Granularity Shock: A Small Perturbation Two-Factor Model

Osadchiy, Maksim

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Maksim Osadchiy[1]

The paper presents a small perturbation two-factor model designed to capture granularity risk, extending the Vasicek Asymptotic Single Risk Factor (ASRF) portfolio loss model. By applying the Lyapunov Central Limit Theorem, we demonstrate that, for small values of the Herfindahl-Hirschman Index (HHI), granularity risk, conditional on market risk, is proportional to a standard normal random variable. Instead of studying the behavior of a heterogeneous portfolio, we examine the behavior of a homogeneous portfolio subjected to a small perturbation induced by granularity risk. We introduce the Vasicek-Herfindahl portfolio loss distribution, which extends the Vasicek portfolio loss distribution for heterogeneous portfolios with low HHI values. Utilizing the Vasicek-Herfindahl distribution, we derive closed-form granularity adjustments for the probability density function and cumulative distribution function of portfolio loss, as well as for Value at Risk (VaR) and Expected Shortfall (ES). We compare the primary results of our approach with established findings and validate them through Monte Carlo simulations.

1   Introduction

The Vasicek (1987) model, which is based on the Law of Large Numbers (LLN), assumes perfect granularity by considering a homogeneous portfolio with equal weights for all exposures. However, the LLN is not suitable in the case of a heterogeneous portfolio with varying weights. In this case, an additional granularity risk arises, which, due to linearity of expectation, is represented only in the residual $L - \mathbb{E}[L|Y]$, where L denotes the portfolio loss and Y represents the market shock. Gordy (2003) demonstrated that, under mild regularity conditions, as the number of positions in the heterogeneous portfolio increases, the portfolio loss converges almost surely to its conditional expectation given the common factor. Additionally, Gordy (2003) emphasized the importance of the Herfindahl-Hirschman Index (HHI) for studying granularity adjustment (GA).

The foundation for further study of GA to VaR was provided by Gouriéroux *et al* (2000), who calculated the first and second derivatives of VaR.

Emmer & Tasche (2005) obtained GA to VaR for both the general case of loss distribution and for the case of the Vasicek model. The formula of GA to VaR for the case of the Vasicek model was refined by Gordy & Lutkebohmert (2013).

Voropaev (2011) then moved on to studying the behavior of the portfolio loss PDF and GA to VaR and ES using a much simpler methodology based on the method of moments.

However, despite considerable efforts, it has not yet been possible to obtain GA to VaR suitable for supervisory purposes. Currently, there is a "granularity gap" in the regulation of credit risk. On one hand, a primitive archaic approach is used that considers the sizes of loans within the

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[1] Independent analyst. E-mail: maxos1us@gmail.com.

portfolio but ignores the correlations between these assets. On the other hand, the more advanced Internal Ratings-Based (IRB) approach accounts for correlations but neglects the varying sizes of loans in the portfolio. This paper aims to fill this gap concerning small values of HHI. The Vasicek-Herfindahl portfolio loss distribution introduced in our paper can be used to regulate a bank's economic capital.

This paper is organized as follows: In Section 2, we introduce the main subject of our study and provide essential information about the Vasicek model. In Section 3, we examine the behavior of the portfolio loss random variable when the HHI is close to zero. In Section 4, we present the Vasicek-Herfindahl portfolio loss distribution and investigate its properties. In Section 5, we calculate VaR and GA to VaR using our methodology and compare it with the results of Emmer & Tasche (2005). In Section 6, we calculate ES and GA to ES using the framework of our approach. In Section 7, we discuss the Vasicek's attempt at assessing the granularity effect. In Section 8, we review the approach of Voropaev (2011). In Section 9, we analyze the method proposed by Emmer & Tasche (2005). Finally, in Section 10, we summarize our findings.

2   A Model Framework

Consider a portfolio consisting of n loans. The weight of loan k is denoted as w_k, such that:

$$\sum_{k=1}^{n} w_k = 1$$

<div align="right">(2.1)</div>

where $w_k \geq 0$ for each k.

Let the portfolio loss be defined as:

$$Loss(\{X_k\}_{k=1}^{n}, Y) = \sum_{k=1}^{n} w_k \, l(X_k, Y)$$

<div align="right">(2.2)</div>

where Y is a standard normal random variable representing market (systematic) risk, and the standard normal random variables X_k represent individual (specific, idiosyncratic) risks. The random variables $\{X_k\}_{k=1}^{n}$ and Y are independent and identically distributed (i.i.d.). The term $l(X_k, Y)$ denotes the loss associated with loan k,

$$\mathbb{E}[l(X_k, Y)] = PD$$

<div align="right">(2.3)</div>

$$var[l(X_k, Y)] = PD(1 - PD)$$

<div align="right">(2.4)</div>

for each k, where PD is the probability of default.

Assume that $l(X_k, y)$ is equal to 1 with probability $p(y)$ (the default) and 0 otherwise, where y is a realization of the random value Y. The random variable $l(X_k, Y)$, conditional on the market shock Y, follows a Bernoulli distribution:

$$l|Y \sim Bernoulli(p(Y))$$

<div align="right">(2.5)</div>

The conditional mean is given by:

<div align="right">2</div>

$$\mathbb{E}[l|Y] = p(Y)$$

The conditional variance is expressed as:

$$\sigma^2(p(Y)) = p(Y)(1 - p(Y))$$

In the model by Vasicek (2002) the conditional probability of loss for any loan is given by:

$$p(Y) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right)$$

where $\rho \in [0,1]$ is the asset correlation, and Φ denotes the standard normal CDF.

The conditional expectation of the portfolio loss given Y is expressed as:

$$p(Y) = \mathbb{E}[Loss|Y]$$

The Vasicek CDF is given by:

$$F_V(x; PD, \rho) = \Phi\big(-p^{-1}(x)\big) = \Phi\left(\frac{\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right)$$

If the weights w_i are the same, then according to the LLN, the random value $Loss|Y$ converges in probability to the conditional mean $p(Y)$:

$$Loss|Y \xrightarrow{P} p(Y)$$

as $n \to \infty$ (Vasicek (2002)).

3   Lyapunov CLT

Suppose ξ_i is a sequence of independent random variables, each with a finite expected value μ_i and variance σ_i. Define

$$s_n{}^2 = \sum_{i=1}^{n} \sigma_i{}^2$$

If Lyapunov's condition

$$\lim_{n\to\infty} \frac{1}{s_n{}^{2+\delta}} \sum_{i=1}^{n} \mathbb{E}\left(|\xi_i - \mu_i|^{2+\delta}\right) = 0$$

is satisfied for some $\delta > 0$, then the sum

$$\frac{1}{s_n} \sum_{i=1}^{n} (\xi_i - \mu_i) \xrightarrow{d} \mathcal{N}(0,1)$$

converges in distribution to a standard normal variable as $n \to \infty$. Let us apply the Lyapunov CLT to our problem. We have:

$$\xi_i = w_i l(X_i, y)$$

$$\mu_i = w_i p(y)$$

$$s_n = \sqrt{h_n} \sigma(p(y))$$

where

$$h_n = \sum_{k=1}^{n} w_k^2$$

is the Herfindahl-Hirschman Index. In the new variables, formula (3.3) is transformed into the following form:

$$\frac{\sum_{i=1}^{n} w_i \left(l(X_i, y) - p(y) \right)}{\sqrt{h_n} \sigma(p(y))} \xrightarrow{d} \mathcal{N}(0,1)$$

Taking into account equation (2.2), this formula can be expressed as:

$$\frac{Loss(\{X_k\}_{k=1}^{n}, y) - p(y)}{\sqrt{h_n} \sigma(p(y))} \xrightarrow{d} \mathcal{N}(0,1)$$

Thus, the portfolio loss converges in distribution to the asymptotic loss L:

$$Loss(\{X_k\}_{k=1}^{n}, Y) \xrightarrow{d} L(Z, Y) = p(Y) + \sqrt{h} \sigma(p(Y)) Z$$

where $Y, Z \sim \mathcal{N}(0,1)$ are i.i.d. random variables, and

$$h = \lim_{n \to \infty} h_n$$

The heterogeneous portfolio loss risk encompasses not only market risk but also the granularity risk, which is represented by the term $\sqrt{h} \sigma(p(Y)) Z$. It is important to note that granularity risk is influenced by market risk.

