On the Extension and Decomposition of a Preorder under Translation Invariance

Mabrouk, Mohamed

Ecole Superieure de Statistique et d'Analyse de l'Information de Tunis

14 December 2018

On the Extension and Decomposition of a Preorder under Translation Invariance

Mohamed ben Ridha Mabrouk[1]
Version: December 14, 2018

Abstract

We prove the existence, for a translation-invariant preorder on a divisible commutative group, of a complete preorder extending the preorder in question and satisfying *translation invariance* (theorem 1). We also prove that the extension may inherit a property of continuity (theorem 2). This property of continuity may lead to *scalar invariance*. By seeking to clarify the relationship between continuity and scalar invariance under translation invariance, we are led to formulate a theorem that asserts the existence of a continuous linear *weak representation* under a certain condition (theorem 3). The application of these results in a space of infinite real sequences shows that this condition is weaker than the axiom *super weak Pareto*, and that the latter is itself weaker than the axiom *monotonicity* for non-constant preorders. Thus, theorem 3 is a strengthening of theorem 4 of Mabrouk 2011. It also makes it possible to show the existence of a sequence of continuous linear preorders whose lexicographic combination constitutes the finest combination coarser than the preorder in question (theorem 4). This decomposition makes it possible to handle continuous functions instead of preoders when one looks for optima, which may be more practical. Finally we apply this decomposition to the preorder *catching-up*. Several examples are provided.

1- Introduction

The present paper establishes the existence, for any preorder on a divisible commutative group satisfying *translation invariance*, of a complete preorder extending the given preorder and satisfying translation invariance (section 3, theorem 1). In Demuynck-Lauwers 2009 the existence of an extension under the conditions translation invariance and *scalar invariance* is proven. However, the result proved here is stronger in the sense that it is freed from the scalar invariance assumption. The proof of theorem 1 follows the same diagram as the proof of Szpilrajn 1930 theorem which may be stated as follows. For any reflexive and transitive binary relation (i.e. a preorder) on a given set, there

[1] I am grateful to two anonymous referees who, when reviewing other papers, suggested me to study the issues of extending a preorder under *translation-invariance* and the relationship between *translation-invariance* and *scalar-invariance*.

exists a complete preorder which is an extension of the given preorder[2]. Starting from a preorder satisfying translation invariance, one adds comparisons on some pairs of alternatives in such a way that translation invariance remains satisfied. Then, an argument based on Zorn's lemma makes it possible to extend the procedure to the whole space. We give two examples, the first of which shows the existence of a complete translation-invariant strict preorder on $\mathbb{R}$ which transgresses scalar invariance and the second shows the existence of a complete translation-invariant preorder satisfying the social choice axioms *strong Pareto* and *fixed–step anonymity* on a set $X^{\mathbb{N}_0}$, where X is a divisible commutative group.

Then, we prove a second extension theorem which asserts that the former extension result (theorem 1) holds under an additional requirement of continuity (section 4, theorem 2). The proof is an adaptation of the proof of Jaffray 1975 to the translation invariance case. It relies on the construction of a relation that is used to "clean" the extended preorder given by theorem 1 from undesirable rankings that transgress the continuity requirement. As application, we prove the existence of a complete, translation-invariant, strong Pareto, fixed–step anonymous and upper-semi-continuous preorder on $\mathbb{R}^{\mathbb{N}_0}$ which is an extension of a given preorder which satisfies the same axioms except completeness. We also prove that the property of continuity under a *norm topology* leads to scalar invariance.

Seeking to better understand the relationship between continuity and scalar invariance under translation invariance, we offer an example of a complete preorder on $\mathbb{R}^{\mathbb{N}_0}$, translation-invariant, continuous with respect to the l_{1-}topology but not scalar invariant. We are then led to propose a continuity requirement, *linear continuity,* equivalent to scalar invariance, and a theorem that asserts the existence of a continuous linear *weak representation* under a certain condition (section 5, theorem 3). The proof is similar to that of theorem 4 in Mabrouk 2011, except that the open convex cone used is different and the condition *super weak Pareto* is replaced by a sufficient condition which can be formulated in more general spaces.

Back in the context of infinite real sequences, it turns out that the sufficient condition in weaker than monotonicity and super weak Pareto (section 6). Thus, theorem 3 is a strengthening of theorem 4 of Mabrouk 2011. Moreover, using theorem 3, we prove that, for a non-trivial preorder, monotonicity is stronger than super weak Pareto.

A successive application of theorem 3 makes it possible to show the existence of a sequence of continuous and linear preorders whose lexicographic combination constitutes the finest *linear continuous combination* coarser than a given preorder (section 7, theorem 4). Although theorem 4 invokes several times theorem 3, which is non-constructive, it may be used along with other specific information to gain some insight on the preorder and to handle continuous functions instead of preoders when one looks for optima. As an example, section 8 studies

[2] See Alcantud-Diaz 2014 for an overview on the applications and extensions of Szpilrajn theorem.

the decomposition of the *catching-up* preorder under two different norms.

2- Preliminaries

$\mathbb{N}_0$ is the set of positive integers. n, i symbolize positive integers. Q is the set of rational numbers. $(X, +)$ is a divisible commutative group. B being a binary relation on X and x, y two elements of X, xBy is denoted $x \succsim_B y$,[xBy and non(yBx)] is denoted $x \succ_B y$ and [xBy and yBx] is denoted $x \sim_B y$. The symbols $\leq, \geq, <, >$ are used for the natural order on $\mathbb{R}$. A reflexive and transitive binary relation on X is a preorder on X. If, on top of that, for all x, y either $x \succsim_B y$ or $x \precsim_B y$, it is a complete preorder. A binary relation B_1 is said to be a subrelation to a binary relation B_2 , or B_2 an extension of B_1, if for all x, y in X,

$$x \succsim_{B_1} y \Longrightarrow x \succsim_{B_2} y$$

and

$$x \succ_{B_1} y \Longrightarrow x \succ_{B_2} y$$

Axiom *Translation Invariance* (TI) A preorder R satisfies translation invariance if:

$$\forall (x, y) \in X \times X, \forall u \in X, [x \succsim_R y \Rightarrow x + u \succsim_R y + u]$$

Axiom *Division Invariance* (DI) A preorder R satisfies division invariance if:

$$\forall x \in X, \forall n \in \mathbb{N}_0, \left[x \succsim_R y \Rightarrow \frac{1}{n} x \succsim_R \frac{1}{n} y \right]$$

Lemma 1 If a preorder R on X satisfies **TI**, then there exists a preorder $\widehat{R}$ on X of which R is a subrelation and such that $\widehat{R}$ satisfies **TI** and **DI**.

Proof: First, notice that under R, it is possible to sum inequalities. Indeed, by **TI,** if a, b, u, v are such that $a \succsim_R b$ and $u \succsim_R v$, then $a + u \succsim_R b + u$ and $b + u \succsim_R b + v$. By transitivity, $a + u \succsim_R b + v$. For each n, consider the binary relation R_n defined by

$$x \succsim_{R_n} y \text{ iff } nx \succsim_R ny$$

If x, y are such that $x \succsim_R y$, we can sum n times this inequality. Thus, $x \succsim_{R_n} y$. Likewise, it is easily seen that $x \succ_R y$ implies $x \succ_{R_n} y$. As a result, R is a subrelation to R_n. Moreover, R_n is reflexive and transitive. It is easily checked that R_n satisfies **TI.**

Consider the binary relation

$$\widehat{R} = \cup_{n \in \mathbb{N}_0} R_n$$

defined on X by $x \succsim_{\widehat{R}} y$ iff there is n such that $x \succsim_{R_n} y$.

R is a subrelation to $\widehat{R}$. Moreover, $\widehat{R}$ is reflexive and transitive. It is a preorder. Since for each n, R_n satisfies **TI**, we deduce that $\widehat{R}$ satisfies **TI.** The lemma is proved if we show that $\widehat{R}$ satisfies **DI**. Let x, y be such that $x \succsim_{\widehat{R}} y$. There exists a positive integer m such that $x \succsim_{R_m} y$. Thus $mx \succsim_R my$. We can write that as $mn(\frac{1}{n}x) \succsim_R mn(\frac{1}{n}y)$. Thus $\frac{1}{n}x \succsim_{R_{mn}} \frac{1}{n}y$, what implies $\frac{1}{n}x \succsim_{\widehat{R}} \frac{1}{n}y$. $\widehat{R}$ satisfies DI.□

Remark 1 (i) It is easily seen that $\widehat{R}$ is the minimal preorder satisfying **TI** and **DI**, of which R is a subrelation. (ii) If R is complete, since R is a subrelation to $\widehat{R}$, we have necessarily $R = \widehat{R}$. This shows that if the preorder is complete, **TI** implies **DI**.◊

3- The Translation-Invariant Extension Theorem

Theorem 1 Let R be a preorder on X satisfying **TI**. Then there exists a complete preorder on X satisfying **TI**, of which R is a subrelation.

Proof: If R is a complete preorder, there is nothing to prove. Suppose that R is not complete. Consider the preorder $\widehat{R}$ built in the proof of lemma 1, and the set $\Re$ of all preorders on X satisfying **TI** and **DI**, and of which R is a subrelation. $\Re$ is not empty since $\widehat{R} \in \Re$. Let (R_α) be a chain in $\Re$, i.e. for any α, α', R_α is a subrelation to $R_{\alpha'}$ or $R_{\alpha'}$is a subrelation to R_α . Notice that (i) the relation $\cup_\alpha (R_\alpha)$ defined on X by: $x [\cup_\alpha (R_\alpha)] y$ iff there is α such that $xR_\alpha y$, is a preorder, (ii) it satisfies **TI** and **DI**, (iii) R is a subrelation to $\cup_\alpha (R_\alpha)$, (iv) for all α, R_α is a subrelation to $\cup_\alpha (R_\alpha)$. Hence, in the set $\Re$, every chain admits an upper bound. According to Zorn's lemma, $\Re$ admits at least a maximal element. Denote M such a maximal element in $\Re$. Suppose we can prove the following claim:

Claim 1 For any non-complete R' in $\Re$ and any pair of R'-incomparable alternatives (x_0, y_0) , there exists a preorder R'_1 in $\Re$ to which R' is a subrelation and such that x_0 and y_0 are R'_1-comparable.

Then, if M were not complete, there would exist a preorder in $\Re$ to which M is a strict subrelation. This would contradict that M is maximal in $\Re$. Therefore, if the claim holds, M would be necessarily complete. M would be the preorder we are looking for.

What remains of the proof is devoted to establish claim 1. This is done through the following 6 steps.

If there is no non-complete preorder in $\Re$, the theorem is proved since $\Re$ is not empty. Let R' be a non-complete preorder in $\Re$ and x_0, y_0 be two $R'-$incomparable elements of X.

Consider the binary relation B on X: $x \succsim_B y$ iff either $x \succsim_{R'} y$ or there is a positive rational q such that $x - y = q(x_0 - y_0)$.

We prove successively that the two clauses of the definition of B are exclusive (step 1), that the indifference relations are equal (step 2), that R' is a subrelation to B (step 3), that B is weakly acyclic (this prepares for transitivity) (step 4), that R' is a subrelation to the transitive closure of B (step 5), that the transitive closure of B satisfies **TI** and **DI** (step 6). The transitive closure of B is then the required preorder.

Step 1: the two clauses are exclusive. If there is a positive rational q such that $x - y = q(x_0 - y_0)$, then x, y are $R'-$incomparable. Suppose not. For instance suppose $x \succsim_{R'} y$. By **TI,** $x - y \succsim_{R'} 0$. By **DI**, for all n, $\frac{1}{n}(x - y) \succsim_{R'} 0$.

Recall that it is possible to sum inequalities (see the proof of lemma 1). We sum m times the inequality $\frac{1}{n}(x-y) \succsim_{R'} 0$, m being a positive integer. We obtain $\frac{m}{n}(x-y) \succsim_{R'} 0$. Take $\frac{n}{m} = q$. It gives $x_0 - y_0 \succsim_{R'} 0$, what contradicts x_0, y_0 being incomparable. The case $y \succsim_{R'} x$ is similar.

