Mixture and Continuous 'Discontinuity' Hypotheses: An Earnings Management Model with Auditor-Required Adjustment

Yim, Andrew

Cass Business School, City University London

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An Earnings Management Model with Auditor-Required Adjustment[†]

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A model emphasizing cookie-jar earnings management and the effect of auditor-required adjustment is formulated, with the optimal misreporting strategy generally characterized and the closed-form solutions for particular functional form assumptions derived. Using simulation results based on the model, I show that the widely documented discontinuity in the earnings triplet distributions (i.e., earnings, earnings change, and earnings surprise) can be partly due to a steep increase in density appearing like a discontinuity when a continuous distribution is plotted in terms of frequency counts in histogram bins. Additionally, I point out the puzzling volcano shape of the earnings triplet distributions that can be found in prior studies. Simulation results show that the model is capable of accommodating this phenomenon, which can arise from the mixture of a spiky distribution of managed earnings with a bell-shaped distribution of unmanaged earnings. This mixture is due to the auditor's adjustment decision, which seems stochastic from the public's or client firm's perspective. Taken together, the results of this paper provide a unified explanation to two perplexing, salient features of the earnings triplet distributions. Potential applications of the model are suggested, including the construction of an earnings manipulation measure distinct from but complementary to abnormal accruals. (*JEL* M43/M49/K42)

Key words: Misreporting, Cookie-jar accounting, Benchmark Reference, Auditor-client Interaction

Contacts. Cass Business School, City University London, Faculty of Finance, 106 Bunhill Row, London EC1Y 8TZ, UK. Phone: +44 20 7040-0933. Fax: +44 20 7040-8881. E-mail: a.yim@city.ac.uk / andrew.yim@aya.yale.edu .

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1. Introduction

Hayn (1995) was the first to point out a discontinuity at zero in the distribution of earnings. Burgstahler and Dichev (1997) systematically showed that the discontinuity occurs in the distribution of earnings change as well as earnings, suggesting that it could be caused by earnings management to avoid earnings decreases and losses. Since then, a number of studies have found similar phenomena for particular types of institutions and for different countries.[1] Degeorge et al. (1999) found a similar discontinuity in the distribution of earnings surprise (i.e., negative forecast error) using forecast data in 1974-1996. Bhojraj et al. (2009) also documented such a discontinuity using forecast data in 1988-2006. Using simulation results based on an analytical model, I show that the widely documented discontinuity can be partly due to a steep increase in density appearing like a discontinuity when a continuous distribution is plotted in terms of frequency counts in histogram bins. The steep increase in density is a consequence of the compression of unmanaged earnings into reported earnings toward an earnings benchmark as a result of cookie-jar earnings management. This provides an alternative explanation to the discontinuity phenomenon, supplementing the prevalent explanation based on upward earnings management by "just-missed" firms.[2]

Besides a discontinuity at zero, the distributions of the earnings triplet (i.e., earnings, earnings change, and earnings surprise) usually have a volcano shape (see, e.g., figure 2 of Bhojraj et al. 2009, reproduced as figure 5b here). The sharp peak of this shape is markedly different from the smoother, flatter top of the bell shape of a normal distribution, or that of similar alternatives such as logistic or extreme value type I distributions. To my knowledge, no prior study has highlighted the volcano shape of the distributions, let alone explain it. Because of the compression of earnings arising from cookie-jar earnings management attempts, even if the distribution of unmanaged earnings were normal (as one might expect), the distribution of managed earnings can have a volcano shape. Auditors sometimes require adjustments to remove earnings management attempts (Nelson et al. 2002).[3] Consequently, the distribution of reported earnings observed by the public is a mixture of the distributions of unmanaged and managed earnings. The

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[1] For example, banks (Beatty et al. 2002), property-casualty insurers (Beaver et al. 2003), nonprofit hospitals (Leone and Van Horn 2005), hotels (Parte Esteban and Such Devesa 2011), Australia (Coulton et al. 2005), European Union (Daske et al. 2006), UK (Gore et al. 2007), Japan (Shuto 2009), and Spain (Parte Esteban and Such Devesa 2011).

[2] A number of studies have examined whether this prevalent explanation is indeed the main cause (Dechow et al. 2003, Durtschi and Easton 2005 and 2009, Beaver et al. 2007, Jacob and Jorgensen 2007, and Kerstein and Rai 2007). While reasons behind the discontinuity are subject to debate, the phenomenon itself is not.

[3] Among the 515 specific attempts of earnings management recalled by audit partners and managers in the Nelson et al. (2002) study, auditors adjusted 44% of the attempts. The rest were not adjusted because the auditor lacked convincing evidence to prove that the client's position was incorrect (17%) or the auditor believed the client demonstrated compliance with GAAP (21%), with the remaining 18% due to other reasons (usually, immateriality).

mixture can retain the volcano-shape characteristics of the unmanaged earnings distribution. I illustrate this explanation using simulation results based on an analytical model that takes into account auditor-required adjustment. Prior models of misreporting have omitted this aspect in order to focus on other issues, such as the interaction between a firm's disclosure and investors' rational expectation.

Many of the studies investigating earnings management and fraud are empirical-based, with only a few of them (e.g., Kedia and Philippon 2009) closely guided by formal theoretical models. Models of misreporting exist in the literature (e.g., Kumar and Langberg 2009, Guttman et al. 2006, Ewert and Wagenhofer 2005, Kirschenheiter and Melumad 2002, Sankar and Subramanyam 2001, and Fischer and Verrecchia 2000).[4] Yet, except Caskey et al. (2010) and Newman et al. (2001), few have explicitly modeled the auditor, which plays an important role in the corporate reporting process (Bollen and Pool 2009, Caramanis and Lennox 2008, Brown and Pinello 2007, and Liang 2003). Some of the models above have provided explanations to the discontinuity phenomenon. However, no model has given a unified explanation to the two salient features of the earnings triplet distributions.

The sort of earnings manipulation discussed in this paper is better understood as borderline misreporting rather than outright fraud. Proving the intention of borderline misreporting is often difficult. Usually what auditors can do is simply requiring adjustments to remove such earnings management attempts. In the unusual circumstances where the extent of misreporting is so outrageous to strongly suggest outright fraud, auditors may notify regulators. Otherwise, rarely would client firms bear any significant consequences of borderline misreporting disallowed by auditors. Therefore, the main consequence of unsuccessful borderline misreporting is the reversal of the attempted manipulations before announcing the earnings to the public.

The distinction between borderline misreporting and outright fraud suggests a novel assumption on a firm's cost of misreporting for downward manipulations, sometimes referred to as cookie-jar accounting. In contrast to the conventional quadratic assumption, I assume that the cost of misreporting is an increasing function of earnings manipulation, even for the negative values representing downward manipulations. In other words, a larger downward manipulation results in a more negative misreporting cost standing for the opportunity benefit of "saving for the future." An auditor-required adjustment reversing a downward manipulation attempt means a reversal of the negative misreporting cost as well.

Downward manipulations look like conservative accounting when viewed with respect to the current period. However, they can be used to build up cookie-jar reserves for subsequent upward manipulations (Cohen et al. 2011, Jackson and Liu 2010, and Moehrle 2002). Cookie-jar accounting has caused serious

[4] I distinguish between models of misreporting that are continuous in the accounting report in concern and other earnings management models that focus on discrete accounting reports (e.g., Gao 2013).

concern to regulators. In his famous 1998 speech entitled *The 'Numbers Game'*, the then Chairman of the U.S. Securities and Exchange Commission (SEC) Arthur Levitt said: "[U]sing unrealistic assumptions to estimate liabilities for such items as sales returns, loan losses or warranty costs … [some companies] stash accruals in cookie jars during the good times and reach into them when needed in the bad times" (http://www.sec.gov/news/speech/speecharchive/1998/spch220.txt).

Recently, the Public Company Accounting Oversight Board (PCAOB) fined Ernst & Young (E&Y) $2 million for failing to properly evaluate in the 2005 to 2007 audits of Medicis Pharmaceutical Corp. the amount set aside to account for the cost of product returns. In addition, it sanctioned three current partners of the audit firm plus one retired, barring some of them from auditing public companies for one to two years or more and imposing fines from $25,000 to $50,000. Medicis corrected the misreporting when it was discovered in 2008 and has restated the financial statements for the years affected. The company said: "It actually revealed that we were more profitable across the overall six-year restatement period." The company also emphasized that "[s]everal independent reviews found that the errors didn't stem from any improper efforts to inflate earnings" (Gordon 2012). This example suggests that regulators disapprove non-inflationary manipulations as much as inflationary ones and companies do use cookie-jar accounting. The model of this paper accommodates cookie-jar accounting as equilibrium behavior under a wide range of circumstances. In contrast, models focusing on upward manipulations usually admit downward manipulations as possible behavior occurring infrequently in equilibrium.

In the model, a firm manipulates the earnings before providing the figure to an auditor for audit. The audit allows the auditor to separate the unmanaged and manipulated components of the pre-audit (managed) earnings. He can require the firm to make an adjustment to remove the manipulation before announcing the post-audit (reported) earnings to the public. However, the auditor must incur a cost, privately known to him, to convince the firm to follow the requirement.[5] Depending on the magnitude of the cost and that of the expected liability cost arising from tolerating the manipulation, the auditor decides whether to require an adjustment or not. From the firm's perspective, this outcome of the auditor's adjustment decision is stochastic. Considering the chance of a required adjustment, the firm chooses the extent of an upward or downward manipulation to balance the benefit and cost of misreporting. If the auditor tolerates the manipulation, the managed earnings are reported to the public. If he requires an adjustment, the manipulation is reversed. So is the firm's cost of misreporting because it captures the opportunity cost (or benefit) of "borrowing from (or saving for) the future." The earnings reported to the

[5] See section 2 and related discussions in McCracken et al. (2008), Gibbins et al. (2001), and Beattie et al. (2004) for explanations about the adjustment requirement cost borne by the auditor.

public are then the unmanaged earnings.

In reality, an auditor influences the earnings reported to the public through a complicated auditor-client negotiation process (see, e.g., Perreault and Kida 2011, Bame-Aldred and Kida 2007, Beattie et al. 2004, and Gibbins et al. 2001). Modeling this process in an elaborated manner requires a separate paper. The moderate goal here is to use a tractable "take-it-or-leave-it" game to capture only the first-order impact of the auditor-client negotiation. All models are wrong, but some are useful (Box and Draper 1987 and Box 1976). The ability to accommodate the two salient features of the earnings triplet distributions suggests that this stylized formulation of the auditor-client negotiation process is useful.

The paper has two main contributions. First, it offers a unified explanation to two puzzling features of the earnings triplet distributions. Second, the model provides a theoretical foundation for using the optimal manipulation characterized here as an alternative measure of earnings management distinct from but complementary to the commonly used measure based on the abnormal accruals model (see the discussion in section 6 for details).

A recent study by Gerakos and Kovrijnykh (2013) has derived another measure of earnings management based on income smoothing, which is similar to the cookie-jar earnings management emphasized here. They find that the unmanaged earnings inverted from their model have higher volatility than reported earnings, which is consistent with the implication of my model. Their model has no auditor-required adjustment and cannot provide related insights like the mixture hypothesis, which is tied to the important role of auditor in the corporate reporting process. Neither have they pointed out the perplexing volcano shape of the earnings triplet distributions. Moreover, the proportional misreporting rule assumed by them is not likely to be capable of explaining the volcano shape and the discontinuity at the same time. They also assume that the unmanaged earnings are persistent, which is not required in my model. However, their model captures not only accounting-based earnings management but also real earnings management, which is more general than the model here.

The rest of the paper is organized as follows. The next section introduces the model setup. In section 3, I provide a general characterization of the optimal misreporting strategy, which is unique under some mild conditions. In section 4, two (effectively four) closed-form solutions are derived for particular functional form assumptions. Section 5 presents the results of two simulation exercises demonstrating the model's capability of generating the two salient features of the earnings triplet distributions. Potential applications of the model and other concluding remarks are discussed in section 6. Maximum likelihood and nonlinear least squares methods to estimate the model parameters are discussed in appendix A. Appendix B suggests a few ways to extend the model, such as allowing analyst forecast dispersion to play a role. Technical proofs are relegated to appendix C (available upon request).

2. Model Setup

In the model, a firm needs to prepare the earnings figure for audit before announcing it to the public. The firm generally has an interest to manipulate the figure away from the *unmanaged earnings* denoted by y. The interest is affected by some *earnings benchmark* denoted by z, which is common knowledge in the model. Real-world examples of earnings benchmarks include the profit/loss cutoff at zero, last year's earnings highlighting earnings increase/decrease, and the analyst consensus forecast reflecting market expectation (see section 3.1.5 of Dechow et al. 2010). In circumstances where a firm's accounting choices are heavily influenced by certain executives, earnings benchmarks can be some internal yardsticks used for performance evaluation related to salary increases, bonuses, promotion, etc. One of the key insights of this paper is that the widely documented discontinuity phenomenon could be partly due to a steep but *continuous* increase in the density of the earnings distribution. Therefore, I will specify a payoff function of the firm that is smooth even at the earnings benchmark. This specification sets the model apart from others (e.g., Degeorge et al. 1999) that rely on a jump (e.g., a lump-sum bonus) to generate a discontinuity in the distribution of reported earnings.

The earnings figure prepared for audit may include manipulation that nonetheless is not outright fraud. Let $a \in (-\infty,\infty)$ denote such manipulation. The auditor either learns y and a during the audit that allows unrestricted access to the firm's accounting system, or in equilibrium can infer them from the *pre-audit earnings* (or *managed earnings*), denoted by $m = y + a$, provided to him for audit. The closed-form solutions derived in section 4 are examples of such cases where m maps into y uniquely; hence the optimal a can be inferred by the auditor in equilibrium.

In the very beginning of the model, the auditor explains the audit plan to the firm. The quality level $q \in [0,1]$ of the audit is known to the firm at that time before the firm sees the unmanaged earnings y and accordingly selects the manipulation a. Denote by $\varepsilon_q + \varepsilon_{1-q}$ the total amount of unintentional errors contained in y. By definition, the firm is unaware of these errors. The components $\varepsilon_q \sim \text{Normal}(0, q\sigma_u^2)$ and $\varepsilon_{1-q} \sim \text{Normal}(0, (1-q)\sigma_u^2)$ are independent random variables, with the parameter $\sigma_u > 0$. The part of unintentional errors discovered and removed by the auditor is ε_q, with ε_{1-q} remaining in y even after the audit. For ease of exposition, q is treated as an exogenous parameter in this paper. Endogenizing it does not critically affect the main results. (I discuss how it can be endogenized in appendix B.) Excluding unintentional errors from the model does not qualitatively change the results either. However, their inclusion helps to understand the model's implications; for example, it allows one to see the effect of the quality parameter q, which is interesting.

