization of actions. In many cases, the observational decision is supposed to be unobservable in costly observation models. Therefore, even if a player observes his opponent, he cannot know whether the opponent observes him. If the continuation strategy of the opponent depends on the observational decision in the previous period, the opponent might randomize actions from the perspective of the player, even though the opponent chooses pure actions in each history. This new approach enables us to construct a nontrivial sequential equilibrium.

The rest of this paper is organized as follows. In Section 2, we discuss previous studies, and in Section 2.1, we focus on some previous literature and explain some difficulties in constructing a cooperative relationship in an infinitely repeated prisoner's dilemma under costly observation. Section 3 introduces a repeated prisoner's dilemma model with costly observation. In Section 4, we present our efficiency result. For efficiency result, we do not utilize an interim public randomization device. After that, applying the efficiency result, we present a folk theorem with an interim public randomization device. Section 5 provides concluding remarks.

2 Literature Review

We review previous studies on repeated games under costly observation.

One of the greatest difficulties in costly observation is observing the observation activity of opponents, because observational behavior under costly observation is often assumed to be unobservable. Each player has to check this unobservable observation behavior to motivate the other player to observe. Some previous studies circumvent the difficulty by assuming that the observational decision is observable. Kandori and Obara (2004) and Lehrer and Solan (2018) assume that players can observe other players' observational decisions.

Ben-Porath and Kahneman (2003) analyze an information acquisition model with communication. They show that players can share their information through explicit communication, and present a folk theorem for any level of observation cost. Ben-Porath and Kahneman (2003) consider randomizing actions on the equilibrium path. In their strategy, players report their observations to each other. Then, each player can check whether the other player observes him by the reports. Therefore, players can check the observation activities of other players.

Miyagawa et al. (2008) consider that communication is not allowed, but players can obtain imperfect private signals about the other player's action even when they do not observe their opponent. They show that players can communicate with each other using private signals, and present a folk theorem for any level of observation cost.

Another approach is introduction of nonpublic randomization device to infinitely repeated prisoner's dilemma. The nonpublic randomization device enables players to correlate their actions. Hino (2019) shows that if a nonpublic randomization device is available before players choose their actions and observational decisions, then players can achieve an efficiency result.

If these assumptions do not hold, that is, if costless information is unavailable, then cooperation is difficult. Two other papers present folk theorems without costless information. Flesch and Perea (2009) consider observation structures similar to our structure. In their model, players can purchase the information about the actions taken in the past if the players incur an additional cost. That is, some organization keeps track of all the sequence of the action profiles, and each player can purchase the information from the organization. Flesch and Perea (2009) present a folk theorem for an arbitrary observation cost. Miyagawa et al. (2003) consider less stringent models. They assume that no organization keeps track of all the sequence of the action profiles for players. Players can observe the opponent's action in the current period, and cannot purchase the information about the actions in the past. Therefore, if a player wants to keep track of actions chosen by the opponent, he has to pay observation cost every period. This observation structure is the same as the one in the current paper. Miyagawa et al. (2003) present a folk theorem with a small observation cost.

The above two studies, Flesch and Perea (2009) and Miyagawa et al. (2003), consider communication through mixed actions. To communicate with each other by mixed actions, the above two papers need more than two actions for each player. This means that their approach cannot be applied to infinitely repeated prisoner's dilemma under costly. We discuss their implicit communication in Miyagawa et al. (2003); Flesch and Perea (2009) in Section 2.1 in more detail.

It is an open question of whether players can achieve an efficiency result and a folk theorem in two-action games, even though the observation cost is sufficiently small. We show an efficiency result without any randomization device using a mixed observation rather than mixed actions when observation cost is small. We will extend the efficiency result using public randomization, and present a folk theorem in infinitely repeated prisoner's dilemma when observation cost is small.

2.1 Cooperation failure in the prisoner's dilemma (Miyagawa et al. (2003))

Consider the bilateral trade game with moral hazard in Bhaskar and van Damme (2002) simplified by Miyagawa et al. (2003).

		Player 2		
		C_2	D_2	E_2
	C_1	$1\ ,\ 1$	$-1\ ,\ 2$	$-1\ ,\ -1$
Player 1	D_1	$2\ ,\ -1$	$0\ ,\ 0$	$-1\ ,\ -1$
	E_1	$-1\ ,\ -1$	$-1\ ,\ -1$	$0\ ,\ 0$

Table 1: Extended prisoner's dilemma

Players choose whether he observes the opponent or not together with his action choice. Miyagawa et al. (2003) consider the following keep-them-guessing automaton strategy to approximate payoff vector $(1, 1)$. There are three states: cooperation, punishment, and defection.

In the cooperation state, each player chooses C_i with a sufficiently high probability and chooses D_i with the remaining probability. Players observe their opponent irrespective of their actions in the cooperation state. If players observe (C_1, C_2) or (D_1, D_2), the state remains the same. When (C_1, D_2) or (D_1, C_2) is realized, the state moves to the punishment state. The state moves to the defection state if player i chooses E_i or observes E_j. In the punishment state, both players choose E_i for some periods, and then the state moves back to a cooperation state. In the defection state, both players choose E_i, and the state remains the same. In both the punishment state and the defection state, the players do not observe their opponent.

Players have an incentive to observe their opponent because their opponent randomizes actions C_j and D_j in the cooperation state. If a player does not observe their opponent, the player cannot know in which state the opponent is in the next period. If the opponent is in the cooperation state, action E_i is a suboptimal because the opponent never chooses action E_j. That is, choosing action E_i has some opportunity cost because the opponent is in the cooperation state with a positive probability. However, if the opponent is the punishment state, then action E_i is a unique optimal action. Choosing actions C_i or D_i also has opportunity costs because the opponent is in the punishment state with a positive probability. To avoid these opportunity costs, players have an incentive to observe.

These ideas do not hold in two-action games. Suppose that action E_i is not available and consider the prisoner's dilemma as an example. If players randomize C_i and D_i in the cooperation state, then one of the best response actions in the cooperation state is action D_i. The best response action in punishment and defection states is also D_i. As a result, irrespective of player i's observation result, one of the optimal continuation strategies is choosing D_i and not observe player j every period. Therefore, Players don't have an incentive to observe.

I consider the following automaton strategy. In the initial state, player i randomizes actions and observe the opponent with a positive probability only when he chooses C_i. If he chooses C_i and observes C_j, he moves to the cooperation state in the next period. Otherwise, he moves to the defection state.[1] In the cooperation state, player i chooses action C_i with probability one, but randomizes observational decision. Only if player i chooses C_i and observes C_j, player i can remain in the cooperation state. Otherwise, player i moves to the defection state.

The reason why our strategy works is that the strategy prescribes pure action of C_i and does not prescribe a mixed actions in the cooperation state. The repetition of D_i from the cooperation state is not prescribed action. However, it causes another problem related to the observation incentive. As player j does not randomize his action in the cooperation state, player i can easily guess player j's action if he knows that player j is in the cooperation state. In such a situation, player i loses the observation incentive again.

Our strategy can overcome this difficulty as well. Since player j randomize his observational decisions in the cooperation state, player i in the cooperation state cannot know whether player j observed player i or not. If player j does not observes player i, player j moves to the defection state and chooses D_j. Player i cannot be certain that player j is in the cooperation state even if he chooses C_i and observes C_j in the previous period. To obtain the latest information about player j's state, player i has an incentive to observe the opponent in the cooperation state. This is why player i has an incentive to observe player j given our strategy.

3 Model

The stage game is a symmetric prisoner's dilemma, but it has two phases: the action phase and the observation phase. In the action phase, each player i ($i = 1, 2$) chooses an action, C_i or D_i. Let $A_i \equiv \{C_i, D_i\}$ be the set of actions for player i. After both players chooses actions, each player i receives a signal z_i costlessly and privately. The set of private signal for player i is finite set and denoted by Z_i. A signal profile $z = (z_1, z_2) =\in Z \equiv Z_1 \times Z_2$ is realized with probability $\rho(z|a)$ given an action profile $a = (a_1, a_2) \in A \equiv A_1 \times A_2$.

Assumption 1. There exists some $\zeta > 0$ such that

$$\rho(z|a) > \zeta, \quad \forall z \in Z,\ \forall a \in A.$$

We define the accuracy η_i of the signal z_i as follows.

$$\eta_i \equiv 1 - \min_{z_i \in Z_i, a, a' \in A} \frac{\rho(z_i|a')}{\rho(z_i|a)}.$$

[1] For the formal proof, we need another state (transition state). Transition state is crucial only when we consider off the equilibrium path. Therefore, it is omitted here.

The base game payoff for player i is given by $\pi_i(a_i, z_i)$. Given an action profile $a \in A$, an expected base game payoff for player i, $u_i(a) \equiv \Sigma_{z_i \in Z_i} P(z_i|a)\pi_i(a_i, z_i)$, is displayed in Table 2.

		Player 2: C_2	Player 2: D_2
Player 1	C_1	$1 \ , \ 1$	$-\ell \ , \ 1+g$
	D_1	$1+g, \ -\ell$	$0 \ , \ 0$

Table 2: Prisoner's dilemma

We make a usual assumption about the above payoff matrix.

Assumption 2. (i) $g > 0$ and $\ell > 0$; (ii) $g - \ell < 1$.

The first condition implies that action C_i is dominated by action D_i for each player i, and the second condition ensures that the payoff vector of action profile (C_1, C_2) is Pareto efficient. We impose an additional assumption.

Assumption 3. $g - \ell > 0$.

Assumption 3 is the same as Assumption 1 in Chen (2010).

Players simultaneously choose their observational decision in the observation phase after they choose their actions in the action phase. Let m_i represent the observational decision for player i. Let $M_i \equiv \{0, 1\}$ be the set of observational decisions for player i, where $m_i = 1$ represents "to observe the opponent," and $m_i = 0$ represents "not to observe the opponent." If player i observes the opponent, he incurs an observation cost $\lambda > 0$, and receives complete information about the action chosen by the opponent at the end of the stage game. If player i does not observe the opponent, he does not incur any observation cost and obtains no information about his opponent's action. We assume that the observational decision for a player is unobservable.

A stage behavior for player i is a pair of base game action a_i for player i and observational decision m_i for player i and is denoted by $b_i = (a_i, m_i)$. An outcome of the stage game is a pair of stage behaviors $b = (b_1, b_2)$. Let $B_i \equiv A_i \times M_i$ be the set of stage-behaviors for player i, and let $B \equiv B_1 \times B_2$ be the set of outcomes of the stage game. Given an outcome $b \in B$, the stage game payoff $\pi_i(b)$ for player i is given by

$$U_i(b) \equiv u_i(a_1, a_2) - m_i \cdot \lambda.$$

For any observation cost $\lambda > 0$, the stage game has a unique stage game Nash equilibrium outcome, $b^* = ((D_1, 0), (D_2, 0))$.

Let $\delta \in (0, 1)$ be a common discount factor. Players maximize their expected average discounted stage game payoffs. Given a sequence of outcomes of the stage games $(b^t)_{t=1}^{\infty}$, player i's payoff is given by average discounted stage game payoff:

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} U_i(b^t).$$

Player i's nonaveraged payoff is given by:

$$\sum_{t=1}^{\infty} \delta^{t-1} U_i(b^t).$$

We assume that an interim public randomization device is available just before players choose their observational decisions. The random variable X is uniformly distributed over $[0, 1)$ independently of the action profile. Each player observes the realized public randomization without any cost.

Let $o_i \in A_j \cup \{\phi_i\}$ be an observation result for player i. Observation result $o_i = a_j \in A_j$ implies that player i chose observational decision $m_i = 1$ and observed a_j. Observation result $o_i = \phi_i$ implies that player i chose $m_i = 0$, that is, he obtains no information about the action chosen by the opponent.

Let h_i^t be a (private) history of player i at the beginning of the action phase in period $t \geq 2$. This history h_i^t is a sequence of his own actions, realized public randomizations, observation results, and private signals up to period $t-1$: $h_i^t = (a_i^k, x^k, o_i^k, z_i^k)_{k=1}^{t-1}$. We omit the observational decisions $m_i^k (k < t)$ from h_i^t because observation result o_i^k implies the observational decision m_i^k for any $k < t$. Let H_i^t denote the set of all the histories for player i at the beginning of the action phase in period $t \geq 1$, where H_i^1 is an arbitrary singleton set. Similarly, a history $\hat{h}_i^t$ at the beginning of the observation phase in period $t \geq 1$ is (h_i^t, a_i^t, x^t).

An action strategy for player i in the repeated game is a function of the history h_i^t of player i in the action phase to his (mixed) actions. An observation strategy for player i in the repeated game is a function of a history $\hat{h}_i^t$ in the observation phase to his (mixed) observational decision. A (behavior) strategy is a pair of action strategy and observation strategy.

The belief ψ_i^t of player i in period t is a function of the history h_i^t in period t to a probability distribution over the set of histories for player j in period t; H_j^t. Let $\psi_i \equiv (\psi_i^t)_{t=1}^{\infty}$ be a belief for player i, and $\psi = (\psi_1, \psi_2)$ denote a system of beliefs.