If $h = 0$ (perfect granularity), then $L(Z, Y) = p(Y)$. Conversely, the case where $h = 1$ indicates full concentration, occurring when the weight of one of the loans is 1 and the weights of all the others are 0.

The range of the function $Loss(\{X_k\}_{k=1}^{n}, Y)$ is the unit interval $[0,1]$, while the range of the function $L(Z, Y)$ is $\mathbb{R}$. However, when $h = 0$, the range of the function $L(Z, Y)$ is narrowed to the unit interval $[0,1]$.

3.1 Applicability of the Lyapunov CLT

The applicability of our approach is constrained by the limits of the Lyapunov CLT. Let $\delta = 1$. We aim to prove that:

$$\lim_{n \to \infty} \frac{1}{s_n^{\ 3}} \sum_{i=1}^{n} \mathbb{E}[|\xi_i - \mu_i|^3] = 0$$

<div align="right">(3.7)</div>

is equivalent to:

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} w_i^{\ 3}}{\left(\sum_{j=1}^{n} w_j^{\ 2}\right)^{3/2}} = 0$$

<div align="right">(3.8)</div>

Proof.

We start with the expression:

$$\mathbb{E}[|\xi_i - \mu_i|^3] = w_i^{\ 3} \mathbb{E}[|l(X_k, y) - p(y)|^3] = w_i^{\ 3} p(y)(1 - p(y))\left((1 - p(y))^2 + p^2(y)\right)$$

<div align="right">(3.9)</div>

Thus, we have:

$$\lim_{n \to \infty} \frac{1}{s_n^{\ 3}} \sum_{i=1}^{n} \mathbb{E}\left[|\xi_i - \mu_i|^3\right] = \frac{\mathbb{E}[|l(X_k, y) - p(y)|^3]}{\sigma^3(p(y))} \lim_{n \to \infty} \frac{\sum_{i=1}^{n} w_i^{\ 3}}{(\sum_{i=1}^{n} w_i^{\ 2})^{3/2}}$$

$$= \frac{(1 - p(y))^2 + p^2(y)}{\sqrt{p(y)(1 - p(y))}} \lim_{n \to \infty} \frac{\sum_{i=1}^{n} w_i^{\ 3}}{\left(\sum_{j=1}^{n} w_j^{\ 2}\right)^{3/2}}$$

<div align="right">(3.10)</div>

Q.E.D.

To simulate the random variable $Loss$, it is necessary to use a set $w_k \geq 0$, $k = 1, \dots, n$, such that:

$$\sum_{k=1}^{n} w_k = 1$$

$$\sum_{k=1}^{n} w_k^{\ 2} \ll 1$$

$$\frac{\sum_{k=1}^{n} w_k^{\ 3}}{\left(\sum_{j=1}^{n} w_j^{\ 2}\right)^{3/2}} \ll 1$$

<div align="right">(3.11)</div>

We use the geometric progression defined as follows:

$$w_k = \frac{s^{k-1}}{\sum_{j=1}^{\infty} s^j} = (1 - s)s^{k-2}$$

<div align="right">(3.12)</div>

where $0 < s < 1$. Let

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$$h = \sum_{k=1}^{\infty} w_k{}^2 = \frac{1-s}{1+s} \Rightarrow s = \frac{1-h}{1+h}$$

Now we can evaluate:

$$\frac{\sum_{k=1}^{\infty} w_k{}^3}{\left(\sum_{j=1}^{\infty} w_j{}^2\right)^{3/2}} = \frac{\sum_{k=1}^{\infty} s^{3k}}{\left(\sum_{j=1}^{\infty} s^{2j}\right)^{3/2}} = \frac{(1-s^2)^{3/2}}{1-s^3}$$

If s is chosen in the left neighborhood of 1, then $\dfrac{\sum_{k=1}^{\infty} w_k{}^3}{\left(\sum_{j=1}^{\infty} w_j{}^2\right)^{3/2}}$ is close to 0.

4   Vasicek-Herfindahl Distribution

Let random variable
$$V = p(Y) \Rightarrow V \sim Vasicek(PD, \rho)$$

where $Vasicek(PD, \rho)$ represents the Vasicek loan loss distribution. Let

$$H = Z\sigma(V)\sqrt{h}$$

where

$$\sigma(x) = \sqrt{x(1-x)}$$

The asymptotic portfolio loss risk L is the sum of the "classical" Vasicek portfolio loss risk V and the granularity risk H:

$$L = V + H = V + Z\sigma(V)\sqrt{h}$$

$h\sigma^2(V)$ represents the conditional variance of portfolio loss:

$$h\sigma^2(V) = var[L|V]$$

If $h \ll 1$, then the granularity risk, $H = Z\sigma(V)\sqrt{h}$, is considered a small perturbation. This is GA to the portfolio loss risk:

$$GA^L = Z\sigma(V)\sqrt{h}$$

Note that

$$H = L - \mathbb{E}[L|V]$$

Equation (4.4) illustrates that it is incorrect to assert that interpreting $\mathbb{E}[L|V]$ as exclusively representing systematic risk and $L - \mathbb{E}[L|V]$ as purely capturing idiosyncratic risk can be

misleading. This is because the residual term, $Z\sigma(V)\sqrt{h} = L - \mathbb{E}[L|V]$, is influenced by the market risk Y.

The PDF of the portfolio loss L is given by:

$$f_L(x) = f_{V+H}(x) = \int_0^1 f_{H|V}(x - v|v)f_V(v)dv = \int_0^1 f_{H|V}(x - v|v)dF_V(v)$$

$$= \int_0^1 \frac{\varphi\left(\dfrac{x - v}{\sqrt{h}\sigma(v)}\right)}{\sqrt{h}\sigma(v)}dF_V(v)$$

(4.8)

where $\varphi(x)$ is the standard normal PDF. Similarly, the CDF of the portfolio loss L is:

$$F_L(x) = F_{V+H}(x) = \mathbb{P}[V + H < x] = \int_0^1 \mathbb{P}[H < x - v|V = v]f_V(v)dv$$

$$= \int_0^1 F_{H|V}(x - v|v)f_V(v)dv = \int_0^1 \Phi\left(\frac{x - v}{\sqrt{h}\sigma(v)}\right)dF_V(v)$$

(4.9)

The PDF of the random variable H, conditional on the random variable V, is given by:

$$f_{H|V}(x|v) = \frac{\varphi\left(\dfrac{x}{\sqrt{h}\sigma(v)}\right)}{\sqrt{h}\sigma(v)}$$

(4.10)

The CDF of the random variable H, conditional on the random variable V, is given by:

$$F_{H|V}(x|v) = \Phi\left(\frac{x}{\sqrt{h}\sigma(v)}\right)$$

(4.11)