Let $x \in \{0,1\}$ denote the auditor's adjustment decision, with 0 standing for no adjustment required and 1 otherwise. Requiring an adjustment to remove the manipulation is not as simple as just saying no. To achieve the objective, the auditor must bear a cost, $X \in [0,\infty)$, privately known to him. Field studies (e.g., McCracken et al. 2008) have documented the stress auditors face and the effort they make during the negotiation process to convince client firms to make adjustments, while not jeopardizing the relationship and the likelihood of being retained as the auditors. Such a process can take weeks, or even months, to finish (Gibbins et al. 2001), potentially distracting the auditors from concentrating on other work like the audits of other clients. Preparation work such as producing sufficient literature research to back up the adjustment requirement, gathering evidence of similar practices by other companies, securing the support from the national office, or even obtaining a second opinion from another audit firm also contributes to the cost of requiring an adjustment. Alternatively, the cost may be viewed as a parsimonious way to model whether the auditor's personality is closer to a crusader type (who never tolerates any manipulation), as opposed to an accommodator type (who is prepared to bend the rules to interpret a manipulation as complying with GAAP).[6]

The cost X is modeled as a random variable independent of ε_q and ε_{1-q}. It follows a probability distribution $G(l) = \Pr\{X \le l\}$ with a differentiable probability density $g(l) = G'(l) > 0$ for all $l > 0$. The hazard rate function $h(l) = [1-G(l)]/g(l)$ is therefore also differentiable. To ensure that the firm's objective function (to be specified shortly) is twice-differentiable in a, I assume the existence of bounded limits $\lim_{l\downarrow 0} l^{1/2} g(l)$ and $\lim_{l\downarrow 0} [g(l) + 2lg'(l)]$. This is a mild condition satisfied by a number of distributions including those discussed in section 4. The auditor learns the realized value of X just before making the adjustment decision. The firm knows only the distributional properties of X without observing the realized value.

The expected liability cost to the auditor is assumed to be a quadratic function of the extent of the manipulation: $L = ka^2/2$, where $k > 0$. (Any expected liability cost that might arise from failing to remove all the unintentional errors is irrelevant to the auditor's adjustment decision and need not be specified.) The optimal adjustment decision, $x^*(a)$, is determined by minimizing the sum of the expected liability cost and the cost of requiring an adjustment:

$$\text{Min}_{x\in\{0,1\}} \quad (1-x)L + x(X).$$

Clearly, $x^*(a) = \mathbf{1}\{X \le L\}$, i.e., equal to 1 when the event $\{X \le L\}$ holds and 0 otherwise. In other words, the auditor will require an adjustment if and only if the cost of doing so is not higher than the expected liability cost from tolerating the manipulation.

The firm's manipulation incentive is driven by the benefit and cost of misreporting (see Jiambalvo 1996 and Marquardt and Wiedman 2004 for discussions on the benefits and costs). Because I focus on cookie-jar earnings management, a dollar of downward manipulation today helps build up the cookie-jar reserve, facilitating future upward manipulations. Similarly, every dollar of upward manipulation today is essentially borrowed from the future. Hence, if in the absence of distortions and discounting, the cost of

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[6] Beattie et al. (2004) identify six types of auditor personality with the crusader and accommodator types as the two extremes observed in their field study.

misreporting can be as simple as $c(a) = a$. Allowing for discounting and potential distortions (e.g., a small psychic cost of misreporting), I assume a more general $c(a)$ with the requirements that $c(0) = 0$, $\lim_{a\uparrow\infty} c(a) = \infty$, $c'(a) > 0$ with $c'(0) = c_0 < \infty$, and $c''(a) \geq 0$.[7] The monotonicity assumption, $c'(a) > 0$, is a departure from the usual specification of a quadratic misreporting cost function (e.g., Guttman et al. 2006 and Fischer and Verrecchia 2000). The assumption is essential for deriving downward manipulations in the model. I will elaborate on this further shortly.

The firm benefits more from reporting earnings that exceed the earnings benchmark to a greater extent and bears more negative consequences from missing the benchmark more. This is captured by a negative exponential benefit function. If the auditor does not require an adjustment to remove the manipulation a, the net benefit to the firm is

$$\left[\frac{1-\exp\left(-\alpha[y+a-\varepsilon_q-z]\right)}{\alpha}\right]-c(a),$$

where $\alpha > 0$, with the limit case of $\alpha = 0$ accommodating a linear benefit function. If an adjustment is required, a is reversed. So is $c(a)$, which captures the opportunity cost (or benefit) of "borrowing from (or saving for)" the future. Accordingly, the net benefit to the firm is simply $[1 - \exp(-\alpha[y - \varepsilon_q - z])]/\alpha$, as though there was no manipulation attempt.

To reduce notation complexity, I will simply write $x^*(a)$ as x to denote the binary random variable induced by the auditor's optimal adjustment decision. The earnings announced to the public after the audit are referred to as the *post-audit earnings* (or *reported earnings*) defined as follows:

$$r = xy + (1 - x)m - \varepsilon_q.$$

The firm's manipulation decision is made before ε_q is known. Anticipating the auditor's response $x^*(a)$, the firm chooses a to maximize the expected net benefit given below:

$$\mathrm{E}\left\{(1-\mathbf{1}\{X\leq L\})\left[\frac{1-\exp\left(-\alpha[y+a-\varepsilon_q-z]\right)}{\alpha}-c(a)\right]+\mathbf{1}\{X\leq L\}\left[\frac{1-\exp\left(-\alpha[y-\varepsilon_q-z]\right)}{\alpha}\right]\right\},$$

which can be simplified as

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[7] Although the model technically is a single-period model, it should be viewed as modeling a representative period of an infinite-period setting, rather than as a one-shot model. The misreporting cost in this reduced-form representation of an infinite-period setting plays a role analogous to the next-period value function in a Bellman equation for dynamic programming. Such a value function is endogenously determined in a Bellman equation but the misreporting cost in the reduced-form representation here is exogenously specified. The reduced-form representation could be formulated as a two-period model like some others in the literature. Such two-period models typically assume that the manipulation in the first period must be reversed in the second period, effectively turning the two-period choices of manipulation into a one-period choice. Therefore, the difference between the two formulations is not as big as they appear. The representative-period formulation, however, seems to be more consistent with the going-concern perspective of accounting.

$$v = \frac{(1-b)}{\alpha} + [1 - G(L)]\left[\frac{b[1-\exp(-\alpha a)]}{\alpha} - c(a)\right],$$

where

$$b = \exp\left[-\alpha(y-z) + \frac{\alpha^2 \sigma_{\mathrm{u}}^2 q}{2}\right].$$ [8]

Note that the parameter b defined above summarizes the impacts of the quality parameter q and the deviation of y from z on the firm's misreporting incentive. For ease of reference, I simply call b the *marginal expected benefit of manipulation*, although strictly speaking it is only one of the contributing factors.

The firm can control a but not ε_q. When ε_q is more uncertain as a result of a higher quality level q, it raises b for any given levels of $y - z$ and a. Consequently, the slope of the expected net benefit function changes, providing a different incentive to manipulate earnings. Likewise, a different benchmark z can also change the incentive.

Whether the optimal manipulation is upward or downward also depends on the marginal cost of manipulation, $c'(a)$, relative to the marginal benefit affected by b. If $c(a)$ were assumed to be quadratic as in other earnings management models, $c'(a)$ would be negative for $a < 0$. Downward manipulation would never constitute an equilibrium. Therefore, the assumption of $c'(a) > 0$ is critical for accommodating both upward and downward manipulations. Given that cookie-jar accounting is an important regulatory concern, it is interesting to explore the implications of this alternative assumption that captures the idea of "saving for the future."

I end this section with table 1 that summarizes the notations used and the timeline in figure 1 that summarizes the sequence of events in the model.

3. Optimal Misreporting Strategy

This section gives a general characterization of the optimal manipulation. In the next section, closed-form solutions for particular adjustment requirement cost distribution and misreporting cost functions are derived. Then in section 5 the solutions are used to conduct two simulation exercises to see how well, or not so well, the model is able to accommodate the two salient features of the earnings triplet distributions.

Central to the characterization of the optimal manipulation is a cutoff point y_0 given by the equation $\exp[-\alpha(y - z) + \alpha^2\sigma_{\mathrm{u}}^2 q/2] = c_0$. That is to say, $y_0 \equiv z - (\ln c_0)/\alpha + \alpha\sigma_{\mathrm{u}}^2 q/2$. This cutoff point of the

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[8] $\mathrm{E}[\ [1 - \mathbf{1}\{X \le L\}][[1 - \exp(-\alpha[y + a - \varepsilon_q - z])]/\alpha - c(a)] + \mathbf{1}\{X \le L\}[1 - \exp(-\alpha[y - \varepsilon_q - z])]/\alpha\ ]$
$= \mathrm{E}[1 - \exp(-\alpha[y - \varepsilon_q - z])]/\alpha + \mathrm{E}[1 - \mathbf{1}\{X \le L\}]\ \mathrm{E}[\ \exp(-\alpha[y - \varepsilon_q - z])[1 - \exp(-\alpha a)]/\alpha - c(a)\ ]$
$= (1 - b)/\alpha + [1 - G(L)][\ b[1 - \exp(-\alpha a)]/\alpha - c(a)\ ],$
where $b = \mathrm{E}[\exp(-\alpha[y - z - \varepsilon_q])] = \exp[-\alpha(y - z) + \alpha^2\sigma_{\mathrm{u}}^2 q/2]$. The first equality above uses the assumption that X and ε_q are independent, and the second utilizes the normality assumption on ε_q.

unmanaged earnings determines whether the optimal manipulation is upward or downward. The proposition stated below proves the existence of the optimal misreporting strategy and characterizes it with several equality and inequality conditions.

Proposition 1 (Existence and Characterization of Optimal Misreporting Strategy): *Let* $b = \exp[-\alpha(y - z) + \alpha^2\sigma_u^2 q/2]$ *and* $y_0 = z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$. *An optimal manipulation of the firm, denoted by* a^*, *exists and satisfies*

(POS) $$b[1 - \exp(-\alpha a)]/\alpha - c(a) \geq 0,$$

which defines a convex set of a, denoted by Φ. *Moreover,*

(i) a^* *is a solution of the following equation:*

(FOC) $$[1-G(L)][\,b\exp(-\alpha a) - c'(a)\,] = kag(L)[\,b[1 - \exp(-\alpha a)]/\alpha - c(a)\,],$$

where $L \equiv ka^2/2$;

(ii) a^* *satisfies the following inequality:*

(SOC) $$[1 - G(L)][\,\alpha b\exp(-\alpha a) + c''(a)\,] + k[\,g(L) + 2Lg'(L) + 4Lg(L)^2/(1 - G(L))\,][\,b[1 - \exp(-\alpha a)]/\alpha - c(a)\,] \geq 0;$$

(iii) unless $y = y_0$, *zero manipulation (i.e.,* $a = 0$*) is suboptimal;*

(iv) for $y > y_0$, *any optimal manipulation is downward (i.e.,* $a^* < 0$*); for* $y < y_0$, *any optimal manipulation is upward (i.e.,* $a^* > 0$*).*

For a manipulation to be optimal, the v for $a > 0$ minus the v for $a = 0$ must be non-negative, i.e., the manipulation must bring a non-negative gain to the firm. Since $[1 - G(L)] > 0$, the non-negativity requirement is simply condition POS. The equality first-order condition FOC can pin down candidates of optimal manipulation, of which some are actually suboptimal. The second-order condition SOC can differentiate between them but oftentimes condition POS can do so more conveniently. The last two parts of the proposition highlight a very simple structure of the optimal manipulation: Downward manipulation is optimal when the unmanaged earnings are sufficiently high (i.e., $y > y_0$); otherwise, upward manipulation is optimal. This result can provide the force to turn normally distributed unmanaged earnings into a volcano-shaped distribution of post-audit earnings. I will explain this further in section 5.

The next result stated below establishes the uniqueness of the optimal manipulation. Other earnings management models often use a signaling approach, leading to multiple equilibria. This limits the potential of using such models to guide *structural* estimation in empirical studies because it is not clear which of the equilibria is observed in data.[9] In contrast, the uniqueness result in the next proposition removes such

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[9] One can choose to focus only on the linear equilibrium when linking signaling-based earnings management models to data. In fact, this is typically the only equilibrium analyzed for such models, as a way to get around the intractability of analyzing all the multiple equilibria. The linear equilibrium specifies a rational price conjecture under which the share price is an affine function of the reported earnings. Although this equilibrium appears to be simplest and most intuitive, taking it to data causes some technical complications. For example, the reported earnings in

ambiguity when linking the model of this paper to data.

Proposition 2 (Uniqueness of Optimal Misreporting Strategy): *Suppose* $1 + 2l[d\ln g(l)/dl] - 4l[d\ln(1 - G(l))/dl] \geq 0$ *(e.g., if* $d\ln g(l)/dl \geq -1$*), or equivalently,* $d\ln[h(l)(1 - G(l))]/dl \leq 1/2l$ *(e.g., if* $h'(l) \leq 0$*). Then any manipulation satisfying conditions FOC and POS also satisfies condition SOC with strict inequality. Consequently, the firm has a unique optimal manipulation* a^* *fully characterized by conditions FOC and POS.*

The proof of this proposition shows that under the conditions on the adjustment requirement cost distribution specified in the proposition, conditions FOC and POS can pin down the unique optimal manipulation. This approach offers the way to derive the closed-form solutions given in the next section.

4. Closed-Form Solutions

I begin with the following lemma showing that several families of the adjustment requirement cost distribution can ensure the uniqueness of the optimal manipulation.

Lemma 1 (Distributions Ensuring Unique Optimal Misreporting Strategy): *Suppose that the adjustment requirement cost distribution belongs to the following families:*

(i) Weibull distribution with $\lambda > 0$ *and* $\theta > 0$*, i.e.,* $g(l) = \theta\lambda^{\theta}l^{\theta-1}\exp[-(\lambda l)^{\theta}]$ *with* $1 - G(l) = \exp[-(\lambda l)^{\theta}]$ *and* $h(l) = l^{1-\theta}/\theta\lambda^{\theta}$*, provided that* $\theta \geq ½$ *(including Exponential when* $\theta = 1$*);*

(ii) Gompertz distribution with $\lambda > 0$ *and* $\theta > 0$*, i.e.,* $g(l) = \lambda\theta\exp(\lambda l)\exp(-\theta[\exp(\lambda l) - 1])$ *with* $1 - G(l) = \exp(-\theta[\exp(\lambda l) - 1])$ *and* $h(l) = \exp(-\lambda l)/\lambda\theta$*;*

(iii) Pareto Type II distribution with $\lambda > 0$ *and* $\theta > 0$*, i.e.,* $g(l) = (\theta/\lambda)[\lambda/(\lambda + l)]^{\theta+1}$ *with* $1 - G(l) = [\lambda/(\lambda + l)]^{\theta}$ *and* $h(l) = (\lambda + l)/\theta$*, provided that* $\theta \geq ½$.

Then the firm has a unique optimal manipulation a^*.

I derive two closed-form solutions under the assumptions of an exponential and a linear misreporting cost function, respectively. This effectively provides two more, with the additional ones corresponding to hybid misreporting cost functions combined from the exponential and linear functions. Before elaborating on this further, let me first present the more specific form of condition FOC under the assumption of a Weibull adjustment requirement cost distribution with $\theta = ½$.[10] This proves to be a useful assumption that greatly simplifies the first-order condition and allows the solution to be expressed in closed form.

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principle can take an arbitrarily large positive or negative value, e.g., as a realization from a normal distribution. However, the share price cannot be negative. Therefore, the linear rational price conjecture cannot be well-defined for all possible values of the reported earnings if the distribution of earnings has the real line as the support, which seems very reasonable. In contrast, the model of this paper does not run into such a complication.

[10] Though not popular in accounting studies, the Weibull distribution is one of the most commonly used distributions in the area of reliability in engineering. The distribution is a member of the exponential family. Other members of the family include familiar distributions such as normal, exponential, gamma, chi-squared, beta, binomial, and Poisson.