Step 2: equivalence of indifferences. Clearly, $x \sim_{R'} y \Rightarrow x \sim_B y$. We show now that $x \sim_B y$ entails $x \sim_{R'} y$. According to the definition of B, it is enough to prove that x and y are necessarily $R'-$comparable. Suppose not. Then $x \succsim_B y$ implies that there is some positive rational q such that $x - y = q(x_0 - y_0)$. We have also $y \succsim_B x$. Thus, for some positive rational q', $y - x = q'(x_0 - y_0)$. We see that this gives $q'(x_0 - y_0) = -q(x_0 - y_0)$, what implies $x_0 - y_0 = 0$ because q, q' are both positive. But that contradicts x_0, y_0 being $R'-$incomparable.

Step 3: R' is a subrelation to B. This is a direct consequence of $x \succsim_{R'} y \Longrightarrow x \succsim_B y$ (definition of B) and $x \sim_B y \Leftrightarrow x \sim_{R'} y$ (step 2).

Step 4: B is weakly-acyclic**.** We show that for all x, y, z in $X : x \succsim_B y$ and $y \succsim_B z \Rightarrow x \succsim_B z$ or non$(z \succsim_B x)$.

One of the four following cases is implied by $x \succsim_B y$ and $y \succsim_B z$. (1) $x \succsim_{R'} y$ and $y \succsim_{R'} z$, (2) there are q, q' such that $x-y = q(x_0-y_0)$ and $y-z = q'(x_0-y_0)$, (3) $x \succsim_{R'} y$ and there is q' such that $y - z = q'(x_0 - y_0)$, (4) there is q such that $x - y = q(x_0 - y_0)$ and $y \succsim_{R'} z$. Consider successively the four cases:

(1) By transitivity of $R' : x \succsim_{R'} z$. Thus, $x \succsim_B z$.

(2) $x-y = q(x_0-y_0)$ and $y-z = q'(x_0-y_0)$ entails $x-z = (q+q')(x_0-y_0)$. Thus $x \succsim_B z$.

(3) Suppose we had $z \succsim_B x$. We would have either $z \succsim_{R'} x$ or $z - x = q''(x_0 - y_0)$. Both possibilities contradict $x \succsim_{R'} y$ and $y - z = q'(x_0 - y_0)$. Indeed, with $x \succsim_{R'} y$, $z \succsim_{R'} x$ gives $z \succsim_{R'} y$ what contradicts $y-z = q'(x_0-y_0)$ (step 1); whereas $y - z = q'(x_0 - y_0)$ with $z - x = q''(x_0 - y_0)$ implies $y - x = (q' + q'')(x_0 - y_0)$, what contradicts $x \succsim_{R'} y$. As a result, we have non$(z \succsim_B x)$.

(4) This case is similar to (3)

Remark 2 Let x, y, z be such that $x \succsim_B y$ and $y \succsim_B z$. Weak acyclicity entails that if one of the comparisons $x \succsim_B y$ and $y \succsim_B z$ is a strict preference, then either the comparison on (x, z) is $x \succ_B z$ or x and z are $B-$incomparable.$\Diamond$

Step 5: R' is a subrelation to the transitive closure of B. Consider $\overline{B}$ the transitive closure of B defined by: $x \succsim_{\overline{B}} y$ if there is a sequence $(z_i)_{i=1}^n$ such that $x \succsim_B z_1, z_1 \succsim_B z_2 ...$ and $z_n \succsim_B y$. It is clear that $x \succsim_{R'} y$ implies $x \succsim_{\overline{B}} y$ (step 3: R' is a subrelation to B). It is enough to prove that $x \succsim_{\overline{B}} y$ implies non$(y \succ_{R'} x)$.

Consider the statement Q_n: "If there is a sequence $(z_i)_{i=1}^n$ such that $x \succsim_B z_1 \succsim_B z_2 .... \succsim_B z_n \succsim_B y$, then non$(y \succ_{R'} x)$." Let's prove by induction that Q_n is true for all positive integers. Notice that when the sequence (z_i) has n terms, there is $n+1$ successive comparisons.

$n = 1 :$ We have $x \succsim_B z_1 \succsim_B y$. By step 4, we have $x \succsim_B y$ or non$(y \succsim_B x)$. Both possibilities contradict $y \succ_{R'} x$. So, we have non$(y \succ_{R'} x)$.

Suppose that Q_n is true and let's show that Q_{n+1} is true. Consider the sequence of $n+2$ comparisons: $x \succsim_B z_1 \succsim_B z_2 .... \succsim_B z_{n+1} \succsim_B y$.

Each one of these comparisons comes either from the clause $x \succsim_{R'} y$ or the clause $x - y = q(x_0 - y_0)$ of the definition of B. If there is two successive comparisons coming from the clause $x \succsim_{R'} y$, say $z_p \succsim_{R'} z_{p+1} \succsim_{R'} z_{p+2}$ (with $p = 0, ..., n+2$ and the convention: $z_0 = x$ and $z_{n+2} = y$), by transitivity of R' we have: $x \succsim_B ...z_p \succsim_B z_{p+2}.... \succsim_B y$ which constitutes a sequence of $n+1$ comparisons. By Q_n we have non$(y \succ_{R'} x)$. If there is two successive comparisons coming from the clause $x - y = q(x_0 - y_0)$, say $z_p \succsim_B z_{p+1} \succsim_B z_{p+2}$, then $z_p - z_{p+1} = q(x_0 - y_0)$ and $z_{p+1} - z_{p+2} = q'(x_0 - y_0)$. Thus, $z_p - z_{p+2} = (q + q')(x_0 - y_0)$ so that $z_p \succsim_B z_{p+2}$. We have again reduced the number of comparisons to $n+1$. Thus, we have also non$(y \succ_{R'} x)$. It remains to consider the cases where the comparisons are alternate. Two cases must be considered: $n+2$ even and $n+2$ odd.

$n+2$ even: The sequence of comparisons either begin or ends with a comparison from R'. Suppose it begins with a comparison from R': $x \succsim_{R'} z_1 \succsim_B z_2.... \succsim_{R'} z_{n+1} \succsim_B y$. Apply Q_n to $z_1 \succsim_B z_2.... \succsim_{R'} z_{n+1} \succsim_B y$. It gives non$(y \succ_{R'} z_1)$. Since $x \succsim_{R'} z_1$, we cannot have $y \succ_{R'} x$. If the sequence of comparisons ends with a comparison from R', the proof is similar. So it is omitted.

$n+2$ odd: If the sequence of comparisons begins with a comparison from R', the proof is also similar. So it is omitted. If the sequence of comparisons begins with a comparison from the clause $x - y = q(x_0 - y_0)$, we have

$$x \succsim_B z_1 \succsim_{R'} z_2.... \succsim_{R'} z_{n+1} \succsim_B y \tag{1}$$

Denote (x, z_1) by (α_1, β_1), (z_2, z_3) by (α_2, β_2) .... $\left(z_{2(p-1)}, z_{2p-1}\right)$ by $\left(\alpha_p, \beta_p\right)$ with $p = 1, ..., \frac{n+1}{2}$ and the convention $z_0 = x$ and $z_{n+2} = y$. Since comparisons $x \succsim_B z_1, z_2 \succsim_B z_3....z_{n-1} \succsim_B z_n, z_{n+1} \succsim_B y$ come from the clause $x - y = q(x_0 - y_0)$, we have $\alpha_p - \beta_p = q_p(x_0 - y_0)$ for $p = 1,...,\frac{n+3}{2}$. Moreover, according to (1), $\beta_p \succsim_{R'} \alpha_{p+1}$ for $p = 1, ..., \frac{n+1}{2}$. Thus

$$\begin{array}{rcl} \alpha_1 - q_1(x_0 - y_0) & \succsim & {}_{R'}\alpha_2 \\ \alpha_2 - q_2(x_0 - y_0) & \succsim & {}_{R'}\alpha_3 \\ & & ..... \\ \alpha_{(n+1)/2} - q_{(n+1)/2}(x_0 - y_0) & \succsim & {}_{R'}\alpha_{(n+3)/2} \end{array}$$

We can sum these inequalities (this is established in the proof of lemma 1). We obtain

$$\alpha_1 + \sum_2^{(n+1)/2} \alpha_p - \sum_2^{(n+1)/2} q_p (x_0 - y_0) \succsim_{R'} \sum_2^{(n+1)/2} \alpha_p + \alpha_{(n+3)/2}$$

By **TI** we obtain

$$\alpha_1 - \sum_1^{(n+1)/2} q_p (x_0 - y_0) \succsim_{R'} \alpha_{(n+3)/2}$$

But $\alpha_1 = x_1$ and $\alpha_{(n+3)/2} = y$. Denote $q = \sum_1^{(n+1)/2} q_p$. Thus

$$x - q(x_0 - y_0) \succsim_{R'} y$$

By **TI**, $x - y \succsim_{R'} q(x_0 - y_0)$. If we had $y \succ_{R'} x$, it would give $0 \succ_{R'} x - y \succsim_{R'} q\,(x_0 - y_0)$. By transitivity of R' and by **TI**, x_0 and y_0 would be $R'-$comparable, which is not the case. As a result, we have non($y \succ_{R'} x$). Step 5 is proved.

Remark 3 R' is a subrelation to $\overline{B}$, but B is not.◊

Step 6: $\overline{B}$ **satisfies TI.** As R' is translation-invariant, B is clearly translation-invariant. It is easily deduced that $\overline{B}$ is also translation-invariant. Likewise, it is easily seen that $\overline{B}$ satisfies **DI.** Thus, $\overline{B}$ is the required preorder.□

Corollary 1 Let B be a reflexive binary relation satisfying **TI**. Then there exists a complete preorder satisfying **TI**, of which B is a subrelation, iff B is a subrelation to its transitive closure.

Proof: Necessity: the condition that B is a subrelation to its transitive closure is necessary and sufficient for the existence of a complete preorder of which B is a subrelation (Suzumura 1976, Bossert 2008). Sufficiency: denote $\overline{B}$ the transitive closure of B. It easily seen that $\overline{B}$ is a preorder satisfying **TI.** Apply theorem 1 to $\overline{B}$ to deduce that there exists a complete preorder satisfying **TI,** of which $\overline{B}$ is a subrelation. Since B is a subrelation to $\overline{B}$, it is also a subrelation to the complete preorder.□

Example 1: A translation-invariant and complete strict preorder on $\mathbb{R}$ with π smaller than 0 and 0 smaller than 1.

Consider the following binary relation $\succsim$ on $\mathbb{R}$:

$$x \precsim y \text{ if there is two nonnegative rationals } q, q' \text{ such that } x - y = -q + q'\pi$$

$\precsim$ is reflexive, transitive and satisfies **TI**. Moreover, $\succsim$ is a strict preorder, which means that $x \precsim y$ and $y \precsim x$ implies $x = y$. Indeed $x - y = -q + q'\pi$ and $y - x = -q_1 + q_1'\pi$ yields $0 = (x - y) + (y - x) = -(q + q_1) + (q' + q_1')\pi$. Thus $(q + q_1) = (q' + q_1')\pi$. We must have $q' + q_1' = 0$ otherwise π would be rational. Thus we have also $q + q_1 = 0$. Since q, q_1, q', q_1' are nonnegative, we have $q = q_1 = q' = q_1' = 0$ and $x = y$.

Theorem 1 asserts the existence of a translation-invariant and complete preorder, say $\succsim_\pi$, of which $\succsim$ is a subrelation. $\succsim_\pi$ is strict like $\succsim$ (i.e. $x \sim_\pi y \Leftrightarrow x = y$). Observe that $\succsim_\pi$respects the natural order of rationals. But it does not coincide with the natural order of reals. Moreover it does not satisfy invariance with respect to multiplication by a positive scalar (*scalar invariance)* since if you multiply $0 \precsim_\pi 1$ by π the inequality is reversed. Finally, $\succsim_\pi$is not continuous. Consider a positive sequence of rational (q_n) such that $\lim q_n = \frac{1}{\pi}$. **TI** allows to multiply an inequality by a positive rational. Multiplying $\pi \precsim_\pi 0$ by q_n yields $q_n\pi \precsim_\pi q_n.0 = 0$ for all n. But $\lim q_n\pi = 1 \succsim_\pi 0$. A question then arises: can Scalar-Invariance still be transgressed under **TI** and continuity? An answer is provided in section 5 and 6.