Now we introduce the Vasicek-Herfindahl PDF of the portfolio loss:

$$f_{VH}(x; PD, \rho, h) = \int_0^1 \frac{\varphi\left(\dfrac{x - v}{\sqrt{h}\sigma(v)}\right)}{\sqrt{h}\sigma(v)}dF_V(v; PD, \rho)$$

(4.12)

and the Vasicek-Herfindahl CDF:

$$F_{VH}(x; PD, \rho, h) = \int_0^1 \Phi\left(\frac{x - v}{\sqrt{h}\sigma(v)}\right)dF_V(v; PD, \rho)$$

(4.13)

Using integration by parts, the function can be transformed into the following form:

$$F_{VH}(x; PD, \rho, h) = \int_0^1 F_V(v; PD, \rho) d\Phi\left(\frac{v - x}{\sqrt{h}\sigma(v)}\right)$$

<div align="right">(4.14)</div>

4.1 Additivity of Granular Shocks

Suppose that the "large" portfolio consists of N "small" portfolios. The weight of the "small" portfolio i is denoted as u_i, and its HHI is represented by h_i. The loss of the "small" portfolio i is given by:

$$L_i = V + \sigma(V)Z_i\sqrt{h_i}$$

where $\{Z_k\}_{k=1}^n$ are i.i.d. random variables, and $Z_i \sim N(0,1)$. The loss of the "large" portfolio can be calculated as:

$$L = \sum_{i=1}^N u_i L_i = V + \sigma(V)\sum_{i=1}^N Z_i u_i \sqrt{h_i} = V + \sigma(V)Z\sqrt{h}$$

where the HHI of the "large" portfolio is:

$$h = \sum_{i=1}^N u_i^2 h_i$$

4.2 Model Validation Using Monte Carlo Simulation

The simulated CDF is given by:

$$F_s(x) = \sum_{k=1}^N I(Loss_k \leq x)$$

<div align="right">(4.15)</div>

where $I(x)$ is the indicator function, $Loss_k$ is the kth Monte Carlo simulation of the random variable $Loss$, and N is the total number of simulations.

The simulated PDF is:

$$f_s(x_j) = \frac{1}{N}\sum_{k=1}^N I(x_j \leq Loss_k < x_{j+1})$$

<div align="right">(4.16)</div>

where

$$x_j = x_0 + j\Delta x$$

<div align="right">(4.17)</div>

and Δx is the bin width.

The theoretical Vasicek-Herfindahl ΔCDF is given by:

$$\Delta F_{VH}(x; PD, \rho, h) = F_{VH}(x; PD, \rho, h) - F_{VH}(x; PD, \rho, 0)$$

<div align="right">(4.18)</div>

The theoretical Vasicek-Herfindahl ΔPDF is defined as:

$$\Delta f_{VH}(x; PD, \rho, h) = f_{VH}(x; PD, \rho, h) - f_{VH}(x; PD, \rho, 0)$$

(4.19)

The simulated ΔCDF is expressed as:

$$\Delta F_s(x; h) = F_s(x; h) - F_s(x; 0)$$

(4.20)

The simulated ΔPDF is represented by:

$$\Delta f_s(x; h) = f_s(x; h) - f_s(x; 0)$$

(4.21)

The difference between the theoretical Vasicek-Herfindahl ΔCDF with the simulated ΔCDF is given by the equation:

$$\Delta\Delta F(x) = \Delta F_{VH}(x) - \Delta F_s(x)$$

(4.22)

Similarly, the difference between the theoretical Vasicek-Herfindahl ΔPDF and the simulated ΔPDF is represented as:

$$\Delta\Delta f(x) = \Delta f_{VH}(x) - \Delta f_s(x)$$

(4.23)

Figure 1 (with $h = 0.01$) and Figure 2 (with $h = 0.1$) illustrate the differences between the theoretical Vasicek-Herfindahl ΔCDF and ΔPDF and their corresponding simulated functions. Both figures demonstrate a decline in model quality as HHI values increase. Nevertheless, even with the relatively large value of $h = 0.1$, the model still accurately represents the shapes of both the PDF and CDF.
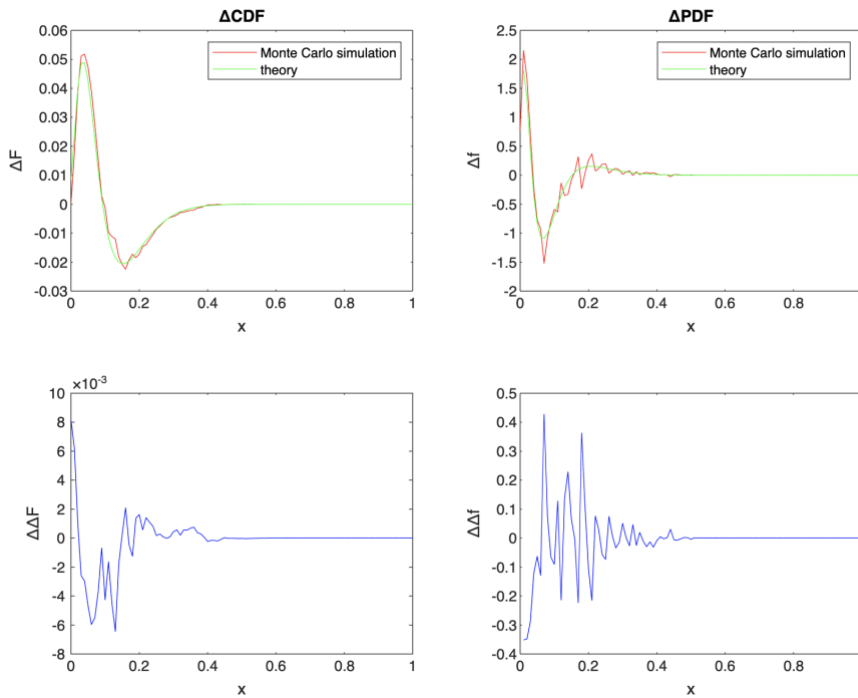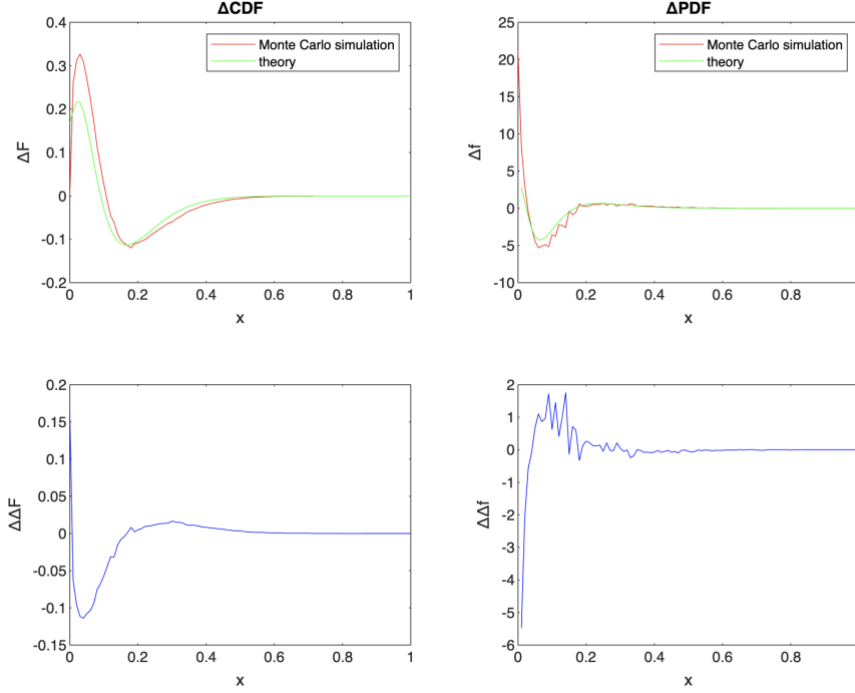

Figure 1. Theoretical Vasicek-Herfindahl ΔCDF and ΔPDF vs simulated functions (top row). Below the plots of the functions are the corresponding plots of the differences between theoretical and simulated functions. Number of Monte Carlo simulations: 20 000. Parameters used: PD=0.1, ρ=0.1, h=0.01, n=20 000.