Corollary 1 (First-Order Condition with Weibull Adjustment Requirement Cost Distribution): *Suppose the distribution of the auditor's adjustment requirement cost is Weibull with* $\lambda > 0$ *and* $\theta = ½$. *Let* $\eta = (k\lambda/2)^{½}$, *which is a parameter capturing the relative importance of the expected liability cost and adjustment requirement cost to the auditor. Moreover, let* $b = \exp[-\alpha(y-z) + \alpha^2\sigma_u^2 q/2]$, $y_0 = z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$, *and* Φ *be the convex set of a defined by condition POS. Then*

(i) for $y > y_0$ *(i.e.,* $b < c_0$*), the optimal manipulation is downward (i.e.,* $a^* < 0$*) and uniquely determined by*

$$\exp(\alpha a)[\, c'(a) + \eta c(a) - (\eta/\alpha)b \,] = (1 - \eta/\alpha)b \qquad \text{for } a \in \Phi;$$

(ii) for $y < y_0$ *(i.e.,* $b > c_0$*), the optimal manipulation is upward (i.e.,* $a^* > 0$*) and uniquely determined by*

$$\exp(\alpha a)[\, c'(a) - \eta c(a) + (\eta/\alpha)b \,] = (1 + \eta/\alpha)b \qquad \text{for } a \in \Phi;$$

(iii) for $y = y_0$ *(i.e.,* $b = c_0$*), the optimal manipulation is zero manipulation (i.e.,* $a^* = 0$*), which solves both of the conditions above regardless of* η.

As mentioned, the cutoff point y_0 plays an important role in dividing the model into two separate regions. This is reflected in the two parts of the simplified condition FOC stated in the corollary above, with the dividing point at y_0 fitting either part. An important implication of this structure is that if two misreporting cost functions are identical on one side of the cutoff point, the optimal manipulation must be identical as well for that side. So when the closed-form solutions for an exponential and a linear misreporting cost function are derived, a mix-and-match of them yields the closed-form solutions for some hybrid misreporting cost functions. These functions are referred to as "left-linear, right-exponential" (LLRE) and "left-exponential, right-linear" (LERL) functions, with $a = 0$ that corresponds to $y = y_0$ as the dividing point:

(LLRE) $c(a) = c_0 a$ for $a < 0$; $c(a) = (c_0/\gamma)[\exp(\gamma a) - 1]$, with $\gamma > 0$, for $a \geq 0$

(LERL) $c(a) = (c_0/\gamma)[\exp(\gamma a) - 1]$, with $\gamma > 0$, for $a < 0$; $c(a) = c_0 a$ for $a \geq 0$.

Using the simplified condition FOC in Corollary 1, I am able to derive the closed-form solutions given in the next two propositions.

Proposition 3 (Solution with Exponential Misreporting Cost and Weibull Adjustment Requirement Cost Distribution): *Suppose the firm's cost of misreporting is exponential, i.e.,* $c(a) = (c_0/\gamma)[\exp(\gamma a) - 1]$ *with* $\gamma > 0$, *and the distribution of the auditor's adjustment requirement cost is Weibull with* $\lambda > 0$ *and* $\theta = ½$. *Let* $\eta = (k\lambda/2)^{½}$. *Moreover, let* $b = \exp[-\alpha(y-z) + \alpha^2\sigma_u^2 q/2]$ *and* $y_0 = z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$. *Then the optimal manipulation* a^* *is given as follows:*

(i) For $\eta = \alpha$, *if* $y \geq y_0$ *(i.e.,* $b \leq c_0$*),* $a^* = (1/\alpha)\ln(\, [(b/c_0) + (\alpha/\gamma)]/[1 + (\alpha/\gamma)] \,)$.

(ii) For $\eta = \gamma$, *if* $y \leq y_0$ *(i.e.,* $b \geq c_0$*),* $a^* = (1/\alpha)\ln(\, [1 + (\alpha/\gamma)]/[1 + (c_0/b)(\alpha/\gamma)] \,)$.

(iii) For $\alpha = \gamma$, *if* $y \geq y_0$ *(i.e.,* $b \leq c_0$*),*

$$a^* = (1/\alpha)\ln(\, [\, (1 + b/c_0) + \{(1 - b/c_0)^2 + 4(\alpha/\eta)^2(b/c_0)\}^{½} \,] / 2(\alpha/\eta + 1) \,);$$

if $y \leq y_0$ *(i.e.,* $b \geq c_0$*),*

$$a^* = (1/\alpha)\ln(\ [\ \{(1 - b/c_0)^2 + 4(\alpha/\eta)^2(b/c_0)\}^{½} - (1 + b/c_0)\ ]\ /\ 2(\alpha/\eta - 1)\ )$$

for $\eta \neq \alpha$, *and* $a^* = (1/\alpha)\ln[2b/(c_0 + b)]$ *for* $\eta = \alpha$.

Parts (i) and (ii) of the proposition consider the optimal manipulation for specific cases with $\eta = \alpha$ or $\eta = \gamma$. Under these cases, the first-order conditions for the relevant regions take a linear form. Solving out a^* in closed form for those regions is thus trivial.

The closed-form solution given in part (iii), with $\alpha = \gamma$, comes from a quadratic equation that conveniently arises under this case for an exponential misreporting cost function assumed. The proof involves considering multiple cases to ensure that the optimal manipulation is well-defined even for certain parameter values that appear to result in an ill-defined first-order condition.

Proposition 4 (Solution with Linear Misreporting Cost and Weibull Adjustment Requirement Cost Distribution): *Suppose the firm's cost of misreporting is linear, i.e.,* $c(a) = c_0a$, *and the distribution of the auditor's adjustment requirement cost is Weibull with* $\lambda > 0$ *and* $\theta = ½$. *Let* $\eta = (k\lambda/2)^{½}$. *Moreover, let* $b = \exp[-\alpha(y - z) + \alpha^2\sigma_u^2q/2]$ *and* $y_0 = z - (\ln c_0)/\alpha + \alpha\sigma_u^2q/2$. *Then the optimal manipulation* a^* *is given as follows:*

(i) For $y \geq y_0$ *(i.e.,* $b \leq c_0$*),*

$$a^* = (1/\alpha)\{\ (b/c_0) - (\alpha/\eta) + W_0(\ (\alpha/\eta - 1)\exp(\alpha/\eta)(b/c_0)\exp(-b/c_0)\ )\ \};$$

(ii) for $y \leq y_0$ *(i.e.,* $b \geq c_0$*),*

$$a^* = (1/\alpha)\{\ (b/c_0) + (\alpha/\eta) + W_{-1}(\ -(\alpha/\eta + 1)\exp(-\alpha/\eta)(b/c_0)\exp(-b/c_0)\ )\ \},$$

where W_0 *and* W_{-1}, *with* $W_0 \geq -1 \geq W_{-1}$, *are the single-valued upper and lower segments of the real branch of the Lambert W function defined on the domains* $[-\exp(-1),\infty)$ *and* $[-\exp(-1),0]$, *respectively.*

The linear misreporting cost function considered in this proposition leads to a simple form of the first-order condition. It has a closed-form solution that can be expressed in terms of the Lambert W function. This special function is defined as the (multi-valued) inverse of the function $f(W) = W\exp(W)$. The real branch of the function has an upper and a lower (single-valued) segment, denoted by W_0 and W_{-1}, defined on the domains $[-\exp(-1),\infty)$ and $[-\exp(-1),0]$, respectively, with $W_0 \geq -1 \geq W_{-1}$. Figure 2 shows the shape of the function. The figure is adopted from Corless et al. (1996) that reviews the function and further develops its properties, making it more widely known than before. However, work related to the function dates back to Johann Lambert (1728-1777) and Leonhard Euler (1707-1783) (Brito et al. 2008).[11]

[11] The Lambert W function finds its role in many fields including mathematics (e.g., combinatorial number theory), computer science (e.g., algorithm and data structures), statistics (e.g., generalized skewed distributions and risk estimation), engineering (e.g., combustion, fuel consumption, and time-delayed systems), geology (e.g., earthquake forecasting), population ecology (e.g., Lotka-Volterra equations for population growth), health science (e.g., epidemic models), chemistry (e.g., enzyme kinetics), and especially physics (e.g., electrostatics, statistical mechanics, general relativity, inflationary cosmology, radiative transfer, and quantum chromodynamics) (Goerg 2011, Brito et al. 2008, Scott et al. 2006, and Corless et al. 1996). A decade ago, an "editorial in *Focus*, the

The shape of the Lambert W function gives a special touch to the optimal manipulation. It can result in a rather drastic change in the density around the cutoff point y_0 in the earnings triplet distributions. This provides a continuous explanation to the discontinuity in histogram documented in the literature (e.g., Burgstahler and Dichev 1997 and Degeorge et al. 1999). I will elaborate on this further in section 5 where two simulation exercises based on solutions involving the Lambert W function are discussed.

Figure 3a illustrates the closed-form solution of the optimal manipulation (a^*) in Proposition 4 based on the Lambert W function, alongside with the marginal expected benefit of manipulation (b), both as functions of the unmanaged earnings y. In figure 3b, the optimal manipulation is added to the unmanaged earnings to depict the pre-audit earnings $m = y + a^*$ as a function of the unmanaged earnings. The shaded areas in the figure indicate the optimal manipulations that constitute the pre-audit earnings.

I end this section with the following result that shows the one-to-one mapping from the pre-audit earnings m back into the unmanaged earnings y. Holding for the two (effectively four) closed-form solutions, this result provides a hope to derive conditions under which such a one-to-one inverse mapping exists generally. The challenge is left for future research.

Proposition 5 (Invertibility of Pre-audit Earnings to Unmanaged Earnings): *Suppose the distribution of the auditor's adjustment requirement cost is Weibull with* $\lambda > 0$ *and* $\theta = ½$. *Let* $\eta = (k\lambda/2)^{½}$ *and* $\mu = \exp(-\alpha(m - z) + \alpha^2\sigma_u^2 q/2\,)/c_0$. *Then the unmanaged earnings y are below (above)* $y_0 = z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$ *if and only if the pre-audit earnings* $m = y + a^*$ *are below (above)* y_0. *Moreover, the auditor can infer y from m using the relation* $y = Y(\mu)$, *with Y defined as follows:*

(i) If the firm's cost of misreporting is exponential with $\gamma = \alpha$, *i.e.,* $c(a) = (c_0/\alpha)[\exp(\alpha a) - 1]$,

$$Y(\mu) \equiv z + \alpha\sigma_u^2 q/2 - (1/\alpha)\ln[\, c_0[\mu + (\alpha/\eta - 1)\mu^2]/(\alpha/\eta + 1 - \mu)\,] \qquad \text{for } m \geq y_0 \text{ (i.e., } \mu \leq 1)$$

$$Y(\mu) \equiv z + \alpha\sigma_u^2 q/2 - (1/\alpha)\ln[\, c_0[(\alpha/\eta + 1)\mu^2 - \mu]/(\alpha/\eta - 1 + \mu)\,] \qquad \text{for } m \leq y_0 \text{ (i.e., } \mu \geq 1);$$

(ii) if the firm's cost of misreporting is linear, i.e., $c(a) = c_0 a$,

$$Y(\mu) = z + \alpha\sigma_u^2 q/2 - (1/\alpha)\ln[\, -c_0 W_0(\, -\exp(-\alpha/\eta)\mu\exp[(\alpha/\eta - 1)\mu]\,)\,] \qquad \text{for } m \geq y_0 \text{ (i.e., } \mu \leq 1)$$

$$Y(\mu) = z + \alpha\sigma_u^2 q/2 - (1/\alpha)\ln[\, -c_0 W_{-1}(\, -\exp(\alpha/\eta)\mu\exp[-(\alpha/\eta + 1)\mu]\,)\,] \qquad \text{for } m \leq y_0 \text{ (i.e., } \mu \geq 1).$$

The unmanaged earnings $y = Y(\mu)$ *are continuous and strictly increasing in the pre-audit earnings* $m \in (-\infty,\infty)$.

In the next section, I present the results of two simulation exercises based on the model. The analysis helps to assess the limit and the potential of the model as a framework for guiding empirical research.

5. Empirical Contents of the Model

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newsletter of the Mathematical Association of America, asked: 'Time for a new elementary function?'," suggesting that the usefulness of the function in diverse fields qualifies it to be considered a candidate member of elementary functions, like the familiar sin, cosine, logarithm, exponential, etc (Hayes 2005).

Before presenting the simulation results in subsections 5.2 and 5.3, I first briefly explain why the model should be capable of accommodating the two salient features of the earnings triplet distributions.

5.1 Why the Model Can Accommodate the Puzzling Features

To elucidate how the model can lead to a phenomenon similar to the discontinuity documented in the literature, I plot the distribution of unmanaged earnings, y, in figure 4a and superimpose on it the distribution of pre-audit earnings, m. In the figure, the unmanaged earnings are assumed to follow a normal distribution. Only the left side of the density line (in gray) appears in the plot given the scale chosen. The earnings benchmark z is marked by the vertical solid line (in black). It is assumed to be 0.02, rather than zero, to highlight that the model does not require the earnings benchmark to be equal to zero.

The vertical dash line (in green) indicates the location of the cutoff point y_0 (assumed to be 0.06), which divides the two directions of earnings manipulation. Unmanaged earnings below y_0 are manipulated upward, pushing the density line to the right toward the cutoff point. Similarly, the density line on the right of y_0 is pushed to the left toward the point. Mathematically, this one-to-one remapping of the density of unmanaged earnings into the "density" of pre-audit earnings is straightforward and represented by the thin solid curve (in black). However, the area under this curve need not add up to one. To ensure that the remapped density meets the "sum to one" requirement of probability theory, the thin solid curve is multiplied by a normalizing factor to obtain the density line of pre-audit earnings. This is represented by the thick solid curve (in blue).

Figure 4a assumes that the firm's misreporting cost function is linear. Consequently, the optimal manipulation can be solved in terms of the Lambert W function. The shape of this function leads to a particularly steep slope around y_0. A zoom-in view of that part of the density lines (surrounded by red dotted rectangles) is provided on the left side of the figure.

Imagine that the probability distribution of pre-audit earnings described by the model is sampled, with the distribution of observations plotted in histogram form. Because of the steep slope around y_0, it is likely to see a sharp difference between the frequencies observed in the immediate left and right bins next to the cutoff point. A "discontinuity" similar to those documented in the literature can thus arise in the continuous model of this paper. Several issues however should be kept in mind before concluding that the model can accommodate the discontinuity phenomenon.

First, the documented discontinuity occurs at zero in the distributions of earnings, earnings change, and earnings surprise. For the model to explain the phenomenon, y_0 must be close to the earnings benchmark in concern, be it the profit/loss cutoff, earnings increase/decrease, or beating/missing an earnings forecast. This proximity between y_0 and z is assumed in figure 4a and in the simulation exercises. Whether it is so in reality is an open question that may be answered empirically. I discuss the estimation of the model parameters in appendix A.

Second, figure 4a only shows a "discontinuity" in the distribution of pre-audit earnings, which are not exactly the same as the reported earnings in audited financial statements. I therefore conduct the two simulation exercises to fill the gap between these earnings concepts.

Third, the documented discontinuity is not about the distribution of earnings from the same firm observed multiple times. Instead, the earnings concerned in the literature come from different firms. Thus, they are likely to be drawn from different distributions, rather than from a single distribution assumed in figure 4a. The simulation exercises make an attempt to address this issue.