Example 2: Existence of a translation-invariant, strong-Pareto, fixed–step-anonymous and complete preorder on $X^{\mathbb{N}_0}$, where X is a divisible commutative group equipped with a complete preorder R satisfying **TI.**

It is possible to prove the existence of such a preorder using the ultrafilter technique, as in Fleurbaey-Michel 2003, Lauwers 2009. We prove here this existence without using ultrafilters, which are highly nonconstructive objects. Although our theorem 1 also makes use of the axiom of choice, one may consider that our method is nevertheless more constructive in the sense that it indicates the concrete steps of adding comparisons.

Let $Y = X^{\mathbb{N}_0}$, let R' be a preorder on Y. We first give the following definitions:

***Fixed-Step Permutation*:** (Fleurbaey-Michel 2003) σ is a fixed-step permutation if there exist $k \in \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$, $\sigma(\{1, ..., kn\}) = \{1, ..., kn\}$.

Axiom *Fixed-Step Anonymity*: Denote $\sigma(x)$ the sequence obtained by permuting the components of $x \in Y$ according to the permutation σ. R' is fixed-step-anonymous if for all $x \in Y$ and fixed-step permutation σ, we have $x \sim_{R'} \sigma(x)$.

Axiom *Strong Pareto*: R' is strong-Pareto if, for all $x, y \in Y$ such that $\forall i \in \mathbb{N}_0\ x_i \succsim_R y_i$ and $x_j \succ_R y_j$ for some j , we have $x \succ_{R'} y$ (x_i, y_i denote the i^{th} component of resp. x, y).

Pareto axioms capture the idea that an increase of the components of a vector must increase the ranking of the vector. Anonymity axioms express a requirement of symmetry in the treatment of individuals or dates.

The *Fixed-Step Catching-up* SC. For all $x, y \in Y$, $x \succsim_{SC} y$ iff there exist $k, m \in \mathbb{N}_0$ such that, for all $n \in \mathbb{N}_0$ with $n > m$, we have

$$\sum_{i=1}^{kn} x_i \geq \sum_{i=1}^{kn} y_i$$

SC is a fixed–step-anonymous preorder (Fleurbaey-Michel 2003).

Proposition 1: There exists a translation-invariant, strong-Pareto, fixed-step-anonymous and complete preorder on Y.

Proof: Apply theorem 1 to SC. There exists a translation-invariant and complete preorder R' on Y of which SC is a subrelation. SC being a subrelation to R' entails that R' satisfies strong Pareto and fixed–step anonymity. R' is the required preorder.□

4- Continuity

For a given nontrivial preorder R on a divisible commutative group X, $\tau_+(R)$ is the associated upper-order-topology, i.e. the topology generated by the base of open intervals: $\beta_+(R) = \{\{x \in X : x \prec_R a\}, a \in X\}$.

Theorem 2: Let R be a preorder on X satisfying **TI**. Then there exists a complete preorder R' on X satisfying **TI**, of which R is a subrelation, and such that $\tau_+(R') \subset \tau_+(R)$.

Proof: The following proof is an adaptation of the proof of Jaffray 1975 to a translation-invariant preorder. We start from a translation-invariant complete preorder which extends R, whose existence is guaranteed by theorem 1. We then apply a clause[3] to "clean up" rankings that do not respect the upper-order-topology. It turns out that this clause is also translation-invariant, which makes it possible to build the desired preorder.

Step 1: Building the complete preorder. Let R_1 be a complete preorder extending R and satisfying **TI**. Let $x, y \in X$. Consider the following clause :

$C(x,y)$: "There exists $B \in \beta_+(R)$ containing x such that, for all $B' \in \beta_+(R)$ containing y,we can find $x' \in B'$ such that for all $z \in B$, we have $z \prec_{R_1} x'$ "

Because R_1 satisfies **TI**, it is easily seen that if $C(x,y)$ is true, $C(x+h, y+h)$ is true for all h in X. Moreover, if $C(x,y)$ is true, it is clear that we cannot have $C(y,x)$ true. Thus, we can define a asymmetric relation R_2 checking **TI** as follows: $x \prec_{R_2} y$ iff $C(x,y)$ is true.

We prove now that R_2 is negatively transitive, i.e.

$$\text{not}(x \prec_{R_2} y) \text{ and } \text{not}(y \prec_{R_2} z) \text{ implies } \text{not}(x \prec_{R_2} z)$$

We have:

Not$(x \prec_{R_2} y) \Longleftrightarrow$ for all $B_1 \in \beta_+(R)$ containing x, there exists $B_1' \in \beta_+(R)$ containing y such that [for all x_1'in B_1', there exists x_1'' in B_1 such that $x_1'' \succsim_{R_1} x_1'$].

Not$(y \prec_{R_2} z) \Longleftrightarrow$ for all $B_2 \in \beta_+(R)$ containing y, there exists $B_2' \in \beta_+(R)$ containing z such that [for all x_2'in B_2', there exists x_2'' in B_2 such that $x_2'' \succsim_{R_1} x_2'$].

Let B_1 be in $\beta_+(R)$ containing x and B_1' be the interval which existence is asserted by the clause "not$(x \prec_{R_2} y)$". Take B_1' as the interval B_2 of the clause "not$(y \prec_{R_3} z)$". Thus, there exists $B_2' \in \beta_+(R)$ containing z such that [for all x_2'in B_2', there exists x_2'' in B_1' such that $x_2'' \succsim_{R_1} x_2'$]. Now apply the clause "not$(x \prec_{R_2} y)$" for x_2'' instead of x_1' and deduce that there exists x_1'' in B_1 such that $x_1'' \succsim_{R_1} x_2''$. By transitivity of R_2, $x_1'' \succsim_{R_1} x_2''$ and $x_2'' \succsim_{R_1} x_2'$ gives $x_1'' \succsim_{R_1} x_2'$.

Summing up: for some B_1 in $\beta_+(R)$ containing x, we have found $B_2' \in \beta_+(R)$ containing z such that [for all x_2'in B_2' there exists x_1'' in B_1 such that $x_1'' \succsim_{R_1} x_1'$]. This is exactly the clause not$(x \prec_{R_2} z)$.

Since asymmetry and negative transitivity imply transitivity, R_2 is transitive.

Now let R' be the following binary relation:

$$x \precsim_{R'} y \text{ iff } [(x \prec_{R_2} y) \text{ or } \text{not}(x \succ_{R_2} y)]$$

The transitivity and negative transitivity of R_2 implies the transitivity of R'. Moreover, R' is complete and satisfies **TI**.

Step 2: R is a subrelation to R'. Let x, y be such that $x \prec_R y$. In the clause $C(x,y)$, take $B = \{z \in X : z \prec_R y\}$. We have $x \in B$ and for all B' containing y,we have $z \prec_{R_1} y$ for all $z \in B$. Hence the clause $C(x,y)$ is true and $x \prec_{R_2} y$.

[3]This clause combines the two clauses proposed by Jaffray 1975 in the proof of his theorem 1, the first of which defines a preorder on $\beta_+(R)$ and the second a preorder on X.

Consequently, $x \prec_{R'} y$. If x, y are such that $x \sim_R y$, the clause $C(x,y)$ cannot be satisfied. To see it, it suffices to notice that an interval containing x necessarily contains y and vice versa. If we take $B' = B$ in the clause $C(x,y)$, there is no x' in B such that for all $z \in B$, we have $z \prec_{R_1} x'$. Thus we have not$(x \prec_{R_2} y)$. In the same way, we have not$(y \prec_{R_2} x)$.Consequently, $x \sim_{R'} y$.

It remains to show that $\tau_+(R') \subset \tau_+(R)$. Let $y \in X$. We show that any subset in $\beta_+(R')$, the base of open intervals generating $\tau_+(R')$, is open with respect to $\tau_+(R)$. Let $x \in B = \{z \in X : z \prec_{R'} y\}$. By the definition of R', there is B_x in $\beta_+(R)$, containing x, such that for all $B_y \in \beta_+(R)$ containing y,we can find $x' \in B_y$ such that for all $z \in B_x$, we have $z \prec_{R_1} x'$. We can see that this implies that for all $z \in B_x$, we have $z \prec_{R'} y$. Hence $B_x \subset B$. Recap: for all x in B, we found B_x in $\beta_+(R)$ containing x such that $B_x \subset B$.As a result, B is a union of open sets of $\tau_+(R)$. It is thus an open set of $\tau_+(R)$.□

Remark 4: Theorem 2 holds if we replace $\tau_+(R)$ and $\tau_+(R')$ respectively by $\tau_-(R)$ and $\tau_-(R')$ the lower-order-topologies.◊

Remark 5: The inclusion $\tau_+(R') \subset \tau_+(R)$ entails the upper semicontinuity of the extension with respect to any topology on X stronger than $\tau_+(R)$. Upper semicontinuity is used here in the sense that lower sections $\{x \in X : x \prec_R a\}$ are open. But it is not necessary for the topology on X to be stronger than $\tau_+(R)$ to have the upper semicontinuity of the extension. For more information on this issue, see Jaffray 1975, section 5.◊

Axiom *Scalar Invariance* (SI): For all nonnegative real α and vectors x, y in a real vector space equipped with a preorder R, Y, $x \succsim_R y \Longrightarrow \alpha x \succsim_R \alpha y$.

Corollary 2: Let Y be a real normed vector space**.** Denote t the topology induced by the norm of Y (i.e. the *norm topology*). Let R be a preorder on Y satisfying **TI** and $\tau_+(R) \subset t$. Let R' be one of the complete preorders which existence is asserted by theorem 2, i.e. a complete preorder of which R is a subrelation, satisfying **TI** and such that $\tau_+(R') \subset \tau_+(R)$. Then R' satisfies **SI**.

Proof: We have $\tau_+(R') \subset t$. Let α be a nonnegative real and x, y two vectors in Y such that $x \succsim_R y$. Using **TI** and **DI** we get $q(x-y) \succsim_{R'} 0$ for any nonnegative rational number q. Let (q_n) be a nonnegative sequence of rationals converging to α. The sequence $q_n(x-y)$ converges to $\alpha(x-y)$. On the other hand, $q_n(x-y) \in P = \{z \in Y : z \succsim_{R'} 0\}$ and P is closed since $\tau_+(R') \subset t$. Thus, the limit of the sequence $(q_n(x-y))$, which is $\alpha(x-y)$, belongs to P. As a result $\alpha(x-y) \succsim_{R'} 0$. What yields, by **TI**, $\alpha x \succsim_{R'} \alpha y$.□

An immediate consequence of corollary 2 is the following:

Corollary 3: Let R be a complete preorder on Y, a real normed vector space**,** satisfying **TI** and $\tau_+(R) \subset t$, where t is the norm topology of Y. Then R satisfies **SI**.