Figure 2. Theoretical Vasicek-Herfindahl ΔCDF and ΔPDF vs simulated functions (top row). Below the plots of the functions are the corresponding plots of the differences between theoretical and simulated functions. Number of Monte Carlo simulations: 20 000. Parameters used: PD=0.1, ρ=0.1, h=0.1, n=20 000.

4.3 Properties of the Vasicek-Herfindahl Distribution

4.3.1 Normalization Property of the PDF

The total area under the PDF curve is equal to 1:

$$\int_{-\infty}^{\infty} f_{VH}(x; PD, \rho, h) dx = 1$$

(4.24)

Proof.

$$\int_{-\infty}^{\infty} f_{VH}(x; PD, \rho, h) dx = \int_{0}^{1} \frac{dF_V(v; PD, \rho)}{\sqrt{h}\sigma(v)} \int_{-\infty}^{\infty} \varphi\left(\frac{x - v}{\sqrt{h}\sigma(v)}\right) dv = \int_{0}^{1} dF_V(v; PD, \rho)$$
$$= F_V(v; PD, \rho)|_0^1 = 1$$

(4.25)

Q.E.D.

4.3.2 Expected Loss

The unconditional mean of the asymptotic loss L is equal to the unconditional probability of default:

$$\mathbb{E}[L] = PD$$

(4.26)

Proof.

$$\mathbb{E}[L] = \mathbb{E}\left[V + Z\sqrt{V(1-V)h}\right] = \mathbb{E}[V] + \sqrt{h}\mathbb{E}[Z]\mathbb{E}\left[\sqrt{V(1-V)}\right] = \mathbb{E}[V] = PD$$

<div align="right">(4.27)</div>

Q.E.D.

4.3.3 Variance

The unconditional variance of the asymptotic loss L is a linear function of h:

$$var[L] = PD(1-PD)h + (1-h)var[V]$$

<div align="right">(4.28)</div>

where
$$var[V] = var\left[\mathbb{E}[L|V]\right] = \Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2$$

<div align="right">(4.29)</div>

Proof.

$$\begin{aligned}
\mathbb{E}[L^2] &= \mathbb{E}\left[V^2 + 2VZ\sqrt{V(1-V)h} + Z^2V(1-V)h\right] \\
&= \mathbb{E}[V^2] + 2\sqrt{h}\mathbb{E}[Z]\mathbb{E}\left[V\sqrt{V(1-V)}\right] + h\mathbb{E}[Z^2]\mathbb{E}[V(1-V)] \\
&= \mathbb{E}[V^2] + h\mathbb{E}[V(1-V)] = (1-h)\mathbb{E}[V^2] + h\mathbb{E}[V] \\
&= (1-h)\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) + hPD
\end{aligned}$$

<div align="right">(4.30)</div>

$$\begin{aligned}
var[L] &= \mathbb{E}[L^2] - \mathbb{E}^2[L] = (1-h)\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) + hPD - PD^2 = \\
&= (1-h)(\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2) + (1-h)PD^2 + hPD - PD^2 \\
&= (1-h)var[V] + PD(1-PD)h
\end{aligned}$$

<div align="right">(4.31)</div>

Q.E.D.

4.3.4 Symmetry Property

The distribution exhibits a symmetry property:

$$F_{VH}(x; PD, \rho, h) = 1 - F_{VH}(1-x; 1-PD, \rho, h)$$

<div align="right">(4.32)</div>

This is similar to the symmetry property presented by (Vasicek O. A., 2002) (p.4):

$$F_V(x; PD, \rho) = 1 - F_V(1-x; 1-PD, \rho)$$

<div align="right">(4.33)</div>

Proof.

$$F_{VH}(x; PD, \rho, h) = \int_0^1 \Phi\left(\frac{x-v}{\sqrt{h}\sigma(v)}\right) dF_V(v; PD, \rho)$$

$$= -\int_0^1 \Phi\left(\frac{x-v}{\sqrt{v(1-v)h}}\right) dF_V(1-v; 1-PD, \rho)$$

$$= \int_0^1 \Phi\left(\frac{x-(1-u)}{\sqrt{(1-u)uh}}\right) dF_V(u; 1-PD, \rho) =$$

$$= 1 - \int_0^1 \Phi\left(\frac{(1-x)-u}{\sqrt{h}\sigma(u)}\right) dF_V(u; 1-PD, \rho) = 1 - F_{VH}(1-x; 1-PD, \rho)$$

(4.34)

Q.E.D.

4.4 Taylor Series of CDF and PDF

Let $z = h\sigma^2(v)$. Given that $0 < z \ll 1$, we expand $\Phi(u/\sqrt{z})$ into a Taylor series around $z = 0$:

$$\Phi\left(\frac{u}{\sqrt{z}}\right) = \theta(u) + \sum_{k=1}^{\infty} (z/2)^k \frac{\delta^{(2k-1)}(u)}{k!}$$

(4.35)

(see proof in Appendix). Hence, the Vasicek-Herfindahl CDF can be written as:

$$F_{VH}(x; PD, \rho, h) = \int_0^1 \Phi\left(\frac{x-v}{\sqrt{h}\sigma(v)}\right) dF_V(v; PD, \rho)$$

$$= F_V(x; PD, \rho)$$

$$+ \sum_{k=1}^{\infty} \frac{(h/2)^k}{k!} \int_0^1 \delta^{(2k-1)}(x-v)\sigma^{2k}(v) f_V(v; PD, \rho) dv$$

$$= F_V(x; PD, \rho)$$

$$- \sum_{k=1}^{\infty} \frac{(h/2)^k}{k!} \int_0^1 \delta^{(2k-1)}(v-x)\sigma^{2k}(v) f_V(v; PD, \rho) dv = F_V(x; PD, \rho)$$

$$+ \sum_{k=1}^{\infty} \frac{(h/2)^k}{k!} \frac{\partial^{2k-1}}{\partial x^{2k-1}}\left(\sigma^{2k}(x) f_V(x; PD, \rho)\right)$$

$$= F_V(x; PD, \rho) + \frac{h}{2}\frac{\partial}{\partial x}\left(\sigma^2(x) f_V(x; PD, \rho)\right) + o(h)$$

(4.36)

We used the Dirac delta function properties:

$$\delta^{(k)}(-x) = (-1)^k \delta^{(k)}(x)$$

(4.37)

$$\int_{-\infty}^{+\infty} \delta^{(k)}(x) f(x) dx = (-1)^k f^{(k)}(0)$$

(4.38)

The Vasicek-Herfindahl PDF is given by:

$$f_{VH}(x; PD, \rho, h) = \frac{\partial}{\partial x} F_{VH}(x; PD, \rho, h) = \sum_{k=0}^{\infty} \frac{(h/2)^k}{k!} \frac{\partial^{2k}}{\partial x^{2k}} \left(\sigma^{2k}(x) f_V(x; PD, \rho) \right)$$

$$= f_V(x; PD, \rho) + \frac{h}{2} \frac{\partial^2}{\partial x^2} \left(\sigma^2(x) f_V(x; PD, \rho) \right) + o(h)$$