A strategy profile σ is a pair of strategies σ_1 and σ_2. Given a strategy profile σ, a sequence of completely mixed behavior strategy profiles $(\sigma^n)_{n=1}^{\infty}$ that converges to σ is called a *tremble*. Each completely mixed behavior strategy profile σ^n induces a unique system of beliefs ψ^n.

The solution concept is a sequential equilibrium. We say that a system of beliefs ψ is consistent with strategy profile σ if a tremble $(\sigma^n)_{n=1}^{\infty}$ exists such that the corresponding sequence of systems of beliefs $(\psi^n)_{n=1}^{\infty}$ converges to ψ. Given the system of beliefs ψ, strategy profile σ is sequentially rational if, for each player i, the continuation strategy from any history in each phase is optimal given his belief and the opponent's strategy. It is defined that a strategy profile σ is a *sequential equilibrium* if a consistent system of beliefs ψ for which σ is sequentially rational exists.

4 Results

In this section, we show our efficiency result. Then, applying the efficiency result, we present a folk theorem with an interim public randomization device.

To prove the desired propositions, first, we assume $\eta_1 = \eta_2 = 0$. It means that a player obtains no information about the action of the opponent if he does not observe the opponent. We present related propositions given $\eta_1 = \eta_2 = 0$. After that, we will show the desired propositions using the related propositions.

4.1 Efficiency

The following proposition shows that the symmetric efficient outcome is approximated by a sequential equilibrium if the observation cost λ is small and the discount factor δ is moderately low.

Proposition 1. *Suppose that $\eta_1 = \eta_2 = 0$, Assumptions 2 and 3 are satisfied. For any $\varepsilon > 0$, there exist $\underline{\delta} \in \left(\frac{g}{1+g}, 1\right)$, $\overline{\delta} \in (\underline{\delta}, 1)$, and $\overline{\lambda} > 0$ such that for any discount factor $\delta \in [\underline{\delta}, \overline{\delta}]$ and for any observation cost $\lambda \in (0, \overline{\lambda})$, there exists a symmetric sequential equilibrium σ^* whose payoff vector (v_1^*, v_2^*) satisfies $v_i^* \geq 1 - \varepsilon$ for each $i = 1, 2$.*

Proof. See Appendix A. □

We here present the main idea. The precise proof will be give in Appendix A.

Strategy

First, we define our strategy σ^*. Fix any $\varepsilon > 0$. We define $\overline{\varepsilon}$, $\underline{\delta}$, $\overline{\delta}$, and $\overline{\lambda}$ as follows.

$$
\begin{aligned}
\overline{\varepsilon} \equiv & \frac{\ell^2}{54(1+2g)^3} \frac{\varepsilon}{1+\varepsilon}, \\
\underline{\delta} \equiv & \frac{g}{1+g} + \overline{\varepsilon}, \\
\overline{\delta} \equiv & \frac{g}{1+g} + 2\overline{\varepsilon} < 1, \\
\overline{\lambda} \equiv & \frac{1}{16} \frac{\ell}{(1+2g)^2} \overline{\varepsilon}^2.
\end{aligned}
$$

We fix an arbitrary discount factor $\delta \in [\underline{\delta}, \overline{\delta}]$ and an arbitrary observation cost $\lambda \in (0, \overline{\lambda})$.

Our strategy σ^* is represented by an automaton independently of private signal z. Let us consider the following automaton who has four types of states: initial state ω_i^1, cooperation state $(\omega_i^{C,t})_{t=2}^{\infty}$, transition state $(\omega_i^{E,t})_{t=2}^{\infty}$, and defection state ω_i^D. In the initial state ω_i^1, player i chooses D_i with probability $\beta_{i,1}$, and chooses C_i with probability $1 - \beta_{i,1}$.[2]When player i chooses C_i, he observes the opponent with probability $1-\beta_{i,2}$. Player i never observes the opponent when player i chooses D_i. In the cooperation state $\omega_i^{C,t}(t \geq 2)$, player i chooses C_i. If player i chooses C_i, he chooses $m_i = 1$ with probability $1 - \beta_{i,t+1}$. When player i chooses D_i, he never observes the opponent. In the transition state $\omega_i^{E,t}(t \geq 2)$ and defection state ω_i^D, player i chooses D_i and does not observe the opponent irrespective of his action. The prescribed actions and observational decisions are summarized in the table below.

State	ω_i^1	$\omega_i^{C,t}$	$\omega_i^{E,t}$	ω_i^D
Action	C_i w.p. $1-\beta_{i,1}$ D_i w.p. $\beta_{i,1}$	C_i	D_i	D_i
m_i given C_i	$m_i = 1$ w.p. $1-\beta_{i,2}$ $m_i = 0$ w.p. $\beta_{i,2}$	$m_i = 1$ w.p. $1-\beta_{i,t+1}$ $m_i = 0$ w.p. $\beta_{i,t+1}$	$m_i = 0$	$m_i = 0$
m_i given D_i	$m_i = 0$			

Table 3: Actions and observational decisions

[2]The probability $\beta_{i,t}(t \geq 1)$ will be defined using (1) and (2) later.

The state transition function is defined as follows. In the initial state ω_i^1, if player i observes $(a_i^t, o_i^t) = (C_i, C_j)$, he moves to the cooperation state $\omega_i^{C,2}$. When player i chooses D_i or observes D_j, the state in the next period is ω_i^D. Only when player i observes $(a_i^t, o_i^t) = (C_i, \phi_i)$, the state moves to the transition state $\omega_i^{E,2}$. In the cooperation state and transition state in period t, player i moves to the cooperation state $\omega_i^{C,t+1}$ if he observes $(a_i^t, o_i^t) = (C_i, C_j)$. If $(a_i^t, o_i^t) = (C_i, \phi_i)$, he moves to the transition state $\omega_i^{E,t+1}$. If player i chooses D_i or observes D_j, the state moves to the defection state ω_i^D. Note that player i moves back to the cooperation state $\omega_i^{C,t+1}$ from the transition state in period t if he observes (C_i, C_j), which is the event off the equilibrium path. The defection state ω_i^D is an absorbing state and player i never moves to another state from the defection state ω_i^D.

The state transition is summarized in Figure 1.

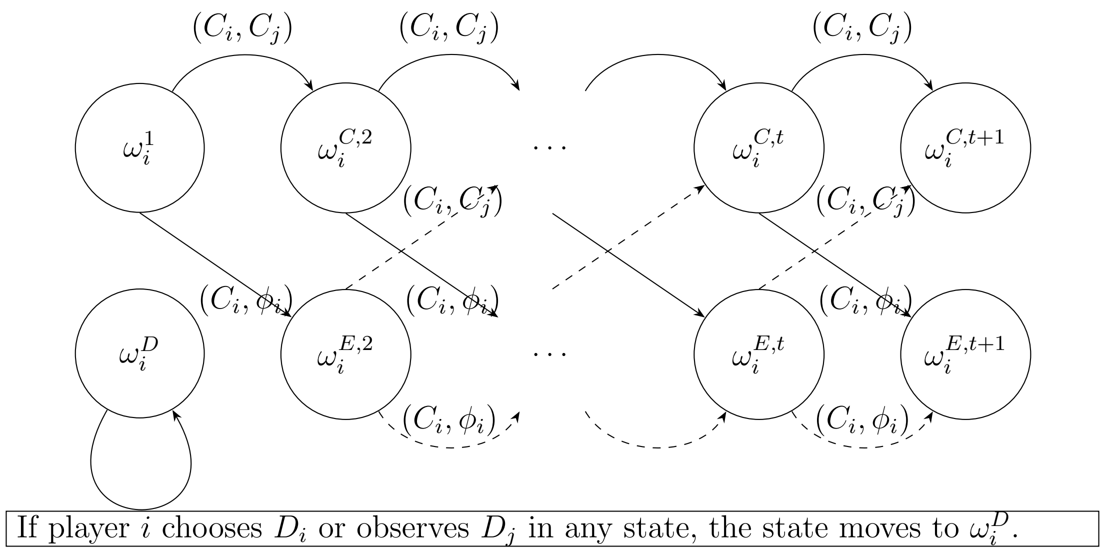


Figure 1: State transition function

Using the above automaton, we fix randomization probabilities in each state. Let us define $\varepsilon' \equiv \delta - \frac{g}{1+g}$. First, we fix a small probability $\beta_{i,1} \equiv \frac{1+g+\ell}{g+\ell}\varepsilon'$. We fix a probability $\beta_{i,2}$ so that player j is indifferent between actions C_j and D_j in the initial state ω_j^1. Hence, $\beta_{i,2}$ is determined as the solution of the following equality.

$$(1-\beta_{i,1})(1+g) = (1-\beta_{i,1}) \cdot 1 - \beta_{i,1} \cdot \ell + \delta(1-\beta_{i,1})(1-\beta_{i,2})(1+g). \tag{1}$$

The left-side is the nonaveraged payoff when player j chooses $(a_j^1, m_j^1) = (D_j, 0)$ in the initial state ω_j^1. The right-side is the one when player j chooses $(a_j^1, m_j^1) = (C_j, 0)$.

Probability $\beta_{i,t+2} (t \geq 1)$ is determined to make player j in state $\omega_j^{C,t}$ indifferent between $m_j = 1$ and $m_j = 0$ given his action C_j. Player j believes that player i is in the cooperation state with probability $1 - \beta_{i,t}$ because he observes C_j in the previous period and he is sure that player j was in the cooperation state $\omega_j^{C,t-1}$ in the previous period $t-1$. Therefore, probability $\beta_{i,t+2}$ is a solution of the following equality.

$$\begin{aligned}
&\delta(1-\beta_{i,t})(1-\beta_{i,t+1})(1+g) \\
=&(1-\beta_{i,t}) \cdot 1 - \beta_{i,t} \cdot \ell - \lambda \\
&+ \delta(1-\beta_{i,t})\left\{(1-\beta_{i,t+1}) \cdot 1 - \beta_{i,t+1} \cdot \ell + \delta(1-\beta_{i,t+1})(1-\beta_{i,t+2})(1+g)\right\}
\end{aligned} \tag{2}$$

The left-side is the nonaveraged payoff when player j chooses $m_j = 0$ in period t. The right-side is the one when player j chooses $m_j = 1$ in period t and chooses $(C_j, 0)$ if he is in the cooperation state $\omega_j^{C,t+1}$ in period $t+1$.

Specifically, $\beta_{i,2}$ is defined by (1), and $\beta_{i,t+2}$ ($t \in \mathbb{N}$) is defined by (2), or

$$\begin{aligned}
\beta_{i,2} =& \frac{(1-\beta_{i,1})\left\{\delta(1+g)-g\right\}-\beta_{i,1}\ell}{\delta(1-\beta_{i,1})(1+g)} \\
=& \frac{g+g^2-\ell^2-(1+g+\ell)(1+g)\varepsilon'}{(g+\ell)\left\{g+(1+g)\varepsilon'\right\}\left(1-\frac{1+g+\ell}{g+\ell}\varepsilon'\right)}\varepsilon' \\
=& \frac{1+g-\frac{\ell}{g}\ell-(1+g+\ell)\frac{1+g}{g}\varepsilon'}{1+\frac{\ell}{g}\frac{1}{g+\ell}\varepsilon'-\frac{(1+g)(1+g+\ell)}{g(g+\ell)}(\varepsilon')^2}\frac{1}{g+\ell}\varepsilon' \\
\beta_{i,t+2} =& \frac{(1-\beta_{i,t+1})\left\{\delta(1+g)-g\right\}-\beta_{i,t+1}\ell-\frac{\lambda}{\delta(1-\beta_{i,t})}}{\delta(1-\beta_{i,t+1})(1+g)}, \quad \forall\, t \in \mathbb{N}.
\end{aligned}$$

The following Lemma 1, which is proved in Appendix B, ensures that any $\beta_{i,t}$ is greater than zero and smaller than one.

Lemma 1. *Suppose that Assumptions 2 and 3 are satisfied. Fix any discount factor $\delta \in [\underline{\delta}, \overline{\delta}]$ and observation cost $\lambda \in (0, \overline{\lambda})$. Then, it holds that*

$$\frac{1}{2}\frac{1+g-\ell}{g+\ell}\varepsilon' < \beta_{i,2} < \beta_{i,4} < \beta_{i,6} \cdots < \beta_{i,5} < \beta_{i,3} < \beta_{i,1} = \frac{1+g+\ell}{g+\ell}\varepsilon'.$$

Strategy σ^* is the strategy defined by the above automaton.

Next we define a consistent system of beliefs with strategy profile σ^*. We consider a sequence of behavioral strategy profiles $(\hat{\sigma}^n)_{n=1}^{\infty}$ such that each strategy profile attaches a positive probability to every move, but puts far greater weights on the trembles on C_i in the defection state ω_i^D compared with other stage behaviors in the other states. These trembles induce a consistent system of beliefs that player i at any defection state ω_i^D is sure that the state of their opponent is the defection state ω_j^D or transition state $\omega_j^{E,t}$ for some $t \geq 2$.