Figure 4b illustrates that even when the unmanaged earnings are distributed normally, the distribution of pre-audit earnings may have a volcano shape with the peak sharper than that of a bell-shaped normal distribution. This may occur if y_0 is located around the peak of the unmanaged earnings distribution, with the left and right sides of the density line pushed toward the peak strongly. For example, imagine that the expected liability cost is low and the adjustment requirement cost tends to be high. So the auditor is reluctant to require an adjustment. The firm will have a strong incentive to manipulate earnings, resulting in a big push of the density line toward the middle and hence a volcano-shaped distribution of pre-audit earnings.

Before presenting the simulation results in the next two subsections, let me first introduce some terminology. Recall that the audit will remove some of the unintentional errors, namely ε_q, before the firm publicly announces the post-audit earnings $r = xy + (1 - x)m - \varepsilon_q$. Without an auditor-required adjustment (i.e., $x = 0$), r is simply the pre-audit earnings m corrected for the discovered unintentional errors. If an adjustment is required (i.e., $x = 1$), r is the unmanaged earnings y corrected for the discovered unintentional errors.

The first simulation exercise focuses on the distribution of excess earnings, defined as the part of earnings exceeding the earnings benchmark z. (Negative excess earnings mean the part of earnings falling short of the benchmark.) Based on this definition, the *post-audit excess earnings* are

$$\delta r = [xy + (1 - x)m - \varepsilon_q] - z.$$

Similarly, the pre-audit and unmanaged excess earnings are $m - z$ and $y - z$, respectively. If the analyst consensus forecast is taken as the earnings benchmark, post-audit excess earnings coincide with the concept of earnings surprise.

To simulate the distribution of earnings change, I consider a simple repetition of the model for two periods. This yields some interesting results. However, one needs to bear in mind a caveat. Repeating the model for multiple periods does not give a truly dynamic model. In a dynamic model, care must be taken to consider the accumulation of past earnings manipulations in the total assets (see, e.g., Barton & Simko 2002 and Baber et al. 2011), which might enhance the auditor's incentive to require an adjustment in the future. Owing to the limited space here, the analysis of a truly dynamic version of the model is left for future research.

Let r_1 denote the *lagged post-audit earnings*, i.e., the post-audit earnings in the earlier period of a two-period repetition of the model. The second simulation exercise assumes that in the current period of the two-period repetition some firms use the profit/loss cutoff as the earnings benchmark (i.e., $z = 0$) while others use the earnings increase/decrease cutoff as the benchmark (i.e., $z = r_1$). Because of this diversity in the benchmarks assumed, a discontinuity can occur in the distributions of earnings and earnings change/difference simultaneously. The *post-audit earnings change* is defined as

$$\Delta r = [xy + (1 - x)m - \varepsilon_q] - r_1.$$

Similarly, the pre-audit and unmanaged earnings differences are defined as $m - r_1$ and $y - r_1$, respectively. For firms using lagged post-audit earnings as the benchmark, the post-audit earnings change is simply their excess earnings. However, for firms using zero earnings as the benchmark, the post-audit earnings change is not the same as excess earnings but the post-audit earnings are.

5.2 Simulating a Volcano-shaped Distribution of Excess Earnings with a Sharp Peak

Figure 5a visualizes the results of the first simulation with $y_0 - z = \$0.011$ and a linear misreporting cost function. The simulation assumes a population of firms each with possibly a different distribution of unmanaged earnings (per share in cents). For simplicity, the distributions are all normal with the same standard deviation but possibly different means. The means are themselves drawn from a normal distribution.

I am interested in simulating the situation where earnings forecasts taken as the earnings benchmarks are equal to the means of the unmanaged earnings distribution. The purpose is to see how far the distribution of the simulated post-audit excess earnings can get close to its counterpart reported in figure 2 of Bhojraj et al. (2009), which is included in figure 5b for ease of reference. Because this simulation is meant to be a first look at the empirical contents of the model, I do not consider the more complicated situation with the forecasts set to the post-audit earnings of an earlier period. The second simulation presented later will include such dynamic considerations.

In figure 5a, the distributions of the pre-audit, post-audit, and unmanaged excess earnings are plotted in histogram form, with the latter overlaid on the former one after another. The bin width of the histogram is \$0.01, to be consistent with the choice in Bhojraj et al. (2009). To provide a better angle in viewing the three distributions, a three-dimensional plot of the distributions is given in figure 5c. The shape of the Lambert W function that characterizes the optimal manipulation behind this simulation induces a huge spike in the frequency distribution of the pre-audit excess earnings (in yellow) in the back. This starkly differs from the nearly flat distribution of the unmanaged excess earnings (in gray) in the front. The stochastic nature of the adjustment decision, together with the correction for the discovered unintentional errors, mixes the two publicly unobservable distributions into the distribution of the post-audit excess earnings (in blue) in the middle, which is observable to the public.

The especially sharp peak of the distribution of earnings surprise is a feature easily noticed in related studies (e.g., figure 6 of Degeorge et al. 1999 and figure 2 of Frankel et al. 2010). The simulation illustrates how the model turns a rather flat distribution of unmanaged excess earnings (originating from normally distributed unmanaged earnings) into a dramatically different distribution of post-audit excess earnings. The assumption of $y_0 - z$ = \$0.011 ensures that the sharp peak of the distribution occurs in the two right bins next to zero excess earnings. The solid stair-step line (in blue) in figure 5a outlines the distribution of the post-audit excess earnings partially hidden behind the distribution of unmanaged excess earnings in the front. The overall shape of the post-audit excess earnings distributions looks quite similar to its counterpart in figure 2 of Bhojraj et al. 2009, which however is fatter at the mid-level and has thinner tails. Given the simple structure of the model, compared to the complex reality it tries to approximate, such mismatches seem unsurprising.

The following hypothesis summarizes the key insight from the first simulation.

The Mixture Hypothesis: *The volcano-shaped distribution of earning surprise with a sharp peak documented in the literature is due to a mixture of a relatively flat distribution of unmanaged excess earnings with a spiky distribution of pre-audit excess earnings.*

5.3 Simulating a 'Discontinuity' in the Distributions of Earnings and Earnings Change

Throughout the second simulation, I assume $y_0 - z$ = \$0.015 and a "left-exponential, right-linear" (LERL) misreporting cost function (see section 4 for the definition). Assuming a linear misreporting cost function as in the first simulation would not change the results critically. However, the overall shapes of the simulated distributions would look less similar to their counterparts based on actual data.

The first simulation directly assumes a distribution of unmanaged earnings per share (in cents). In contrast, the second simulation uses a more complicated procedure to simulate the distribution of unmanaged earnings based on the actual data of total assets in 1988-2006, assuming a relation between the unmanaged earnings and total assets. The distribution of unmanaged earnings per share is then computed using the actual shares outstanding data associated with the total assets data. I use this unmanaged earnings distribution as the "seed" for simulating the distribution of post-audit earnings for an earlier period, assuming zero earnings as the benchmark for manipulation. The earnings increase/decrease cutoff is not a choice at this point because there are no lagged post-audit earnings yet.

After obtaining the post-audit earnings for the earlier period, they become the lagged post-audit earnings for the current period of the simulation. The beginning value of the total assets for the current period is updated from that of the earlier period using the simulated post-audit earnings, assuming a 75% payout ratio. The updated total assets are used as a basis to simulate the current period's unmanaged earnings, assuming a relation augmented by some "natural growth."

There are three main sources for such "natural growth." First, it can come from technological advancement that improves the productivity of any given asset base. Second, it can arise from an expansion of the asset base due to new investment opportunities discovered. (But this is not captured in the simple clean-surplus updating of the total assets assumed above.) Third, the earnings concerned in the literature are in nominal value. Inflation can contribute to the "natural growth" of earnings even when the productivity is fixed and the asset base is constant.

Allowing for some "natural growth" is important. Otherwise, the peak of the distribution of the simulated unmanaged earnings difference would not be so far to the right as in figure 7a. Consequently, the peak of the distribution of the simulated post-audit earnings change would be much sharper, unlike its counterpart in figure 7b based on actual earnings data in 1988-2006. It is important to keep in mind that the model of this paper does not explain the shape of the (unobservable) unmanaged earnings distributions. It only explains how the distribution may be transformed into the (observable) post-audit earnings with a distinctly different look. Therefore, the purpose of assuming some "natural growth" is to come up with something reasonably close to the reality and let the model explain the remaining difference, which otherwise is perplexing.

With the unmanaged earnings for the current period simulated, the model then converts them into pre-audit and post-audit earnings. Unlike in the first simulation, now firms can differ in the earnings benchmarks assumed: some use $z = 0$, while others use $z = r_1$. For simplicity, I assign these two benchmarks randomly with 40% of the chance setting $z = 0$ and 60% setting $z = r_1$. The distributions of the pre-audit, post-audit, and unmanaged earnings are plotted in figure 6a, with the latter overlaid on the former one after another. Again, the bin width of the histogram is \$0.01. I use small solid circles (in blue) to outline the distribution of the post-audit earnings partially hidden behind the distribution of unmanaged earnings (in gray) in the front. Whenever the frequency of the post-audit earnings exceeds that of the unmanaged earnings, the exceeding part can be clearly seen as a blue bar segment on top of a gray histogram bar. The key difference between the distributions of the post-audit and unmanaged earnings is the higher frequencies in the several right bins next to zero earnings. A close-up view of that part (surrounded by red dotted rectangles) is provided on the left side of the figure. The distribution of the actual earnings in 1998-2006 is given in figure 6b for comparison.

Figure 7a shows the distributions of the pre-audit earnings difference (in yellow) in the back, the post-audit earnings change (in blue) in the middle, and the unmanaged earnings difference (in gray) in the front. Again, small solid circles (in blue) are used to outline the distribution of the post-audit earnings change partially hidden behind the distribution of the unmanaged earnings difference. Note that 40% of the simulated observations in the figure use $z = 0$ (rather than $z = r_1$) as the benchmark for earnings manipulation. Still the remaining 60% are sufficient to induce the noticeably higher frequencies in the several right bins next to zero earnings change indicated by the solid vertical line (in black). The left side of the figure provides a zoom-in view for that part of the post-audit earnings change distribution (surrounded by red dotted rectangles).

The key insight from the second simulation is summarized as the following hypothesis:

The Continuous 'Discontinuity' Hypothesis: *The simultaneous existence of a "discontinuity" in* *the distributions of earnings and earnings change documented in the literature is due to a continuous but drastic increase in the density of the distributions around the respective earnings benchmarks.*

6. Concluding Remarks

The model of this paper can accommodate the two salient features of the earnings triplet distributions, despite the simple representation of the auditor-client negotiation process as a "take-it-or-leave-it" game. Aside from some mismatches that seem unsurprising given the simple model structure, the simulation results demonstrate the model's potential to capture the driving forces behind the two salient features.

Two caveats for the simulation results are worth noting. In the second simulation, I assume for simplicity that firms stochastically choose the profit/loss or the earnings increase/decrease cutoffs as the benchmarks for manipulation. In unreported analysis, I also consider a behavioral model of benchmark selection based on the unmanaged earnings' proximity to different benchmarks. The results are broadly similar. Because constructing such a selection model is not the focus of this paper, in the interest of space it is not reported here. However, endogenizing the benchmark selection is an interesting extension of the model.

The second simulation also tries to be dynamic. But the model here is only "pseudo-dynamic," i.e., it uses an exogenously specified misreporting cost function to represent in reduced form any future benefits and costs of the current-period manipulation. A truly dynamic version of the model requires carefully considering the connection between periods through a misreporting cost function endogenously determined in equilibrium.

A major approach used in the literature to identify earnings management is the abnormal accruals model. It is an empirical model that defines the unexplained residuals of a linear regression model as measuring earnings management. The model has been used widely with success. However, owing to imperfections explained below, there remains room for alternative models to complement this major approach.

While the linear regression in the abnormal accruals model is convenient, simple, and easy to understand, it seems quite unlikely that relations governing earnings management activities are indeed linear. This has an impact on the model because it defines anything not captured by the linear regression as abnormal accruals. Aside from this, any unrelated random noise affecting the total accruals are defined as part of the abnormal accruals. These imperfections suggest that a nonlinear model differentiating between random noise and earnings manipulation can complement the abnormal accruals model.

The simulation results of this paper indicate that the nonlinear solutions involving the Lambert W function help explain the two salient features of the earnings triplet distributions, suggesting that the solutions capture some important aspects of the reality. Using the estimation procedures discussed in appendix A, one can obtain estimates of the model parameters that nonetheless would not fit the data perfectly. The unexplained residuals of the observed post-audit earnings are better viewed as the discovered unintentional errors, rather than as part of the earnings manipulation. With the parameter estimates and the predicted post-audit earnings, one can use the model to infer the underlying pre-audit earnings and unmanaged earnings. The difference between the two provides an alternative measure of earnings manipulation that can complement the abnormal accruals measure.

The model of this paper when interpreted strictly according to its intent also differs from the abnormal accruals model in terms of the type of earnings management captured. For firms making conservative or liberal accounting choices that nevertheless are reasonable and allowed under GAAP, if the effect is firm-specific rather than common across many firms, it would be picked up by the abnormal accruals model as earnings management. In contrast, the model here focuses on earnings manipulation unacceptable under GAAP. Thus, for empirical studies that need to separate earnings manipulation from acceptable conservative/liberal accounting choices, the model here together with the abnormal accruals model can provide a distinction between the two types of earnings management activities.

There are other potential applications of the model. I will mention three here. The first is to use the estimated model for policy analysis. In Heckman's (2010) view, "[p]olicy analysis is all about identifying counterfactual states. Counterfactual policy states are possible outcomes in different hypothetical states of the world. … Causal comparisons entail contrasts between outcomes in alternative possible states holding factors other than the features of the policy being analyzed the same across the contrasts." (p. 359) He further elaborates and says: "The goal of the structural econometrics literature, like the goal of all science, is to understand the causal mechanisms producing effects so that one can use empirical versions of models to forecast the effects of interventions never previously experienced, to calculate a variety of policy counterfactuals and to use theory to guide choices of estimators to interpret evidence and to cumulate evidence across studies." (p. 361)

By hypothesizing how regulation changes might affect certain model parameters (or the distribution of unmanaged earnings), one can use the model to predict the effect of the changes on earnings manipulation. This provides an assessment useful for comparing the anticipated effect to the implementation cost of a regulation change, helping regulators to make informed decisions.

In addition, the model can be used as a framework for examining the effectiveness of certain corporate governance mechanisms in curbing earnings manipulation. For such mechanisms to be useful, they must change the model parameters related to the benefit and cost of misreporting (e.g., c_0). By including corporate governance measures as explanatory variables in the estimation procedures, one can test whether the relevant parameters are sensitive to these variables. Conclusions can then be drawn on the effectiveness of the variables in discouraging earnings manipulation.

Besides viewing the model from a positive perspective, it may be used as a framework for developing decision aids to help auditors improve adjustment decisions. In the model, the auditor is assumed to know all the parameters without doubt. In reality, this seems unlikely. Instead, auditors might have a subconscious assessment of the factors corresponding to the model parameters and make decisions based on "professional judgment." Less experienced auditors might misjudge and make mistakes against their own interest. With the decision made in a judgmental manner, even experienced auditors might make occasional mistakes due to subconscious psychological biases. Systematically collecting information useful for estimating the model parameters and developing a formal decision aid based on the model can raise the awareness of the auditors making adjustment decisions. By highlighting the strategic considerations in the model, the auditors can carefully balance such considerations with other unmodeled factors. The decision quality can thus be improved.