<div align="right">(4.39)</div>

Hence, the GA to CDF is given by:

$$GA^{CDF} = \frac{h}{2} \frac{\partial}{\partial x} \left(\sigma^2(x) f_V(x; PD, \rho) \right)$$

<div align="right">(4.40)</div>

and the GA to PDF is expressed as:

$$GA^{PDF} = \frac{h}{2} \frac{\partial^2}{\partial x^2} \left(\sigma^2(x) f_V(x; PD, \rho) \right)$$

<div align="right">(4.41)</div>

5 Value at Risk (VaR)

The Value at Risk $VaR_\alpha(L(h)) = x(\alpha, h)$ is the root of the integral equation:

$$1 - \alpha = \int_0^1 \Phi \left(\frac{x(\alpha, h) - v}{\sqrt{h} \sigma(v)} \right) dF_V(v; PD, \rho)$$

<div align="right">(5.1)</div>

where α is the confidence level. The numerical value of this root can be easily determined using known parameters PD, ρ, h, and α. For brevity, when writing $x(\alpha, h)$, we omit including the parameters PD and ρ.

$x(\alpha, h)$ is the α-quantile of $L(h)$:

$$x(\alpha, h) = q_\alpha(L(h))$$

where, for any random variable X,

$$q_\alpha(X) = inf\{x \in \mathbb{R} : \mathbb{P}[X \leq x] \geq \alpha\}$$

Since a closed-form solution for $x(\alpha, h)$ from the integral equation (5.1) is generally unavailable, we consider expanding $x(\alpha, h)$ into a Taylor series around $h = 0$.

Differentiating the equality

$$1 - \alpha = F_{VH}(x(\alpha, h), h)$$

<div align="right">(5.2)</div>

with respect to h, we obtain:

$$\frac{\partial}{\partial \xi} F_{VH}(v, h) \bigg|_{v=x(\alpha,h)} \frac{\partial x(\alpha, h)}{\partial h} + \frac{\partial}{\partial h} F_{VH}(v, h) \bigg|_{v=x(\alpha,h)} = 0$$

<div align="right">(5.3)</div>

Note: parameters PD and ρ were omitted for brevity. Using the relation

$$\frac{\partial}{\partial v}F_{VH}(v,h) = f_{VH}(v,h)$$

we derive:

$$\left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0} = -\left.\frac{\frac{\partial}{\partial h}F_{VH}(v,h)}{f_{VH}(v,h)}\right|_{v=x(\alpha),h=0}$$

where the α-quantile of $L(h=0)$

$$x(\alpha) = q_\alpha(L(h=0)) = VaR_\alpha(L(h=0)) = \Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(1-\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)$$

is the root of the equation

$$1 - \alpha = F_V(x(\alpha); PD, \rho)$$

Since the following equalities hold:

$$\left.\frac{\partial F_{VH}(x; PD, \rho, h)}{\partial h}\right|_{h=0} = \frac{1}{2}\frac{\partial}{\partial x}\left(\sigma^2(x)f_V(x; PD, \rho)\right)$$

(see (4.36)) and

$$f_{VH}(x, PD, \rho, h=0) = f_V(x; PD, \rho)$$

then

$$\left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0} = -\left.\frac{\frac{\partial}{\partial v}\left(\sigma^2(v)f_V(v; PD, \rho)\right)}{2f_V(v; PD, \rho)}\right|_{v=x(\alpha)}$$

Using the same methods applied in deriving formula (4.37), we can obtain the following expression for the partial derivative:

$$\frac{\partial x(\alpha,h)}{\partial h} = -\left.\frac{\sum_{k=0}^{\infty}\frac{(h/2)^k}{k!}\frac{\partial^{2k+1}}{\partial v^{2k+1}}\left(\sigma^{2(k+1)}(v)f_V(v; PD, \rho)\right)}{2\sum_{k=0}^{\infty}\frac{(h/2)^k}{k!}\frac{\partial^{2k}}{\partial v^{2k}}\left(\sigma^{2k}(v)f_V(v; PD, \rho)\right)}\right|_{v=x(\alpha,h)}$$

Taylor series expansion at $h = 0$:

$$VaR_\alpha(L(h)) = x(\alpha, h) = x(\alpha) + h\left.\frac{\partial x(\alpha, \chi)}{\partial \chi}\right|_{\chi=0} + o(h)$$

$$= x(\alpha) - h\left.\frac{\frac{\partial}{\partial v}\left(\sigma^2(v)f_V(v; PD, \rho)\right)}{2f_V(v; PD, \rho)}\right|_{v=x(\alpha)} + o(h)$$

(5.12)

It follows from equations (5.10) and $\sigma^2(x) = x(1-x)$ that:

$$\left.\frac{\partial x(\alpha, h)}{\partial h}\right|_{h=0} = -\frac{1}{2}(1 - 2v + v(1-v)\frac{\partial}{\partial v}ln\left(f_V(v; PD, \rho)\right)\Big|_{v=x(\alpha)}$$

(5.13)

Since the Vasicek PDF is given by:

$$f_V(x; PD, \rho) = \sqrt{\frac{1-\rho}{\rho}}exp\left\{-\frac{1}{2\rho}\left(\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD)\right)^2 + \frac{1}{2}\left(\Phi^{-1}(x)\right)^2\right\}$$

(5.14)

it follows that:

$$\frac{\partial ln f_V(x; PD, \rho)}{\partial x} = \frac{(2\rho - 1)\Phi^{-1}(x) + \sqrt{1-\rho}\Phi^{-1}(PD)}{\rho\varphi\left(\Phi^{-1}(x)\right)}$$

(5.15)

From equations (2.10), (5.6) and (5.15), we have:

$$\left.\frac{\partial}{\partial v}ln\left(f_V(v; PD, \rho)\right)\right|_{v=x(\alpha)} = \frac{\Phi^{-1}\left(x(\alpha)\right) - \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(1-\alpha)}{\varphi\left(\Phi^{-1}(x)\right)}$$

(5.16)

It follows from equations (5.13) and (5.16) that:

$$\left.\frac{\partial x(\alpha, h)}{\partial h}\right|_{h=0} = -\frac{1}{2}\left(1 - 2x(\alpha) + \frac{x(\alpha)\left(1 - x(\alpha)\right)}{\varphi\left(\Phi^{-1}(x(\alpha))\right)}\left(\Phi^{-1}(x(\alpha)) - \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(1-\alpha)\right)\right)$$

(5.17)

Hence, the GA to VaR is:

$$GA^{VaR} = -\frac{h}{2}\left(1 - 2x(\alpha) + \frac{x(\alpha)\left(1 - x(\alpha)\right)}{\varphi\left(\Phi^{-1}(x(\alpha))\right)}\left(\Phi^{-1}(x(\alpha)) - \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(1-\alpha)\right)\right)$$

(5.18)