Remark 6: $\tau_+(R) \subset t$ is a continuity requirement. Under that continuity requirement and **TI**, **SI** is, in a sense, satisfied since every complete preorder extending the original preorder and satisfying the same axiom of continuity and **TI** must satisfy **SI**.◊

Remark 7: Demuynck-Lauwers 2009 showed that a given preorder satisfying **TI** and **SI** can be extended into a complete preorder satisfying **TI** and **SI**. Corollary 2 shows that if, in addition, the initial preorder satisfies upper semicontinuity, then it admits an extension which also satisfies upper semicontinuity in addition to the axioms **TI** and **SI**.$\Diamond$

Remark 8: On the relationship between **SI** and continuity, while corollary 3 presents **SI** as a consequence of **TI** and a condition of continuity, Weibull 1985 theorem A has shown that under conditions **TI**, **SI**, and two other conditions C4 and C5, a complete preorder verifies a strong condition of continuity that results in continuous representability, i.e. the existence of a real-valued order-preserving continuous function. On the other hand, Mitra-Ozbek 2013 introduce another continuity condition called *scalar continuity* which, under monotonicity (see definition in section 6), implies representability (Mitra-Ozbek 2013, proposition 2). If we add condition **TI**, it is not difficult to see that **SI** follows, as well as Weibull 1985 condition C4. Proposition 3 below gives a weaker continuity condition which turns out to be equivalent to **SI**.$\Diamond$

Example 3: Consider the following relation SC' on $\mathbb{R}^{\mathbb{N}_0}$:

$$x \succsim_{SC'} y \text{ iff } \exists k \in \mathbb{N}_0, \liminf_n \sum_{i=1}^{kn} (x_i - y_i) \geq 0$$

SC' is a preorder verifying strong Pareto and fixed-step anonymity. In their lemma 3, Fleurbaey-Michel 2003 show that SC' is continuous with respect to the l_1-norm.

Proposition 2: There exists a complete preorder SC'' on $\mathbb{R}^{\mathbb{N}_0}$, of which SC' is a subrelation, and which is translation-invariant, strong Pareto, fixed–step-anonymous and upper semi-continuous with respect to the l_{1-}topology.

Proof: The l_{1-}topology is induced by the l_1-norm: $\sum_{i=1}^{+\infty} |x_i|$. By theorem 2, there exists a translation-invariant, upper-semi-continuous and complete preorder SC'' on $\mathbb{R}^{\mathbb{N}_0}$ of which SC' is a subrelation. SC' being a subrelation to SC'' entails that SC'' satisfies strong Pareto and fixed–step anonymity. SC'' is the required preorder.$\Box$

Remark 9: In accordance with remark 4, proposition 2 holds if we replace "upper-semi-continuous" by "lower-semi-continuous".$\Diamond$

5- Scalar Invariance and the Continuous Linear Weak Representation

Let Y be a real vector space and R a preorder on Y satisfying **TI**. Remark 6 and Remark 8 show that, under **TI**, there is a relationship between **SI** and the concept of continuity. Notice that theorem 2 of Mitra-Ozbek 2010 also suggests that[4]. However, under **TI,** although continuity according to the norm topology implies **SI** (corollaries 2 and 3), this is not true for l_1-topology as shown in the following example.

[4] In the finite-dimensional case, they show that **TI,** strong Pareto and *minimal individual symmetry* are not sufficient to warrant linear representability, hence continuity. To that aim, they construct an interesting example of a complete preorder on $\mathbb{R}^n$, $n \geq 2$ satisfying these axioms but not **SI** (Mitra-Ozbek 2010, section 3).

Example 4: A translation-invariant preorder, l_1-continuous but not scalar-invariant. We build a preorder on $Y = \mathbb{R}^{\mathbb{N}_0}$, in the manner of Svensson 1980. Consider the equivalence relation $r : x \Longleftrightarrow_r y$ iff $\sum_{i=1}^{+\infty} |x_i - y_i| < +\infty$. Denote $x_\alpha = (\alpha, \alpha, ...)$ where $\alpha \in \mathbb{R}$. Each x_α can be considered as a representative of a distinct equivalence class X_α. Denote (X_β) the remaining equivalence classes and for each X_β, denote x_β a representative. Consider the set $M = \{x_\alpha...\} \cup \{x_\beta...\}$. As the addition of equivalence classes for the relation r is well defined, let R be the preorder on M defined by $x \precsim_R y$ if there is $\alpha \in \mathbb{R}$ such that $\alpha \precsim_\pi 0$ and $x - y = x_\alpha$ (see example 1 for the definition of $\precsim_\pi$). R is **TI.** According to theorem 1 there is a complete preorder R' on M satisfying **TI** of which R is a subrelation. Define a complete preorder R_γ on each equivalence class X_γ (where $\gamma = \alpha$ or β) exactly as in Svensson 1980 page 1255, (iv). x, y being two elements of Y, denote $\widetilde{x}, \widetilde{y}$ the representatives in M of their respective classes. Consider the preorder R'' defined as follows: (i) (if $\widetilde{x} \neq \widetilde{y}$ then $x \succsim_{R''} y$ if $\widetilde{x} \succsim_{R'} \widetilde{y}$), (ii) (if $\widetilde{x} = \widetilde{y} = x_\gamma$ then $x \succsim_{R''} y$ if $x \succsim_{R_\gamma} y$). R'' is **TI.** As in Svensson 1980, one shows that R'' is continuous with respect to the $l1-$topology. However R'' is not **SI.**

It can thus be said that continuity with respect to the l_1-topology is not a sufficiently strong requirement to guarantee **SI**. Conversely, we may try to find out what level of continuity is verified if **SI** is verified. For example, the lexicographic order on $Y = \mathbb{R}^2$ is translation-invariant and scalar-invariant but it is not continuous. If Y is the space of the real bounded sequences l_∞, the preorder SC' is translation-invariant and scalar-invariant but it is not continuous under the norm $\|x\| = \sup |x_i|$.

The following notion of continuity is proposed:

Linear Continuity: A preorder on Y satisfies *linear continuity* if the preorder induced on every straight line of Y equipped with the canonical topology of the real line, is continuous.

Proposition 3: Let R' be a complete and translation-invariant preorder on Y. R satisfies **SI** iff R satisfies linear continuity.

Proof: If R' satisfies linear continuity, take a nonzero vector u in $P = \{v : v \succsim_{R'} 0\}$. Using **TI** and **DI** we show that for every nonnegative rational q we have $qu \succsim_{R'} 0$. By continuity of the induced order on the straight line generated by u, we obtain that for every nonnegative real α we have $\alpha u \succsim_{R'} 0$. Thus, for any x, y in Y such that $x \succsim_{R'} y$, we can take $u = x - y$ and conclude that $\alpha x \succsim_{R'} \alpha y$ for every nonnegative real α. This establishes that R' satisfies **SI**. Conversely,if R' satisfies **SI**, take a straight line D in Y. There is two possible situations. Either $u \sim_{R'} 0$ for all $u \in D$ or there is $u \in D$ such that $u \succ_{R'} 0$. In the first situation, the induced order on D is indifference. It is continuous. In the second situation, by **SI**, for all real δ we have $\delta u \succsim_{R'} 0 \Leftrightarrow \delta \geq 0$. Let $\lambda_n \to \lambda$ in $\mathbb{R}$ and $\lambda_n u \succsim_{R'} \lambda' u$ for all n. By **TI**, $(\lambda_n - \lambda')\, u \succsim_{R'} 0$ and, consequently, by **SI**, $\lambda_n - \lambda' \geq 0$. We deduce that $\lambda - \lambda' \geq 0$ and $\lambda u \succsim_{R'} \lambda' u$. The induced order on D is thus upper-semi-continuous. We show in the same way that it is lower-semi-continous.□

Remark 10: Proposition 3 remains valid for a non-complete preorder verifying **TI** and **DI**.$\Diamond$

Norm continuity (i.e. continuity with respect to the norm topology) implies *scalar continuity* as defined by Mitra-Ozbek 2013, which, in turn, implies linear continuity. A preorder may satisfy **SI** without norm continuity, as SC' in l_∞, or without scalar continuity, as the lexicographic order in $\mathbb{R}^2$. But of course both of them satisfy linear continuity since they satisfy **SI**.

We would now like to know to what extent we can approach a translation-invariant preorder by a coarser preorder and respecting **TI** and **SI**. A preorder R_1 is said to be finer than a preorder R_2, or R_2 coarser than R_1, if $x \succsim_{R_1} y \Rightarrow x \succsim_{R_2} y$. As a first step, the following theorem establishes the existence of a continuous linear *weak representation*[5]. Let $P_+(R) = \{x \in Y : x \succ_R 0\}$ and $P'_+(R) = \{x \in Y : \text{ for all positive real } \lambda, \lambda x \succ_R 0\}$. We have $P'_+(R) \subset P_+(R)$. Moreover, if $P'_+(R)$ is not empty, it is stable by positive scalar multiplication and by addition. Thus, it is a convex cone.

Theorem 3: Let Y be a normed vector space and R' be a complete translation-invariant preorder on Y. Suppose that the interior of $P'_+(R')$, denoted $\overset{o}{P'_+}(R')$, is not empty. Then there exists a non-zero, continuous linear functional φ on Y such that for all x, y in Y, $\varphi(x) > \varphi(y) \Rightarrow x \succ_{R'} y$.

Proof: The proof is similar to that of theorem 4 in Mabrouk 2011, except that the set Q is here $\cup_{x \succsim_{R'} 0} \left(x + \overset{o}{P'_+}(R') \right)$ and the condition *super weak Pareto* is replaced by the condition $\overset{o}{P'_+}(R') \neq \varnothing$. The argument is based on the geometrical form of Hahn-Banach theorem who asserts the existence of a continuous linear functional on a real normed vector space which separates a convex set having a non-empty interior from a point out of the interior of the convex set. First we show that Q is open and convex. $x + \overset{o}{P'_+}(R')$ is open as a translation of a non-empty open set. Q is open as a union of open sets. Since the sets $\{x : x \succsim_{R'} 0\}$ and $\overset{o}{P'_+}(R')$ are closed under addition, Q is closed under addition. It remains to show that for all positive real μ and $x + p$ in Q, we have $\mu(x + p) \in Q$. Let k_n, m_n two sequences of positive integers such that $\lim \frac{k_n}{m_n} = \mu$. Let $p_n = \left(\mu - \frac{k_n}{m_n} \right) x + \mu p$. We have $\mu p \in \overset{o}{P'_+}(R')$ [6] and $\lim p_n = \mu p$. Thus, $\overset{o}{P'_+}(R')$ being open, there is an integer N such that p_N is in $\overset{o}{P'_+}(R')$. Since $\mu(x + p) = \frac{k_N}{m_N} x + p_N$ and $\frac{k_N}{m_N} x \succsim_{R'} 0$ and $p_N \in \overset{o}{P'_+}(R')$, we have $\mu(x + p) \in Q$. This proves that Q is convex. By invoking Hahn-Banach theorem, there is a continuous linear functional φ which separates Q from 0, i.e. $\varphi(x) > 0$ for all x in Q. Now take x, y in Y such that $x \succsim_{R'} y$

[5] According to the terminology in Mitra-Ozbek 2013.