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Appendix A: Estimation of Model Parameters

In this appendix, I discuss issues related to estimating the parameters of the model. The first issue to deal with is sample observation homogeneity. Only firms similar to each other are expected to have similar parameter values and hence may be pooled together for estimation. In the extreme case, only time-series observations of a firm may be used for estimating firm-specific parameters. This is usually impractical because the majority of firms have relatively short histories. The sample size is unlikely to be large enough to obtain reliable estimates of the firm-specific parameters, especially when the structure of the model demands nonlinear estimation.

To get around the difficulty, some strong assumptions are required to pool firm observations together for cross-sectional / panel estimation. One commonly used approach is to group firms together based on their similarities in certain aspects, e.g., industry membership, firm size proxied by total assets / market value / sales, risk level proxied by beta, etc. As far as the model is concerned, industry membership is exogenously determined, unlikely to be affected by earnings management activities. However, past earnings manipulations can affect total assets. Moreover, earnings manipulations and total assets can affect sales (e.g., by early revenue recognition) and market value. Therefore, the usual method of ranking firms by decile based on the above-mentioned characteristics may not be able to group truly similar firms together.

One way that might improve the grouping is to classify firms iteratively as follows. After the initial grouping that allows estimation of the parameters, one can compute the predicted unmanaged earnings and adjust the total assets accordingly. If this leads to a significantly different grouping of the firms based on the adjusted total assets, re-estimating the parameters after re-grouping might be necessary.

Suppose the sample observation homogeneity issue has been addressed. The next issue is about the estimation method. Given the nonlinear nature of the model, the maximum likelihood (ML) method seems to be the natural choice. However, the correction for the discovered unintentional errors causes a complication. In the following, I will first discuss the likelihood function conditional on ε_q. Then integrating the conditional likelihood function with the assumed normal distribution of ε_q gives the expected likelihood function for maximization.

Suppose that the parametric assumptions of a linear misreporting cost function and a Weibull adjustment requirement cost distribution $G(l; \lambda, ½) = 1 - \exp[-(\lambda l)^{½}]$ are made. Moreover, assume that the unmanaged earnings y are normally distributed with a density function $n(y; \bar{y}, \bar{\sigma}^2)$, where the unknown mean $\bar{y}$ and variance $\bar{\sigma}^2$ are to be estimated as well. Conditional on ε_q, the observed post-audit earnings r are simply $y - \varepsilon_q$ or $m - \varepsilon_q$, depending on whether an adjustment is required or not. The probabilities of these events are $G(ka^{*2}/2; \lambda, ½) = 1 - \exp(-\eta a^*)$ and $1 - G(ka^{*2}/2; \lambda, ½) = \exp(-\eta a^*)$, respectively.

Let

$$P(y; z, c_0, \alpha, \sigma_u^2 q, \eta) = \exp[-\eta(a^* \circ b(y; z, c_0, \alpha, \sigma_u^2 q, \eta))],$$

where $a^* \circ b$ is the composite function of a^* as a function of b and b as a function of y. The function P is the probability of no adjustment expressed directly as a function of y. Depending on whether an adjustment is required or not, one can infer that $y = r + \varepsilon_q$ or $m = r + \varepsilon_q$, respectively. Since m can be mapped back into y through $Y(\mu)$, the latter case means

$$y = Y \circ \mu(r + \varepsilon_q; z, c_0, \alpha, \sigma_u^2 q, \eta),$$

where $Y \circ \mu$ is the composite function of Y as a function of μ and μ as a function of m.

Conditional on ε_q, the likelihood of observing post-audit earnings r is

$$\begin{aligned}
&\Lambda(r; z, c_0, \alpha, \sigma_u^2 q, \eta, \bar{y}, \bar{\sigma}^2 \mid \varepsilon_q) \\
&\quad = n(r + \varepsilon_q; \bar{y}, \bar{\sigma}^2)[1 - P(r + \varepsilon_q; z, c_0, \alpha, \sigma_u^2 q, \eta)] \\
&\qquad + n(Y \circ \mu(r + \varepsilon_q; z, c_0, \alpha, \sigma_u^2 q, \eta); \bar{y}, \bar{\sigma}^2)P(Y \circ \mu(r + \varepsilon_q; z, c_0, \alpha, \sigma_u^2 q, \eta); z, c_0, \alpha, \sigma_u^2 q, \eta).
\end{aligned}$$

Let

$$\bar{\Lambda}(r; z, c_0, \alpha, \sigma_u^2 q, \eta, \bar{y}, \bar{\sigma}^2) = \int_{-\infty}^{\infty} \Lambda(r; z, c_0, \alpha, \sigma_u^2 q, \eta, \bar{y}, \bar{\sigma}^2 \mid \varepsilon_q) n(\varepsilon_q; 0, q\sigma_u^2) d\varepsilon_q,$$

which is the expected likelihood of observing post-audit earnings r. Moreover, denote by r_j the jth firm-year observation. Under the assumption of independent observations, maximizing

$$\textstyle\sum_j \ln[\, \bar{\Lambda}(r_j; z, c_0, \alpha, \sigma_u^2 q, \eta, \bar{y}, \bar{\sigma}^2)\,]$$

with respect to the parameters gives the ML estimates.

Clearly, the k and λ that constitute η cannot be separately identified. Neither can σ_u^2 and q. Whether the remaining parameters can be reliably identified depends on the sensitivity of the log expected likelihood function to the parameters. As usual, there might be multiple local maxima. So care must be taken to increase the chance of identifying the global ML estimates.

Suppose q differs systematically between big and non-big auditors. By hypothesizing that q is a linear function of a BIG dummy variable for the auditor type, the variable can be used as an explanatory variable for estimating the parameters.

In addition, suppose one can determine a restricted sample where firms are believed to use the analyst consensus forecast as the earnings benchmark. Then instead of estimating z as a parameter, the analyst consensus forecast may be used as an explanatory variable for estimation. Moreover, if a benchmark selection model could be formulated to describe the decision to choose a benchmark from the three common alternatives, the selection model could be used for joint estimation together with the expected likelihood function.

Besides using r for estimation, the audit fee F may also be used. Even for a sample with observations from comparable firms only, the observed audit fees may differ from the F specified by the model. This is due to unmodeled factors that may contribute to the differences. By choosing the parameters to minimize the sum of the squared differences, nonlinear least squares (LS) estimates of the parameters can be obtained.

To understand the nonlinear LS estimation procedure, first note that the audit fee is determined before the auditor knows the pre-audit earnings m, or any clue of the unmanaged earnings y. Under the assumption of a competitive audit market, the audit fee F is set to equate with the sum of the expected adjustment requirement, audit, and liability costs to the auditor:

$$\mathrm{E}_y[\ \mathrm{E}_X[\ (X)\mathbf{1}\{X \le L\} + (L)[1 - \mathbf{1}\{X \le L\}]\ |\ a = a^*\ ]\ ] + [c_\mathrm{u}(q) + L_\mathrm{u}],$$

where $L_\mathrm{u} = \mathrm{E}[k_\mathrm{u}\varepsilon_{1-q}{}^2/2] = k_\mathrm{u}(1-q)\sigma_\mathrm{u}{}^2/2$, with $k_\mathrm{u} > 0$, is assumed to be the expected liability cost arising from failing to remove all the unintentional errors and $c_\mathrm{u}(q)$ is assumed to be the cost of conducting an audit with a quality level q. Note that

$$\begin{aligned}
&\mathrm{E}_X[\ (X)\mathbf{1}\{X \le L\}\ |\ a = a^*\ ] \\
&= \int_0^{\frac{ka^{*2}}{2}} lg(l;\lambda,½)dl \\
&= \int_0^{\frac{ka^*\circ b(y;z,c_0,\alpha,\sigma_u^2 q,\eta)^2}{2}} lg(l;\lambda,½)dl,
\end{aligned}$$

where $g(l;\lambda,½) = (½)(\lambda/l)^{½}\exp[-(\lambda l)^{½}]$, and

$$\begin{aligned}
&\mathrm{E}_X[\ [1 - \mathbf{1}\{X \le L\}]\ |\ a = a^*\ ] \\
&= 1 - G(ka^{*2}/2;\lambda,½) \\
&= \exp(-\eta a^*) \\
&= \exp[-\eta(\ a^*\circ b(y; z, c_0, \alpha, \sigma_\mathrm{u}{}^2 q, \eta)\ )] \\
&= P(y; z, c_0, \alpha, \sigma_\mathrm{u}{}^2 q, \eta).
\end{aligned}$$

Moreover, recall that $\eta = (k\lambda/2)^{½}$, and y is assumed to be normally distributed with a density function $n(y; \bar{y}, \bar{\sigma}^2)$. Hence, the audit fee specified by the model is given by the following function:

$$\begin{aligned}
&F(z, c_0, \alpha, \sigma_\mathrm{u}{}^2 q, k, \lambda, \bar{y}, \bar{\sigma}^2, c_\mathrm{u}(q)+L_\mathrm{u}) \\
&= \int_{-\infty}^{\infty}\Bigg[\ \int_0^{\frac{ka^*\circ b(y;z,c_0,\alpha,\sigma_u^2 q,(k\lambda/2)^{½})^2}{2}} lg(l;\lambda,½)dl \\
&\qquad + [k\,a^*\circ b(y; z, c_0, \alpha, \sigma_\mathrm{u}{}^2 q, \eta)^2/2]P(y; z, c_0, \alpha, \sigma_\mathrm{u}{}^2 q, (k\lambda/2)^{½})\Bigg]n(y; \bar{y}, \bar{\sigma}^2)dy + [c_\mathrm{u}(q)+L_\mathrm{u}],
\end{aligned}$$

where $c_\mathrm{u}(q)+L_\mathrm{u}$ is treated as a single parameter for estimation, assuming that all firms in the sample have auditors of similar qualities. A change of the variable l in the inner integral cannot group k and λ together to appear as a product as in η. So in principle k and λ can be separately identified when the audit fee is used for estimation.

For the model parameters to be identifiable, at least one explanatory variable must be included besides the observed audit fee, denoted by F_j, as the dependent variable. For example, suppose that it is reasonable to believe that the firms use the analyst consensus forecasts, denoted by z_j's, as the earnings benchmarks. The nonlinear LS estimates can then be obtained by minimizing

$$\sum_j [\, F_j - F(z_j, c_0, \alpha, \sigma_{\mathrm{u}}^2 q, k, \lambda, \bar{y}, \bar{\sigma}^2, c_{\mathrm{u}}(q)+L_{\mathrm{u}}) \,]^2$$

with respect to the parameters.

Besides the possibility of being trapped in a local extremum, another challenge of using nonlinear estimation methods (like ML or nonlinear LS discussed above) is that the iterative process of finding an extremum might not converge. Initializing the process with good starting values raises the odds of convergence but is more art than science.

Finally, suppose moment conditions could be derived to capture the discontinuity and volcano shape of the earnings triplet distributions. Then the generalized method of moments (GMM) could also be used to estimate the model parameters.

Appendix B: Extensions

In the main text, I treat the quality level of the audit as an exogenous parameter. In this appendix, I discuss how it can be endogenized. Moreover, if the firm takes the earnings forecast by a particular analyst (e.g., a star analyst) as the earnings benchmark, the model can be extended to include the analyst's decision on setting the forecast. This possibility and the further extension of allowing for multiple analysts are also discussed below.

B.1 Endogenous Audit Quality

For simplicity, suppose the auditor can observe z when choosing q. Assuming instead that he knows at that time only a signal of the z to be used by the firm would not change the modeling critically. Alternatively, if the value of z is simultaneously chosen by another player, say, a star analyst (see subsection B.2), the following approach still goes through as long as the auditor can at the time of choosing q correctly anticipate the equilibrium value of z.

Given any (conjectured) z and the anticipated optimal manipulation $a^*\circ b$ and optimal adjustment decision $x^*\circ a^*\circ b$, both expressed as functions of y directly, the auditor chooses q to minimize the sum of the expected adjustment requirement, audit, and liability costs below (see appendix A for the derivation):

$$\int_{-\infty}^{\infty}\left[\int_{0}^{\frac{ka^*\circ b(y;z,c_0,\alpha,\sigma_u^2 q,(k\lambda/2)^{1/2})^2}{2}} lg(l;\lambda,\tfrac{1}{2})dl + [ka^*\circ b(y;z,c_0,\alpha,\sigma_u^2 q,\eta)^2/2]P(y;z,c_0,\alpha,\sigma_u^2 q,(k\lambda/2)^{1/2})\right]n(y;\bar{y},\bar{\sigma}^2)dy + [c_u(q)+L_u],$$

where $c_u(0) = 0$, $c_u'(q) > 0$, and $c_u''(q) \geq 0$, $L_u = E[k_u\varepsilon_{1-q}^2/2] = k_u(1-q)\sigma_u^2/2$, with $k_u > 0$, $n(y;\bar{y},\bar{\sigma}^2)$ is the density function of the unmanaged earnings y assumed to follow a normal distribution, and $P(y; z, c_0, \alpha, \sigma_u^2 q, \eta) = \exp[-\eta(a^*\circ b(y; z, c_0, \alpha, \sigma_u^2 q, \eta))]$. Although the objective function above looks quite complicated, the intuition behind it is simple. The auditor's choice of q will have direct impacts on the audit cost $c_u(q)$ and on the expected liability cost L_u arising from failing to discover all unintentional errors. In addition, the choice can affect the marginal expected benefit of manipulation b and thereby influence the optimal manipulation a^*. Consequently, the probability of no adjustment P is affected. So is the expected adjustment requirement cost conditional on an adjustment, as well as the expected liability cost related to earnings manipulation conditional on no adjustment. These two costs are represented by the two terms in the large square brackets in the auditor's objective function.

Analytically solving for the optimal q^* is challenging because of the complexity of a^*. Numerical exploration seems to be a more viable route to gain insights about q^*. If an analytical solution is still desirable, one can consider finding out q^* using a first-order "mechanism design" approach. The idea is to let the auditor choose q and also the manipulation a, subject to the incentive compatibility (IC) constraint given by condition FOC that characterizes the optimal manipulation selected by the firm. The optimal a^* for this formulation minimizes the total costs to the auditor given the optimal q^* and the IC constraint. Similarly, the optimal q^* for this formulation minimizes the total costs given the optimal a^* for this formulation and the IC constraint. The approach allows finding out q^* without directly dealing with the firm-chosen a^* as a function of q. This seems to be analytically less difficult than the original formulation of the auditor's decision on the quality level.

B.2 Analyst Forecast as Earnings Benchmark

Consider the case where the value of z is set by a star analyst. It seems reasonable to suppose that the analyst cannot observe q^* when he sets the value of z. However, if the structure and parameters of the model are common knowledge, in equilibrium he would be able to anticipate q^* correctly.

Given any conjectured q and the anticipated distribution of r conditional on z, the analyst chooses z to minimize

$$\mathrm{E}_r[\ \omega z - (z - r)^2 \mid z\ ],$$

where $(z - r)^2$ is the squared forecast error and $\omega > 0$. This objective function assumes that the analyst receives more benefit from a higher (i.e., more optimistic) forecast but also bears more cost due to a larger squared forecast error. The parameter ω captures the relative importance of the benefit and cost to the analyst. Solving the analyst's problem gives an optimal forecast z^*. When evaluated at the equilibrium q^*, the optimal forecast is a function of ω.

Suppose the model is extended further to allow for multiple analysts following the firm. Their differences can be modeled as a distribution of the parameter ω. The analyst giving the mean forecast, often referred to as the consensus forecast, can be defined as the star analyst assumed in the single-analyst extension.