Based on the approach outlined by Gouriéroux et al (2000), Emmer & Tasche (2005, Remark 2.3) with a correction Gordy & Marrone (2012) (minus before $\sqrt{\frac{1-\rho}{\rho}}$ instead of plus) derived the

same formula with precision up to notation ($q_{1-\alpha}(X) = -\Phi^{-1}(1-\alpha)$, $\Phi\left(\frac{c-\sqrt{\rho}q_{1-\alpha}(X)}{\sqrt{1-\rho}}\right) = x(\alpha)$).
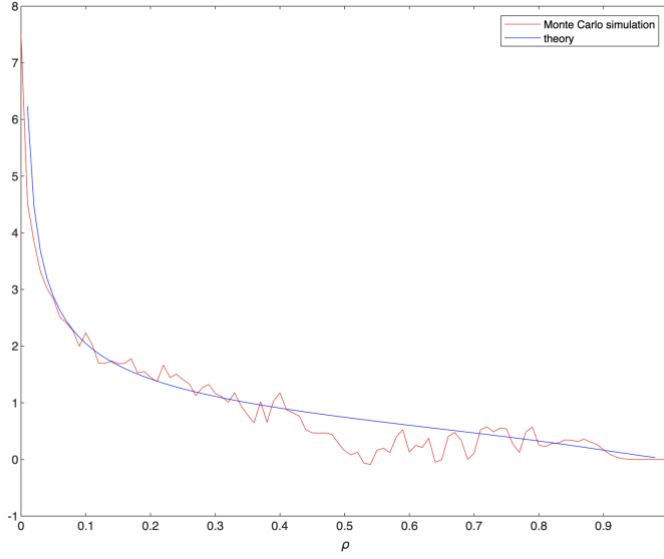

Figure 3. Comparison of the dependence on ρ of the simulated function $\frac{x(\alpha,h)-x(\alpha)}{h}$ (red line) and of the theoretical function $\frac{\partial x(\alpha,h)}{\partial h}\Big|_{h=0}$ (blue line). Number of Monte Carlo simulations: 20 000. The parameters used: $PD = 0.1$, $n = 15\,000$, $h = 0.01$, $\alpha = 0.01$.

6 Expected Shortfall

The Expected Shortfall at level α is defined as:

$$ES_\alpha(L) = \mathbb{E}[L|L > VaR_\alpha(L)] = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma(L)d\gamma$$

(6.1)

We expand $VaR_\gamma(L(h))$ into a Taylor series around $h = 0$:

$$VaR_\gamma(L(h)) = VaR_\gamma(L(h=0)) + h\frac{\partial VaR_\gamma(L(h))}{\partial h}\Bigg|_{h=0} + o(h)$$

(6.2)

where $VaR_\gamma(L(h=0)) = x(\gamma)$ is the root of the equation:

$$1 - \gamma = F_V(x(\gamma); PD, \rho)$$

(6.3)

Differentiating equation (6.3) with respect to γ, we obtain:

$$-1 = \frac{dx(\gamma)}{d\gamma}\frac{\partial F_V(v; PD, \rho)}{\partial v}\Bigg|_{v=x(\gamma)} = \frac{dx(\gamma)}{d\gamma}f_V(x(\gamma); PD, \rho)$$

(6.4)

which leads to:

$$\frac{dx(\gamma)}{d\gamma} = -\frac{1}{f_V(x(\gamma); PD, \rho)}$$

(6.5)

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By the chain rule, the formula (5.10)

$$\frac{\partial x(\gamma, h)}{\partial h}\bigg|_{h=0} = -\frac{\frac{\partial}{\partial v}\left(\sigma^2(v) f_V(v; PD, \rho)\right)}{2 f_V(v; PD, \rho)}\bigg|_{v=x(\gamma)}$$

(6.6)

is transformed into the following form:

$$\frac{\partial x(\gamma, h)}{\partial h}\bigg|_{h=0} = \frac{1}{2}\frac{dx(\gamma)}{d\gamma}\left(\frac{\partial}{\partial v}\left(\sigma^2(v) f_V(v; PD, \rho)\right)\right)\bigg|_{v=x(\gamma)} = \frac{1}{2}\frac{\partial}{\partial \gamma}\left(\sigma^2(x(\gamma)) f_V(x(\gamma); PD, \rho)\right)$$

(6.7)

Hence,

$$\int_0^\alpha \frac{\partial x(\gamma, h)}{\partial h}\bigg|_{h=0} d\gamma = \frac{1}{2}\sigma^2(x(\gamma)) f_V(x(\gamma); PD, \rho)\bigg|_{\gamma=0}^{\gamma=\alpha} = \frac{1}{2}\sigma^2(x(\alpha)) f_V(x(\alpha); PD, \rho)$$

(6.8)

It follows from equations (6.1) and (6.8) that

$$\frac{\partial ES_\alpha(L(h))}{\partial h}\bigg|_{h=0} = \frac{1}{2\alpha}\sigma^2(x(\alpha)) f_V(x(\alpha); PD, \rho)$$

(6.9)

Therefore, the Taylor expansion of ES around $h = 0$ becomes:

$$ES_\alpha(L(h)) = ES_\alpha(L(0)) + \frac{h}{2\alpha}\sigma^2(x(\alpha)) f_V(x(\alpha); PD, \rho) + o(h)$$

(6.10)

$$\frac{ES_\alpha(L(h)) - ES_\alpha(L(0))}{h} \approx \frac{1}{2\alpha}\sigma^2(x(\alpha)) f_V(x(\alpha); PD, \rho)$$

(6.11)

The GA to ES is

$$GA^{ES} = \frac{h}{2\alpha}\sigma^2(x(\alpha)) f_V(x(\alpha); PD, \rho)$$

(6.12)

For Monte Carlo simulations of conditional expectation, we use the formula

$$ES_\alpha(L) = \mathbb{E}[L|L > VaR_\alpha(L)] = \frac{\sum_{k=1}^N L_k I(L_k > VaR_\alpha(L))}{\sum_{k=1}^N I(L_k > VaR_\alpha(L))}$$
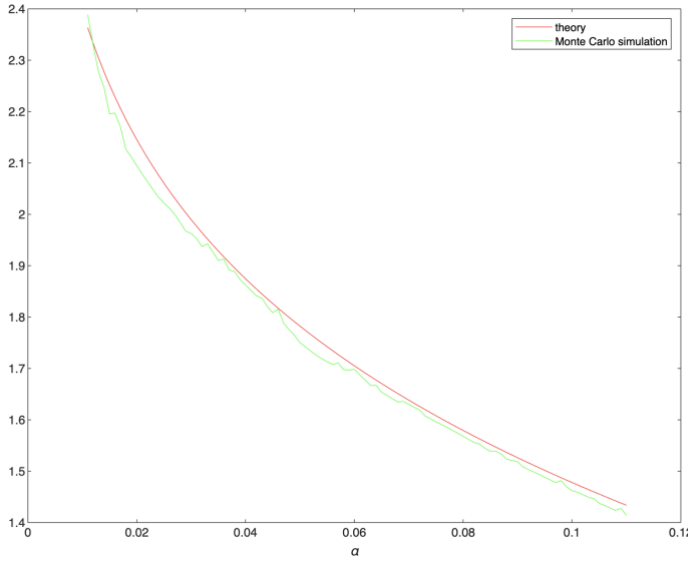
(6.13)

Figure 4. Comparison of the dependence on α of the simulated function $\frac{ES_\alpha(L(h)) - ES_\alpha(L(0))}{h}$ (green line) and of the theoretical function $\frac{\partial ES_\alpha(L(h))}{\partial h}\Big|_{h=0}$ (red line). Number of Monte Carlo simulations: 25 000. The parameters used: PD=0.1, $\rho = 0.1$, n=25 000, h=0.01.