[6] Clearly, $\mu p \in P'_+(R')$. It is an interior point of $P'_+(R')$ because if B is an open sphere of center p and radius τ and $B \subset P'_+(R')$, then $B' = \mu B$ is an open sphere of center μp and radius $\mu\tau$ and $B' \subset P'_+(R')$.

and p in $\overset{o}{P'_+}(R)$. Let (α_n) be a sequence of positive reals decreasing to 0. We have $x - y + \alpha_n p \in Q$. Thus $\varphi(x - y + \alpha_n p) > 0$. The continuity of φ yields $\lim_n \varphi(x - y + \alpha_n p) = \varphi(x - y) \geq 0$. We have shown that for all x, y in Y, $x \succsim_{R'} y$ implies $\varphi(x) \geq \varphi(y)$. Consequently, $\varphi(x) > \varphi(y)$ implies $x \succ_{R'} y$.□

Remark 11: φ is unique up to a positive multiplicative factor. Indeed, let φ_1, φ_2 be two continuous linear weak representations of R'. We have

$$\varphi_1(x) > \varphi_1(y) \Rightarrow x \succ_{R'} y \Rightarrow \varphi_2(x) \geq \varphi_2(y) \tag{2}$$

It is known that the sets $\ker \varphi_1 = \{x : \varphi_1(x) = 0\}$ and $\ker \varphi_2 = \{x : \varphi_2(x) = 0\}$ are hyperplanes. Taking $y = 0$ in (2) yields $\varphi_1(x) > 0 \Rightarrow \varphi_2(x) \geq 0$. If $\ker \varphi_1$ were different from $\ker \varphi_2$, there would exist h_1 in $\ker \varphi_1$ such that $\varphi_2(h_1) > 0$ and h_2 in $\ker \varphi_2$ such that $\varphi_1(h_2) > 0$. Hence $\varphi_1(h_2 - h_1) = \varphi_1(h_2) > 0$ and $\varphi_2(h_2 - h_1) = -\varphi_2(h_1) < 0$. This contradicts the implication $\varphi_1(x) > 0 \Rightarrow \varphi_2(x) \geq 0$. Consequently, $\ker \varphi_1 = \ker \varphi_2$. Denote $H = \ker \varphi_1 = \ker \varphi_2$. Let $v \notin H$. Every x in Y can be written in a unique manner $x = \alpha v + u$, where $u \in H$. Thus

$$\varphi_2(x) = \varphi_2(\alpha v + u) = \alpha \varphi_2(v) = \frac{\varphi_2(v)}{\varphi_1(v)} \varphi_1(\alpha v) = \frac{\varphi_2(v)}{\varphi_1(v)} \varphi_1(\alpha v + u) = \frac{\varphi_2(v)}{\varphi_1(v)} \varphi_1(x)$$

The factor $\frac{\varphi_2(v)}{\varphi_1(v)}$ is positive since φ_1, φ_2 have always the same sign.◊

6- Pareto and Monotonicity Axioms

We are now in the space $l_\infty^r = \left\{(x_1, x_2, ...) : x_i \in \mathbb{R} \text{ and } \sup |x_i| e^{-ri} < +\infty\right\}$, where r is a real. The norm is $\|x\| = \sup |x_i| e^{-ri}$. This space is suitable for studying economic decisions in discrete time, infinite horizon and exponentially growing economy (if $r > 0$). Denote $l_{\infty+}^r = \{x \in l_\infty^r : \text{for all } i, x_i > 0\}$. If $r = 0$, the economy remains bounded. Let R' be a translation-invariant and complete preorder on l_∞^r. Theorem 4 of Mabrouk 2011 and the present theorem 3 differ in that the axiom *super weak Pareto* used in theorem 4 of Mabrouk 2011, is replaced by the assumption $\overset{o}{P'_+}(R') \neq \varnothing$ in theorem 3. Proposition 4 below shows that the former condition is stronger than the latter when the space is l_∞^r. The present formulation is therefore more general. In addition, we are interested in the condition of monotonicity because, on the one hand, it is a minimum requirement of efficiency for any preorder intended to rank economic alternatives. On the other hand, it turns out that in l_∞^r, monotonicity fulfills the sufficient condition of theorem 3 for the existence of a weak representation (proposition 5).

Axiom *Super Weak Pareto*: if $\inf(x_i - y_i)e^{-ri} > 0$ then $x \succ_{R'} y$ (or $\varphi(x) > \varphi(y)$ if the axiom is applied to a functional φ).

Axiom *Monotonicity*: If $x_i - y_i \geq 0$ for all i, then $x \succsim_{R'} y$ (or $\varphi(x) \geq \varphi(y)$ if the axiom is applied to a functional φ).

Proposition 4: Let R' be a translation-invariant, super weak Pareto and complete preorder on l_∞^r. Then $\overset{o}{P'_+}(R') \neq \varnothing$.

Proof: The interior of $l^r_{\infty+}$ is $\overset{o}{l^r_{\infty+}} = \{x \in l^r_\infty : \inf x_i e^{-ri} > 0\}$ (see Mabrouk 2011, page 8). Super weak Pareto implies that for all x in $\overset{o}{l^r_{\infty+}}$ and all $\alpha > 0$, we have $\alpha x \succ_{R'} 0$. Thus $x \in P'_+(R')$. Consequently $\overset{o}{l^r_{\infty+}} \subset P'_+(R')$, what yields $\overset{o}{l^r_{\infty+}} \subset \overset{o}{P'_+}(R')$. So $\overset{o}{P'_+}(R') \neq \varnothing$.□

Proposition 5: Let R' be a non-constant, translation-invariant, monotone and complete preorder on l^r_∞. Then $\overset{o}{P'_+}(R') \neq \varnothing$.

Proof: R' non-constant and translation-invariant entails that $P_+(R') \neq \varnothing$. Let $x \in P_+(R')$ and $\varepsilon > 0$. Monotonicity entails that the vector y defined by $y_i = x_i + \|x\| e^{ri} + \varepsilon$ is in the set $A = P_+(R') \cap l^r_{\infty+}$. Hence $A \neq \varnothing$. For all x in A, the set $x + l^r_{\infty+}$ is included in A. Consequently, the interior of $x + l^r_{\infty+}$, which is $x + \overset{o}{l^r_{\infty+}}$, is included in $\overset{o}{A}$ and we have $\overset{o}{A} \neq \varnothing$. By monotonicity, for all x in A and all real $\lambda > 1$, we have $\lambda x \succsim_{R'} x \succ_{R'} 0$. Let $x \in A$. The set $a = \{\alpha > 0 : \forall \lambda \geq \alpha, \lambda x \succ_{R'} 0\}$ contains 1 and admits 0 as lower bound. Thus it admits an infimum, say α_0. Suppose $\alpha_0 > 0$. Let $\beta \in]0, \alpha_0[$. We would have $\beta \notin a$. In other words, there exists $\lambda' \geq \beta$ such that $\lambda' x \precsim_{R'} 0$. By **TI**, for all positive rational q, we would have $q\lambda' x \precsim_{R'} 0$. We may choose q such that $q\lambda' > 1$. However the inequality $q\lambda' x \precsim_{R'} 0$ contradicts the fact that for all real $\lambda > 1$, we have $\lambda x \succ_{R'} 0$. Consequently, $\alpha_0 = 0$. In other words, for all $\lambda > 0$, we have $\lambda x \succ_{R'} 0$. Thus $A \subset P'_+(R')$. What entails $\overset{o}{A} \subset \overset{o}{P'_+}(R')$. Since $\overset{o}{A} \neq \varnothing$, we must have $\overset{o}{P'_+}(R') \neq \varnothing$.□

The following lemma is needed to prove corollary 4.

Lemma 2: Let φ be a nonzero linear functional on l^r_∞. Then φ is monotone iff φ is super weak Pareto.

Proof: (i) non super weak Pareto $\Rightarrow$ non monotone: φ nonzero entails that there is some p in l^r_∞ such that $\varphi(p) > 0$. φ non-super-weak-Pareto $\Rightarrow$ there is some u in l^r_∞ such that $\inf u_i e^{-ri} > 0$ and $\varphi(u) \leq 0$. Let $\delta = \frac{1}{2}\frac{\inf u_i e^{-ri}}{\|p\|}$. We check easily that $v_i = u_i - \delta p_i > 0$ for all i. But $\varphi(v) = \varphi(u) - \delta\varphi(p) < 0$. All components of v are positive and its image is negative. φ is not monotone. (ii) non monotone $\Rightarrow$ non super weak Pareto: There is some u in l^r_∞ such that $u_i \geq 0$ and $\varphi(u) < 0$. If there is no p in l^r_∞ such that $\inf p_i e^{-ri} > 0$ and $\varphi(p) > 0$, then φ is not super weak Pareto. If there is such a vector p in l^r_∞, let $\delta = -\frac{\varphi(u)}{\varphi(p)}$ and $v = u + \delta p$. We have $\inf v_i e^{-ri} = \inf(u_i + \delta p_i)e^{-ri} > 0$ and $\varphi(v) = \varphi(u) + \delta\varphi(p) = 0$. Thus φ is not super weak Pareto.□

Corollary 4: Let R' be a non-constant, translation-invariant and complete preorder on l^r_∞. If R' is monotone, R' is super weak Pareto.

Proof: According to proposition 5, R' non-constant, translation-invariant, monotone and complete implies $\overset{o}{P'_+}(R') \neq \varnothing$. According to theorem 3, there exists a non-zero, continuous linear functional φ on Y such that for all x, y in l^r_∞, $\varphi(x) > \varphi(y) \Rightarrow x \succ_{R'} y$, or, equivalently, $x \succsim_{R'} y \Rightarrow \varphi(x) \geq \varphi(y)$. Consequently, If x, y are such that $x_i - y_i \geq 0$ for all i, then $x \succsim_{R'} y$ and $\varphi(x) \geq \varphi(y)$. Consequently, φ is monotone. By lemma 2, φ is super-weak-

Pareto. If $\inf(x_i - y_i)e^{-ri} > 0$ then $\varphi(x) > \varphi(y)$ and $x \succ_{R'} y$.□

In short, for a non-constant, translation-invariant and complete preorder R' on l_∞^r we have the following implications:

$$R' \text{ monotone} \Rightarrow R' \text{ super weak Pareto} \Rightarrow \overset{o}{P'_+}(R') \neq \varnothing$$

Remark 12: These implications and theorem 3 show that a monotone and non-constant preorder on l_∞^r has a weak representation. So we find proposition 1 of Mitra-Ozbek 2013, in another context and following a different path.◊

Remark 13: The results of this section hold if we replace the space l_∞^r by the finite dimensional space $\mathbb{R}^n$. In this case the axiom super weak Pareto amounts to the axiom *weak Pareto* defined as follows: if for all $i, x_i > y_i$ then $x \succ_{R'} y$.◊

Remark 14: Monotonicity implies super weak Pareto for a non-constant translation-invariant and complete preorder but the converse is not true. Consider the following preorder R on l_∞^r : the lexicographic combination of $(\varphi, -x_1)$ where φ is a positive linear limit (see definition and existence in Mabrouk 2011, pages 8 and 9). R is defined by $x \succsim_R y$ if (i) $\varphi(x) > \varphi(y)$ or (ii) $\varphi(x) = \varphi(y)$ and $-x_1 \geq -y_1$. R inherits super-weak-Pareto from φ. But $(-1, 0, 0...) \succ_R (0, 0, 0...)$, what shows that R' is not monotone.◊

Remark 15: Proposition 4 and proposition 5 hold if the preorder is not complete. Concerning theorem 3, if the preorder is not complete, one can only assert the existence of a continuous linear functional φ such that $\varphi(x) > \varphi(y)$ implies non$(x \precsim_R y)$, where R is the preorder in question. Moreover, φ may not be unique as shown in the following preorder R on $\mathbb{R}^2$: $x \succsim_R y$ iff $x_1 \geq y_1$ and $x_2 \geq y_2$. Every functional of the form $\varphi(x) = a.x_1 + b.x_2$, where a, b are two positive reals, satisfies $x \succsim_R y \Rightarrow \varphi(x) \geq \varphi(y)$.◊

Remark 16: Let R be a non-constant, monotone and translation-invariant preoder that is not complete. Every translation-invariant preorder R' of which R is a subrelation inherits monotonicity. As a result, by corollary 4, R' is super weak Pareto. Hence, one may say that super weak Pareto is in a sense satisfied since every translation-invariant and complete preorder extending the original preorder must satisfy super weak Pareto.◊

7- Linear Continuous Lexicographic Decomposition:

Let Y be a real normed vector space. If R_1 and R_2 are two preorders on Y, denote (R_1, R_2) their lexicographic combination defined as in Remark 14.

Linear Continuous Lexicographic Combination **(LCLC):** A preorder R on Y is said to be a **LCLC** if there exists a sequence, finite or infinite, of continuous linear functionals $(\varphi_1, \varphi_2, ...)$ on Y such that $R = \left(\succsim_{\varphi_1}, \succsim_{\varphi_2}, ...\right)$ where $\succsim_{\varphi_n}$ is the preorder defined by $x \succsim_{\varphi_n} y$ iff $\varphi_n(x) \geq \varphi_n(y)$.

Notice that what is important for the preorder is the value of φ_n on the subspace H_{n-1} defined by $H_0 = Y$ and $H_n = \ker \varphi_n \cap H_{n-1}$ for $n \geq 1$. If R is non-constant and if φ_n is zero on H_{n-1}, it plays no role in the definition of R and can be wiped out. Hence, if R is non-constant we will suppose that every functional φ_n is non-zero on H_{n-1}. By convention, if R is constant, the sequence (φ_n) amounts to φ_1 which is zero. A **LCLC** is complete, translation-invariant and scalar-invariant**.** By proposition 3, a **LCLC** is linear-continuous. But a linear-continuous translation-invariant and scalar-invariant preorder is not necessarily a **LCLC** even if it is complete (see the example of section 8 and particularly remark 19). It should be noted that a **LCLC** is linear-continuous but generally not continuous, as the standard lexicographic preorder on $\mathbb{R}^2$.