Suppose that in the multiple-analyst extension what the firm really cares is an updated forecast of the star analyst, which is not known when choosing the manipulation. However, the firm knows that the updated forecast is a random variable distributed normally with a mean equal to the initial consensus forecast known to the firm and with a variance depending on the initial dispersion of the analyst forecasts. Then the forecast dispersion would have an impact on the optimal manipulation much like the effect of ε_q's variance $q\sigma_u^2$ that influences a^* through b. This way, analyst forecast dispersion could also be incorporated into the model.

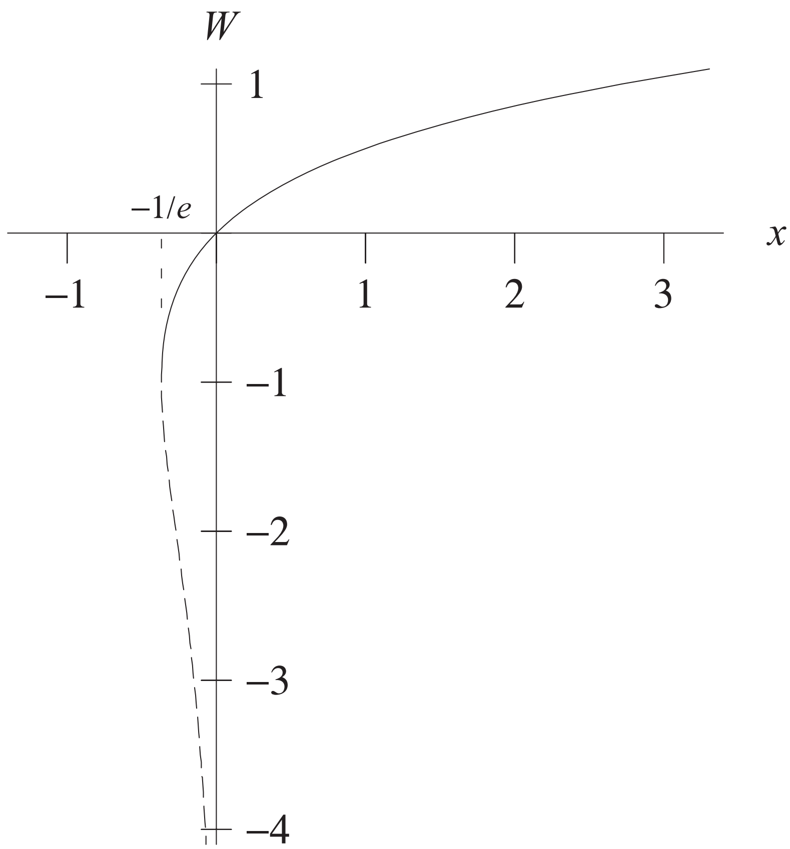
Table 1: Notations

y	$\in$	$(-\infty,\infty)$ is the unmanaged earnings of the firm
z	$\in$	$(-\infty,\infty)$ is the earnings benchmark affecting the firm's incentive to manipulate earnings
a	$\in$	$(-\infty,\infty)$ is the earnings manipulation chosen by the firm
m	$=$	$y + a$ is the pre-audit earnings (or managed earnings) provided to the auditor for audit.
q	$\in$	$[0,1]$ is the quality level of the audit.
ε_q	$\sim$	Normal$(0, q\sigma_u^2)$, with $\sigma_u > 0$, is the part of the unintentional errors contained in y that is discovered and removed by the auditor.
ε_{1-q}	$\sim$	Normal$(0, (1-q)\sigma_u^2)$ is the part of the unintentional errors remaining in y even after the audit.
L	$=$	$ka^2/2$, where $k > 0$, is the expected liability cost to the auditor arising from tolerating the earnings manipulation.
x	$\in$	$\{0,1\}$ is the auditor's adjustment decision.
X	$\in$	$[0,\infty)$ is the auditor's cost of requiring an adjustment, which follows a probability distribution $G(l) = \Pr\{X \le l\}$, with a differentiable probability density $g(l) = G'(l) > 0$ for all $l > 0$ and a differentiable hazard rate function $h(l) = [1-G(l)]/g(l)$. The existence of bounded limits $\lim_{l\downarrow 0} l^{½}g(l)$ and $\lim_{l\downarrow 0} [g(l) + 2lg'(l)]$ is assumed.
$x^*(a)$	$=$	$\mathbf{1}\{X \le L\}$ is the optimal adjustment decision
$c(a)$	$=$	the firm's misreporting cost, where $c(0) = 0$, $\lim_{a\uparrow\infty} c(a) = \infty$, $c'(a) > 0$ with $c'(0) = c_0 < \infty$, and $c''(a) \ge 0$.
b	$=$	$E[\exp(-\alpha[y - z - \varepsilon_q])] = \exp[-\alpha(y - z) + \alpha^2\sigma_u^2 q/2]$ is the "marginal expected benefit of manipulation" that summarizes the impacts of the quality parameter q and the deviation of y from z on the firm's misreporting incentive.
v	$=$	$(1 - b)/\alpha + [1 - G(L)][\, b[1 - \exp(-\alpha a)]/\alpha - c(a)\,]$, with $\alpha > 0$, is the firm's expected net benefit from earnings manipulation.
y_0	$=$	$z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$ is the cutoff of the unmanaged earnings that determines whether the optimal manipulation is upward or downward.
λ	$>$	0 is the reciprocal of the scale parameter of a Weibull adjustment requirement cost distribution.
θ	$=$	½ is the shape parameter of a Weibull adjustment requirement cost distribution that simplifies the first-order condition for the optimal manipulation, allowing the solution to be expressed in closed form.
η	$=$	$(k\lambda/2)^{½}$ is a parameter capturing the relative importance of the expected liability cost and adjustment requirement cost to the auditor.
W	$=$	the Lambert W function, which is the (multi-valued) inverse of the function $f(W) = W\exp(W)$. The real branch of the function has an upper and a lower (single-valued) segment defined on the domains $[-\exp(-1),\infty)$ and $[-\exp(-1),0]$, respectively. The two segments are denoted by W_0 and W_{-1}, respectively, with $W_0 \ge -1 \ge W_{-1}$.
μ	$=$	$\exp(\,-\alpha(m - z) + \alpha^2\sigma_u^2 q/2\,)/c_0$ is the counterpart of b for $Y(\mu)$, the inverse mapping from m back to y.
r	$=$	$xy + (1 - x)m - \varepsilon_q$ is the post-audit earnings (or reported earnings) announced to the public.
δr	$=$	$[xy + (1 - x)m - \varepsilon_q] - z$ is the post-audit excess earnings.
r_1	$=$	the lagged post-audit earnings.
Δr	$=$	$[xy + (1 - x)m - \varepsilon_q] - r_1$ is the post-audit earnings change.

Figure 1. Timeline of the model

The auditor explains the audit plan to the firm, letting it know the quality level q of the audit.	The firm assumes an earnings benchmark z (e.g., analyst consensus forecast), which is common knowledge in the model.	The firm learns the unmanaged earnings y, chooses the manipulation a, and provides the pre-audit earnings $m = y + a$ to the auditor for audit.	The auditor conducts the audit and removes the discovered part ε_q of the unintentional errors in y, with the undiscovered part ε_{1-q} remaining in y.	After the audit, the auditor knows the components y and a of m.	The auditor decides whether to incur a cost X, privately known to him, in order to require an adjustment ($x = 1$) to remove the manipulation a, or not ($x = 0$).	The firm announces to the public the post-audit earnings $r = xy + (1 - x)m - \varepsilon_q = y + (1 - x)a - \varepsilon_q$.

Figure 2. The two real branches of the Lambert W function
(Source: Figure 1 of Corless et al 1996)



The two real branches of $W(x)$. ——, $W_0(x)$; − − −, $W_{-1}(x)$.

Figure 3a. Optimal manipulation and marginal expected benefit of manipulation
as functions of unmanaged earnings