Let us confirm this property of the belief. There are two cases where player i moves to the defection state ω_i^D; (1) player i observes D_j, (2) player i chooses D_i. The property is obvious in the first case. In any state of player j, player j moves to the defection state ω_j^D after he chooses D_j. Furthermore, the defection state ω_j^D is an absorbing state. Therefore, player i is certain that player j is in the defection state ω_j^D after player i observes D_j. The property is not obvious in the second case; $a_i = D_i$. Let us consider the following history of player i in period 3. Player i chooses $a_i = D_i$ and $m_i = 0$ in period 1, and he chooses C_i and $m_i = 1$ (by mistakes) and observes C_j in period 2. We can consider the following two types of player j's histories which are consistent with the history of player i. The first type of history is that player j chooses $a_j = D_j$ in period 1, and he chooses $a_j = C_j$ (by mistake) at the defection state ω_j^D in period 2. The second type of history is that player j chooses $a_j = C_j$ and $m_j = 0$ in period 1, and he chooses $a_j = C_j$ (by mistake) at the transition state $\omega_j^{E,2}$ in period 2. As we put far greater weights on the trembles on C_j in the defection state ω_j^D, player i is sure that the first type of history is realized, and player j is in the defection state ω_j^D. A similar argument holds even if player i observes $(a_i, o_i) = (C_i, C_j)$ many times after he chooses D_i.

An illustration

We here explain that the strategy σ^* is a sequential equilibrium whose payoff vector (v_1^*, v_2^*) satisfies $v_i^* \geq 1 - \varepsilon$ for each $i = 1, 2$.

Let us consider sequential rationality in each state. First, we consider the defection state ω_i^D. As we have considered above, player i in the defection state ω_i^D is certain that player j is in the defection state ω_j^D. Therefore, action D_i is optimal because player i is sure that player j does not observe player i. As player i is certain that player j is in the defection state ω_j^D and chooses D_j, observational decision $m_i = 0$ is also optimal.

Let us consider sequential rationality in the initial and cooperation states. By the definition of $\beta_{j,2}$ and $\beta_{j,3}$, player i is indifferent among $(C_i, 1)$, $(C_i, 0)$, and $(D_i, 0)$. Furthermore, if player i chooses D_i in the initial state ω_i^1, player j moves to the transition state $\omega_j^{E,2}$ or defection state ω_j^D. In either case, the continuation strategy of player j is a repetition of $(a_j, m_j) = (D_j, 0)$. As the observation result has no effect on the conjecture over the continuation strategy, player i has no incentive to choose $m_i = 0$ when he chooses action D_i. Therefore, it is optimal for player i to follow strategy σ^* in the initial state ω_i^1.

In the cooperation state $\omega_i^{C,t}(t \geq 2)$, player i is indifferent to his observational decisions by the definition of $\beta_{j,t+2}$. It is also suboptimal to choose $(a_i^t, m_i^t) = (D_i, 1)$ as in the initial state ω_i^1. Furthermore, the definition of $\beta_{j,t+1}$ ensures that player i strictly prefers action C_i to D_i in the cooperation state $\omega_i^{C,t}$. Using (2) for $t - 1$, we obtain the following equation.

$$(1 - \beta_{j,t}) - \beta_{j,t+1}\ell + \delta(1 - \beta_{j,t})(1 - \beta_{j,t+1})(1 + g) - \delta(1 + g) = \frac{\lambda}{\delta(1 - \beta_{j,t-1})} \tag{3}$$

The first three terms on the right-hand side represent the nonaveraged payoff when player i chooses C_i and $m_i = 0$ in the cooperation state $\omega_i^{C,t}$. The last term on the right-hand side is the nonaveraged payoff when player i chooses $(a_i, m_i) = (D_i, 0)$ in the cooperation state $\omega_i^{C,t}$. Therefore, (3) shows that choosing D_i at the cooperation $\omega_i^{C,t}$ state is not optimal. Sequential rationality at the cooperation state $\omega_i^{C,t}$ is satisfied.

Another explanation is as follows. Suppose that player i weakly prefers action D_i at the cooperation state $\omega_i^{C,t}$ in period t. As player j moves to the transition state $\omega_i^{E,t+1}$ or the defection state ω_j^D after player i chooses D_i in period t, the assumption implies that player i weakly prefers $(a_i, m_i) = (D_i, 0)$ from period t onwards. One of the optimal continuation strategies from the cooperation state $\omega_i^{C,t}$ coincides with the one from the defection state ω_i^D. Then, player i has no incentive to observe player j in the cooperation state ω_i^{t-1} because the repetition of $(a_i, m_i) = (D_i, 0)$ is one of his optimal continuation strategies irrespective of the observation result. It contradicts the definition of $\beta_{j,t+1}$. Therefore, player i strictly prefers action C_i in the cooperation state $\omega_i^{C,t}$.

Next, let us consider the transition state $\omega_i^{E,t}$. In the transition state $\omega_i^{E,t}$, player i does not know the action chosen by the opponent in the previous period $t - 1$. If $(a_i, a_j) = (C_i, C_j)$ is realized in the previous period, player i should be at the cooperation state $\omega_i^{C,t}$ and action D_i is suboptimal.

Although action D_i is suboptimal in the cooperation state $\omega_i^{C,t}$, the payoff when player i chooses D_i at $\omega_i^{C,t}$ is close enough to the one when he chooses D_i at $\omega_i^{C,t}$ when the observation cost λ is sufficiently small. If the payoffs are not close to each other, player i strictly prefers $m_i = 1$ at the cooperation state $\omega_i^{C,t-1}$ to know which state he should move to because the observation cost is small.

The loss from choosing D_i in the transition state $\omega_i^{E,t}$ is small. The loss from choosing C_i is strictly positive. Player j is in the transition state $\omega_j^{E,t}$ or defection state ω_j^D with probability at least $(1-\beta_{j,t-1})(1-\beta_{j,t})$, and then choosing C_i makes a loss of $-\ell$. Therefore, choosing C_i is suboptimal at the transition state $\omega_i^{E,t}$. We will prove this fact in Appendix A.

Next, let us consider the observation decision in the transition state $\omega_i^{E,t}$. It is straightforward that if player i chooses D_i, then $m_i = 0$ is optimal. Assume that player i chooses C_i. If player j chooses C_j in the previous period, then player i should have been at the cooperation state $\omega_i^{C,t}$ and one of the optimal stage behaviors given action C_i was $m_i = 0$. If player j chooses D_j in the previous period, then one of player i's optimal stage behaviors was $m_i = 0$. In each case, $m_i = 0$ is optimal. Therefore, $m_i = 0$ is optimal in the transition state $\omega_i^{E,t}$.

Lastly, let us consider the payoff. As player 1 prefers action D_i in the initial state ω_i^1, his payoff is given by

$$\begin{aligned} v_i^* &= (1-\delta)(1-\beta_{j,1})(1+g) \\ &= \{1-(1+g)\varepsilon'\}\left(1-\frac{1+g+\ell}{g+\ell}\varepsilon'\right) \\ &> 1-\left(1+g+\frac{1+g+\ell}{g+\ell}\right)\varepsilon' \\ &> 1-\varepsilon. \end{aligned}$$

Therefore, we have obtained Proposition 1.

Remark 1. In our strategy σ^*, the observation result in the current period determines the state in the next period independently of the past observation result (on the path of σ^*). Thus, each player has no incentive to acquire information in the past. Therefore, even if we allow players to purchase information in the past, our efficiency result holds.

Remark 2. As we do not use interim public randomization, the assumption that each player chooses an observational decision after he chooses his action is not crucial. Even if each player chooses his action and observational decision together, we can define a strategy and belief in a similar manner to strategy σ^* and belief ψ.

Proposition 2. *Fix any positive $\zeta > 0$. Suppose that Assumptions 2 and 3 are satisfied. For any $\varepsilon > 0$, there exist $\underline{\delta} \in \left(\frac{g}{1+g}, 1\right)$, $\overline{\delta} \in (\underline{\delta}, 1)$, $\overline{\lambda} > 0$, and $\overline{\eta} > 0$ such that for any discount factor $\delta \in [\underline{\delta}, \overline{\delta}]$, any observation cost $\lambda \in (0, \overline{\lambda})$, and any $\eta_1, \eta_2 \in [0, \overline{\eta})$, there exists a symmetric sequential equilibrium σ^* whose payoff vector (v_1^*, v_2^*) satisfies $v_i^* \geq 1-\varepsilon$ for each $i = 1, 2$.*

Proof of Proposition 2 . We show that the strategy σ^* in the proof of Appendix A is a sequential equilibrium under small η_1 and η_2. If player i is in the cooperation state $\omega_i^{C,t}$, he observed C_j in the previous period. Thus, private signal z_i has no effect on player i's belief. The best response stage-behavior in the cooperation state $\omega_i^{C,t}$ is unchanged. Let us consider the transition state $\omega_i^{E,t}$ or the defection state ω_i^D. In the proof of Appendix A, it has been proved that player i strictly prefers D_i and $m_i = 0$ in those states given $\eta_1 = \eta_2 = 0$. Therefore, because of continuity of expected utility function, player i strictly prefers prefers D_i and $m_i = 0$ when η_1 and η_2 is sufficiently close to zero. Hence, the strategy σ^* is a sequential equilibrium when η_1 and η_2 is sufficiently small. □

Next, we extend Proposition 2 using Lemma 2.

Lemma 2. *Fix any payoff vector v and any $\varepsilon > 0$. Suppose that there exist $\underline{\delta} \in \left(\frac{g}{1+g}, 1\right)$, $\overline{\delta} \in (\underline{\delta}, 1)$ such that for any discount factor $\delta \in [\underline{\delta}, \overline{\delta}]$, there exists a sequential equilibrium whose payoff vector (v_1^*, v_2^*) satisfies $|v_i^* - v_i| \leq \varepsilon$ for each $i = 1, 2$. Then, there exists $\underline{\delta}^* \in \left(\frac{g}{1+g}, 1\right)$ such that for any discount factor $\delta \in [\underline{\delta}^*, 1)$, there exists a sequential equilibrium whose payoff vector (v_1^*, v_2^*) satisfies $|v_i^* - v_i| \leq \varepsilon$ for each $i = 1, 2$.*

Proof of Lemma 2 . We use the technique of Lemma 2 in Ellison (1994). We define $\underline{\delta}^* \equiv \underline{\delta}/\overline{\delta}$, and choose any discount factor $\delta \in (\underline{\delta}^*, 1)$. Then, we choose some integer n^* that satisfies $\delta^{n^*} \in [\underline{\delta}, \overline{\delta}]$. Then there exists a strategy $\sigma^{*'}$ whose payoff vector is (v_1^*, v_2^*) given δ^{n^*}. We divide the repeated game into n^* distinct repeated games. The first repeated game is played in period $1, n^*+1, 2n^*+1 \ldots$, the second repeated game is played in period $2, n^*+1, 2n^*+2 \ldots$, and so on. Each repeated game can be regarded as a repeated game with discount factor δ^{n^*}. Let us consider the following strategy σ^L. In the 1st game, players follow strategy $\sigma^{*'}$. In the 2nd game, players follow strategy $\sigma^{*'}$. In the $n(n \leq n^*)$th game, players follow strategy $\sigma^{*'}$. Then, strategy σ^L is a sequential equilibrium because strategy $\sigma^{*'}$ is a sequential equilibrium in each game. As the equilibrium payoff vector in each game satisfies $|v_i^* - v_i| \leq \varepsilon$ for each $i = 1, 2$, the equilibrium payoff of strategy σ^L also satisfies $|v_i^* - v_i| \leq \varepsilon$ for each $i = 1, 2$. □

We obtain efficiency for a sufficiently high discount factor.

Proposition 3. *Fix any $\zeta > 0$. Suppose that Assumptions 2 and 3 are satisfied. For any $\varepsilon > 0$, there exist $\underline{\delta}^* \in (0, 1)$, $\overline{\lambda} > 0$, and $\overline{\eta} > 0$ such that for any discount factor $\delta \in (\underline{\delta}^*, 1)$, any $\lambda \in (0, \overline{\lambda})$, and any $\eta_1, \eta_2 \in [0, \overline{\eta})$, there exists a sequential equilibrium whose payoff vector (v_1^*, v_2^*) satisfies $v_i^* \geq 1 - \varepsilon$ for each $i = 1, 2$.*

Proof of Proposition 3 . Apply Lemma 2 to Proposition 1. □

Remark 3. Proposition 3 shows monotonicity of efficiency on the discount factor. If efficiency holds given some ε, observation cost λ, η_1, η_2 and discount factor δ, then efficiency holds given a sufficiently large discount factor $\delta' > \delta$.

4.2 Folk theorem

In what follows, we introduce an interim public randomization device at the end of the action phase. Public signal x is uniformly distributed over $[0, 1)$ independently of the action profile chosen. Each player observes the interim public signal without cost. The purpose of interim public randomization is to prove a folk theorem (Theorem 1).

Let

$$\mathcal{F} \equiv \text{convex hull of} \{u(a)|\ a \in A\},$$
$$\mathcal{F}^* \equiv \{v \in \mathcal{F}|\ v_1 \geq 0 \text{ and } v_2 \geq 0\}.$$

Theorem 1 (Approximate folk theorem)**.** *Suppose that an interim public randomization is available, and Assumptions 2 and 3 are satisfied. Fix any positive $\zeta > 0$. Fix any interior point $v = (v_1, v_2)$ of $\mathcal{F}^*$. Fix any $\varepsilon > 0$. There exist a discount factor $\underline{\delta} \in \left(\frac{g}{1+g}, 1\right)$, observation cost $\overline{\lambda} > 0$, and $\overline{\eta} > 0$ such that for any $\delta \in [\underline{\delta}, 1)$, any $\lambda \in (0, \overline{\lambda})$, and any $\eta_1, \eta_2 \in [0, \overline{\eta})$, there exists a sequential equilibrium whose payoff vector $v^F = (v_1^F, v_2^F)$ satisfies $|v_i^F - v_i| \leq \varepsilon$.*

To prove Theorem 1, we prove the following proposition first.