Lemma 3: If L is a **LCLC** on a real vector space Z, then the preorder L' induced by L on a subspace Z' of Z is also a **LCLC.**

Proof: Obvious. However it must be noticed that the sequence of linear functionals may not be the same if one of the functionals is zero on its new corresponding subspace. In that case, that functional is wiped out.□

Lemma 4: If a **LCLC** L on a real vector space Z is not constant. Then

$$\overset{o}{P'_+}(L) = \{x \in Z : \varphi(x) > 0\}$$

where φ is the functional defining the first component of L.

Proof: L not constant entails that φ is non zero. Clearly $\{x \in Z : \varphi(x) > 0\} \subset P_+(L) = P'_+(L)$. Since $\{x \in Y : \varphi(x) > 0\}$ is open, $\{x \in Y : \varphi(x) > 0\} \subset \overset{o}{P'_+}(L)$. Conversely, every point of $P'_+(L)$ satisfies $\varphi(x) \geq 0$. If $\varphi(x) = 0$, let u be such that $\varphi(u) > 0$ and α a positive real. Denote $y = x - \alpha u$. We have $\varphi(y) = \varphi(x - \alpha u) = -\alpha\varphi(u) < 0$. Thus $y \prec_L 0$. For a given neighborhood of x, one can make α as small as necessary for y to be in that neighborhood. This proves that if $\varphi(x) = 0$, x in not in the interior of $P'_+(L)$. Therefore $\overset{o}{P'_+}(L) = \{x \in Y : \varphi(x) > 0\}$.□

Lemma 5: Let L_1, L_2 be two **LCLC**s on a real vector space Z such that L_2 is coarser than L_1. Then either L_2 is constant or it has the same first component than L_1.

Proof: If L_1 is constant, L_2 must obviously be constant. If L_1 is not constant and L_2 is constant, there is nothing to prove. The remaining case is L_1 and L_2 not constant. Let φ_1 be the first component of L_1 and φ_2 be the first component of L_2. According to lemma 4 $\overset{o}{P'_+}(L_j) = \left\{x \in Z : \varphi_j(x) > 0\right\}$, $j = 1, 2$. Moreover L_2 coarser than L_1 entails $P'_+(L_2) \subset P'_+(L_1)$. Consequently $\overset{o}{P'_+}(L_2) \subset \overset{o}{P'_+}(L_1)$. What yields $\{x \in Z : \varphi_2(x) > 0\} \subset \{x \in Z : \varphi_1(x) > 0\}$. Taking the closure of these two spaces: $\{x \in Z : \varphi_2(x) \geq 0\} \subset \{x \in Z : \varphi_1(x) \geq 0\}$. We must also have $\{x \in Z : \varphi_1(x) \geq 0\} \subset \{x \in Z : \varphi_2(x) \geq 0\}$ because if there was $y \in Y$ such that $\varphi_1(y) \geq 0$ and $\varphi_2(y) < 0$, we would have $\varphi_1(-y) \leq 0$ and $\varphi_2(-y) > 0$, what would contradicts $\{x \in Z : \varphi_2(x) \geq 0\} \subset \{x \in Z : \varphi_1(x) \geq 0\}$. Consequently $\{x \in Z : \varphi_2(x) \geq 0\} = \{x \in Z : \varphi_1(x) \geq 0\}$. With the same argument as in Remark 11, we conclude that $\varphi_1 = \varphi_2$ up to a positive multiplicative factor.□

Theorem 4: Let R be a complete translation-invariant preorder on Y. Consider the sequences of preorders $(R_n)_{1 \leq n \leq 1+n_{\max}}$, functionals $(\varphi_n)_{1 \leq n \leq n_{\max}}$ and subspaces $H_0 = Y$ and $(H_n)_{1 \leq n \leq n_{\max}}$ built by a successive application of theorem 3 as follows:

- stage 1: $R_1 = R$. If $\overset{o}{P'_+}(R) = \varnothing$, then $\varphi_1 = 0$, $n_{\max} = 1$, $H_1 = H_0$ and the construction of the sequences $(R_n), (\varphi_n)$ and (H_n) stops. If $\overset{o}{P'_+}(R) \neq \varnothing$, then φ_1 is the functional, unique up to a positive multiplicative factor, given by theorem 3 applied to the preorder R_1 on $H_0 = Y$, H_1 is defined as $H_1 = \ker \varphi_1 \cap H_0$ and R_2 is the preorder induced by R on H_1.

- stage 2: In the subspace H_1 equipped with the relative topology, if $\overset{o}{P'_+}(R_2) = \varnothing$, then $\varphi_2 = 0$, $n_{\max} = 2$, $H_2 = H_1$ and the construction of the sequences $(R_n), (\varphi_n)$ and (H_n) stops. If $\overset{o}{P'_+}(R_2) \neq \varnothing$, then φ_2 is the functional, unique up to a positive multiplicative factor, given by theorem 3 applied to the preorder R_2 on H_1 , $H_2 = \ker \varphi_2 \cap H_1$ and R_3 is the preorder induced by R on H_2.

...

- stage n: In the subspace H_{n-1} equipped with the relative topology, if $\overset{o}{P'_+}(R_n) = \varnothing$, then $\varphi_n = 0$, $n_{\max} = n$, $H_n = H_{n-1}$ and the construction of the sequences $(R_n), (\varphi_n)$ and (H_n) stops. If $\overset{o}{P'_+}(R_n) \neq \varnothing$, then φ_n is the functional, unique up to a positive multiplicative factor, given by theorem 3 applied to the preorder R_n on H_{n-1} , $H_n = \ker \varphi_n \cap H_{n-1}$ and R_{n+1} is the preorder induced by R on H_n.

...

Then, the **LCLC** $L(R) = \left(\succsim_{\varphi_1}, \succsim_{\varphi_2}, ..., \succsim_{\varphi_{n\max}}\right)$ [7], with $n_{\max}$ possibly infinite, is the finest **LCLC** coarser than R and every **LCLC** on Y coarser than R is either constant or of the form $\left(\succsim_{\varphi_1}, \succsim_{\varphi_2}, ..., \succsim_{\varphi_n}\right)$, with $n \leq n_{\max}$.

Proof: Let L' be a **LCLC** on Y coarser than R. By lemma 5, either L' is constant, or it has the same first component than $L(R)$. If $\varphi_1 \neq 0$, apply again lemma 5 to the preorders L'_2 and $L(R)_2$ induced respectively by L' and $L(R)$ on the space $H_1 = \ker \varphi_1 \cap H_0$ equipped with the relative topology. Again, either L'_2 is constant, or it has the same first component than $L(R)_2$, which is the second component of $L(R)$. Repeat this operation until the component of rank $n_{\max}$. This shows that $L' = \left(\succsim_{\varphi_1}, \succsim_{\varphi_2}, ..., \succsim_{\varphi_n}\right)$, with $n \leq n_{\max}$.□

$L(R)$ may be referred to as the *linear continuous lexicographic decomposition* of R. Although theorem 4 invokes several times theorem 3 which is non-constructive, it may be used along with other specific information to gain some insight on the preorder, as in section 8.

Corollary 5: Let R^1, R^2 be two non-constant **LCLC**s on Y. Denote $R^1 = \left(\succsim_{\varphi_1^1}, \succsim_{\varphi_2^1}, ... \succsim_{\varphi_{n_1}^1}\right)$ and $R^2 = \left(\succsim_{\varphi_1^2}, \succsim_{\varphi_2^2}, ... \succsim_{\varphi_{n_2}^2}\right)$. If R^2 is coarser than R^1, then $n_1 \geq n_2$ and for all $n \leq n_2$, $\varphi_n^1 = \varphi_n^2$ on the subspace H_{n-1}^1 (defined as in theorem 4 for the preorder R^1) up to a positive multiplicative factor. If $n_1 = n_2$ then $R^1 = R^2$.

Proof: It is a direct application of Theorem 4.□

[7] In fact, φ_n is well defined on H_{n-1} not on Y. To be fully in line with the definition of a **LCLC**, one should consider extensions of φ_n on Y. But this will play no role in the calculus of the preorder. In order not to overload the text unnecessarily, I keep φ_n in the definition of $L(R)$.

Remark 17: If the preorder is not complete, then the sequence

$$\left(\succsim_{\varphi_1}, \succsim_{\varphi_2}, ..., \succsim_{\varphi_{n\max}}\right)$$

may not be unique. See Remark 15.◊

Example 5: Consider the following linear functionals on $\mathbb{R}^3$, $\varphi_1(x) = x_1 + x_2 + x_3$, $\varphi_2(x) = -x_1$. Denote $\succsim_\pi^3$ the preorder $\succsim_\pi$(defined in example 1) applied to the third component. Consider the complete preorder on $\mathbb{R}^3$, $R = \left(\succsim_{\varphi_1}, \succsim_{\varphi_2}, \succsim_\pi^3\right)$. The first step is to determine $\overset{o}{P'_+}(R)$. We have $x \succ_R 0$ iff (i) $\varphi_1(x) > 0$ or (ii) $\varphi_1(x) = 0$ and $\varphi_2(x) > 0$ or (iii) $\varphi_1(x) = 0$ and $\varphi_2(x) = 0$ and $x \succ_\pi^3 0$. Therefore $x \in P'_+(R)$ iff (i) $\varphi_1(x) > 0$ or (ii) $\varphi_1(x) = 0$ and $\varphi_2(x) > 0$. Thus $\overset{o}{P'_+}(R) = \left\{x \in \mathbb{R}^3 : \varphi_1(x) > 0\right\}$. We can apply theorem 3. φ_1 is a continuous linear functional that separates $\overset{o}{P'_+}(R)$ from 0. Thus $H_1 = \left\{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\right\}$. The second step is to apply again theorem 3 in the vector space H_1 equipped with the induced topology. The induced preorder on H_1 is R_2 defined by $x \succsim_{R_2} y$ iff $x_1+x_2+x_3 = y_1+y_2+y_3 = 0$ and (i) $x_1 < y_1$ or (ii) $x_1 = y_1$ and $x_3 \succsim_\pi^3 y_3$. Thus $P'_+(R_2) = \{x \in H_1 : x_1 < 0\}$. This subspace being open in H_1, we have $\overset{o}{P'_+}(R_2) = P'_+(R_2)$. The continuous and linear functional φ_2 separates $\overset{o}{P'_+}(R_2)$ from 0. Thus $H_2 = \{x \in H_1 : x_1 = 0\}$. The induced order in H_2 is R_3 defined by $x \succsim_{R_3} y$ iff $x_1+x_2+x_3 = y_1+y_2+y_3 = 0$ and $x_1 = y_1 = 0$ and $x_3 \succsim_\pi^3 y_3$. Therefore $P'_+(R_3) = \varnothing$. The preorder $\left(\succsim_{\varphi_1}, \succsim_{\varphi_2}\right)$ is the finest **LCLC** coarser than R.

8- Decomposition of the Catching-up Preorder

8-1- Decomposition under the Supnorm

Consider the catching-up preorder C defined on l_∞ by $x \succsim_C y$ iff there is n_0 such that $n \geq n_0 \Rightarrow \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} y_i$. The norm is the supnorm $\|x\| = \sup |x_i|$. Because C is scalar-invariant, the set Q used in the proof of theorem 3 is equal to $\overset{o}{P_+}(C) = \overset{o}{P'_+}(C)$, that is, the interior of $\{x \in l_{+\infty} : x \succ_C 0\}$.