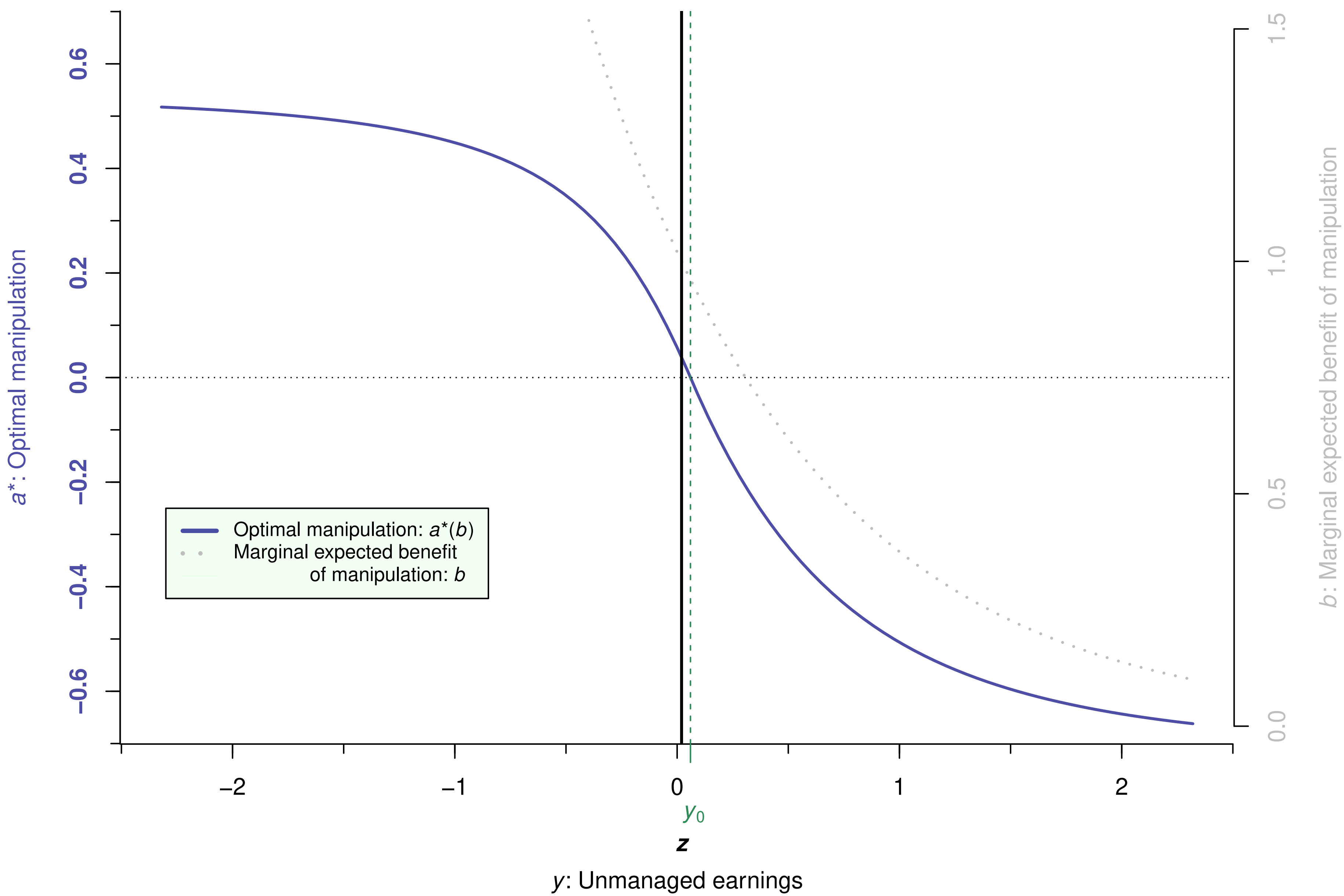


Figure 3b. Pre−audit earnings as a function of unmanaged earnings

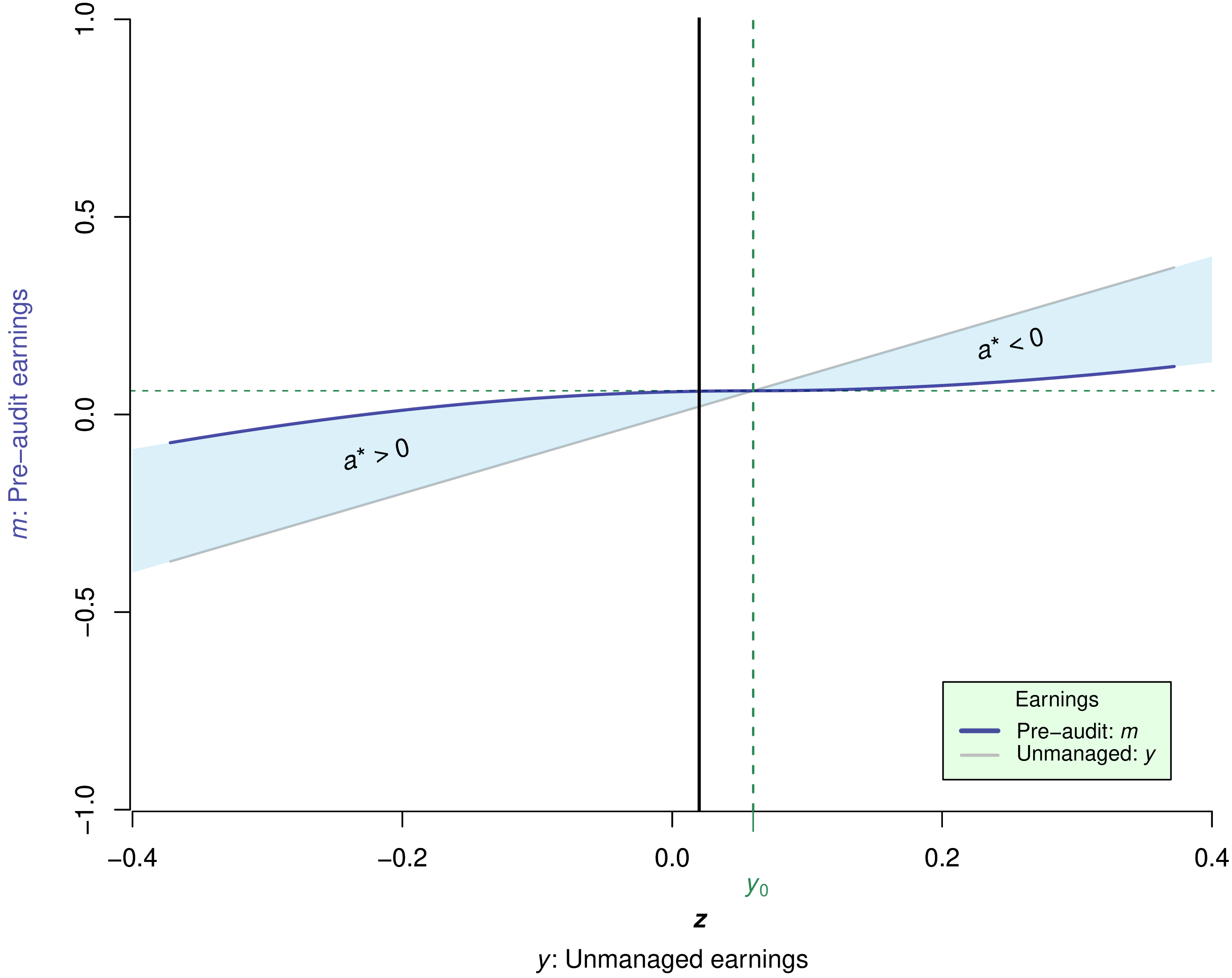


Figure 4a. Distributions of pre−audit and unmanaged earnings

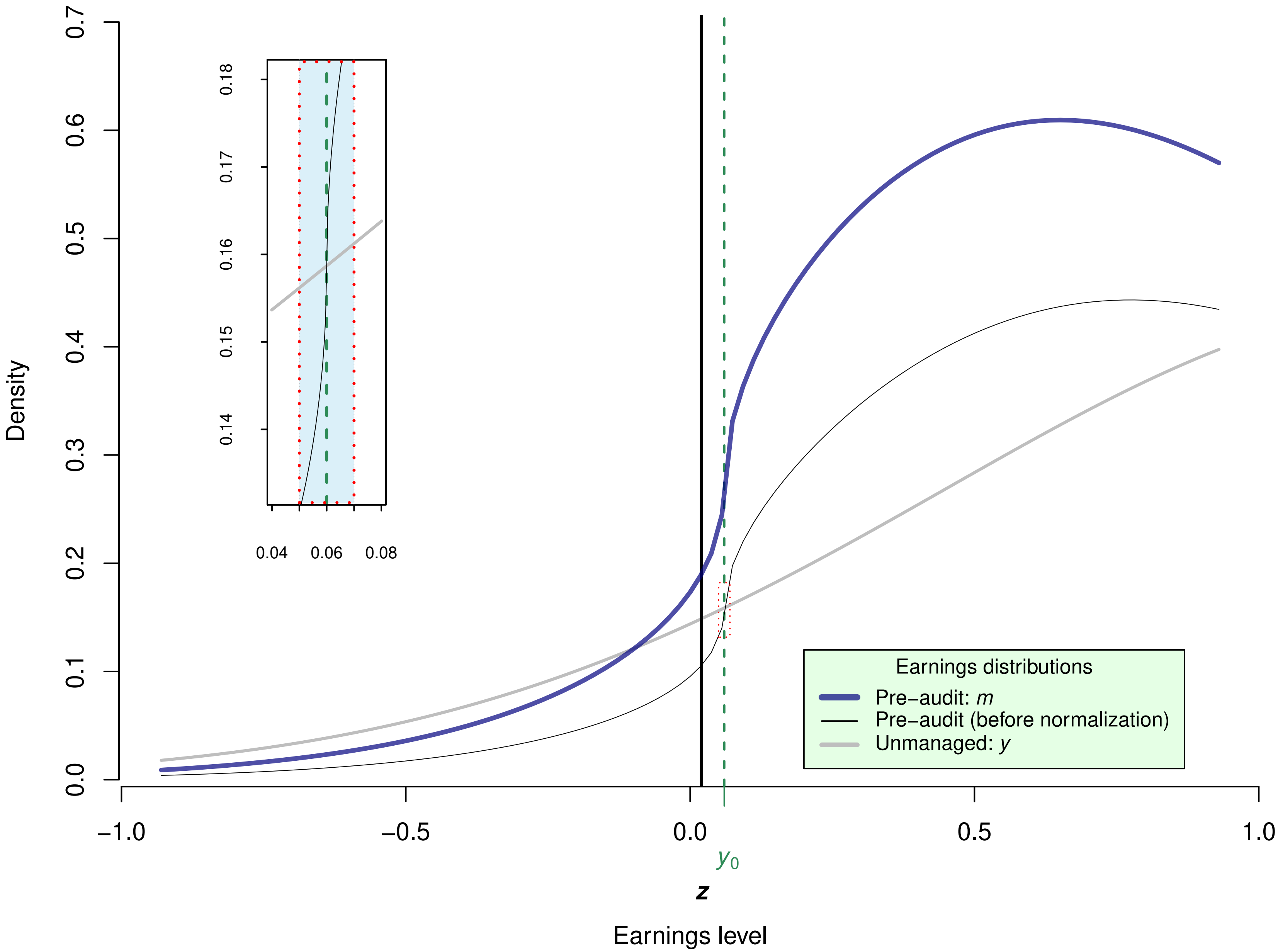


Figure 4b. Normal distribution of unmanaged earnings transformed into volcano−shaped distribution of pre−audit earnings

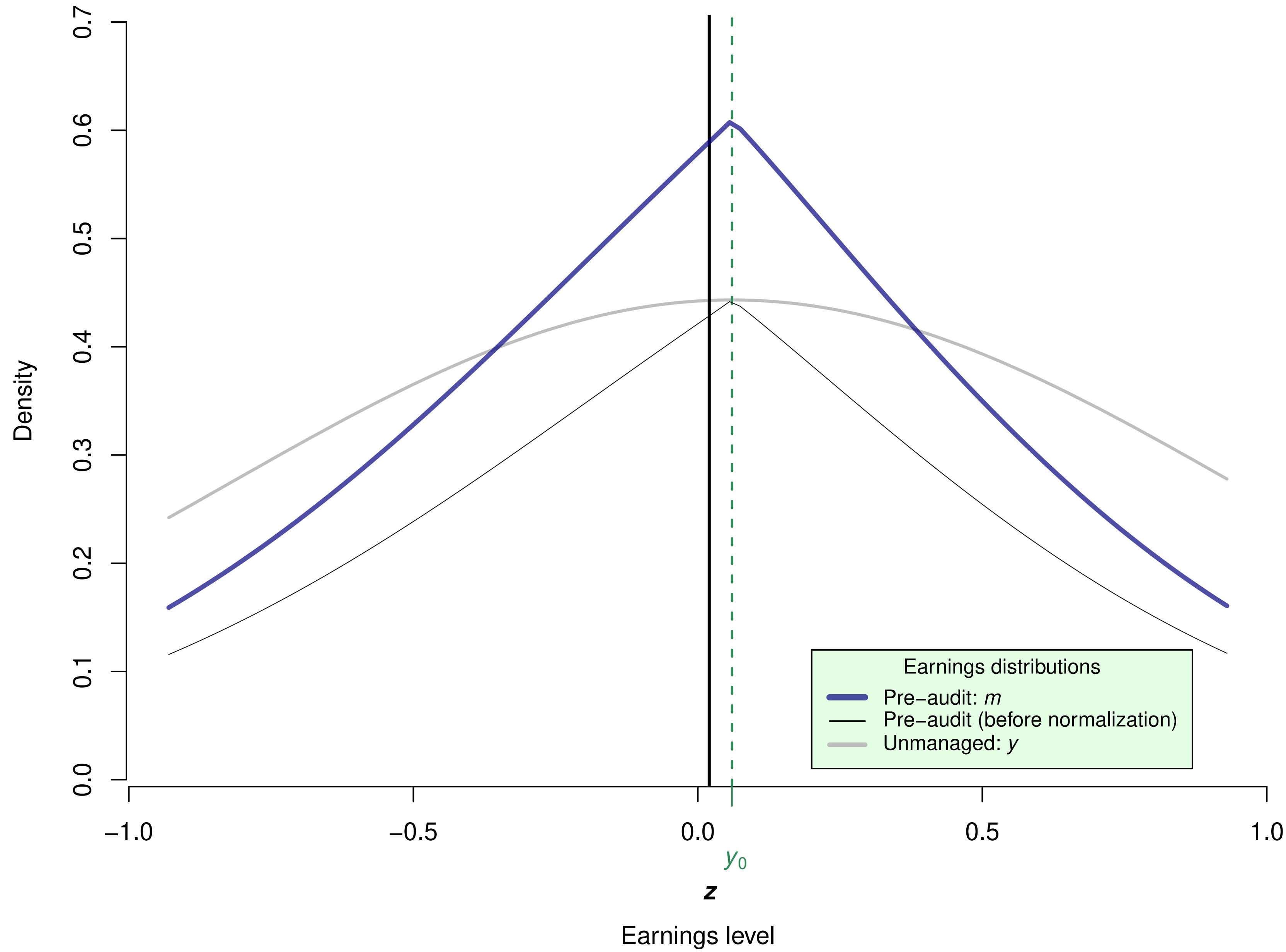


Figure 5a. Frequency distributions of simulated excess earnings

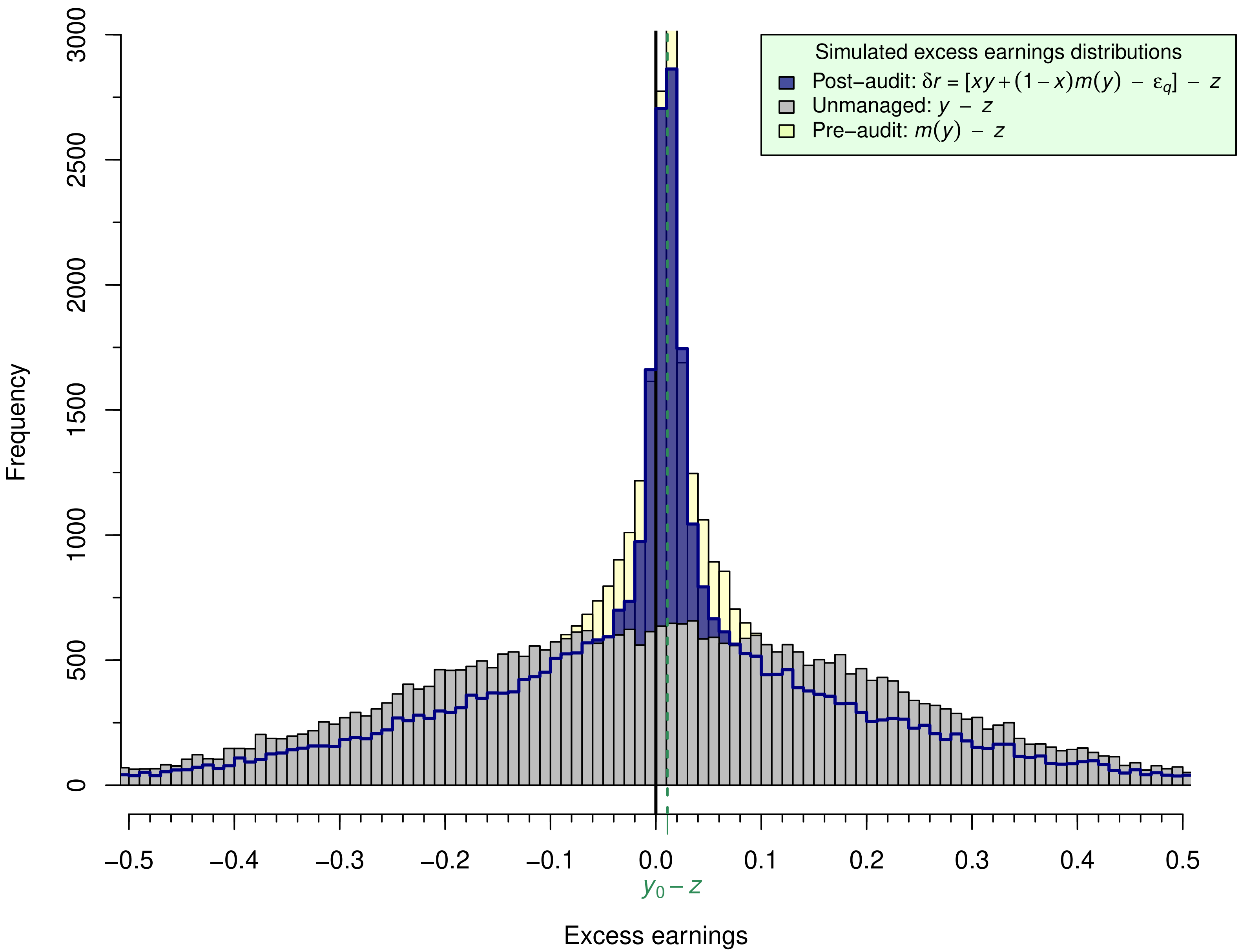
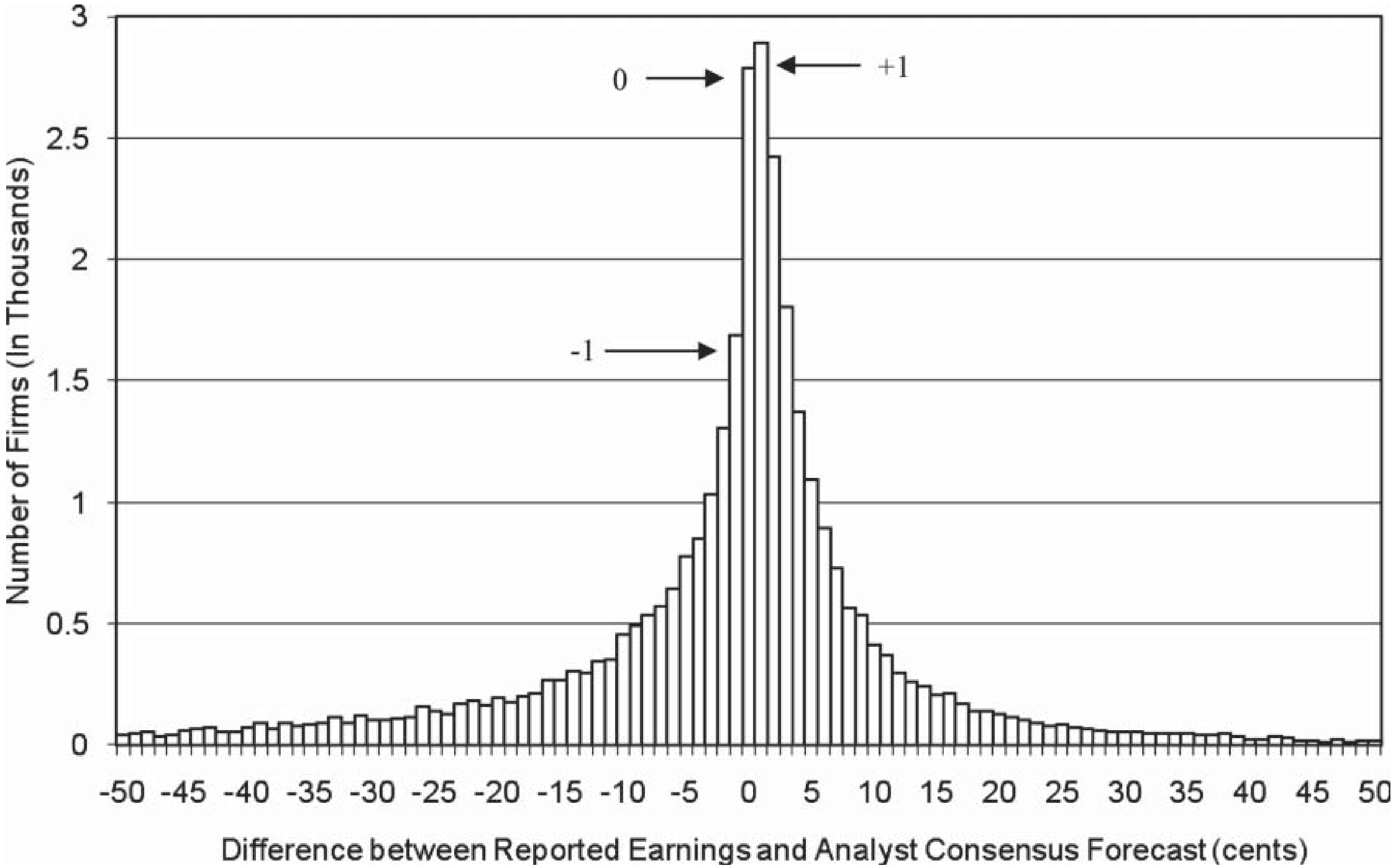


Figure 5b. Frequency distributions of excess earnings in 1988–2006
(Source: Figure 2 of Bhojraj et al 2009)



Excess earnings defined as earnings surprises relative to analysts' consensus forecast (in cents)