Proposition 4. *Suppose that a public randomization device is available, and* $\eta_1 = \eta_2 = 0$*, Assumptions 2 and 3 are satisfied. For any* $\varepsilon > 0$*, there exist* $\underline{\delta} \in \left(\frac{g}{1+g}, 1\right)$*,* $\overline{\delta} \in (\underline{\delta}, 1)$*, and* $\overline{\lambda} > 0$ *such that for any discount factor* $\delta \in [\underline{\delta}, \overline{\delta}]$ *and for any observation cost* $\lambda \in (0, \overline{\lambda})$*, there exists a sequential equilibrium* σ^{**} *whose payoff vector* (v_1^{**}, v_2^{**}) *satisfies* $v_1^{**} = 0$ *and* $v_2^{**} \geq \frac{1+g+\ell}{1+\ell} - \varepsilon$*.*

Strategy

First, we define strategy σ^{**} independently of private signal z, which will be used to present Proposition 4.

Fix any $\varepsilon > 0$. We define $\overline{\varepsilon}$, $\underline{\delta}$, $\overline{\delta}$, and $\overline{\lambda}$ as follows.

$$\begin{aligned}
\overline{\varepsilon} \equiv & \frac{\ell^2}{54(1+g)^3} \frac{\varepsilon}{1+\varepsilon}, \\
\underline{\delta} \equiv & \frac{g}{1+g} + \overline{\varepsilon}, \\
\overline{\delta} \equiv & \frac{g}{1+g} + 2\overline{\varepsilon} < 1, \\
\overline{\lambda} \equiv & \frac{1}{16} \frac{\ell}{(1+2g)^2} \overline{\varepsilon}^2.
\end{aligned}$$

We fix an arbitrary discount factor $\delta \in [\underline{\delta}, \overline{\delta}]$ and an arbitrary observation cost $\lambda \in (0, \overline{\lambda})$. We show that there exists a sequential equilibrium whose payoff vector (v_1^{**}, v_2^{**}) satisfies $v_1^{**} = 0$ and $v_2^{**} \geq \frac{1+g+\ell}{1+\ell} - \varepsilon$.

Applying the strategy in Section 4.1, let us consider another automaton strategy profile σ^{**}. Player 1 has five types of states: Initial state $\hat{\omega}_1^1$, adjustment state ω_1^A, cooperation states $(\omega_1^{C,t})_{t=3}^{\infty}$, transition states $(\omega_1^{E,t})_{t=3}^{\infty}$, and defection state ω_1^D. Player 2 also has five types of states: Initial state $\hat{\omega}_2^1$, adjustment state ω_2^A, cooperation states $(\omega_2^{C,t})_{t=3}^{\infty}$, transition states $(\omega_2^{E,t})_{t=1}^{\infty}$, and defection state ω_2^D.

The stage behaviors and transition functions in the cooperation states $(\omega_1^{C,t})_{t=3}^{\infty}$ and $(\omega_2^{C,t})_{t=2}^{\infty}$, transition states $(\omega_1^{E,t})_{t=3}^{\infty}$ and $(\omega_2^{E,t})_{t=2}^{\infty}$, and the defections state ω_i^D are the same as those given in strategy σ^*. Note that $\hat{\omega}_i^1$, ω_i^A, and $\omega_2^{E,1}$ are new states.

To define the stage behaviors and transition functions in the new states, we use the sequence $(\beta_{i,t})_{i=1,2,t=1}^{\infty}$, which is defined in the proof of Proposition 1. Let us define

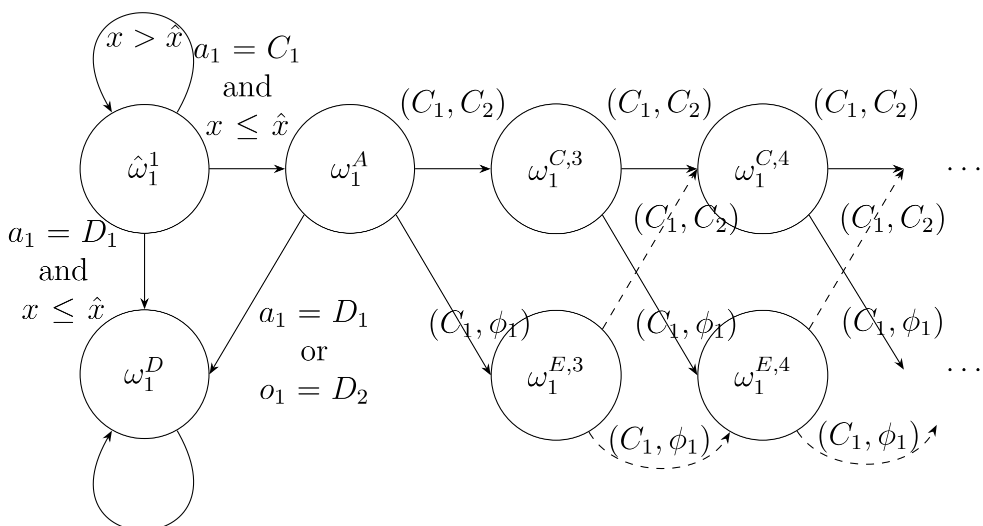
$$\hat{x} \equiv \frac{\ell}{\delta(1-\beta_{2,2})(1+g)}.$$

Player 1 chooses stage behavior C_1 with probability $\beta_{1,1}$, and D_1 with probability $1-\beta_{1,1}$ in the initial state $\hat{\omega}_1^1$. Irrespective of player 1's action, he chooses $m_1 = 0$. The state remains the same if realized x is greater than $\hat{x}$. Player 1 moves to the adjustment state ω_1^A if player 1 chose C_1 and realized x is smaller than $\hat{x}$. Player 1 moves to the defection state ω_1^D if player 1 chose C_1 and realized x is smaller than $\hat{x}$. In the adjustment state ω_1^A, player 1 chooses C_1 with probability $1-\beta_{1,2}$. If player 1 chooses C_i, he chooses $m_1 = 1$ with probability $1-\beta_{1,3}$. When player 1 chooses D_1, he never observes the opponent. The transition function in the adjustment state ω_1^A is the same as the one in the transition state $\omega_1^{E,2}$.

The prescribed actions and observational decisions, and state transition function are summarized in the table and figure below.

State	$\hat{\omega}_1^1$	ω_1^A	$\omega_1^{C,t}$	$\omega_1^{E,t}$	ω_1^D
Action	C_1 w.p. $1-\beta_{1,1}$ D_1 w.p. $\beta_{1,1}$	C_1 w.p. $1-\beta_{1,2}$, D_1 w.p. $\beta_{1,2}$	Same as in strategy σ^*		
m_1 given C_1	$m_1 = 0$	$m_1 = 1$ w.p. $1-\beta_{1,3}$ $m_1 = 0$ w.p. $\beta_{1,3}$			
m_1 given D_1	$m_1 = 0$				

Table 4: Actions and observational decisions of player 1



If player 1 chooses D_1 or observes D_2 in states ω_1^A, $\omega_1^{C,t}$, or $\omega_1^{E,t}$ for some $t \ge 3$, the state moves to ω_1^D.

Figure 2: State transition function of player 1

Player 2 chooses D_2 in the initial state $\hat{\omega}_2^1$. Player 2 observes player 1 with probability $1-\beta_{2,1}$ irrespective of her action when realized x is not greater than $\hat{x}$. The state remains the same if realized x is smaller than $\hat{x}$. Player 2 moves to the adjustment state ω_2^A if she observes C_1 and realized x is smaller than $\hat{x}$. Player 1 moves to the defection state ω_1^D if she observes C_1 and realized x is smaller than $\hat{x}$. Player 2 moves to the adjustment state $\omega_2^{E,1}$ if she chooses $m_2 = 0$ and realized x is smaller than $\hat{x}$. In the adjustment state ω_2^A, player 2 chooses C_2. When player 2 chooses C_2, she observes player 1 with probability $1-\beta_{2,2}$ If player 2 chooses D_2, she does not observe the opponent. In the transition state $\omega_2^{E,1}$, player 2 chooses D_2 and $m_2 = 0$ irrespective of her action. The transition functions in the adjustment state ω_2^A and the transition state $\omega_2^{E,1}$ are the same as the one in the initial state ω_2^1 given strategy σ^*. That is, if player 2 observes (C_2, C_1), she moves to the cooperation state $\omega_2^{C,2}$. If player 2 chooses C_2 but does not observe, she moves to the transition state $\omega_2^{E,2}$. When player 2 chooses D_2 or observes D_1, she moves to the defection state ω_2^D.

The prescribed actions and observational decisions, and state transition function are summarized in the table and figure below.

State	$\hat{\omega}_2^1$	ω_2^A	$\omega_2^{E,1}$	$\omega_2^{C,t}$	$\omega_2^{E,t}$ ($t \geq 2$)	ω_2^D
Action	D_2	C_2	D_2	Same as in strategy σ^*		
m_2 given C_2	$m_2 = 1$ w.p. $1 - \beta_{2,1}$ $m_2 = 0$ w.p. $\beta_{2,1}$	$m_2 = 1$ w.p. $1 - \beta_{2,2}$ $m_2 = 0$ w.p. $\beta_{2,2}$	$m_2 = 0$			
m_2 given D_2	$m_i = 0$					

Table 5: Actions and observational decisions of player 2

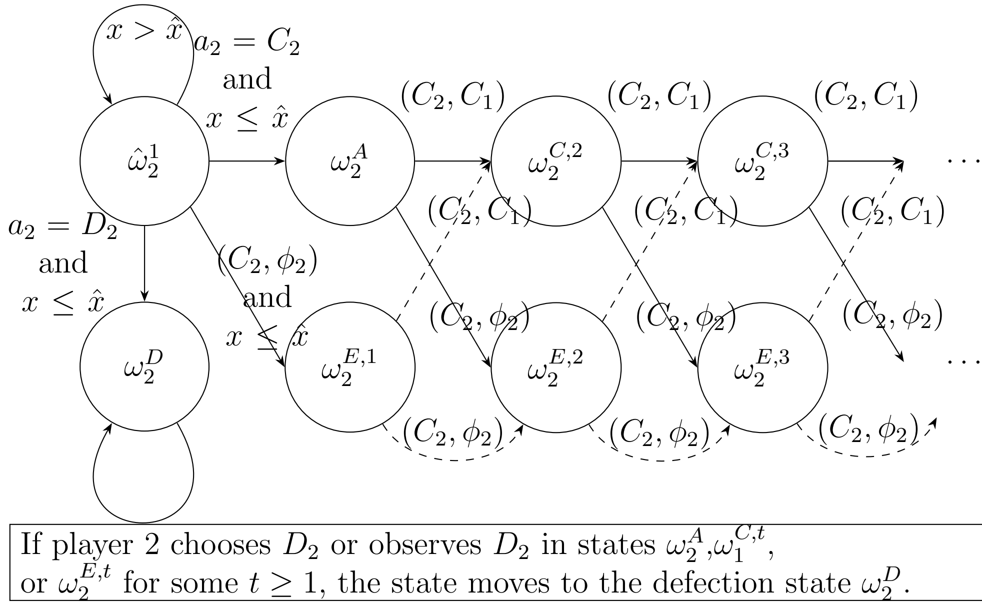


Figure 3: State transition function of player 2

Let strategy σ^{**} be the strategy defined by the above automaton. Next, we define a consistent system of beliefs with strategy profile σ^{**}. We consider a sequence of behavioral strategy profiles $(\hat{\sigma}^n)_{n=1}^{\infty}$ such that each strategy profile attaches a positive probability to every move, but puts far greater weights on the trembles on C_i in the defection state ω_i^D compared with other stage behaviors in the other states. These trembles induce a consistent system of beliefs that player i at any defection state ω_i^D is sure that the state of their opponent is the defection state ω_j^D or transition state $\omega_j^{E,t}$ for some $t \geq 2$.

Proof of Proposition 4 . Here we prove Proposition 4 using strategy σ^{**}.

Let us consider the sequential rationality of player 1. We consider the defection state ω_1^D. As in the proof of Proposition 1, player 1 in the defection state ω_1^D is certain that player 2 is in the defection state ω_2^D or transition state $\omega_2^{E,t}$ for some $t \geq 1$. Therefore, it is optimal for player 1 to choose action D_1 and choose $m_1 = 0$ irrespective of his action.