Proposition 7: $\overset{o}{P_+}(C) = \left\{x \in l_{+\infty} : \liminf \frac{1}{n} \sum_{i=1}^{n} x_i > 0\right\}$

Proof: Condition $x \succ_C 0$ is equivalently written

$$\exists n_0 \text{ such that } n \geq n_0 \Rightarrow \sum_{i=1}^{n} x_i \geq 0$$

$$\text{and for all } n \text{ there is } m \geq n \text{ such that } \sum_{i=1}^{n} x_i > 0$$

where n_0, n, m are positive integers. $x \in \overset{o}{P_+}(C)$ means that there is a positive real r such that the sphere of center x and radius r, $B(x, r)$, is in $P_+(C)$, that is, for all y in $l_{+\infty}$ such that $\|y - x\| \leq r$, we have

$$\exists n_0 \text{ such that } n \geq n_0 \Rightarrow \sum_{i=1}^{n} y_i \geq 0$$
$$\text{and for all } n \text{ there is } m \geq n \text{ such that } \sum_{i=1}^{n} y_i > 0$$

For x to be in $\overset{o}{P_+}(C)$, it is enough that there is $r > 0$ such that

$$\exists n_0 \text{ such that } n \geq n_0 \Rightarrow \sum_{i=1}^{n} (x_i - r) \geq 0$$
$$\text{and for all } n \text{ there is } m \geq n \text{ such that } \sum_{i=1}^{n} (x_i - r) > 0$$

This last condition is satisfied if $\liminf \frac{1}{n} \sum_{i=1}^{n} x_i > 0$. Conversely, suppose $\liminf \frac{1}{n} \sum_{i=1}^{n} x_i = r' > 0$. Let $r = \frac{r'}{2}$. Then, clearly, $B(x, r)$ is in $P_+(C)$. As a result, $\overset{o}{P_+}(C) = \left\{ x \in l_{+\infty} : \liminf \frac{1}{n} \sum_{i=1}^{n} x_i > 0 \right\}$.□

Denote $\overline{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $a = \{x \in l_\infty \text{ such that the sequence } \overline{x}_n \text{ converges}\}$. a is a subspace of l_∞. Denote $\overline{x}_\infty = \lim \overline{x}_n$ and α_∞ the linear functional on a defined by $\alpha_\infty(x) = \overline{x}_\infty$. Let $x \in \overset{o}{P_+}(C)$.

Let φ_1 be a non-zero continuous linear functional as given by theorem 3 and Remark 15. Denote c the set of converging sequences, l_1 the set of absolutely converging sequences and δ_∞ the linear functional on c defined by $\delta_\infty(x) = x_\infty = \lim x_n$.

Proposition 8 φ_1 is equal to α_∞ on a, up to a positive multiplicative factor.

Proof: There is unique linear continuous functionals φ_1^1 and φ_1^2 on l_∞ such that $\varphi_1 = \varphi_1^1 + \varphi_1^2$, with $\varphi_1^1 \in l_1$ and the restriction of φ_1^2 to c is proportional to δ_∞ (Yosida-Hewitt theorem[8]). In the other hand, the preorder C satisfies the axiom *finite anonymity* which states that the ranking of a sequence does not change if one permutes two terms of the sequence. Since the preorder $\succsim_{\varphi_1}$ is coarser than C, it inherits finite anonymity. Thus, the terms of the sequence φ_1^1 are equal. Since their limit is 0, we have necessarily $\varphi_1^1 = 0$ and $\varphi_1 = \varphi_1^2$. Consider the sequences y, z in a defined by $y_n = \liminf \overline{x}_k$ and $z_n = \limsup \overline{x}_k$ for all n. For all positive real ε, there is a positive integer n_0 such that $n \geq n_0 \Rightarrow \liminf \overline{x}_k - \varepsilon \leq \overline{x}_n \leq \limsup \overline{x}_k + \varepsilon$. Therefore $y - \varepsilon.e \precsim_C x \precsim_C z + \varepsilon.e$ where $e = (1, 1...)$, what entails $\varphi_1(y - \varepsilon.e) \leq \varphi_1(x) \leq \varphi_1(z + \varepsilon.e)$. Since φ_1 is proportional to δ_∞ on c, there is a real λ such that

$$\lambda(\lim y_n - \varepsilon) = \lambda(\liminf \overline{x}_k - \varepsilon) \leq \varphi_1(x) \leq \lambda(\lim z_n + \varepsilon) = \lambda(\limsup \overline{x}_k + \varepsilon)$$

[8]For the convenience of the reader, a statement and a proof of the theorem is given in the appendix with the objects and notations used in the present paper.

These inequalities being satisfies for all ε, we have $\lambda \liminf \overline{x}_k \leq \varphi_1(x) \leq \lambda \limsup \overline{x}_k$. We have $\lambda \geq 0$ because φ_1 inherits monotonicity from C and $\lambda \neq 0$ because φ_1 is non-zero. Hence, on a, φ_1 is equal to α_∞ up to a positive multiplicative factor.□

Remark 18: It is obvious that, on the set a, $\alpha_\infty(x) > \alpha_\infty(y) \Rightarrow x \succ_C y$. But it is less obvious that the restriction on a of every continuous linear functional that separates $\overset{o}{P_+}(C)$ from 0, is equal to α_∞ up to a positive multiplicative factor.◊

The second stage consist in studying the preorder C in $H_1 = \ker \varphi_1$, denoted C_2. We have $P_+(C_2) = \{x \in H_1 : x \succ_{C_2} 0\}$. We prove by contradiction that $\overset{o}{P_+}(C_2) = \varnothing$. Thus, by theorem 4, the preorder C has no second component.

Proposition 9: $\overset{o}{P_+}(C_2) = \varnothing$.

Proof: Suppose not. Let φ_2 be a non-zero continuous linear functional as given by theorem 3 and Remark 15 applied to C_2 in H_1. Denote s the vector subspace of H_1 : $s = \left\{ x \in H_1 : \ \lim \sum_1^n x_i \text{ exists} \right\}$ and σ the linear functional on s defined by $\sigma(x) = \lim \sum_1^n x_i$. Note that σ is not continuous. For all x, y in s, we have $\sigma(x) > \sigma(y) \Rightarrow x \succ_{C_2} y \Rightarrow \varphi_2(x) \geq \varphi_2(y)$. With the same reasoning than that of Remark 11 applied on σ and φ_2 in the subspace s, we arrive to the result that σ is equal to φ_2 on s, up to a positive multiplicative factor. But φ_2 is continuous and σ is not. A contradiction.□

8-2- Decomposition under a Stronger Norm

It is possible to propose a more precise decomposition of C by using a stronger norm, hence a stronger topology. However, this will be at the cost of a narrower domain than l_∞. For example let's take the norm $\|x\| = \sup |x_n| + \lim \sum_1^n |x_i - \overline{x}_i|$ defined on the vector space

$$c_1 = \left\{ x \in l_\infty : \sup |x_n| + \lim \sum_1^n |x_i - \overline{x}_i| < +\infty \right\}$$

For $n \geq 2$, we have $x_n = n\overline{x}_n - (n-1)\overline{x}_{n-1}$. Thus

$$x_n - \overline{x}_n = (n-1)\left(\overline{x}_n - \overline{x}_{n-1}\right) \tag{3}$$

Lemma 6: The space c_1 is included in the space c of converging sequences.

Proof: Denote $u_n = \overline{x}_n - \overline{x}_{n-1}$ for $n \geq 2$ and $u_1 = \overline{x}_1 = x_1$. We have $\sum_1^n |x_i - \overline{x}_i| = \sum_1^n (i-1)|u_i|$. Thus, $\sum_1^n |u_i| \leq \sum_1^n (i-1)|u_i| < +\infty$ and $\overline{x}_n = \sum_1^n u_i$ converges. Moreover, $\sum_1^n |x_i - \overline{x}_i| < +\infty$ entails $\lim(x_n - \overline{x}_n) = 0$. As a result, x_n converges.□

In exactly the same way as for the space l_∞ equipped with the supnorm, we prove that in c_1 equipped with the norm $\|x\| = \sup |x_n| + \lim \sum_1^n |x_i - \overline{x}_i|$, we have $P_+ \overset{o}{(C)} = \{x \in c_1 : \lim x_n > 0\}$. The linear functional $\delta_\infty(x) = \lim x_n$, or strictly speaking $\succsim_{\delta_\infty}$,is clearly the first component of the preorder C on the space c_1.

To determine the second component, φ_2, let's consider the preorder C_2 , restriction of C on $H_1 = \ker \delta_\infty$, and calculate $P_+ \overset{o}{(C_2)}$ in H_1.

Proposition 10: The second component of C is the functional σ defined on H_1 by $\sigma(x) = \sum_1^{+\infty} x_n$, up to a positive multiplicative factor.

Proof: In H_1 we have $\lim \overline{x}_n = \lim x_n = 0$. Together with (3), the equality $\overline{x}_n = \sum_{i=1}^n u_i$ yields:

$$\sum_{i=1}^n \overline{x}_i + \sum_{i=1}^n (i-1)\, u_i = n \sum_{i=1}^n u_i \tag{4}$$

On the other hand $\lim \sum_{i=1}^n u_i = \lim \overline{x}_n = 0$. Thus $\sum_{i=1}^n u_i + \sum_{i=n+1}^{+\infty} u_i = 0$. Replacing $\sum_{i=1}^n u_i$ by $-n \sum_{i=n+1}^{+\infty} u_i$ in (4), we get $\sum_{i=1}^n \overline{x}_i + \sum_{i=1}^n (i-1)\, u_i = -n \sum_{i=n+1}^{+\infty} u_i$. Furthermore $n \left| \sum_{n+1}^{+\infty} u_i \right| \leq n \sum_{n+1}^{+\infty} |u_i| \leq \sum_{n+1}^{+\infty} (i-1)\, |u_i| \to 0$ when $n \to +\infty$. Since $\sum_1^{+\infty} (i-1)\, |u_i| < +\infty$, equation (4) yields

$$\sum_1^{+\infty} \overline{x}_n + \sum_1^{+\infty} (n-1)\, u_n = 0 \tag{5}$$

It follows that $\sum_1^{+\infty} \overline{x}_n$ converges (i.e. it has a finite limit). Since $\sum_1^{+\infty} |x_n - \overline{x}_n| < +\infty$, $\sum_1^{+\infty} (x_n - \overline{x}_n)$ converges. Therefore the sum $\sigma(x) = \sum_1^{+\infty} x_n$ converges on H_1.

We prove now that σ is continuous on H_1. Let x, x' be in H_1 such that $x' \to x$. Using (5) and (3) we get

$$\begin{aligned} \left| \sum_1^{+\infty} (\overline{x}'_n - \overline{x}_n) \right| &= \left| \sum_{n=1}^{+\infty} (n-1)(u'_n - u_n) \right| \leq \sum_{n=1}^{+\infty} (n-1)\, |(u'_n - u_n)| \\ &= \sum_1^{+\infty} |(x'_n - \overline{x}'_n) - (x_n - \overline{x}_n)| = \sum_1^{+\infty} |(x'_n - x_n) - (\overline{x}'_n - \overline{x}_n)| \end{aligned}$$

By definition, $\sum_1^{+\infty} |(x'_n - x_n) - (\overline{x}'_n - \overline{x}_n)| \to 0$ when $x' \to x$.

Thus $\left|\sum_{1}^{+\infty}(\overline{x}'_n-\overline{x}_n)\right|\to 0$. Moreover

$$\begin{aligned}\left|\left|\sum_{1}^{+\infty}(x'_n-x_n)\right|-\left|\sum_{1}^{+\infty}(\overline{x}'_n-\overline{x}_n)\right|\right| &\leq \left|\sum_{1}^{+\infty}(x'_n-x_n)-\sum_{1}^{+\infty}(\overline{x}'_n-\overline{x}_n)\right| \\ &\leq \sum_{1}^{+\infty}\left|(x'_n-x_n)-(\overline{x}'_n-\overline{x}_n)\right|\to 0\end{aligned}$$

As a result, $\sum_{1}^{+\infty}x'_n\to\sum_{1}^{+\infty}x_n$ and σ is continuous on H_1. The set $\sigma^{-1}(]0,+\infty[)$ is clearly not empty, open and included in $P_+(C_2)$. Hence $P_+\overset{o}{(C_2)}\neq\varnothing$.