Figure 5c. 3D plot of frequency distributions of simulated excess earnings

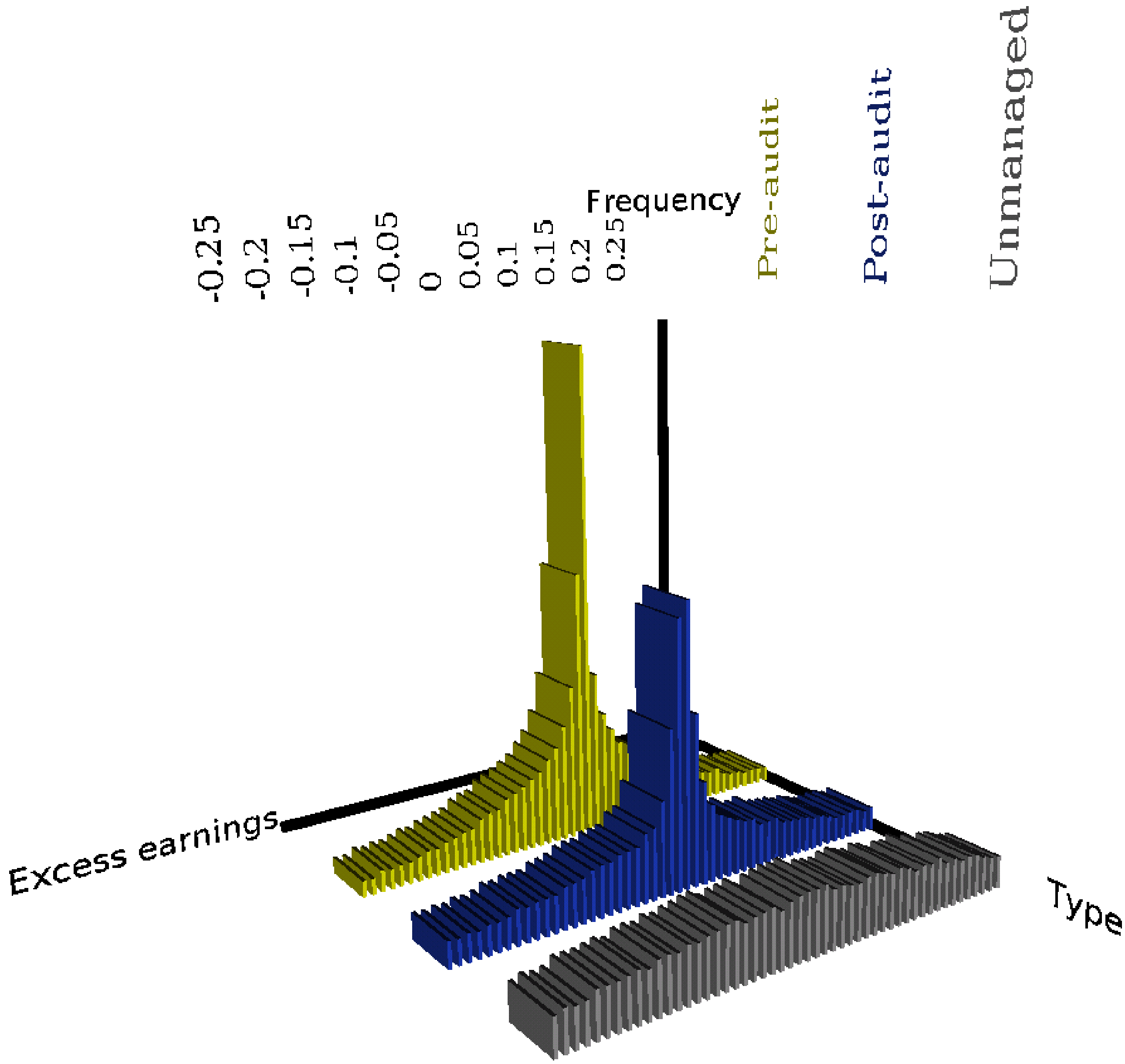


Figure 6a. Frequency distributions of simulated earnings

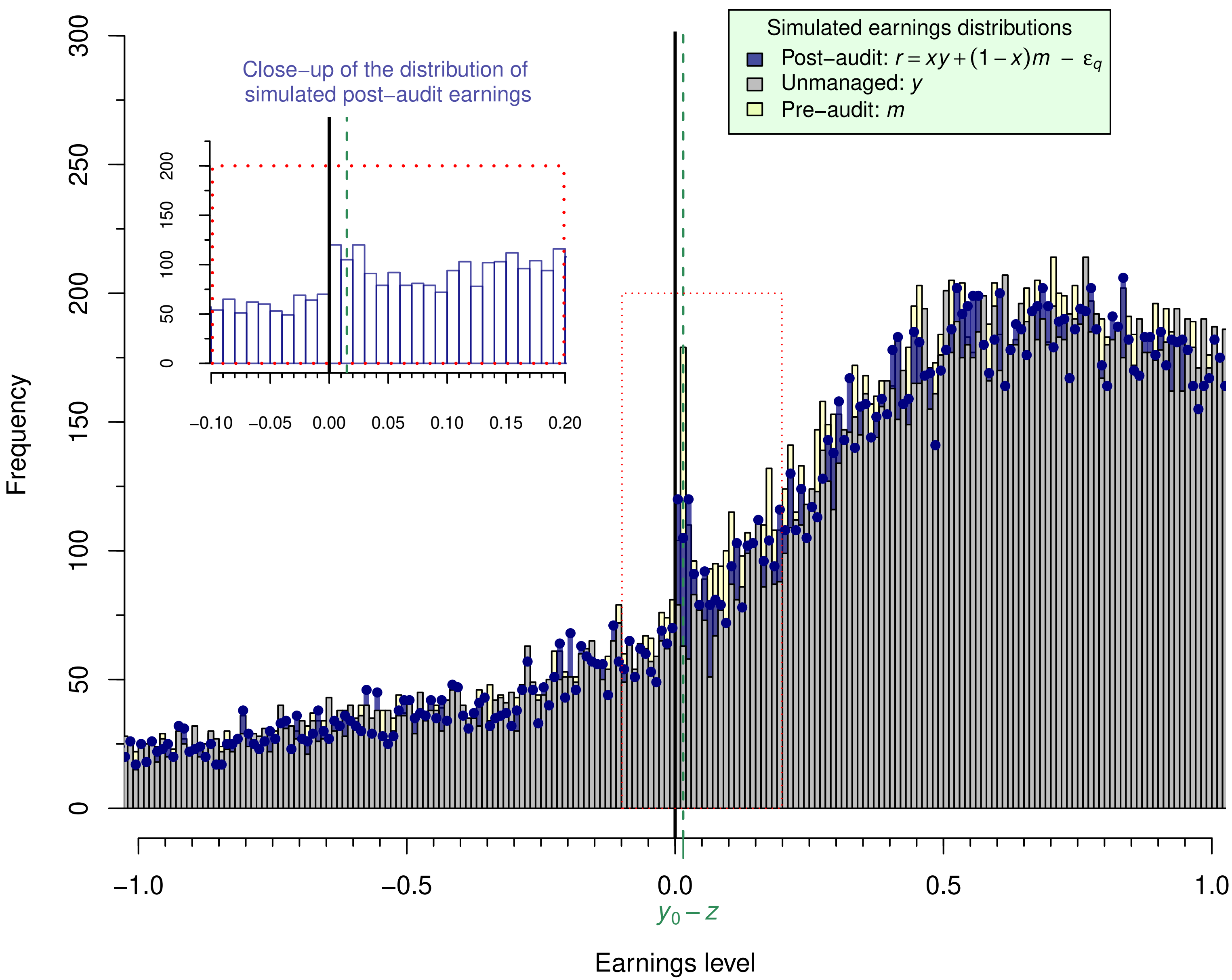


Figure 6b. Frequency distributions of earnings in 1988−2006

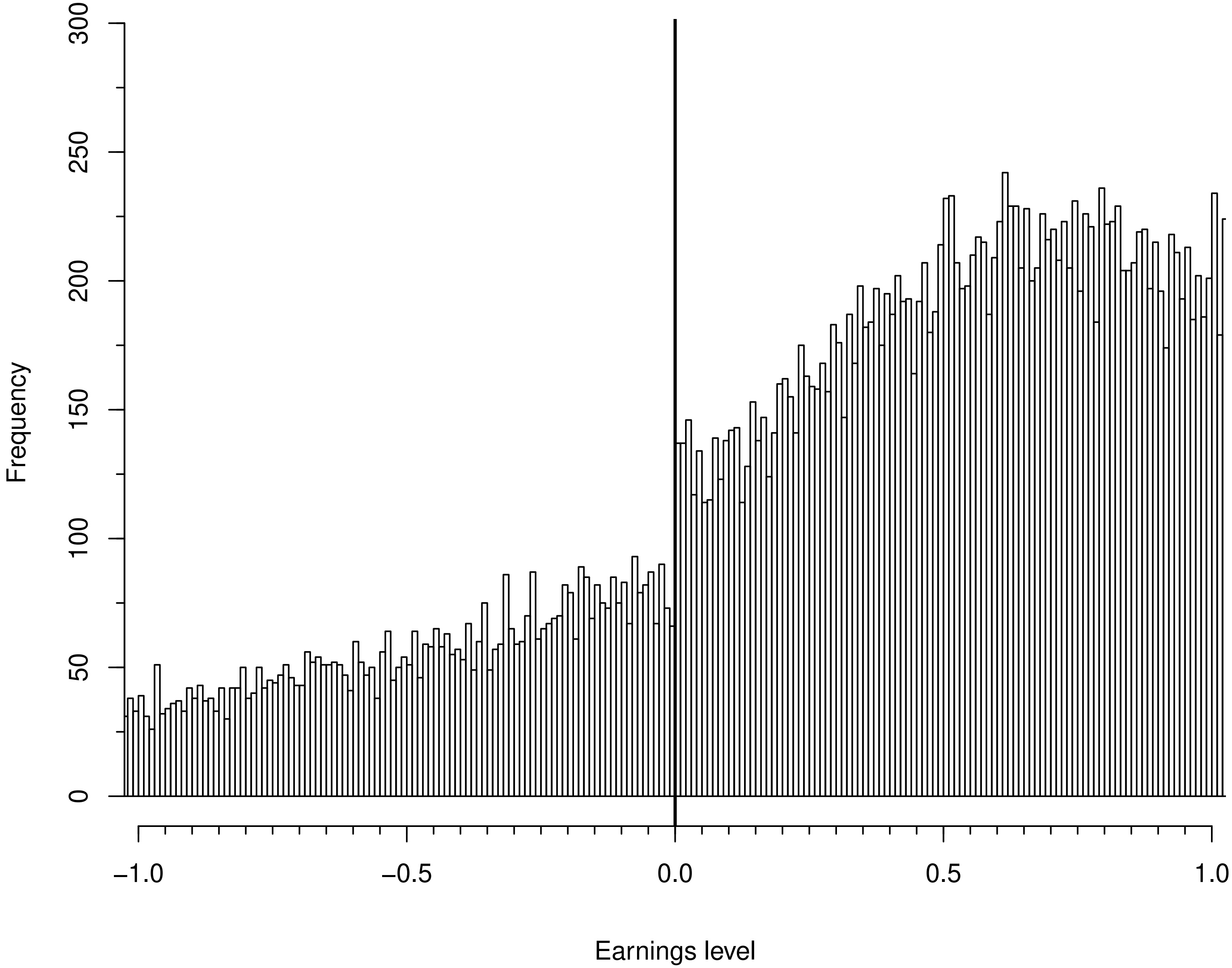


Figure 7a. Frequency distributions of simulated earnings differences

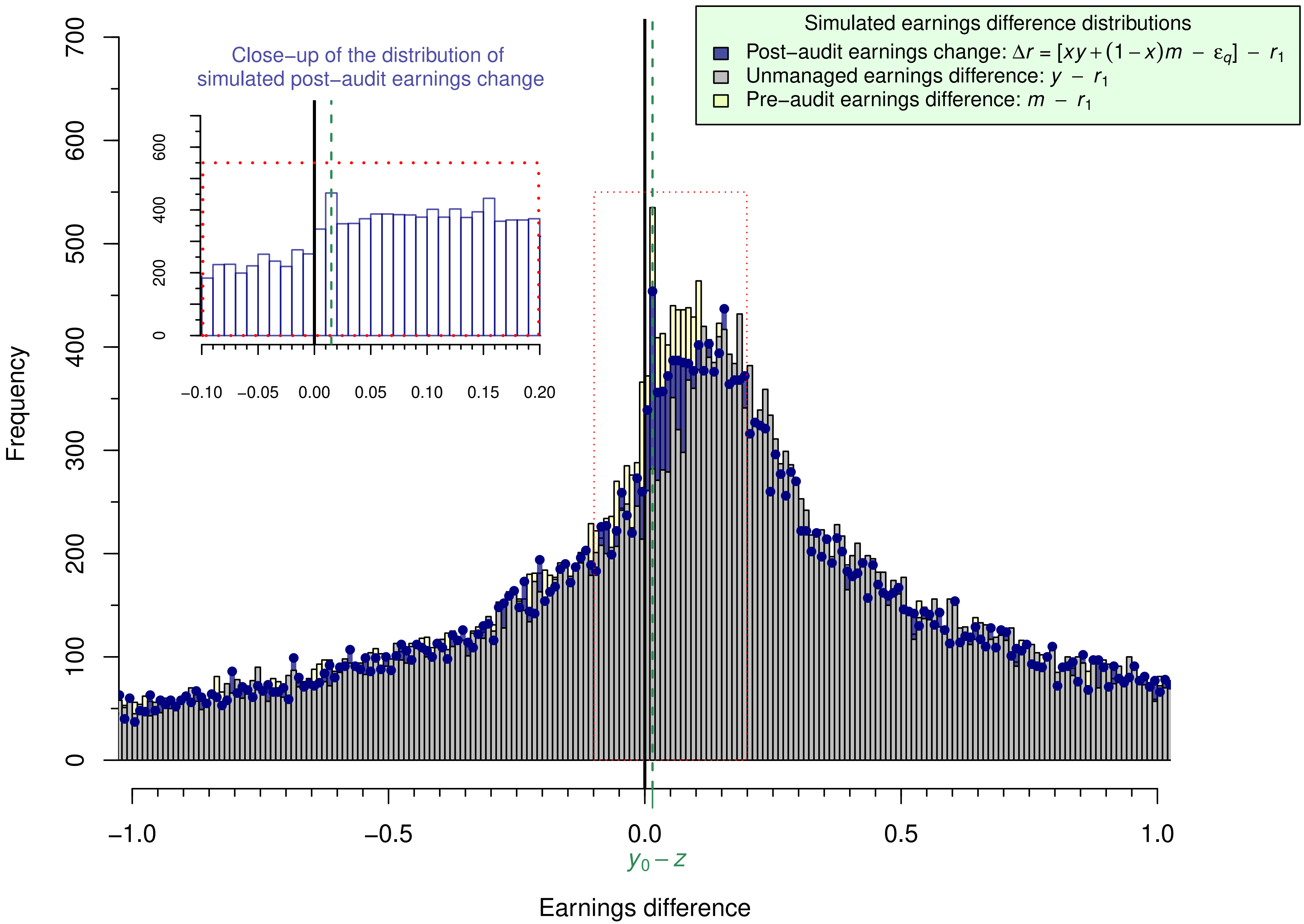


Figure 7b. Frequency distributions of earnings change in 1988−2006