Next, let us consider a cooperation state $\omega_1^{C,t} (t \geq 3)$. Player 1 believes that player 2 is in the cooperation state $\omega_1^{C,t-1}$ with probability $1 - \beta_{2,t-1}$ and the transition state $\omega_1^{E,t-1}$ with the remaining probability $\beta_{2,t-1}$. This is the same belief over the opponent's state as the one that player 1 has in the cooperation state $\omega_1^{C,t-1}$ given strategy σ^*. Hence, the optimal stage behavior is also the same as the one of the cooperation state $\omega_1^{C,t-1}$ given strategy σ^*. Therefore, it is optimal for player 1 to choose C_1. When player 1 chooses C_1, player 1 is indifferent to his observational decision. Using the same argument, the sequential rationality in the transition state $\omega_1^{E,3}(t \geq 3)$ is also straightforward.

Let us consider the adjustment state ω_1^A. Player 2 is in the adjustment state ω_2^A with probability $1 - \beta_{2,1}$. Player 2 observes player 1 with probability $1 - \beta_{2,3}$ given action C_2 in the adjustment state ω_2^A. Furthermore, the state transition functions in the adjustment state ω_2^A and the state transition state $\omega_2^{E,1}$ are the same as the one in the initial state ω_2^1 given strategy σ^*. This conjecture over the continuation play of player 1 is the same as the one in the initial state ω_1^1 given strategy σ^*. Therefore, player 1 is indifferent among $(C_1, 1)$, $(C_1, 0)$, and $(D_1, 0)$. When player 1 chooses D_1, player 1 prefers $m_1 = 0$.

Finally, let us consider the initial state $\hat{\omega}_1^1$. If player 1 chooses D_1, he obtains zero payoff. If player 1 chooses C_1 and $x \leq \hat{x}$ is realized, player 1 will move to the adjustment state. Then, choosing $(D_1, 0)$ in the adjustment state, player 1 obtains $(1-\delta)(1-\beta_{2,1})(1+g)$. Therefore, the indifference condition between action C_1 and action D_1 is given by

$$0 = -\ell + \hat{x}\delta(1-\delta)(1-\beta_{2,1})(1+g).$$

This condition is ensured by the definition of $\hat{x}$. In addition, $m_1 = 0$ is optimal irrespective of his actions because player 2 chooses D_2 with certainty. Therefore, it is optimal for player 1 to follow the strategy σ^{**}.

Next, let us consider player 2. Applying similar arguments of player 1 to states ω_2^D, $\omega_2^{C,t}(t \geq 2)$, and $\omega_2^{E,t}(t \geq 2)$, we can show the sequential rationality in those states. The sequential rationality in the defection state ω_2^D is straightforward because player 2 is sure that player 1 is in the transition state or the defection state. In the cooperation state $\omega_2^{C,t}$, player 1 is in the cooperation state $\omega_1^{C,t+1}$ with probability $1 - \beta_{1,t+1}$. This belief over the continuation play of player 1 is the same as the one that player 2 has in the cooperation state $\omega_2^{C,t+1}$ given strategy σ^*. Therefore, choosing C_2 is optimal and player 2 is indifferent to her observational decision given C_2. When player 2 chooses D_2, she prefers $m_2 = 0$. Similarly, it is obvious that D_2 and $m_2 = 0$ irrespective of his action are optimal in the transition state $\omega_2^{E,t}$.

Let us consider the adjustment state ω_2^A. Then, player 2 is certain that player 1 is in the adjustment state ω_1^A. Then, player 1 chooses C_1 with probability $1 - \beta_{1,2}$, and observes player 2 with probability $1 - \beta_{1,3}$ given C_1. Furthermore, the state transition function of player 1 is the same as the one in the cooperation state $\omega_1^{C,2}$. The conjecture is the same as the one in the cooperation state $\omega_2^{C,2}$ given strategy σ^*. Therefore, choosing C_2 is optimal, and player 2 is indifferent to her observation decisions given C_2. When player 2 chooses D_2, she prefers $m_2 = 0$. We apply the same argument to the transition state $\omega_2^{E,1}$ and obtain that it is optimal for player 2 to choose D_2 and $m_2 = 0$ irrespective of her action.

Using similar arguments again, we can consider the initial state $\hat{\omega}_2^1$ as well. Consider observation phase after $x \leq \hat{x}$ is realized. If player 2 observes C_1, player 1 moves to the adjustment state ω_1^A for sure. As we confirmed before, the belief in the adjustment state ω_2^A is the same as the one in the cooperation state $\omega_2^{C,2}$ given strategy σ^*. If player 2 observes D_1, player 1 moves to the adjustment state ω_1^D with certainty. This conjecture is the same as the one player 2 faces in the observation phase given C_2 in the initial state ω_2^1 given strategy σ^*. Therefore, player 2 is indifferent between $m_2 = 1$ and $m_2 = 0$. Furthermore, it is obvious that player 2 has no incentive to choose D_2 in the action phase in the initial state $\hat{\omega}_2^1$ because player 1 does not observe player 2 in the initial state $\hat{\omega}_1^1$. It has been proved that this strategy σ^{**} is a sequential equilibrium.

Finally, let us consider the equilibrium payoff. It is obvious that v_1^{**} equals zero because player 1 (weakly) prefers action D_1 in the initial state $\hat{\omega}_1^1$. Player 2 prefers action D_2 in the initial state $\hat{\omega}_2^1$. In the adjustment state ω_2^A, one of the best responses is choosing C_2 and $m_2 = 0$, and the payoff is bounded below by the one of choosing D_2 and $m_2 = 0$. Therefore, player 2's payoff is bounded below by

$$\begin{aligned} v_2^{**} >&(1-\delta)\left\{(1-\beta_{1,1})(1+g) - \hat{x}\lambda\right\} + \delta\hat{x}(1-\delta)(1-\beta_{1,2})(1+g) + \delta(1-\hat{x})v_2^{**} \\ >&(1-\delta)\left\{(1-\beta_{1,1})(1+g) - \lambda\right\} + \hat{x}(1-\delta)(1-\beta_{1,2})g + \delta(1-\hat{x})v_2^{**} \\ >&(1-\delta)\left(1-\beta_{1,1}\right)\left(1+g+\hat{x}g\right) - (1-\delta)\lambda + \delta(1-\hat{x})v_2^{**}. \end{aligned}$$

The second inequality holds because $\delta > \underline{\delta} > \frac{g}{1+g}$ and $\hat{x} < 1$ hold. Lemma 1 ensures $\beta_{1,2} < \beta_{1,1}$ and the third inequality.

Subtracting $\delta(1-\hat{x})v_2^{**}$ from both sides, we obtain

$$v_2^{**} > \frac{\left(1 - \frac{1+g+\ell}{g+\ell}\varepsilon'\right)(1+g+\hat{x}g) - \lambda}{1+\frac{\delta}{1-\delta}\hat{x}} > \frac{1+g+\hat{x}g - 2(1+g)\frac{1+g+\ell}{g+\ell}\varepsilon'}{1+\frac{\delta}{1-\delta}\hat{x}}.$$

In what follows, we often use the following lemma.

Lemma 3. *For any $y \in \left(0, \frac{1}{2}\right)$, it holds that*

$$\begin{aligned} 1+y <&\frac{1}{1-y} < 1+2y, \\ 1-y <&\frac{1}{1+y} < 1. \end{aligned}$$

Proof of Lemma 3. This can be shown with simple calculations. □

Let us consider the denominator.

$$\begin{aligned} 1+\frac{\delta}{1-\delta}\hat{x} =&1+\frac{g+(1+g)\varepsilon'}{1-(1+g)\varepsilon'}\hat{x} = 1+\left(\frac{1+g}{1-(1+g)\varepsilon'} - 1\right)\hat{x} \\ <&1+\left\{(1+g)(1+2(1+g)\varepsilon') - 1\right\}\hat{x} \\ =&1+\left\{g+2g(1+g)^2\varepsilon'\right\}\hat{x} \end{aligned}$$

Lemma 3 ensures the inequality.

The value of $\hat{x}$ is bounded above by

$$\hat{x} = \frac{\ell}{\delta(1-\beta_{2,2})(1+g)} < \frac{1}{1-\frac{1+g+\ell}{g+\ell}\varepsilon'}\frac{\ell}{g} < \left(1+2\frac{1+g+\ell}{g+\ell}\varepsilon'\right)\frac{\ell}{g}.$$

Lemma 3 ensures the last inequality. Therefore, we have an upper bound of the denominator as follows.

$$\begin{aligned} 1+\frac{\delta}{1-\delta}\hat{x} <&1+\left\{g+2(1+g)^2\varepsilon'\right\}\left(1+2\frac{1+g+\ell}{g+\ell}\varepsilon'\right)\frac{\ell}{g} \\ <&1+\left\{\ell+2(1+g)^2\varepsilon'\right\}\left(1+2\frac{1+2g}{g}\varepsilon'\right) \\ <&1+\ell+2(1+g)^2\varepsilon' + 2(1+2g)\varepsilon' + (1+g)^2\varepsilon' \\ <&1+\ell+5(1+2g)^2\varepsilon'. \end{aligned}$$

The third inequality follows from Assumption 3 and $\varepsilon' < 2\overline{\varepsilon}$.

Next, let us consider a lower bound of the numerator.

$$1 + g + \hat{x}g - 2(1+g)\frac{1+g+\ell}{g+\ell}\varepsilon' > 1 + g + \hat{x}g - 2\frac{(1+2g)^2}{g}\varepsilon'.$$

The value of $\hat{x}$ has the following lower bound.

$$\hat{x} > \frac{\ell}{g+(1+g)\varepsilon'} = \frac{1}{1+\frac{1+g}{g}\varepsilon'}\frac{\ell}{g} > \left(1 - \frac{1+g}{g}\varepsilon'\right)\frac{\ell}{g} = \frac{\ell}{g} - \frac{1+g}{g}\varepsilon'.$$

Thus, the numerator is bounded below by

$$1 + g + \left(\frac{\ell}{g} - \frac{1+g}{g}\varepsilon'\right)g - 2\frac{(1+2g)^2}{g}\varepsilon' > 1 + g + \ell - 3\frac{(1+2g)^2}{g}\varepsilon'.$$

The last inequality is ensured by Lemma 3.

Therefore, we obtain a lower bound of v_2^{**} as follows.

$$\begin{aligned}
v_2^{**} >& \frac{1+g+\ell-3\frac{(1+2g)^2}{g}\varepsilon'}{1+\ell+5(1+2g)^2\varepsilon'} \\
>& \frac{1+g+\ell}{1+\ell}\left(\frac{1-3\frac{(1+2g)^2}{g(1+g+\ell)}\varepsilon'}{1+5\frac{(1+2g)^2}{1+\ell}\varepsilon'}\right) \\
>& \frac{1+g+\ell}{1+\ell}\left(1-3\frac{(1+2g)^2}{g(1+g+\ell)}\varepsilon'\right)\left(1-5\frac{(1+2g)^2}{1+\ell}\varepsilon'\right) \\
>& \frac{1+g+\ell}{1+\ell}\left(1-3\frac{(1+2g)^2}{g}\varepsilon'\right)\left(1-5\frac{(1+2g)^2}{g}\varepsilon'\right) \\
>& \frac{1+g+\ell}{1+\ell}\left(1-8\frac{(1+2g)^2}{g}\varepsilon'\right) \\
>& \frac{1+g+\ell}{1+\ell} - 8\frac{(1+2g)^3}{g}\varepsilon' > 1-\varepsilon.
\end{aligned}$$

□

Let us explain why we need an interim public randomization device and why we cannot use a public randomization device at the end of the observation phase instead of interim public randomization. In our strategy, the defection state ω_i^D is an absorbing state. It is also obvious that the payoff vector of (D_1, D_2) is Pareto inefficient. Therefore, to achieve a nearly Pareto-efficient outcome, the probability that each player i moves to the defection state ω_i^D must be small enough. It means that the observation probability of player 2 in the initial state $\hat{\omega}_2^1$ and the probability of C_1 in the initial state $\hat{\omega}_1^1$ must be high enough. However, taking Assumption 3 into account, player 1 has a stronger incentive to choose C_1 given strategy σ^{**} than given strategy σ^*, and does not randomize actions C_1 and D_1. To mitigate this strong incentive, we need a public randomization device. It is well known that we can decrease the efficient discount factor by dividing the game into several games (e.g., Ellison (1994)). Moving back to the initial state irrespective of stage behavior with a certain probability, player i considers the continuation payoff to be less important. Let $\hat{\delta}$

be an efficient discount factor in the initial state. If player 1 chooses D_1 in the initial state, he obtains 0. If player 1 chooses C_1 in the initial state, he obtains a nonaveraged payoff $-\ell + \hat{\delta}(1+g)$. Therefore, to make player 1 indifferent between actions C_1 and D_1 in the initial state $\hat{\omega}_1^1$, the efficient discount factor must be close to $\frac{\ell}{1+g}$.

It will affect not only player 1 but also player 2's incentive. As the continuation payoff is less important, player 2's observation incentive decreases. To keep the right-hand side of (3) unchanged, the probability $\gamma_{1,1}$ of D_1 in the initial state $\hat{\omega}_1^1$ must satisfy the following equation.

$$\delta(1-\beta_{1,2}) = \hat{\delta}(1-\gamma_{1,1})$$

or,

$$\gamma_{1,1} = 1 - \frac{\delta}{\hat{\delta}}(1-\beta_{1,2}).$$

Taking $\delta \sim \frac{g}{1+g}$, $\hat{\delta} \sim \frac{\ell}{1+g}$, Assumption 3, and $\beta_{1,2} \sim 0$ into account, we find that $\gamma_{1,1} \sim 1 - \frac{\ell}{g}$ is negative. Therefore, we cannot make player 2 indifferent to her observational decisions when player 1 is indifferent between actions C_1 and D_1. We need an interim public randomization device to mitigate player 1's incentive independently of player 2's incentive.