To prove the unicity of σ, notice that $\sigma(x)>\sigma(y)\Rightarrow x\succ_{C_2}y$. Let φ_2 be a non-zero continuous linear functional as given by theorem 3 and Remark 15. We thus have

$$\sigma(x)>\sigma(y)\Rightarrow x\succ_{C_2}y\Rightarrow\varphi_2(x)\geq\varphi_2(y)$$

Hence, the implication (2) holds and we can apply Remark 11. φ_2 is equal to σ on H_1 up to a positive multiplicative factor.□

The third stage consists in calculating $P_+\overset{o}{(C_3)}$ in $H_2=\ker\sigma$, where C_3 is the restriction of C to H_2. Note that $P_+(C_3)$ is not empty. For example the sequence $x_n=-\frac{1}{n(n-1)}$ for $n\geq 2$ and $x_1=1$ is in $P_+(C_3)$.

Proposition 11: $P_+\overset{o}{(C_3)}=\varnothing$.

Proof: Let x be in $P_+(C_3)$. We prove now that in every neighborhood of x in H_2 one can find y such that not$(y\succ_C 0)$. To that end, we build y by adding a small term such that the sum becomes episodically negative while remaining in H_2 and in the neighborhood of x. Since x is in $P_+(C_3)$, for all n there is an even integer $k>n$ such that $\sum_{1}^{k}x_i>0$. Denote $k(n)=k$. Let δ be a positive real and m and p be two positive integers such that $p>m$. There is an integer $p'>m$ such that $\left|\sum_{1}^{k(p')}x_i\right|<\frac{1}{2^p}$. Denote $k_m(p)=p'$. It is always possible to choose k and p' such that functions k and k_m are increasing. Define y^m as follows: if there is p such that $n=k(k_m(p))$ then $y_n^m=x_n-(1+\delta)\sum_{1}^{n}x_i$; if there is p such that $n=1+k(k_m(p))$ then $y_n^m=x_n+(1+\delta)\sum_{1}^{n-1}x_i$; else $y_n^m=x_n$. For $n=k(k_m(p))$ we have $\sum_{1}^{n}y_i^m=\sum_{1}^{n}x_i-(1+\delta)\sum_{1}^{n}x_i=-\delta\sum_{1}^{n}x_i<0$. Therefore, $\sum_{1}^{n}y_i^m$ is episodically negative, which yields not$(y^m\succ_C 0)$. We check easily that y^m is bounded and that $\sum_{1}^{+\infty}y_n^m=0$. Moreover, if there is p such that $n=k(k_m(p))$ then $y_n^m-\overline{y}_n^m=x_n-\overline{x}_n-(1-\frac{1}{n})(1+\delta)\left(\sum_{1}^{n}x_i\right)$; if there is p

such that $n = 1 + k(k_m(p))$ then $y_n^m - \overline{y}_n^m = x_n - \overline{x}_n - (1+\delta)\left(\sum_1^{n-1} x_i\right)$; else $y_n^m - \overline{y}_n^m = x_n - \overline{x}_n$. Therefore, for N sufficiently large:

$$\begin{aligned}\sum_1^N |y_n^m - \overline{y}_n^m| &\leq \sum_1^N |x_n - \overline{x}_n| + 2(1+\delta) \sum_{\substack{k(k_m(p))\leq N \\ m<p}} \left| \sum_1^{k(k_m(p))} x_i \right| \\ &\leq \sum_1^N |x_n - \overline{x}_n| + 2(1+\delta) \sum_{\substack{k(k_m(p))\leq N \\ m<p}} \frac{1}{2^p}\end{aligned}$$

Hence, $\sum_1^{+\infty} |y_n - \overline{y}_n|$ converges, what shows that y is in H_2. Using again the expression of $y_n - \overline{y}_n$, we get:

$$\begin{aligned}\sum_1^N |(y_n^m - x_n) - (\overline{y}_n^m - \overline{x}_n)| &\leq 2(1+\delta) \sum_{\substack{k(k_m(p))\leq N \\ m<p}} \left| \sum_1^{k(k_m(p))} x_i \right| \\ &\leq 2(1+\delta) \sum_{\substack{k(k_m(p))\leq N \\ m<p}} \frac{1}{2^p}. \text{ Thus} \\ \sum_1^{+\infty} |(y_n^m - x_n) - (\overline{y}_n^m - \overline{x}_n)| &\leq 2(1+\delta) \sum_{m<p} \frac{1}{2^p} \to 0 \text{ when } m \to +\infty\end{aligned}$$

and

$$\sup_n |y_n^m - x_n| = \sup_{n\geq m} |y_n^m - x_n| \leq \sup_{n\geq m} |y_n^m| + \sup_{n\geq m} |x_n| \to 0 \text{ when } m \to +\infty$$

□

Hence, by theorem 4, C has not a third component in c_1 equipped with the norm $\|x\| = \sup |x_n| + \lim \sum_1^n |x_i - \overline{x}_i|$.

Remark 19: Since either under the supnorm or c_1-norm, the decomposition of the catching-up preorder C is strictly coarser than C, so is the decomposition of every complete preorder extending C. Such a complete preorder provides an example of a complete, translation-invariant and scalar-invariant preorder which is not reducible to a **LCLC**.◊

Appendix: Decomposition of an element $y \in l_\infty^*$

The following is a statement and proof of the Yosida-Hewitt theorem without resorting to concepts related to measures.

Recall that δ_∞ is the linear functional defined on c, the space of real converging sequences, by $\delta_\infty(x) = \lim x_n$, and l_∞^* is the dual of l_∞, the space of real bounded sequences.

Theorem*:* Let $y \in l_\infty^*$. Then we can write in a unique manner:

$$y = y_1 + y_2$$

where $y_1$verifies:

$$\sum_{i=1}^{+\infty} |y_{1i}| \prec +\infty$$

and y_2 is such as its restriction to c is proportional to δ_∞.

Proof:

Step 1: Projection from l_∞^* on l_1. For $i \geq 1$, let e_i be the element of l_∞ such that all its components are zero except the i^{th} which is 1. Let $y \in l_\infty^*$. Let $x \in c_0$. We have $\sum_1^n x_i e_i \longrightarrow x$, so $y \mid (\sum_1^n x_i e_i) \longrightarrow y \mid x$, then $\sum_1^{+\infty} x_i \left(y \mid e_i\right) = y \mid x$. One the other hand, y continuous$\Leftrightarrow \frac{|y|x|}{\|x\|} \leq \|f\|$. Since $\|e_i\| = 1$, we get $|(y \mid e_i)| \leq \|y\|$ for all $i \geq 1$. Let $\alpha \in]0,1[$. Take $x_n = sign(y \mid e_n) \cdot \frac{1}{n^\alpha}$. We have $x = (x_n)_{n \geq 1} \in c_0$ and

$$\sum_1^{+\infty} \frac{|y \mid e_n|}{n^\alpha} = |f(x)| \leq \|x\| . \|f\| = \|f\|$$

Let $\varphi(\alpha) = \sum_1^{+\infty} \frac{|y|e_n|}{n^\alpha}$. Then φ is bounded and decreasing on $]0,1[$. Hence, it has a finite limit as $\alpha \longrightarrow 0$. We can show easily that this limit is $\sum_1^{+\infty} |(y \mid e_n)|$. Thus the sequence $(y \mid e_n)_{n \geq 1}$ is in l_1. Denote Φ the mapping from l_∞^* to l_1 which associates to y the sequence $(y \mid e_i)_{i \geq 1}$. Φ is a projection from l_∞^* to l_1. Indeed, it is a linear transformation and, considering $l_1$as a subset of l_∞^*, if $y \in l_1$then $\Phi\left(y\right) = y$.

Step 2: Decomposition of an element $y \in l_\infty^*$ by Φ. Consider the mapping Identity I from c_0 to l_∞

$$I : c_0 \underset{x \longrightarrow x}{\longrightarrow} l_\infty$$

We can verify easily that Φ is the adjoint operator of I, what we write: $\Phi = I^*$. Given that the adjoint of a continuous linear operator is continuous and that I is linear and continuous, Φ is a continuous linear operator. Furthermore, we have (Luenberger p155) $R(I)^\perp = N(I^*)$ where $R(I) = \{y \in l_\infty \diagup \exists x \in c_0 : I(x) = y\} = c_0$ and $N(I^*) = \{x \in c_0 \diagup I^*(x) = 0\}$ which means $N(\Phi) = c_0^\perp$. For $y \in l_\infty^*$, define $k = \Phi(y) - y$. We can write $y = \Phi(y) + k$, with $\Phi(y) \in l_1$ and $k \in c_0^\perp$. We have decomposed an element y of l_∞^* as a sum of an element of l_1 and an element of $c_0^\perp$. We easily show that this decomposition is unique.

Step 3: Study of $c_0^{\perp}$. We have:

$$\|\delta_\infty\| = \sup_{x\in c}\frac{|\lim x_n|}{\|x\|} = \sup_{x\in c}\frac{|\lim x_n|}{\sup|x_n|} = 1$$

and

$$\forall\alpha \in R : \|\alpha\delta_\infty\| = |\alpha| \, \|\delta_\infty\| = |\alpha|$$

so we can apply Hahn-Banach theorem, and extend $\alpha\delta_\infty$ with an element of l_∞^*, say β. Denote B the set of such linear functionals. We now show that $c_0^{\perp} = B$. We see easily that B is a vector subspace of l_∞^* included in $c_0^{\perp}$. Reciprocally, let $\beta \in c_0^{\perp}$ and $x \in c$. Denote $e = (1, 1\cdots)$. We have $x - (\delta_\infty \mid x)e \in c_0$, so $\langle\beta \mid (x - (\delta_\infty \mid x)e)\rangle = 0$. Thus $\beta \mid x = (\beta \mid e)(\delta_\infty \mid x)$. This proves that the restriction of β to c is proportional to δ_∞. Then $\beta \in B$ and $c_0^{\perp} \subset B$.□

References

Alcantud, J.C.R., Diaz, R. (2014) "Conditional Extensions of Fuzzy Preorders", Fuzzy Sets and Systems, doi:10.2016/j.fss.2014.09.009

Bossert, W. (2008) "Suzumura Consistency", In: Pattanaik P.K., Tadenuma K., Xu Y., Yoshihara N. (eds) Rational Choice and Social Welfare. Studies in Choice and Welfare. Springer, Berlin, Heidelberg, 159-179

Demuynck, T. and Lauwers, L. (2009) "Nash rationalizability of collective choice over lotteries", Mathematical Social Sciences, 57:1-15

Fleurbaey, M. and Michel, P. (2003) "Intertemporal equity and the extension of the Ramsey criterion", Journal of Mathematical Economics, 39:777-802

Jaffray, J.Y. (1975) "Semicontinuous extensions of a partial order", Journal of Mathematical Economics 2, 395-406

Lauwers, L. (2009) "Ordering infinite utility streams comes at the cost of a non-Ramsey set", Journal of Mathematical Economics, 46, 1: 32-37

Luenberger, D. (1997) "Optimization by vector spaces", Wiley-Interscience, New Ed edition 1997

Mabrouk, M.B.R., (2011) "Translation-Invariance when utility streams are infinite and unbounded" , International Journal of Economic Theory, 7, 1-13

Mitra, T., Ozbek, M.K. (2010) "On representation and weighted utilitarian representation of preference orders on finite utility streams", Cornell University, CAE working paper #10-05

Mitra, T., Ozbek, M.K. (2013) "On representation of monotone preference orders in a sequence space", Social Choice and Welfare, 41:473–487 doi:10.1007/s00355-012-0693-z

Svensson, L., (1980) "Equity among generations", Econometrica, vol. 48, 5:1251-1256

Suzumura, K. (1976) "Remarks on the theory of collective choice", Economica 43:381-389

Szpilrajn, E. (1930) "Sur l'extension de l'ordre partiel", Fundamenta mathematicae 16:386-38

Weibull, J.W. (1985) "Discounted-value representations of temporal preferences," Mathematics of Operational Research 10, 244-250