7   Vasicek's Attempt

From formula (10) in Vasicek (2002, p. 8), after obvious transformations, the formula

$$var[L] = h\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); 1) + (1-h)\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2$$

(7.1)

follows for the unconditional variance, taking into account:

$$a = 1$$
$$b = \Phi^{-1}(PD)$$
$$H = T$$
$$\Phi_2(x, x; 1) = \Phi(x)$$

However, the following formula from Vasicek (2002) is erroneous:

$$var[L] \approx (\rho + (1-\rho)h)\varphi^2(\Phi^{-1}(PD))$$

(7.2)

Let's demonstrate how this error occurred.

Vasicek used the tetrachoric expansion of the bivariate normal CDF:

$$\Phi_2(x, x; \rho) \approx \Phi^2(x) + \rho\varphi^2(x)$$

(7.3)

$$\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) \approx PD^2 + \rho\varphi^2(\Phi^{-1}(PD))$$

(7.4)

Applying this expansion to the case $\rho = 1$ yields the incorrect result:

$$\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); 1) \approx PD^2 + \varphi^2(\Phi^{-1}(PD))$$

(7.5)

whereas, in fact,

$$\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) \xrightarrow[\rho \to 1^-]{} PD$$

As a result, Vasicek arrived at the incorrect approximation:

$$var[L] \approx h\left(PD^2 + \varphi^2(\Phi^{-1}(PD))\right) + (1 - h)\left(PD^2 + \rho\varphi^2(\Phi^{-1}(PD))\right) - PD^2$$
$$= (\rho + (1 - \rho)h)\varphi^2(\Phi^{-1}(PD))$$

instead of the correct approximation:

$$var[L] \approx hPD + (1 - h)\left(PD^2 + \rho\varphi^2(\Phi^{-1}(PD))\right) - PD^2$$
$$= hPD(1 - PD) + (1 - h)\rho\varphi^2(\Phi^{-1}(PD))$$

Furthermore, on page 8, Vasicek presented equation (12):

$$\mathbb{P}[L \le x] = F_V(x; p, \rho + h(1 - \rho))$$

without proper justification. The fallacy of this formula is demonstrated in Figure 5, where the function

$$dF_V(x) = \frac{F_V(x; p, \rho + h(1 - \rho)) - F_V(x; p, \rho)}{h}$$

is compared to the corresponding simulated function

$$dF_s(x) = \frac{f_s(x; p, \rho, h) - f_s(x; p, \rho, 0)}{h}$$

as well as the function

$$dF_{VH}(x) = \frac{F_{VH}(x; p, \rho, h) - F_{VH}(x; p, \rho, 0)}{h} = \frac{F_{VH}(x; p, \rho, h) - F_V(x; p, \rho)}{h}$$
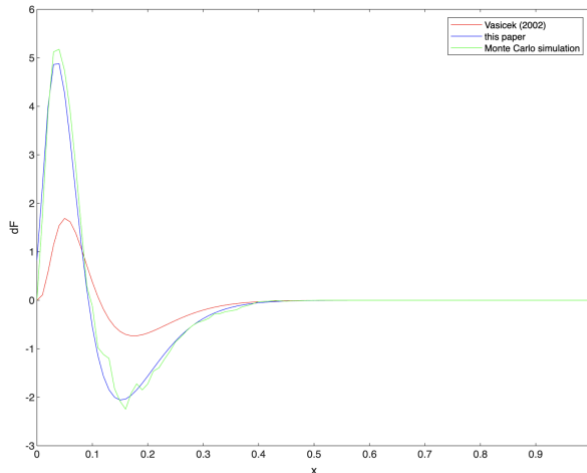
Figure 5. Comparison of $dF_V(x)$ (red line) with the corresponding simulated function (green line) and with $dF_{VH}(x)$ (blue line). Number of Monte Carlo simulations: 20 000. The parameters used: PD = 0.1, ρ = 0.1, n = 20 000, h = 0.01.

The poor quality of Vasicek's attempt to assess the granularity effect is evident in Figure 5.

8 Approach of (Voropaev, 2011)

Based on the information obtained in our paper about the behavior of the portfolio loss distribution near zero of the HHI, let us consider the approach of Voropaev (2011).

Transformation of variables $v = x - u$ allows us to write the Vasicek-Herfindahl function PDF (4.12) in the form of formula (3.1) of Voropaev (2011):

$$f^*(x) = \int_{-\infty}^{\infty} g(u|x-u)f(x-u)\,du$$

(8.1)

where

$$g(u|x) = \frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)}\theta(x)\theta(1-x)$$

(8.2)

$$f^*(x) = f_{VH}(x; PD, \rho, h)$$
$$f(x) = f_V(x; PD, \rho)$$

(8.3)

By expanding the integrand into a Taylor series at $u = x$, we obtain

$$g(u|x-u)f(x-u) = \sum_{k=0}^{\infty}(-1)^k \frac{u^k}{k!}\frac{\partial^k}{\partial x^k}\big(g(u|x)f(x)\big)$$

(8.4)

$$f^*(x) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!}\frac{d^k}{dx^k}\big(f(x)m_k(x)\big)$$

(8.5)

where $m_k(x)$ is the kth moment of distribution H conditional on V:

$$m_k(x) = \int_{-\infty}^{\infty} u^k g(u|x)\,du$$

(8.6)

Since $g(u|x)$ was unknown to Voropaev, he was unable to clarify the meaning of these moments, which renders his work unsuitable for practical applications. We will go further and show that his approach leads to the same results as our method.

It follows from equations (8.2) and (8.6) that

$$m_k(x) = \theta(x)\theta(1-x)\int_{-\infty}^{\infty} u^k \frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)}\,du$$

Since $\dfrac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)}$ is the even function of u, odd moments are equal to zero. Hence,

$$f^*(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{d^{2k}}{dx^{2k}}\big(f(x)m_{2k}(x)\big)$$

Since

$$\int_{-\infty}^{\infty} u^{2k} exp(-au^2)\, du = \sqrt{\frac{\pi}{a}} \frac{(2k-1)!!}{(2a)^k}$$

then

$$m_{2k}(x) = \theta(x)\theta(1-x) \int_{-\infty}^{\infty} u^{2k} \frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)}\, du = \theta(x)\theta(1-x)\big(h\sigma^2(x)\big)^k (2k-1)!!$$

Hence, by employing Voropaev's approach, we derive the equation:

$$f^*(x) = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!} h^k \frac{\partial^{2k}}{\partial x^{2k}}\big(\sigma^{2k}(x)f_V(x; PD, \rho)\big)$$

In contrast, our approach leads to the formula (4.39). Since

$$\frac{(2k-1)!!}{(2k)!} = \frac{1}{k!}\left(\frac{1}{2}\right)^k$$

both equalities coincide, leading to the result:

$$f_{VH}(x; PD, \rho, h) = f^*(x)$$

9 Approach of Emmer & Tasche (2005)

The approach proposed by Emmer & Tasche (2005) is based on the transformation:

$$\Lambda(h, c) = cL(h) + (1-c)\mathbb{E}(L(h)|V)$$

along with the expansion of $VaR_\alpha(\Lambda(h, c)) = x(\alpha, h, c)$ into a Taylor series at $c = 0$.