Corollary 4.1. *Suppose that an interim public randomization device is available, and Assumptions 2 and 3 are satisfied. Fix any positive $\zeta > 0$. For any $\varepsilon > 0$, there exist $\underline{\delta} \in \left(\frac{g}{1+g}, 1\right)$, $\overline{\delta} \in (\underline{\delta}, 1)$, $\overline{\lambda} > 0$, and $\overline{\eta} > 0$ such that for any discount factor $\delta \in [\underline{\delta}, \overline{\delta}]$, any observation cost $\lambda \in (0, \overline{\lambda})$, and any $\eta_1, \eta_2 \le \overline{\eta}$, there exists a sequential equilibrium σ^{**} whose payoff vector (v_1^{**}, v_2^{**}) satisfies $v_1^{**} = 0$ and $v_2^{**} \ge \frac{1+g+\ell}{1+\ell} - \varepsilon$.*

Proof of Corollary 4.1 . Let us show that strategy σ^{**} is a sequential equilibrium if η_1 and η_2 is sufficiently small. If player i is in the cooperation state $\omega_i^{C,t}$, the private signal z_i has no effect on the belief of player i because player i directly observed player j's action, C_j, in the previous period. In the adjustment state ω_1^A, player 1 is certain that player 2 chose D_2 in the initial state $\hat{\omega}_2^1$. Hence, the private signal z_i does not change the belief and best response stage-behavior of player 1 when player 1 is in the cooperation or adjustment states. In the transition or defection states, player 1 strictly prefers action D_1 and $m_1 = 0$ when $\eta_2 = 0$. Therefore, because of continuity of expected utility function, it is optimal for player 1 to choose action D_1 and $m_1 = 0$ if η_2 is sufficiently small. Thus, it is optimal for player 1 to follow strategy σ^{**} if η_2 is sufficiently small.

Let us consider player 2. In any transition state $\omega_2^{E,t}(t \ge 1)$, player 2 strictly prefers action D_2 and $m_2 = 0$ when $\eta_1 = 0$. Thus, it is optimal for player 2 choose action D_2 and $m_2 = 0$ when η_1 is sufficiently small. In adjustment and cooperation states, the private signal z_2 has no effect to player 2's belief because player 2 observed C_1 in the previous period. Hence, it is optimal for player 2 to follow strategy σ^{**} if η_1 is sufficiently small. Thus, the strategy σ^{**} is a sequential equilibrium if η_1 and η_2 are sufficiently small. □

Corollary 4.2. *Suppose that an interim public randomization device is available, and Assumptions 2 and 3 are satisfied. Fix any $\zeta > 0$. For any $\varepsilon > 0$, there exist $\underline{\delta} \in \left(\frac{g}{1+g}, 1\right)$, $\overline{\lambda} > 0$, and $\overline{\eta} > 0$ such that for any discount factor $\delta \in [\underline{\delta}, 1)$, any observation cost $\lambda \in (0, \overline{\lambda})$, and any $\eta_1, \eta_2 \in [0, \overline{\eta})$, there exists a sequential equilibrium σ^{**} whose payoff vector (v_1^{**}, v_2^{**}) satisfies $v_1^{**} = 0$ and $v_2^{**} \ge \frac{1+g+\ell}{1+\ell} - \varepsilon$.*

Proof of Corollary 4.2 . Use Lemma 2. □

We have shown that two payoff vectors can be approximated by sequential equilibria (Propositions 1 and 4) when the discount factor is sufficiently large and the observation cost is sufficiently small. It is straightforward that a payoff vector $\left(\frac{1+g+\ell}{1+\ell}, 0\right)$ can be approximated by a sequential equilibrium exchanging the roles of player 1 and player 2.

Using the technique in Ellison (1994) again and alternating four strategies σ^*, σ^{**}, and the repetition of the stage game Nash equilibrium, we can approximate any payoff vector in $\mathcal{F}^*$.

Proof of Theorem 1. See Appendix C. □

Remark 4. As Miyagawa et al. (2008) mentioned, some previous literature requires a very complicate strategy and a very high discount factor for their results. On the other hand, our strategy is much simpler than theirs and a required discount factor is not high. For the payoff vector $(1, 1)$ or $\left(\frac{1+g+\ell}{1+\ell}, 0\right)$, a slightly larger discount factor than $\frac{g}{1+g}$ is required (See Propositions 1 and 4). If we can use a public randomization device at the beginning of the repeated game, our folk theorem holds with the same level of discount factor.

Remark 5. Let us discuss what happens if the prisoner's dilemma is asymmetric, as in Table 6.

		Player 2	
		C_2	D_2
Player 1	C_1	$1 \quad , \quad 1$	$-\ell_1\ ,\ 1+g_2$
	D_1	$1+g_1,\ -\ell_2$	$0 \quad , \quad 0$

Table 6: Asymmetric prisoner's dilemma

In the proofs of the propositions and theorems, we require that the discount factor δ is sufficiently close to $\frac{g}{1+g}$. This condition is needed to approximate a Pareto-efficient payoff vector. If $g_1 \neq g_2$, it is impossible to ensure that the discount factor δ is sufficiently close to both $\frac{g_1}{1+g_1}$ and $\frac{g_2}{1+g_2}$. Therefore, we have to confine our attention to the case of $g_1 = g_2 = g$.

Let us consider Propositions 1 and 3. In the construction of the strategy, the randomization probability of player i is defined based on the incentive constraint of the opponent only, or, it is determined based on δ, g, ℓ_j and is independent of ℓ_i. Hence, if $g_1 = g_2$ and Assumptions 2 and 3 for each ℓ_i $(i = 1, 2)$ hold, our approximate efficiency result and approximate folk theorem hold under a small observation cost. Symmetricity of ℓ_1 and ℓ_2 is not important for our strategy although symmetricity of $g_1 = g_2$ is crucial.

5 Concluding Remarks

Prisoner's dilemma is a minimal model to describe cooperation because it has only two actions: cooperation and uncooperation. Prisoner's dilemma under costly observation has some difficulties in cooperation.

First, the number of actions is limited. This means that players cannot communicate using a variety of actions. If more than two actions are available, we can consider an equilibrium strategy where each player randomizes some two actions on the equilibrium path. If a player has an incentive to randomize actions C_i and D_i on the path in infinitely repeated prisoner's dilemma, it means that the repetition of D_i is one of the optimal strategies. Player i loses an incentive to observe because one of his optimal strategies is unchanged irrespective of his observation result.

Second, the number of players is limited. If there are three players A, B, and C, it is easy to check the observation deviation of the opponents. Player A can monitor the observational decisions of players B and C by comparing their actions. If players B and C choose inconsistent actions toward each other, player A finds that players B or C do not observe some of the players. Third, there is no free-cost informative signal. To obtain information about the actions chosen by their opponents, players have to observe. Despite the above limitations, we have shown our efficiency without randomization device.

We considered an interim public randomization device and obtained a folk theorem. It is worth mentioning that our folk theorem holds in some asymmetric prisoner's dilemma. Our results might be applied to more general games.

Appendix

A Proof of Proposition 1

Proof. We prove Proposition 1. Now, let us show that the strategy profile σ^* is a sequential equilibrium. The equilibrium payoff and the sequential rationalities in the initial, cooperation, and defection states have already been shown in Section 4. We consider th sequential rationality in the transition state $\omega_i^{E,t}$ in detail.

We consider any history in period t (≥ 2) associated with the transition state. Strategy σ^* prescribes D_i and $m_i = 0$ irrespective of his actions in the transition state. Let us consider a nonaveraged continuation payoff when player i chooses action C_i. Let p be the belief of player i that his opponent is in the cooperation state $\omega_j^{C,t}$. Therefore, if player i observes his opponent in period t, then $(a_i^t, o_i^t) = (C_i, C_j)$ is realized with probability p and the state moves to the cooperation state $\omega_i^{C,t+1}$. Let

$$W_{i,t} \equiv \{(1-\beta_{j,t}) \cdot 1 - \beta_{j,t} \cdot \ell\} + \delta(1-\beta_{j,t})(1-\beta_{j,t+1})(1+g). \tag{4}$$

The value of $W_{i,t}$ is the nonaveraged continuation payoff from the cooperation state $\omega_i^{C,t}$ when player i follows strategy σ_i^*. Therefore, the upper bound of the nonaveraged payoff when player i chooses action C_i in period t is given by

$$p - (1-p)\ell + \delta p W_{i,t+1}.$$

The nonaveraged payoff when player i chooses D_i is bounded above by $p(1+g)$. Therefore, action D_i is profitable if the following value is negative.

$$p - (1-p)\ell + \delta p W_{i,t+1} - p(1+g).$$

We can rewrite the above value as follows.

$$
\begin{aligned}
& p-(1-p)\ell+\delta p W_{i,t+1}-p(1+g) \\
=& (1-\beta_{j,t})-\beta_{j,t}\ell-\lambda+\delta(1-\beta_{j,t})W_{i,t+1}-(1-\beta_{j,t})(1+g) \\
& \quad +\lambda+\{p-(1-\beta_{j,t})\}\{1+\ell+\delta W_{i,t+1}-(1+g)\} \\
=& W_{i,t}-(1-\beta_{j,t})(1+g)+\lambda+\{p-(1-\beta_{j,t})\}\{\delta W_{i,t+1}-(g-\ell)\} \\
=& \frac{\lambda}{\delta(1-\beta_{j,t-1})}+\lambda+\{p-(1-\beta_{j,t})\}\{\delta W_{i,t+1}-(g-\ell)\}. \qquad (5)
\end{aligned}
$$

The second equality follows from equation (4) in period t. The last equality is ensured by (3) in period $t-1$.

Using equation (3), we obtain the lower bound of $\delta W_{t+1}-(g-\ell)$ as follows.

$$
\begin{aligned}
\delta W_{i,t+1}-(g-\ell) \geq & \delta(1-\beta_{j,t+1})(1+g)-(g-\ell) \\
\geq & \{g+(1+g)\varepsilon'\}\left(1-\frac{1+g+\ell}{g+\ell}\varepsilon'\right)-(g-\ell) \\
\geq & \frac{\ell}{2}. \qquad (6)
\end{aligned}
$$

The second inequality follows from $\beta_{i,t} \leq \frac{1+g+\ell}{g+\ell}\varepsilon'$ by Lemma 1. The last inequality is ensured by $\varepsilon' \leq 2\overline{\varepsilon}$. The maximum value of p is $(1-\beta_{j,t-1})(1-\beta_{j,t})$. Taking (6) into account, we show that (5) is negative as follows.

$$
\begin{aligned}
& \frac{\lambda}{\delta(1-\beta_{j,t-1})}+\lambda-\{(1-\beta_{j,t})-p\}\{\delta W_{j,t+1}-(g-\ell)\} \\
\leq & \frac{\lambda}{\delta(1-\beta_{j,t-1})}+\lambda-(1-\beta_{j,t})\beta_{j,t-1}\frac{\ell}{2} \\
\leq & \frac{1+g}{g}\frac{1}{1-\frac{1+g+\ell}{g+\ell}\varepsilon'}\lambda+\lambda-\left(1-\frac{1+g+\ell}{g+\ell}\varepsilon'\right)\frac{1}{2}\frac{1+g-\ell}{g+\ell}\varepsilon'\frac{\ell}{2}<0.
\end{aligned}
$$

The second inequality is ensured by $\delta \in [\underline{\delta}, \overline{\delta}]$ by Lemma 1 and $\beta_{j,t}, \beta_{j,t-1} \in \left[\frac{1}{2}\frac{1+g-\ell}{g+\ell}\varepsilon', \frac{1+g+\ell}{g+\ell}\varepsilon'\right]$. Therefore, player i prefers D_i to C_i. Hence, it has been proven that it is optimal for player i to follow strategy σ^*. The strategy σ^* is a sequential equilibrium. Proposition 1 has been proved. □

B Proof of Lemma 1

Proof of Lemma 1. To prove Lemma 1, we will use the following Lemma 4 holds.

Lemma 4. *Suppose that Assumptions 2 and 3 are satisfied. Fix any discount factor $\delta \in [\underline{\delta}, \overline{\delta}]$ and observation cost $\lambda \in (0, \overline{\lambda})$. Then, $\beta_{i,1}-\beta_{i,2} \geq \frac{\ell}{g+\ell}\varepsilon'$ holds and, for any $t \in \mathbb{N}$, it holds that*

$$
0<\frac{\ell}{2g}<-\frac{\beta_{i,t+2}-\beta_{i,t+1}}{\beta_{i,t+1}-\beta_{i,t}}<1.
$$

Assume that Lemma 4 holds. Using $\beta_{i,t}$, $\beta_{i,t+1}$, and $-\frac{\beta_{i,t+2}-\beta_{i,t+1}}{\beta_{i,t+1}-\beta_{i,t}}$, we can express $\beta_{i,t+2}$ as follows.

$$\begin{aligned}\beta_{i,t+2} =&\beta_{i,t} + (\beta_{i,t+1} - \beta_{i,t}) + (\beta_{i,t+2} - \beta_{i,t+1})\\ =&\beta_{i,t} + (\beta_{i,t+1} - \beta_{i,t}) \left\{1 - \left(-\frac{\beta_{i,t+2} - \beta_{i,t+1}}{\beta_{i,t+1} - \beta_{i,t}}\right)\right\}\\ =&\left(-\frac{\beta_{i,t+2} - \beta_{i,t+1}}{\beta_{i,t+1} - \beta_{i,t}}\right)\beta_{i,t} + \left\{1 - \left(-\frac{\beta_{i,t+2} - \beta_{i,t+1}}{\beta_{i,t+1} - \beta_{i,t}}\right)\right\}\beta_{i,t+1}.\end{aligned}$$

Therefore, if $\beta_{i,t}, \beta_{i,t+1} \in [0,1]$, and $\frac{\ell}{2g} < -\frac{\beta_{i,t+2}-\beta_{i,t+1}}{\beta_{i,t+1}-\beta_{i,t}} < 1$ hold, we obtain $\beta_{i,t+2} \in (\min\{\beta_{i,t}, \beta_{i,t+1}\}, \max\{\beta_{i,t}, \beta_{i,t+1}\})$ because $\beta_{i,t+2}$ is a convex combination of $\beta_{i,t}$ and $\beta_{i,t+1}$.

Let us compare $\beta_{i,1}$, $\beta_{i,2}$, and $\beta_{i,3}$. By Lemma 4, $\beta_{i,1} - \beta_{i,2}$ is greater than $\frac{\ell}{g+\ell}\varepsilon'$. Furthermore, we have $\beta_{i,2} < \beta_{i,3} < \beta_{i,1}$ because $-\left(-\frac{\beta_{i,t+2}-\beta_{i,t+1}}{\beta_{i,t+1}-\beta_{i,t}}\right) \in (0,1)$ by Lemma 4 and, then, $\beta_{i,3}$ is a convex combination of $\beta_{i,1}$ and $\beta_{i,2}$. Next, let us compare $\beta_{i,2}$, $\beta_{i,3}$, and $\beta_{i,4}$. As we find, $\beta_{i,2}$ is smaller than $\beta_{i,3}$. Therefore, we have $\beta_{i,2} < \beta_{i,4} < \beta_{i,3}$ because $\beta_{i,4}$ is a convex combination of $\beta_{i,2}$ and $\beta_{i,3}$. Similarly, for any $s \in \mathbb{N}$, it holds that $(\beta_{i,2s} <)\beta_{i,2s+1} < \beta_{i,2s-1}$, and $\beta_{i,2s} < \beta_{i,2s+2}(< \beta_{i,2s+1})$. □

Next, we prove Lemma 4.

Proof of Lemma 4. First, let us derive $-\frac{\beta_{i,3}-\beta_{i,2}}{\beta_{i,2}-\beta_{i,1}}$. By (1), we have

$$0 = -(1-\beta_{i,1})g - \beta_{i,1}\ell + \delta(1+g)(1-\beta_{i,1})(1-\beta_{i,2}). \tag{7}$$

Furthermore, by (3), we have

$$\frac{\lambda}{\delta(1-\beta_{i,1})} = -(1-\beta_{i,2})g - \beta_{i,2}\ell + \delta(1+g)(1-\beta_{i,2})(1-\beta_{i,3}) \tag{8}$$

By (7) and (8), we obtain

$$(\beta_{i,2}-\beta_{i,1})(g-\ell) - \delta(1+g)(1-\beta_{i,2})\left\{(\beta_{i,3}-\beta_{i,2}) + (\beta_{i,2}-\beta_{i,1})\right\} = \frac{\lambda}{\delta(1-\beta_{i,1})}. \tag{9}$$

Let us consider the lower bound of $\beta_{i,2}$. As $\varepsilon' \in [\overline{\varepsilon}, 2\overline{\varepsilon}]$ and $0 < \frac{\ell}{g} < 1$ hold, we have

$$\begin{aligned}\beta_{i,2} =&\frac{1+g-\frac{\ell}{g}\ell - (1+g+\ell)\frac{1+g}{g}\varepsilon'}{1+\frac{\ell}{g}\frac{1}{g+\ell}\varepsilon' - \frac{(1+g)(1+g+\ell)}{g(g+\ell)}(\varepsilon')^2}\frac{1}{g+\ell}\varepsilon'\\ >&\frac{\frac{3}{4}(1+g-\ell)}{\frac{3}{2}}\frac{1}{g+\ell}\varepsilon' > \frac{1}{2}\frac{1+g-\ell}{g+\ell}\varepsilon'.\end{aligned}$$

Next, let us consider the upper bound of $\beta_{i,2}$.

$$\begin{aligned}\beta_{i,2} =&\frac{1+g-\frac{\ell}{g}\ell - (1+g+\ell)\frac{1+g}{g}\varepsilon'}{1+\frac{\ell}{g}\frac{1}{g+\ell}\varepsilon' - \frac{(1+g)(1+g+\ell)}{g(g+\ell)}(\varepsilon')^2}\frac{1}{g+\ell}\varepsilon'\\ <&\frac{1+g-\frac{\ell}{g}\ell}{1-\frac{(1+g)(1+g+\ell)}{g(g+\ell)}(\varepsilon')^2}\frac{1}{g+\ell}\varepsilon' < \frac{1+g}{g+\ell}\varepsilon'.\end{aligned}$$

The last inequality is ensured by $\varepsilon' < 2\bar{\varepsilon}$. Thus, we obtain

$$\frac{1}{2}\frac{1+g-\ell}{g+\ell}\varepsilon' < \beta_{i,2} < \frac{1+g}{g+\ell}\varepsilon'.$$

As $\beta_{i,2} < \frac{1+g}{g+\ell}\varepsilon' < \beta_1 = \frac{1+g+\ell}{g+\ell}\varepsilon'$, we can divide both sides of (9) by $\beta_{i,2} - \beta_{i,1}$ and obtain $-\frac{\beta_{i,3}-\beta_{i,2}}{\beta_{i,2}-\beta_{i,1}}$.

$$-\frac{\beta_{i,3}-\beta_{i,2}}{\beta_{i,2}-\beta_{i,1}} = \frac{\ell + \delta(1+g)(1-\beta_{i,2}) - g + \frac{\lambda}{\delta(1-\beta_{i,1})(\beta_{i,2}-\beta_{i,1})}}{\delta(1+g)(1-\beta_{i,2})}.$$

As Assumption 3, $\beta_{i,1}, \beta_{i,2} < 1$, and $\beta_{i,2} - \beta_{i,1} < 0$ hold, we find an upper bound of $-\frac{\beta_{i,3}-\beta_{i,2}}{\beta_{i,2}-\beta_{i,1}}$.

$$-\frac{\beta_{i,3}-\beta_{i,2}}{\beta_{i,2}-\beta_{i,1}} \leq \frac{\delta(1+g)(1-\beta_{i,2}) + \frac{\lambda}{\delta(1-\beta_{i,1})(\beta_{i,2}-\beta_{i,1})}}{\delta(1+g)(1-\beta_{i,2})} < 1.$$

Taking $\beta_{i,1} = \frac{1+g+\ell}{g+\ell}\varepsilon'$, $\beta_{i,2} < \frac{1+g}{g+\ell}\varepsilon'$, and $-(\beta_{i,2} - \beta_{i,1}) > \frac{\ell}{g+\ell}\varepsilon' > \frac{\ell}{2g}\varepsilon'$ into account, we have a lower bound of $-\frac{\beta_{i,3}-\beta_{i,2}}{\beta_{i,2}-\beta_{i,1}}$ as follows.

$$\begin{aligned}
-\frac{\beta_{i,3}-\beta_{i,2}}{\beta_{i,2}-\beta_{i,1}} &> \frac{\ell + g\left(1 - \frac{1+g}{g+\ell}\varepsilon'\right) - g - \frac{\lambda}{\frac{g}{1+g}\left(1-\frac{1+g+\ell}{g+\ell}\varepsilon'\right)\frac{\ell}{2g}\varepsilon'}}{\left(\frac{g}{1+g}+\varepsilon'\right)(1+g)} \\
&> \frac{\ell - \frac{1+g}{g+\ell}g\varepsilon' - \frac{4(1+g)}{\ell}\frac{\lambda}{\varepsilon'}}{g+(1+g)\varepsilon'} > \frac{\frac{3}{4}\ell}{\frac{3}{2}g} > \frac{\ell}{2g}.
\end{aligned}$$

The first inequality follows from $\delta = \frac{g}{1+g} + \varepsilon' > \frac{g}{1+g}$. The third inequality is ensured by $\varepsilon' < 2\bar{\varepsilon}$ and $\lambda < \bar{\lambda}$. Therefore, we have obtained $\frac{\ell}{2g} < -\frac{\beta_{i,3}-\beta_{i,2}}{\beta_{i,2}-\beta_{i,1}} < 1$ and $\beta_{i,3} \in (\beta_{i,2}, \beta_{i,2})$. That is, $\beta_{i,3} - \beta_{i,2} > 0$.

Next, let us derive $-\frac{\beta_{i,t+3}-\beta_{i,t+2}}{\beta_{i,t+2}-\beta_{i,t+1}}$ inductively. Suppose that $\frac{\ell}{2g} < -\frac{\beta_{i,s+2}-\beta_{i,s+1}}{\beta_{i,s+1}-\beta_{i,s}} < 1$ and $\beta_{i,s+2} \in (\min\{\beta_{i,s}, \beta_{i,s+1}\}, \max\{\beta_{i,s}, \beta_{i,s+1}\})$ hold for period $s = 1, 2, \ldots, t$. We have shown that this supposition holds for $t = 1$. We show that $\frac{\ell}{2g} < -\frac{\beta_{i,t+3}-\beta_{i,t+2}}{\beta_{i,t+2}-\beta_{i,t+1}} < 1$ and $\beta_{i,t+3} \in (\min\{\beta_{i,t+1}, \beta_{i,t+2}\}, \max\{\beta_{i,t+1}, \beta_{i,t+2}\})$ hold.

By (3) for $t+1$ and $t+2$, we have

$$\begin{cases}
\frac{\lambda}{\delta(1-\beta_{i,t})} = -(1-\beta_{i,t+1})g - \beta_{i,t+1}\ell + \delta(1-\beta_{i,t+1})(1-\beta_{i,t+2})(1+g), \\
\frac{\lambda}{\delta(1-\beta_{i,t+1})} = -(1-\beta_{i,t+2})g - \beta_{i,t+2}\ell + \delta(1-\beta_{i,t+2})(1-\beta_{i,t+3})(1+g),
\end{cases}$$

or,

$$\begin{aligned}
&-\frac{\beta_{i,t+1}-\beta_{i,t}}{\delta(1-\beta_{i,t})(1-\beta_{i,t+1})}\lambda \\
&= -(\beta_{i,t+2}-\beta_{i,t+1})(g-\ell) + \delta(1-\beta_{i,t+2})\left\{(\beta_{i,t+3}-\beta_{i,t+2}) + (\beta_{i,t+2}-\beta_{i,t+1})\right\}(1+g).
\end{aligned}$$

The suppositions ensure $\beta_{i,t+2} - \beta_{i,t+1} \neq 0$. Divide both sides of the above equation by $\beta_{i,t+2} - \beta_{i,t+1}$. Then, we obtain

$$-\frac{\beta_{i,t+3}-\beta_{i,t+2}}{\beta_{i,t+2}-\beta_{i,t+1}} = \frac{\ell + \delta(1+g)(1-\beta_{i,t+2}) - g + \frac{1}{\delta(1-\beta_{i,t})(1-\beta_{i,t+1})\frac{\beta_{i,t+2}-\beta_{i,t+1}}{\beta_{i,t+1}-\beta_{i,t}}}\lambda}{\delta(1+g)(1-\beta_{i,t+2})}.$$

As Assumption 3 and $\frac{\beta_{i,t+2}-\beta_{i,t+1}}{\beta_{i,t+1}-\beta_{i,t}} < 0$ hold, $-\frac{\beta_{i,t+3}-\beta_{i,t+2}}{\beta_{i,t+2}-\beta_{i,t+1}}$ is bounded above by

$$-\frac{\beta_{i,t+3}-\beta_{i,t+2}}{\beta_{i,t+2}-\beta_{i,t+1}} \leq \frac{\delta(1+g)(1-\beta_{i,t+2}) + \frac{1}{\delta(1-\beta_{i,t})(1-\beta_{i,t+1})\frac{\beta_{i,t+2}-\beta_{i,t+1}}{\beta_{i,t+1}-\beta_{i,t}}}\lambda}{\delta(1+g)(1-\beta_{i,t+2})} < 1.$$

Taking $0 < \beta_{i,t+1}, \beta_{i,t+2} < \frac{1+g+\ell}{g+\ell}\varepsilon' = \beta_{i,1}$, and $\frac{\ell}{2g} < -\frac{\beta_{i,t+2}-\beta_{i,t+1}}{\beta_{i,t+1}-\beta_{i,t}} < 1$ into account, we find the following lower bound of $-\frac{\beta_{i,t+3}-\beta_{i,t+2}}{\beta_{i,t+2}-\beta_{i,t+1}}$.