Since

$$L(h) = V + Z\sigma(V)\sqrt{h}$$

it follows that

$$\mathbb{E}(L(h)|V) = V$$

and

$$\Lambda(h, c) = V + c\sqrt{h}\sigma(V)Z$$

Thus, this transformation results in replacing the Herfindahl-Hirschman Index h with c^2h:

$$\Lambda(h, c) = L(c^2h)$$

The right-hand side of equation

$$\frac{\partial x(\alpha, h, c = 1)}{\partial h}\bigg|_{h=0} = \frac{\partial x(\alpha, h)}{\partial h}\bigg|_{h=0} = -\frac{\frac{\partial}{\partial v}\left(\sigma^2(v)f_V(v; PD, \rho)\right)}{2f_V(v; PD, \rho)}\bigg|_{v=x(\alpha)}$$

$$= -\frac{\frac{\partial}{\partial v}\left(h\sigma^2(v)f_V(v; PD, \rho)\right)}{2hf_V(v; PD, \rho)}\bigg|_{v=x(\alpha)}$$

$$= -\frac{\frac{\partial}{\partial v}\left(var[L(h)|V = v]f_V(v; PD, \rho)\right)}{2hf_V(v; PD, \rho)}\bigg|_{v=x(\alpha)}$$

coincides, up to a factor of $(2h)^{-1}$, with the right-hand side of equation (7) from Emmer & Tasche (2005):

$$\frac{\partial^2 x(\alpha, h, c)}{\partial c^2}\bigg|_{c=0} = -\frac{\frac{\partial}{\partial v}\left(var[L(h)|V = v]f_V(v; PD, \rho)\right)}{f_V(v; PD, \rho)}\bigg|_{v=x(\alpha)}$$

Emmer & Tasche (2005) denote the density of random variable $V = \mathbb{E}(L(h)|V)$ as γ_L instead of our notation $f_V(v; PD, \rho)$. Note that

$$var[L|V] = h\sigma^2(V)$$

Hence

$$\frac{\partial x(\alpha, h, 1)}{\partial h}\bigg|_{h=0} = \frac{1}{2h}\frac{\partial^2 x(\alpha, h, c)}{\partial c^2}\bigg|_{c=0}$$

Let us prove that this equality holds true.

Based on the Taylor series expansion of the Value at Risk (5.12):

$$VaR_\alpha\left(\Lambda(h, c)\right) = x(\alpha, h, c) = VaR_\alpha\left(L(c^2h)\right) = x(\alpha, c^2h)$$
$$= x(\alpha) + c^2h\frac{\partial x(\alpha, \chi)}{\partial \chi}\bigg|_{\chi=0} + o(c^2h)$$

we have:

$$\frac{\partial x(\alpha, h, c)}{\partial c}\bigg|_{c=0} = 0$$

$$\left.\frac{\partial^2 x(\alpha, h, c)}{\partial c^2}\right|_{c=0} = 2h \left.\frac{\partial x(\alpha, \chi)}{\partial \chi}\right|_{\chi=0}$$

Q.E.D.

10 Conclusion

The approach by Emmer & Tasche (2005) is based on the decomposition:

$$Loss = \mathbb{E}[Loss|Y] + c(Loss - \mathbb{E}[Loss|Y])$$

where $c = 1$, and is built upon the method proposed by Gouriéroux *et al* (2000) to calculate the derivatives of VaR. As a result, Emmer & Tasche (2005) obtained GA for VaR.

Our approach exploits the fact that, for small values of HHI, granularity risk is merely a small perturbation. We use the approximation:

$$Loss - \mathbb{E}[Loss|Y] \approx Z\sigma(p(Y))\sqrt{h}$$

which is obtained using the Lyapunov CLT. The random value $Z\sigma(p(Y))\sqrt{h}$ is the GA to the random value $Loss$. Consequently, in addition to the first factor, the market shock Y, a second factor, the granularity shock $Z\sigma(p(Y))\sqrt{h}$, is added to the model. Rather than studying the behavior of a heterogeneous portfolio, we focus on the behavior of a homogeneous portfolio influenced by this granularity shock.

The expansion of VaR into a Taylor series up to the first order of HHI yields the same result using both approaches. However, the approach proposed by (Emmer & Tasche, 2005) is specifically designed to expand VaR and ES, while our approach is more general and, in particular, allows us to obtain PDF and CDF of portfolio loss, taking into account the granularity effect.

We see that the HHI is a crucial factor for modeling the granularity effect and can be utilized to regulate a bank's economic capital.

The GA to VaR is given by:

$$GA^{VaR} = -\frac{h}{2}\left(1 - 2x(\alpha) + \frac{x(\alpha)(1 - x(\alpha))}{\varphi\left(\Phi^{-1}(x(\alpha))\right)}\left(\Phi^{-1}(x(\alpha)) - \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(1-\alpha)\right)\right)$$

where

$$x(\alpha) = VaR_\alpha\big(Loss(h = 0)\big) = \Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(1-\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)$$

The formula for GA^{VaR} is quite complex and lacks clarity. The IRB approach uses the formula

$$VaR_\alpha\big(Loss(h = 0)\big) = F_V^{-1}(1 - \alpha; PD, \rho)$$

We recommend adopting a more transparent and intuitively understandable formula for supervisory purposes:

$$VaR_\alpha\big(Loss(h)\big) = F_{VH}^{-1}(1-\alpha; PD, \rho, h)$$

where

$$F_{VH}(x; PD, \rho, h) = \int_0^1 F_V(v; PD, \rho)d\Phi\left(\frac{v-x}{\sqrt{h}\sigma(v)}\right)$$

This formulation explicitly incorporates the granularity parameter h, providing a clearer and more flexible framework for supervisory risk assessment.

11 Appendix

The function $\Phi\big(u/\sqrt{z}\big)$ satisfies the classical heat conduction (diffusion) equation:

$$\Phi_z\big(u/\sqrt{z}\big) = \frac{1}{2}\Phi_{uu}\big(u/\sqrt{z}\big)$$

The generalized heat equation

$$\frac{\partial^k}{\partial z^k}\Phi\big(u/\sqrt{z}\big) = 2^{-k}\frac{\partial^{2k}}{\partial u^{2k}}\Phi\big(u/\sqrt{z}\big)$$

can be proved by induction:

$$\frac{\partial^{k+1}}{\partial z^{k+1}}\Phi\big(u/\sqrt{z}\big) = 2^{-k}\frac{\partial}{\partial z}\frac{\partial^{2k}}{\partial u^{2k}}\Phi\big(u/\sqrt{z}\big) = 2^{-k}\frac{\partial^{2k}}{\partial u^{2k}}\frac{\partial}{\partial z}\Phi\big(u/\sqrt{z}\big)$$
$$= 2^{-(k+1)}\frac{\partial^{2(k+1)}}{\partial u^{2(k+1)}}\Phi\big(u/\sqrt{z}\big)$$

Given the limit representation of the Heaviside step function:

$$\Phi(x/\varepsilon) \xrightarrow[\varepsilon \to 0^+]{} \theta(x)$$

and relation between the Dirac delta function and the Heaviside step function:

$$\delta(x) = \theta'(x)$$

we have:

$$\lim_{z\to 0^+}\frac{\partial^k}{\partial z^k}\Phi\big(u/\sqrt{z}\big) = 2^{-k}\lim_{z\to 0^+}\frac{\partial^{2k}}{\partial u^{2k}}\Phi\big(u/\sqrt{z}\big) = 2^{-k}\frac{\partial^{2k}}{\partial u^{2k}}\theta(u) = 2^{-k}\delta^{(2k-1)}(u)$$

Using the above, the function $\Phi\big(u/\sqrt{z}\big)$ can be expanded into a Taylor series around $z = 0$:

$$\Phi\big(u/\sqrt{z}\big) = \theta(u) + \sum_{k=1}^{\infty}\frac{(z/2)^k}{k!}\delta^{(2k-1)}(u)$$

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