$$\begin{aligned}
-\frac{\beta_{i,t+3}-\beta_{i,t+2}}{\beta_{i,t+2}-\beta_{i,t+1}} &= \frac{\ell + \delta(1-\beta_{i,t+2})(1+g) - g + \frac{1}{\delta(1-\beta_{i,t})(1-\beta_{i,t+1})\frac{\beta_{i,t+2}-\beta_{i,t+1}}{\beta_{i,t+1}-\beta_{i,t}}}\lambda}{\delta(1+g)(1-\beta_{i,t+2})} \\
&> \frac{\ell + g\left(1 - \frac{1+g+\ell}{g+\ell}\varepsilon'\right) - g - \frac{1}{\left(\frac{g}{1+g}+\varepsilon'\right)\left(1-\frac{1+g+\ell}{g+\ell}\varepsilon'\right)^2\frac{2g}{\ell}}\lambda}{\left(\frac{g}{1+g}+\varepsilon'\right)(1+g)} \\
&> \frac{\ell - \frac{1+g+\ell}{g+\ell}g\varepsilon' - \frac{1}{\frac{g}{1+g}\cdot\frac{1}{4}\cdot 2}\varepsilon'}{g+(1+g)\varepsilon'} > \frac{\frac{3}{4}\ell}{\frac{3}{2}g} > \frac{\ell}{2g}.
\end{aligned}$$

Therefore, we obtain $\frac{\ell}{2g} < -\frac{\beta_{i,t+3}-\beta_{i,t+2}}{\beta_{i,t+2}-\beta_{i,t+1}} < 1$ and
$\beta_{i,t+3} \in (\min\{\beta_{i,t+1}, \beta_{i,t+2}\}, \max\{\beta_{i,t+1}, \beta_{i,t+2}\})$. □

C Proof of Theorem 1

Proof. Let us fix $\overline{n}$ such that:

$$\overline{n} \geq \frac{4+2g}{\varepsilon}.$$

We use the same technique as in Lemma 2. We divide the repeated game into $\overline{n}$ distinct repeated games. The first repeated game is played in period 1, $\overline{n}+1$, $2\overline{n}+1\ldots$, the second repeated game is played in period 2, $\overline{n}+1$, $2\overline{n}+2\ldots$, and so on. Each repeated game can be regarded as a repeated game with discount factor $\delta^{\overline{n}}$.

We can find a sequential equilibrium strategy $\hat{\sigma}^*$ whose payoff vector $\hat{v}^* = (v_1^*, v_2^*)$ satisfies $|\hat{v}_i^* - 1| < \frac{1}{\overline{n}}$ when discount factor $\delta^{\overline{n}}$ is sufficiently large by Proposition 3. By Corollary 4.2, there exists a sequential equilibrium strategy $\hat{\sigma}^{**}$ whose payoff vector $\hat{v}^{**} = (v_1^{**}, v_2^{**})$ satisfies $\hat{v}_1^{**} = 0$ and $\left|\hat{v}_2^{**} - \frac{1+g+\ell}{1+\ell}\right| < \frac{1}{\overline{n}}$ when discount factor $\delta^{\overline{n}}$ is sufficiently large.

Let us assume that $v_1^F \leq v_2^F$. We choose sufficiently large discount factor δ so that we can use Proposition 4 and Corollary 4.2, and the discount factor δ satisfies the following condition:

$$\frac{1-\delta}{1-\delta^{\overline{n}}} \leq \frac{2}{\overline{n}}.$$

The desired payoff vector v can be expressed uniquely as a convex combination of $\hat{v}^*$, $\hat{v}^{**}$ and $(0,0)$ with some $\alpha_1, \alpha_2 \in (0,1)$ as below.

$$v = \alpha_1\delta\hat{v}^* + \alpha_2\delta\hat{v}^{**} + (1-\alpha_1-\alpha_2)\cdot 0.$$

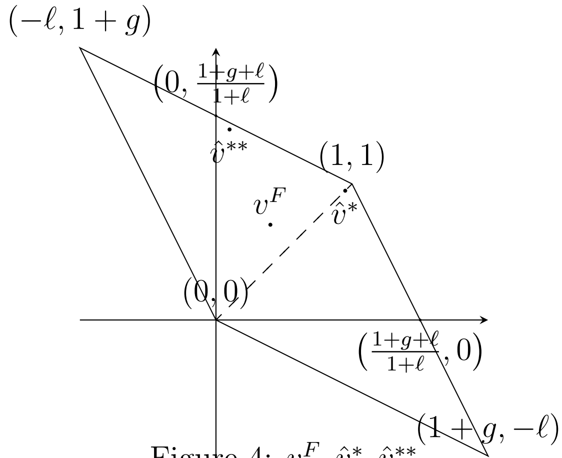


Figure 4: $v^F, \hat{v}^*, \hat{v}^{**}$

Let us define n_1 and n_2 as follows.

$$n_1 \equiv \arg\min_{n \in \mathbb{N}\cup\{0\}} \left| \frac{1-\delta^n}{1-\delta^{\overline{n}}} - \alpha_1 \right|, \qquad n_2 \equiv \arg\min_{n \in \mathbb{N}\cup\{0\}} \left| \frac{\delta^{n_1}-\delta^{n_1+n}}{1-\delta^{\overline{n}}} - \alpha_2 \right|.$$

Then, n_1 and n_2 satisfy

$$\left| \frac{1-\delta^{n_1}}{1-\delta^{\overline{n}}} - \alpha_1 \right| \leq \left(\frac{1-\delta}{1-\delta^{\overline{n}}} \leq \right) \frac{2}{\overline{n}}, \qquad \left| \frac{\delta^{n_1}-\delta^{n_1+n_2}}{1-\delta^{\overline{n}}} - \alpha_2 \right| \leq \frac{2}{\overline{n}}.$$

Let us consider the following strategy σ^F. In the first n_1-th games, players play strategy $\hat{\sigma}^*$. From the n_1+1-th game to the n_1+n_2-th game, players play strategy $\hat{\sigma}^{**}$. From the n_1+n_2+1-th to $\overline{n}$-th game, players play the stage game Nash equilibrium repetitively. It is straightforward that the strategy σ^F is a sequential equilibrium.

The payoff v_i^F for player i is given by

$$v_i^F = \frac{(1-\delta^{n_1})\hat{v}_i^* + (\delta^{n_1}-\delta^{n_1+n_2})\hat{v}_i^{**} + (\delta^{n_1+n_2}-\delta^{\overline{n}}) \cdot 0}{1-\delta^{\overline{n}}}.$$

Therefore, we have

$$\begin{aligned} \left| v_i^F - v \right| <& \left| \frac{1-\delta^{n_1}}{1-\delta^{\overline{n}}} \hat{v}_i^* - \alpha_1 \hat{v}_i^* \right| + \left| \frac{\delta^{n_1}-\delta^{n_1+n_2}}{1-\delta^{\overline{n}}} \hat{v}_i^{**} - \alpha_2 \cdot v_i^{**} \right| + 0 \\ <& \frac{2}{\overline{n}} \cdot 1 + \frac{2}{\overline{n}} \cdot (1+g) = \frac{4+2g}{\overline{n}} < \varepsilon. \end{aligned}$$

We obtain that the payoff vector v can be approximated by a sequential equilibrium payoff vector when $v_1 \leq v_2$ holds.

By symmetricity of the payoff matrix, it is straightforward that the payoff vector v can be approximated by a sequential equilibrium payoff vector when $v_1 \geq v_2$ also holds. □

References

[1] Ben-Porath, Elchanan and Michael Kahneman (2003) “Communication in repeated games with costly monitoring,” *Games and Economic Behavior*, Vol. 44, No. 2, pp. 227–250, DOI: https://doi.org/10.1016/S0899-8256(03)00022-8.

[2] Bhaskar, V. and Eric van Damme (2002) "Moral hazard and private monitoring," *Discussion Paper*, Vol. 102, No. 1, pp. 16–39, DOI: https://doi.org/10.1006/jeth.2001.2861.

[3] Bhaskar, V. and Ichiro Obara (2002) "Belief-Based Equilibria in the Repeated Prisoners' Dilemma with Private Monitoring," *Journal of Economic Theory*, Vol. 102, No. 1, pp. 40–69, DOI: https://doi.org/10.1006/jeth.2001.2878.

[4] Chen, Bo (2010) "A belief-based approach to the repeated prisoners' dilemma with asymmetric private monitoring," *Journal of Economic Theory*, Vol. 145, No. 1, pp. 402–420, DOI: https://doi.org/10.1016/J.JET.2009.05.006.

[5] Ellison, G. (1994) "Cooperation in the Prisoner's Dilemma with Anonymous Random Matching," *The Review of Economic Studies*, Vol. 61, No. 3, pp. 567–588, DOI: https://doi.org/10.2307/2297904.

[6] Ely, Jeffrey C. and Juuso Välimäki (2002) "A Robust Folk Theorem for the Prisoner's Dilemma," *Journal of Economic Theory*, Vol. 102, No. 1, pp. 84–105, DOI: https://doi.org/10.1006/JETH.2000.2774.

[7] Ely, Jeffrey C., Johannes Horner, and Wojciech Olszewski (2005) "Belief-Free Equilibria in Repeated Games," *Econometrica*, Vol. 73, No. 2, pp. 377–415, DOI: https://doi.org/10.1111/j.1468-0262.2005.00583.x.

[8] Flesch, János and Andrés Perea (2009) "Repeated Games with Voluntary Information Purchase," *Games and Economic Behavior*, Vol. 66, No. 1, pp. 126–145, DOI: https://doi.org/10.1016/j.geb.2008.04.015.

[9] Hino, Yoshifumi (2019) "An efficiency result in a repeated prisoner's dilemma game under costly observation with nonpublic randomization," *Mathematical Social Sciences*, Vol. 101, pp. 47–53, DOI: https://doi.org/10.1016/J.MATHSOCSCI.2019.06.005.

[10] Hörner, Johannes and Wojciech Olszewski (2006) "The Folk Theorem for Games with Private Almost-Perfect Monitoring," *Econometrica*, Vol. 74, No. 6, pp. 1499–1544, DOI: https://doi.org/10.1111/j.1468-0262.2006.00717.x.

[11] ——— (2009) "How Robust Is the Folk Theorem?," *Quarterly Journal of Economics*, Vol. 124, No. 4, pp. 1773–1814, DOI: https://doi.org/10.1162/qjec.2009.124.4.1773.

[12] Kandori, Michihiro and Ichiro Obara (2004) "Endogenous Monitoring," *UCLA Economics Online Papers*, Vol. 398, URL: http://www.econ.ucla.edu/people/papers/Obara/Obara398.pdf.

[13] Lehrer, Ehud and Eilon Solan (2018) "High Frequency Repeated Games with Costly," *Theoretical Economics*, Vol. 13, No. 1, pp. 87–113, DOI: https://doi.org/10.3982/TE2627.

[14] Mailath, George J. and Stephen Morris (2002) "Repeated Games with Almost-Public Monitoring," *Journal of Economic Theory*, Vol. 102, No. 1, pp. 189–228, DOI: https://doi.org/10.1006/jeth.2001.2869.

[15] Mailath, George J and Stephen Morris (2006) "Coordination failure in repeated games with almost-public monitoring," *Theoretical Economics*, Vol. 1, pp. 311–340, URL: https://econtheory.org/ojs/index.php/te/article/viewFile/20060311/791/30.

[16] Mailath, George J. and Wojciech Olszewski (2011) "Folk theorems with bounded recall under (almost) perfect monitoring," *Games and Economic Behavior*, Vol. 71, No. 1, pp. 174–192, DOI: https://doi.org/10.1016/j.geb.2010.11.002.

[17] Miyagawa, Eiichi, Yasuyuki Miyahara, and Tadashi Sekiguchi (2003) "Repeated Games with Observation Costs," *Columbia University Academic Commons*, No. 203-14, DOI: https://doi.org/https://doi.org/10.7916/D8VX0TRW.

[18] ——— (2008) "The folk theorem for repeated games with observation costs," *Journal of Economic Theory*, Vol. 139, No. 1, pp. 192–221, DOI: https://doi.org/10.1016/j.jet.2007.04.001.

[19] Piccione, Michele (2002) "The Repeated Prisoner's Dilemma with Imperfect Private Monitoring," *Journal of Economic Theory*, Vol. 102, No. 1, pp. 70–83, DOI: https://doi.org/10.1006/JETH.2001.2819.

[20] Sekiguchi, Tadashi (1997) "Efficiency in Repeated Prisoner's Dilemma with Private Monitoring," *Journal of Economic Theory*, Vol. 76, No. 2, pp. 345–361, DOI: https://doi.org/10.1006/jeth.1997.2313.

[21] Yamamoto, Yuichi (2007) "Efficiency results in N player games with imperfect private monitoring," *Journal of Economic Theory*, Vol. 135, No. 1, pp. 382–413, DOI: https://doi.org/10.1016/J.JET.2006.05.003.

[22] ——— (2009) "A limit characterization of belief-free equilibrium payoffs in repeated games," *Journal of Economic Theory*, Vol. 144, No. 2, pp. 802–824, DOI: https://doi.org/10.1016/J.JET.2008.07.005.